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Existence of Solutions for Fractional Klein-Gordon-Maxwell Systems

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Abstract

In this paper, we study the existence of solutions to the fractional Klein-Gordon-Maxwell equations. We use the Lions lemma and the mountain pass theorem to prove the existence of solutions.

Subject Areas

Functional Analysis

Keywords

Klein-Gordon-Maxwell Equations, Fractional Laplacian, Variational Method

1. Introduction

In recent years, by studying the nonlinear problems related to fractional Laplacian, many practical problems have been solved. For example, in the financial market problem, phase transformation problem, anomalous diffusion problem, crystal dislocation problem, semi-permeable film problem, soft film problem, minimal surface problem (see [1] and references for more details). As it involves more and more fields, the research on the problem is more and more in-depth, and people keep putting forward new problems at the same time, also keep producing new ways to solve the problem. In this paper, we study the following fractional Klein-Gordon-Maxwell system on $\mathbb{R}^3$

$$\begin{cases} (-\Delta)^s u + V(x)u - (2w+\phi)\phi u = f(u) + (u^+)^{2_s^*-1} \\ \Delta^s \phi = (w+\phi)u^2, \end{cases} \tag{1.1}$$

where $s \in \left(\frac{3}{4}, 1\right)$ is a fixed constant and $(-\Delta)^s$ is the fractional Laplacian operator, defined as

$$(-\Delta)^s u(x) = C_{3,s} P.V.\int_{\mathbb{R}^3} \frac{u(x)-u(y)}{|x-y|^{3+2s}} \mathrm{d}y, \quad x, y \in \mathbb{R}^3, \tag{1.2}$$

where $C_{3,s}$ is a constant, dependent on s can be expressed as

$$C_{3,s} = \left(\int_{\mathbb{R}^3} \frac{1-\cos(\xi_1)}{|\xi|^{3+2s}} \mathrm{d}\xi \right)^{-1}, \tag{1.3}$$

and *P. V.* stands the principal value. $u \in H^s(\mathbb{R}^3)$, $\phi \in D^{s,2}(\mathbb{R}^3)$, where $H^s(\mathbb{R}^3)$ and $D^{s,2}(\mathbb{R}^3)$ are defined in (1.9) and (1.11), $2_s^* = \frac{2N}{N-2s}$ is the fractional Sobolev critical exponent. Next, let us mention some illuminating work (1.1) related to this problem. In [2], the critical Klein-Gordon-Maxwell system with external potential is not only studied when the potential well is steep,

$$\begin{cases} -\Delta u + \mu V(x)u - (2\omega+\phi)\phi u = \lambda f(u) + (u^+)^5 & \text{in } \mathbb{R}^3, \\ \Delta\phi = (\omega+\phi)u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.4}$$

where μ and λ are positive parameters, $\omega > 0$, where $V(x)$ and $f(u)$ satisfy the following hypotheses:

(V) $V(x) \in C(\mathbb{R}^3, \mathbb{R})$ and there exists $V_0 > 0$ such that the set $\{x \in \mathbb{R}^3 : V(x) \le V_0\}$ is bounded;

(V') the set $\Omega_0 = \{x \in \mathbb{R}^3 : V(x) = 0\}$ is non-empty and has smooth boundary with $\bar{\Omega}_0 = V^{-1}(0)$;

(f_1') $f \in C(\mathbb{R}^+, \mathbb{R})$, $f(u) \ge 0$ and $\lim_{u\to 0+} \frac{f(u)}{u} = \lim_{u\to+\infty} \frac{f(u)}{u^5} = 0$;

(f_2') $\frac{1}{4} f(u)u - F(u) \ge 0$, where $F(u) = \int_0^u f(s)\mathrm{d}s$. Moreover, there exist $\theta_0 \in (4,6)$, $D_0 > 0$ and $\rho_0 > 0$ such that $F(u) \ge \frac{D_0}{\rho_0} u^{\theta_0}$ for $u \ge \rho_0$.

The existence of the solution and the phenomenon of concentration are proved by using the penalized technique and the elliptic estimation. In addition, the existence of the solution is proved when the potential well is not steep, that is to investigate whether the problem has a solution without any restrictions on μ and λ, that is, consider the following problem

$$\begin{cases} -\Delta u + V(x)u - (2\omega+\phi)\phi u = a(x) f(u) + (u^+)^5 & \text{in } \mathbb{R}^3, \\ \Delta\phi = (\omega+\phi)u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{1.5}$$

In [3], when the nonlinearity exhibits critical growth, the existence of a positive ground state solution to the problem is proved by the Nehari method,

$$\begin{cases} (-\Delta)^s u + V(x)u - (2\omega+\phi)\phi u = \lambda |u|^{\alpha-2} u + |u|^{2_s^*-2} u & \text{in } \mathbb{R}^N, \\ (-\Delta)^s \phi + \phi u^2 = -\omega u^2 & \text{in } \mathbb{R}^N, \end{cases} \tag{1.6}$$

where $\lambda > 0$, $\omega > 0$, $N > 2s$ with $s \in (0,1)$, $\phi \in D^s(\mathbb{R}^N, \mathbb{R})$, and $u \in H^s(\mathbb{R}^N, \mathbb{R})$ are functions, where $V(x) \in C(\mathbb{R}^N, \mathbb{R})$ satisfies some of the following hypotheses:

($\hat{V}_1$) V is periodic in $x_i\,(i=1,\cdots,N)$;

($\hat{V}_2$) There exists $V_*>0$, such that $V(x)\geq V_*$,

$2<\alpha<4$ and $\frac{V_*}{\omega^2}>\frac{(\alpha-4)^2}{4(\alpha-2)}$, $4\leq\alpha<2_s^*$ and $V_*>0$ are studied respectively in both cases is the ground state of existence. Benci and Fortunato first studied the following system in [4],

$$\begin{cases}-\Delta u+\left[m^2-(\omega+\phi)^2\right]u=f(x,u) & \text{in } \mathbb{R}^3,\\ \Delta\phi=(\omega+\phi)u^2 & \text{in } \mathbb{R}^3.\end{cases} \tag{1.7}$$

They proved infinitely many radially symmetric solutions using the variational method when $|m|>|\omega|$ and for sub-critical exponents p satisfying $4<p<2^*$. Based on the nonlinear Klein-Gordon field and electrostatic field of the relationship between research, many researchers on the system of the existence, nonexistence and diversity some results are obtained. In [5], the existence of nontrivial solutions is investigated separately for different $f(x,u)$ cases by means of the Ekeland's variational principle and the mountain pass theorem. Carriao, Cunha and Miyagaki in [6] such as periodic potential $V(x)$ to replace the constant $m_0^2-\omega^2$, considered the critical problem of the existence of the ground state solutions accordingly. After this, more attention was paid to the following Klein-Gordon-Maxwell system

$$\begin{cases}-\Delta u+V(x)u-(2\omega+\phi)\phi u=f(x,u) & \text{in } \mathbb{R}^3,\\ \Delta\phi=(\omega+\phi)u^2 & \text{in } \mathbb{R}^3.\end{cases} \tag{1.8}$$

Inspired by the above literature, the existence of a non-trivial solution of system (1.1) will be discussed in this paper. To illustrate our results, we set the potential functions $V(x)$ and $f(u)$ satisfy the following assumptions:

(V_1) $V(x)\in C(\mathbb{R}^3,\mathbb{R})$, $V(x)\geq 0$ for all $x\in\mathbb{R}^3$ and $\lim_{|x|\to+\infty}V(x)=V_\infty>0$;

(V_2) there exist $C_v>0, R_1>0$ and $h_0>0$ such that $V(x)\leq V_\infty+C_v e^{-h_0|x|}$ for $|x|\geq R_1$;

(f_1) $f\in C(\mathbb{R}^+,\mathbb{R})$ and $\lim_{u\to0^+}\frac{f(u)}{u}=\lim_{|u|\to+\infty}\frac{f(u)}{u^{2_s^*-1}}=0$;

(f_2) there exist $c_0>0$ and $p_0\in(4,2_s^*)$ such that $f(u)\geq c_0u^{p_0-1}$ for $u\geq0$;

(f_3) the function $\frac{f(u)}{u^3}$ is increasing for $u>0$.

Notations:

In this paper, the norm of fractional Sobolev space $H^s(\mathbb{R}^3)$ is defined

$$H^s(\mathbb{R}^3):=\left\{u\in L^2(\mathbb{R}^3):\frac{|u(x)-u(y)|}{|x-y|^{\frac{3}{2}+s}}\in L^2(\mathbb{R}^3\times\mathbb{R}^3)\right\}, \tag{1.9}$$

and define $X=\left\{u\in H^s(\mathbb{R}^3):\int_{\mathbb{R}^3}V(x)u^2<\infty\right\}$, endowed the norm on X by

$$\|u\|_H^2 = \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}} \mathrm{d}x\mathrm{d}y + \frac{C_{3,s}}{2}\int_{\mathbb{R}^3} V(x)|u(x)|^2 \mathrm{d}x, \tag{1.10}$$

and the corresponding inner product is

$$(u,v)_H = \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{3+2s}} \mathrm{d}x\mathrm{d}y + \frac{C_{3,s}}{2}\int_{\mathbb{R}^3} V(x)u(x)v(x)\mathrm{d}x.$$

Therefore, $\|u\|^2 := \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}} \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^3} V_\infty u^2(x)\mathrm{d}x$ is equivalent to the usual norm on $H^s(\mathbb{R}^3)$, where V_∞ is referred to in (V_1). Consider the following fractional critical Sobolev space $D^{s,2}(\mathbb{R}^3)$ is defined by

$$D^{s,2}(\mathbb{R}^3) := \left\{ u \in L^2(\mathbb{R}^3) : \frac{|u(x)-u(y)|}{||x-y|^{\frac{3}{2}+s}} \in L^2(\mathbb{R}^3\times\mathbb{R}^3) \right\}, \tag{1.11}$$

with the norm

$$\|u\|_{D^{s,2}}^2 := \frac{C_{3,s}}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}} \mathrm{d}x\mathrm{d}y, \tag{1.12}$$

where $D^{s,2}(\mathbb{R}^3)$ is the completeness of $C_0^\infty(\mathbb{R}^3)$. For $1 \le p < \infty$, we let

$$|u|_p = \left(\int_{\mathbb{R}^N} |u(x)|^p \mathrm{d}x \right)^{\frac{1}{p}}, \quad u \in L^p(\mathbb{R}^3), \tag{1.13}$$

and $\|u\|_{L^p(B_r(y))} = \left(\int_{B_r(y)} |u|^p \mathrm{d}x \right)^{\frac{1}{p}}$ for $p \ge 1$, where $B_r(y) = \{x \in \mathbb{R}^3 : |x-y| < r\}$. For any $s \in \left(\frac{3}{4}, 1\right)$, the embedded $D^{s,2}(\mathbb{R}^3) \hookrightarrow L^{2_s^*}(\mathbb{R}^3)$ is continuous, exist for the best fractional critical Sobolev constant

$$S := \inf_{u \in H^s(\mathbb{R}^3)\setminus\{0\}} S(u), \tag{1.14}$$

for any $u \in H^s(\mathbb{R}^3)\setminus\{0\}$,

$$S(u) := \frac{\int_{\mathbb{R}^6} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}} \mathrm{d}x\mathrm{d}y}{\left(\int_{\mathbb{R}^3} |u(x)|^{2_s^*} \mathrm{d}x \right)^{2/2_s^*}} \tag{1.15}$$

For this paper, taking C uniformly represents all normal numbers. The main research results can be summarized as follows:

Theorem 1.1. *If* (V_1)-(V_2), *and* (f_1)-(f_3) *hold with* $0 < h_0 < 2\sqrt{V_\infty}$, *then there exists* $\omega_0 > 0$ *such that for* $\omega \in (0, \omega_0)$, *problem* (1) *admits a nontrivial solution* $(u,\phi) \in H^s(\mathbb{R}^3) \times D^{s,2}(\mathbb{R}^3)$.

2. Preliminary Lemmas

By (V_1), there exist V_M such that $|V(x)| \le V_M$ for $x \in \mathbb{R}^3$. Moreover, there exists $\tilde{R}_0 > 0$ such that $V(x) \ge \frac{V_\infty}{2}$ for $|x| \ge \tilde{R}_0$. Then

$$
\begin{aligned}
&\iint_{\mathbb{R}^6} \frac{(u(x)-u(y))^2}{|x-y|^{3+2s}} \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^3} V_\infty u^2(x)\mathrm{d}x \\
&\le \iint_{\mathbb{R}^6} \frac{(u(x)-u(y))^2}{|x-y|^{3+2s}} \mathrm{d}x\mathrm{d}y + \int_{|x|\le \tilde{R}_0} V_\infty u^2(x)\mathrm{d}x + 2\int_{|x|\ge \tilde{R}_0} V(x)u^2(x)\mathrm{d}x \\
&\le \iint_{\mathbb{R}^6} \frac{(u(x)-u(y))^2}{|x-y|^{3+2s}} \mathrm{d}x\mathrm{d}y + 2\int_{|x|\ge \tilde{R}_0} V(x)u^2(x)\mathrm{d}x \\
&\quad + V_\infty \left(\int_{|x|\le \tilde{\mathbb{R}}_0} |u|^{2_s^*} \mathrm{d}x \right)^{\frac{2}{2_s^*}} \left| \int_{|x|\le \tilde{\mathbb{R}}_0} 1\mathrm{d}x \right|^{\frac{2_s^*-2}{2_s^*}} \\
&\le \max\left\{ 1 + \frac{V_\infty \left| \int_{|x|\le \tilde{\mathbb{R}}_0} 1\mathrm{d}x \right|^{\frac{2_s^*-2}{2_s^*}}}{S}, 2 \right\} \|u\|_H^2 .
\end{aligned}
\tag{2.1}
$$

By the embedding $X \hookrightarrow H^s(\mathbb{R}^3)$ is continuous, so in the same way, we get

$$
\int_{\mathbb{R}^3} u^2(x)\mathrm{d}x \le \max\left\{ \frac{\left| \int_{|x|\le \tilde{\mathbb{R}}_0} 1\mathrm{d}x \right|^{\frac{2_s^*-2}{2_s^*}}}{S}, \frac{2}{V_\infty} \right\} \|u\|_H^2 . \tag{2.2}
$$

The purpose of this paper is to find the solution of (1.1). To this end, we give the weak formula of (1.1) by the following questions:

$$
\begin{aligned}
&\int_{\mathbb{R}^6} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{3+2s}} \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^3} V(x)u(x)\varphi(x)\mathrm{d}x \\
&= \int_{\mathbb{R}^3} |u(x)|^{2_s^*-2} u(x)\varphi(x)\mathrm{d}x + \int_{\mathbb{R}^3} f(u(x))\varphi(x)\mathrm{d}x, \\
&\qquad \forall \varphi \in H^s(\mathbb{R}^3), \quad u \in H^s(\mathbb{R}^3).
\end{aligned}
\tag{2.3}
$$

The relevant functional can be defined by (1.1):

$$
\begin{aligned}
\mathcal{J}(u,\phi_u) &= \frac{1}{2}\|u\|_H^2 - \frac{1}{2}\|\phi_u\|_{D^{s,2}}^2 - \frac{1}{2}\int_{\mathbb{R}^3} (2\omega + \phi_u)\phi_u u^2 \mathrm{d}x \\
&\quad - \int_{\mathbb{R}^3} F(u)\mathrm{d}x - \frac{1}{2_s^*}\int_{\mathbb{R}^3} |u^+|^{2_s^*} \mathrm{d}x
\end{aligned}
\tag{2.4}
$$

We take the derivative of that and we get

$$
\mathcal{J}_\phi'(u,\phi_u) = -\|\phi_u\|_{D^{s,2}}^2 - \int_{\mathbb{R}^3} \omega\phi_u u^2 \mathrm{d}x - \int_{\mathbb{R}^3} \phi_u^2 u^2 \mathrm{d}x,
$$

for any $(u,\phi) \in H^s(\mathbb{R}^3) \times D^{s,2}(\mathbb{R}^3)$, we have

$$
\mathcal{J}'(u,\phi_u) = \mathcal{J}_u'(u,\phi_u) + \mathcal{J}_\phi'(u,\phi_u)\phi_u' = \mathcal{J}_u'(u,\phi_u), \tag{2.5}
$$

Next up, we define $\mathcal{G}(u) := \mathcal{J}(u,\phi_u)$, where $u, v \in H^s(\mathbb{R}^3)$, the function $\mathcal{G} : H^s(\mathbb{R}^3) \to \mathbb{R}$ defined as

$$\mathcal{G}(u)=\frac{1}{2}\int_{\mathbb{R}^6}\frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y+\frac{1}{2}\int_{\mathbb{R}^3}V(x)|u(x)|^2\,\mathrm{d}x-\frac{1}{2}\int_{\mathbb{R}^3}\omega\phi_u u^2(x)\mathrm{d}x$$
$$-\frac{1}{2_s^*}\int_{\mathbb{R}^3}|u^+(x)|^{2_s^*}\,\mathrm{d}x-\int_{\mathbb{R}^3}F(u(x))\mathrm{d}x, \tag{2.6}$$

and

$$\mathcal{G}_\infty(u)=\frac{1}{2}\int_{\mathbb{R}^6}\frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y+\frac{1}{2}\int_{\mathbb{R}^3}V_\infty u^2(x)\mathrm{d}x-\frac{1}{2}\int_{\mathbb{R}^3}\omega\phi_u u^2(x)\mathrm{d}x$$
$$-\frac{1}{2_s^*}\int_{\mathbb{R}^3}|u^+(x)|^{2_s^*}\,\mathrm{d}x-\int_{\mathbb{R}^3}F(u(x))\mathrm{d}x. \tag{2.7}$$

Critical points of $\mathcal{G}(u)$ are weak solutions of (1.1). We will prove the existence of the critical points of the functional $\mathcal{G}(u)$.

Lemma 2.1. ([7]) *If* (f_1) *and* (f_3) *is true, then*

1) $\frac{1}{4}f(u)u-F(u)\geq 0$, where $F(u)=\int_0^u f(s)\mathrm{d}s$;

2) $\frac{1}{4}f(u)u-F(u)$ is increasing for $u>0$.

Lemma 2.2. *Let assume* $f:\Omega\times\mathbb{R}\to\mathbb{R}$ *be a Carathéodory function verifying conditions* (f_1), *we get that for any* $\varepsilon>0$, *there exists* $C_\varepsilon>0$ *such that*

$$|f(u)|\leq\varepsilon|u|+C_\varepsilon|u|^{2_s^*-1},\quad u\geq 0. \tag{2.8}$$

$$|F(u)|\leq\varepsilon|u|^2+C_\varepsilon|u|^{2_s^*},\quad u\geq 0. \tag{2.9}$$

Let $s_0\in(2,2_s^*)$, *we also derive that for any* $\varepsilon>0$, *there exists* $C_\varepsilon>0$ *such that*

$$\max\{|F(u)|,|f(u)u|\}\leq\varepsilon|u|^2+\varepsilon|u|^{2_s^*}+C_\varepsilon|u|^{s_0},\quad u\geq 0. \tag{2.10}$$

Lemma 2.3. ([8]) *For any* $u\in H^s(\mathbb{R}^3)$, *there exists a unique* $u=\phi_u\in D^{s,2}(\mathbb{R}^3)$ *satisfying*

$$\Delta^s\phi=(\phi_u+\omega)u^2.$$

And the map $\Phi:u\in H^s(\mathbb{R}^3)\to\Phi[u]=\phi_u\in D^{s,2}(\mathbb{R}^3)$ is continuously differentiable and for any $u\in H^s(\mathbb{R}^3)$,

1) $-\omega\leq\phi_u\leq 0$ on $\{x\in\mathbb{R}^3:u(x)\neq 0\}$;

2) $\|\phi_u\|_{D^{s,2}}\leq C_1\|u\|_{H^s}^2$ and $\int_{\mathbb{R}^3}|\phi_u|u^2\mathrm{d}x\leq C_2\|u\|_{\frac{12}{3+2s}}^4\leq C_3\|u\|_{H^s}^4$,

where C_1,C_2 and C_3 are positive constants.

Lemma 2.4. ([9]) *Let* $s\in(0,1)$ *and* $n>2s$. *Then, the following estimates hold true:*

$$\int_{\mathbb{R}^n}|u_\varepsilon(x)|^2\,\mathrm{d}x\geq\begin{cases}C_s\varepsilon^{2s}+\mathcal{O}(\varepsilon^{n-2s}) & \text{if } n>4s,\\ C_s\varepsilon^{2s}|\log\varepsilon|+\mathcal{O}(\varepsilon^{2s}) & \text{if } n=4s,\\ C_s\varepsilon^{n-2s}+\mathcal{O}(\varepsilon^{2s}) & \text{if } n<4s,\end{cases} \tag{2.11}$$

$$\int_{\mathbb{R}^N} \left|u_\varepsilon(x)\right|^{2_s^*} \mathrm{d}x = S^{n/(2s)} + \mathcal{O}\left(\varepsilon^n\right), \tag{2.12}$$

and

$$\int_{\mathbb{R}^{2n}} \frac{\left|u_\varepsilon(x) - u_\varepsilon(y)\right|^2}{\left|x-y\right|^{n+2s}} \mathrm{d}x\mathrm{d}y \le S^{n/(2s)} + \mathcal{O}\left(\varepsilon^{n-2s}\right), \tag{2.13}$$

as $\varepsilon \to 0$, *for some positive constant* C_s *depending on s.*

Lemma 2.5. ([10]) *If* $u_n \rightharpoonup u$ *in X, then, up to subsequences,* $\phi_{u_n} \rightharpoonup \phi_u$ *in* $D^s\left(\mathbb{R}^3\right)$ *as* $n \to \infty$.

3. The Proof of Theorem 1.1

Lemma 3.1. *Let* $\tau_\infty = \inf\left\{\mathcal{G}_\infty(u) : u \in H^s \setminus \{0\}, \mathcal{G}'_\infty(u) = 0\right\}$. *Define*

$$c_\infty = \inf_{P \in \mathcal{P}} \max_{0 \le t \le 1} \mathcal{G}_\infty\left(P(t)\right),$$

where $\mathcal{P} = \left\{P \in C\left([0,1], H^s\left(\mathbb{R}^3\right)\right) : P(0) = 0, \mathcal{G}_\infty\left(P(1)\right) < 0\right\}$. *The minimum* τ_∞ *is given by a non-negative function* $\tilde{\tau}_\infty$. *Moreover,* $\tau_\infty = \mathcal{G}_\infty\left(\tilde{\tau}_\infty\right) < \frac{s}{3} S^{\frac{3}{2s}}$.

Proof. By (f_1), we get that for $\varepsilon \in \left(0, \frac{1}{4} V_\infty\right)$, there exists $C_\varepsilon > 0$, such that $\left|F(u) + \frac{1}{2_s^*}\left|u^+\right|^{2_s^*}\right| \le \varepsilon |u|^2 + C_\varepsilon |u|^{2_s^*}$, for $u \ge 0$.

$$\begin{aligned}
\mathcal{G}_\infty(u) &\ge \frac{1}{2}\|u\|^2 - \left[\int_{\mathbb{R}^3} F(u) + \frac{1}{2_s^*}\left|u^+\right|^{2_s^*} \mathrm{d}x\right] \\
&\ge \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^3}\left[\varepsilon |u|^2 + C_\varepsilon |u|^{2_s^*}\right]\mathrm{d}x \\
&\ge \frac{1}{4}\|u\|^2 - C\|u\|^{2_s^*}.
\end{aligned}$$

So exist $r_\infty > 0$, such that $\mathcal{G}_\infty(u) \ge \alpha_\infty > 0$, for $\|u\| = r_\infty$. Choose $\varphi_\infty \in H^s\left(\mathbb{R}^3\right) \setminus \{0\}$ and $\varphi_\infty \ge 0$. By (f_2) and Lemma 2.3, we get $F\left(t\varphi_\infty\right) \ge 0$ and $\mathcal{G}_\infty\left(t\varphi_\infty\right) \le \frac{t^2}{2}\left\|\varphi_\infty\right\|^2 + Ct^4\left\|\varphi_\infty\right\|^4_{\frac{12}{3+2s}} - \frac{t^{2_s^*}}{2_s^*}\int_{\mathbb{R}^3}\left|\varphi_\infty\right|^{2_s^*} \mathrm{d}x$. So we derive that $\lim_{t \to +\infty} \mathcal{G}_\infty\left(t\varphi_\infty\right) = -\infty$ and $\mathcal{G}_\infty(0) = 0$. By the mountain pass theorem in [11], there is $\{v_n\} \subset H^s\left(\mathbb{R}^3\right)$ satisfying $\mathcal{G}_\infty(v_n) \to c_\infty > 0$ and $\mathcal{G}'_\infty(v_n) \to 0$.

Step 1: $\{v_n\}$ is bounded in *X*. Then by lemma 2.1 and $s \in \left(\frac{3}{4}, 1\right)$, we get

$$\begin{aligned}
c_\infty + o_n(1)\|v_n\| &= \mathcal{G}_\infty(v_n) - \frac{1}{4}\left\langle \mathcal{G}'_\infty(v_n), v_n\right\rangle \\
&\ge \frac{1}{4}\|v_n\|^2 + \frac{1}{4}\int_{\mathbb{R}^3}\phi_{v_n}^2 v_n^2 \ \mathrm{d}x \\
&\ge \frac{1}{4}\|v_n\|^2.
\end{aligned}$$

So $\|v_n\|$ is bounded.

Step 2: $0 < c_\infty < \frac{s}{3} S^{\frac{3}{2s}}$. By lemma 2.3, there exists $\varepsilon \in (0, \varepsilon')$ and $0 < t' < 1 < t''$ such that

$$\begin{aligned}\sup_{0\le t\le t'} \mathcal{G}_\infty(tu_\varepsilon) &\le \sup_{0\le t\le t'} \frac{1}{2}t^2\left(\|u_\varepsilon\|^2 + C_\omega \|u_\varepsilon\|_H^4\right) \\ &\le \frac{(t')^2}{2}\left[S^{\frac{3}{2s}} + \mathcal{O}\left(\varepsilon^{3-2s}\right) + CS^{\frac{6}{2s}} + \mathcal{O}\left(\varepsilon^{6-4s}\right)\right] \\ &< \frac{s}{3} S^{\frac{3}{2s}}\end{aligned}$$

and

$$\begin{aligned}\sup_{t\ge t''} \mathcal{G}_\infty(tu_\varepsilon) &\le \sup_{t\ge t''}\left[\frac{t^2}{2}\|u_\varepsilon\|^2 + \frac{Ct^4}{2}\|u_\varepsilon\|_H^4 - \frac{t^{2_s^*}}{2_s^*}\int_{\mathbb{R}^3}|u_\varepsilon|^{2_s^*}\,\mathrm{d}x\right] \\ &\le \sup_{t\ge t''}\left[\frac{t^2}{2}S^{\frac{3}{2s}} + \mathcal{O}\left(\varepsilon^{3-2s}\right) + Ct^4 S^{\frac{6}{2s}} + \mathcal{O}\left(\varepsilon^{6-4s}\right) - \frac{t^{2_s^*}}{2_s^*}S^{\frac{3}{2s}} + \mathcal{O}\left(\varepsilon^3\right)\right] \\ &< \frac{s}{3}S^{\frac{3}{2s}}.\end{aligned}$$

By Lemma 2.3 and (f_2),

$$\begin{aligned}\sup_{t\in[t',t'']} \mathcal{G}^\infty(tu_\varepsilon) \le \sup_{t\ge 0}&\left[\frac{t^2}{2}\|u_\varepsilon\|^2 + Ct^4\|u_\varepsilon\|^4_{\frac{12}{3+2s}} - \frac{t^{2_s^*}}{2_s^*}\int_{\mathbb{R}^3}|u_\varepsilon|^{2_s^*}\,\mathrm{d}x\right] \\ &- C(t')^{p_0}\int_{\mathbb{R}^3}|u_\varepsilon|^{p_0}\,\mathrm{d}x.\end{aligned}$$

And by Lemma 2.4, there exists $\varepsilon'' \in (0, \varepsilon')$ such that for $\varepsilon \in (0, \varepsilon'')$,

$$\begin{aligned}&\sup_{t\in[t',t'']} \mathcal{G}_\infty(tu_\varepsilon) \\ &\le \frac{s}{3}\left[\frac{\int_{\mathbb{R}^6}\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^3}V_\infty|u_\varepsilon|^2\,\mathrm{d}x}{\left(\int_{\mathbb{R}^3}|u_\varepsilon|^{2_s^*}\,\mathrm{d}x\right)^{\frac{2}{2_s^*}}}\right]^{\frac{3}{2s}} + O(\varepsilon) - C\varepsilon^{\frac{6-p_0}{4}} \\ &\le \frac{s}{3}S^{\frac{3}{2s}} + O\left(\varepsilon^{\frac{1}{2}}\right) - C\varepsilon^{\frac{6-p_0}{4}} < \frac{s}{3}S^{\frac{3}{2s}}\end{aligned}$$

in consideration of $\frac{6-p_0}{4} < \frac{1}{2}$. Then by the definition of c_∞, we have $0 < c_\infty \le \sup_{t\ge 0}\mathcal{G}_\infty(tu_\varepsilon) < \frac{s}{3}S^{\frac{3}{2s}}$. So we have $0 < c_\infty < \frac{s}{3}S^{\frac{3}{2s}}$.

Step 3: $\tau_\infty \le \mathcal{G}_\infty(w_0) < \frac{s}{3}S^{\frac{3}{2s}}$. Let $v_n^- = \min\{v_n, 0\}$, $u_n = v_n^+$. And $\left(\mathcal{G}'(v_n), v_n^-\right) = o_n(1)$, we get $\|v_n^-\| = o_n(1)$, $\|u_n\|$ is bounded, $\mathcal{G}_\infty(u_n) \to c_\infty$ and $\mathcal{G}'_\infty(u_n) \to 0$, so

$$\left(\mathcal{G}'_\infty(u_n), u_n\right) = o_n(1) = \|u_n\|^2 - \int_{\mathbb{R}^3}\left(2\omega + \phi_{u_n}\right)\phi_{u_n}u_n^2\mathrm{d}x - \int_{\mathbb{R}^3}|u_n|^{2_s^*}\,\mathrm{d}x \tag{3.1}$$

We assume $u_n \rightharpoonup u_0$ weakly in $H^s(\mathbb{R}^3)$. If $\limsup_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_2(y)} |u_n|^2 \mathrm{d}x = 0$, by the Lions Lemma, we have $u_n \to 0$ in $L^p(\mathbb{R}^3)$ for any $p\in(2,2_s^*)$. So $\int_{\mathbb{R}^3} F(u_n)\mathrm{d}x = \int_{\mathbb{R}^3} f(u_n)u_n \mathrm{d}x = o_n(1)$.

$$
\begin{aligned}
c_\infty + o_n(1) &= \mathcal{G}_\infty(u_n) \\
&= \frac{1}{2}\|u_n\|^2 - \frac{1}{2}\int_{\mathbb{R}^3} \omega\phi_{u_n} u_n^2 \mathrm{d}x - \frac{1}{2_s^*}\int_{\mathbb{R}^3} |u_n|^{2_s^*}\mathrm{d}x \\
&\geq \frac{1}{2}\|u_n\|^2 - \frac{1}{2_s^*}\int_{\mathbb{R}^3} |u_n|^{2_s^*}\mathrm{d}x + o_n(1)
\end{aligned}
$$

Because of $c_\infty > 0$, we assume $\lim_{n\to\infty}\|u_n\|^2 = l$, where $l\in(0,+\infty)$. By (3.1) and Lemma 2.3, we have $\lim_{n\to\infty}\int_{\mathbb{R}^3}|u_n|^{2_s^*}\mathrm{d}x \geq l$. Then by the Sobolev embedding theorem

$$
|u|_q \leq C\|u\|, \quad 2\leq q\leq 2_s^*
$$

and when we take the limit of both sides, we have

$$
c_\infty \geq \left(\frac{1}{2} - \frac{1}{2_s^*}\right) l + o_n(1) = \frac{s}{3} l + o_n(1) \geq S^{\frac{3}{2s}} + o_n(1), \tag{3.2}
$$

this is in contradiction with $c_\infty < \frac{s}{3}S^{\frac{3}{2s}}$. Thus, we assume that there exists $\delta_0 > 0$, such that $\limsup_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_2(y)}|u_n|^2\mathrm{d}x \geq \delta_0 > 0$. Therefore we deduce that there exists $\{z_n\}\subset\mathbb{R}^3$ satisfying $w_n = u_n(.+z_n) \rightharpoonup w_0 \neq 0$ weakly in *X*, thus, $w_0 \geq 0$, $\mathcal{G}_\infty(w_n) \to c_\infty$ and $\mathcal{G}'_\infty(w_n)\to 0$. Go through again with Lemma 2.5, we get $\mathcal{G}'_\infty(w_0) = 0$ and

$$
\begin{aligned}
c_\infty + o_n(1) &= \mathcal{G}_\infty(w_n) - \frac{1}{4}\left(\mathcal{G}'_\infty(w_n), w_n\right) \\
&= \frac{1}{4}\|w_n\|^2 + \frac{1}{4}\int_{\mathbb{R}^3}\phi_{w_n}^2 w_n^2\mathrm{d}x + \int_{\mathbb{R}^3}\left(\frac{1}{4}f(w_n)w_n - F(w_n)\right)\mathrm{d}x \\
&\quad + \frac{4s-3}{12}\int_{\mathbb{R}^3}|w_n|^{2_s^*}\mathrm{d}x.
\end{aligned}
$$

By Fatou's lemma

$$
\int_{\mathbb{R}^3} \liminf_{n\to\infty} f_n(x)\mathrm{d}x \leq \liminf_{n\to\infty}\int_{\mathbb{R}^3} f_n(x)\mathrm{d}x,
$$

and $\mathcal{G}'_\infty(w_0) = 0$, we get

$$
\frac{s}{3}S^{\frac{3}{2s}} > c_\infty \geq \mathcal{G}_\infty(w_0) - \frac{1}{4}\left(\mathcal{G}'_\infty(w_0), w_0\right) = \mathcal{G}_\infty(w_0).
$$

By the definition of τ_∞, we get $m_\infty \leq \mathcal{G}_\infty(w_0) < \frac{s}{3}S^{\frac{3}{2s}}$.

Step 4: τ_∞ is attained by w_∞. There exists $\{\tilde{w}_n\}\subset H^s(\mathbb{R}^3)$ such that $\mathcal{G}_\infty(\tilde{w}_n)\to\tau_\infty$ and $\mathcal{G}'_\infty(\tilde{w}_n) = 0$. We put $\tilde{w}_n$ in there, so we get $\left(\mathcal{G}'_\infty(\tilde{w}_n), \tilde{w}_n\right) = 0$, with Lemma 2.2 we get

$$
\begin{aligned}
\|\tilde{w}_n\|^2 &= \int_{\mathbb{R}^3}\left(2\omega + \phi_{\tilde{w}_n}\right)\phi_{\tilde{w}_n}\tilde{w}_n^2\mathrm{d}x + \int_{\mathbb{R}^3} f(\tilde{w}_n)\tilde{w}_n + \int_{\mathbb{R}^3}|\tilde{w}_n|^{2_s^*}\mathrm{d}x \\
&\leq \int_{\mathbb{R}^3}\varepsilon|\tilde{w}_n|^2 + (C_\varepsilon + 1)|\tilde{w}_n|^{2_s^*}\mathrm{d}x.
\end{aligned}
$$

In addition, by the Sobolev embedding theorem, we have $\|\tilde{w}_n\|^2 \geq CS^{\frac{3}{2s}}$. Then

$$\begin{aligned}\tau_\infty + o_n(1)\|\tilde{w}_n\| &= \mathcal{G}_\infty(\tilde{w}_n) - \frac{1}{4}(\mathcal{G}'_\infty(\tilde{w}_n), \tilde{w}_n) \\ &= \frac{1}{4}\|\tilde{w}_n\|^2 + \frac{1}{4}\int_{\mathbb{R}^3}\phi^2_{\tilde{w}_n}\tilde{w}_n^2 \mathrm{d}x + \int_{\mathbb{R}^3}\left(\frac{1}{4}f(\tilde{w}_n)\tilde{w}_n - F(\tilde{w}_n)\right)\mathrm{d}x \\ &\quad + \frac{4s-3}{12}\int_{\mathbb{R}^3}|\tilde{w}_n|^{2_s^*}\,\mathrm{d}x \\ &\geq \frac{1}{4}\|\tilde{w}_n\|^2 \geq CS^{\frac{3}{2s}}.\end{aligned} \tag{3.3}$$

So $\|\tilde{w}_n\|$ is bounded and $0 < \tau_\infty < \frac{s}{3}S^{\frac{3}{2s}}$.

Since $\mathcal{G}_\infty(\tilde{w}_n) \to \tau_\infty$ and $\mathcal{G}'_\infty(\tilde{w}_n) = 0$, we can assume that $\tilde{w}_n \geq 0$. Similarly, we deduce that there exists $y_n \in \mathbb{R}^3$ such that $\tilde{w}_n(.+y_n) \rightharpoonup w_\infty \neq 0$ weakly in $H^s(\mathbb{R}^3)$, where w_∞ is non-negative. By Lemma 2.5, we have $\mathcal{G}'_\infty(w_\infty) = 0$. So by (31) and Fatou's lemma,

$$\begin{aligned}m_\infty &\geq \frac{1}{4}\|w_\infty\|^2 + \frac{1}{4}\int_{\mathbb{R}^3}\phi^2_{w_\infty} w_\infty^2 \mathrm{d}x + \int_{\mathbb{R}^3}\left(\frac{1}{4}f(w_\infty)w_\infty - F(w_\infty)\right)\mathrm{d}x \\ &\quad + \frac{4s-3}{12}\int_{\mathbb{R}^3}|w_\infty|^{2_s^*}\,\mathrm{d}x. \\ &= \mathcal{G}_\infty(w_\infty) - \frac{1}{4}(\mathcal{G}'_\infty(w_\infty), w_\infty) = \mathcal{G}_\infty(w_\infty)\end{aligned}$$

However, by definition of τ_∞, we get a contradiction with $\tau_\infty \leq \mathcal{G}_\infty(w_\infty)$. □

Lemma 3.2. ([2]) *For any* $\delta \in (0,1)$, *there exists* $C_\delta > 0$ *such that*

$$w_\infty(x) \leq C_\delta \mathrm{e}^{-(1-\delta)\sqrt{V_\infty}|x|}.$$

Recall that a sequence $\{u_n\} \subset Y$ is a Ceramisequence sequence for the functional $\mathcal{G}$ if $\mathcal{G}(u_n) \to c$ and $(1+\|u_n\|_Y)\|\mathcal{G}'(u_n)\| \to 0$ as $n \to \infty$. We need the following variation of the mountain pass lemma in [12].

Theorem 3.1. *Let X be a real Banach space and assume* $K \in C^1(X, \mathbb{R})$ *satisfies*

$$\max\{K(0), K(u_1)\} \leq \alpha_2 < \alpha_1 \leq \inf_{\|u\|_X = \rho} K(u)$$

for some $\rho > 0$ and $u_1 \in X$ with $\|u_1\|_X > \rho$. Let

$$c := \inf_{\gamma\in\Gamma}\max_{0\leq t\leq 1} K(\gamma(t)), \tag{3.4}$$

where $\Gamma := \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = u_1\}$. Then there exists a Ceramisequence sequence $\{u_n\}$ for the functional K satisfying $c \geq \alpha_1$.

Proof of Theorem 1.1. Let $\varepsilon < \dfrac{1}{4\max\left\{\dfrac{\left|\int_{|x|\leq\bar{\mathbb{R}}_0} 1\mathrm{d}x\right|^{\frac{2_s^*-2}{2_s^*}}}{S}, \dfrac{2}{V_\infty}\right\}}$. Since $|V(x)| \leq V_M$, Lemma 2.2 and (17), we get there exists $C_\varepsilon > 0$ such that

$$\begin{aligned}\mathcal{G}(u) &\geq \frac{1}{2}\|u\|_H^2 - \frac{1}{2}\int_{\mathbb{R}^3}\omega\phi_u u^2 \mathrm{d}x - \int_{\mathbb{R}^3}\left[\varepsilon u^2 + C_\varepsilon |u|^{2_s^*}\right]\mathrm{d}x - \frac{1}{2_s^*}\int_{\mathbb{R}^3}|u|^{2_s^*}\mathrm{d}x \\ &\geq \frac{1}{4}\|u\|_H^2 + C - \left(C_\varepsilon + \frac{1}{2_s^*}\right)\frac{\|u\|_H^{2_s^*}}{S^{\frac{3}{3-2s}}}.\end{aligned} \tag{3.5}$$

Therefore, there exists $r_V > 0$ such that $\mathcal{G}(u) \geq \alpha_V > 0$ for $\|u\|_H = r_V$. Choose $\varphi_V \in X \setminus \{0\}$ such that $\varphi_V \geq 0$. By (2.10), there exists $L_1 > 0$ is a constant such that $|F(t\varphi_V)| \leq \frac{4s-3}{12}\left(t^2|\varphi_V|^2 + t^{2_s^*}|\varphi_V|^{2_s^*}\right) + L_1 t^{s_0}|\varphi_V|^{s_0}$ for $x \in \mathbb{R}^3$. Because of $\frac{1}{2}\int_{\mathbb{R}^3}|\omega\phi_{t\varphi_V}||t\varphi_V|^2 \ \mathrm{d}x \leq \frac{\omega C t^4}{2}\|\varphi_V\|_{\frac{12}{3+2s}}^4$, we get that $\lim_{t\to+\infty}\mathcal{G}(t\varphi_V) = -\infty$. Therefore, there exists $t_V > 0$ such that $\mathcal{G}(t_V\varphi_V) \leq 0$. At the same time, and we get $\mathcal{G}(0) = 0$. Let $u_V = t_V\varphi_V$. By Theorem 3.1, there exists a sequence $\{u_n\} \subset X$ is a Ceramisequence sequence for the functional $\mathcal{G}$, such that $\mathcal{G}(u_n) \to c_V > 0$ and $\left(1 + \|u_n\|_H\right)\|\mathcal{G}'(u_n)\| \to 0$ as $n \to \infty$, where

$$c_V = \inf_{P\in\mathcal{P}_V}\max_{0\leq t\leq 1}\mathcal{G}(P(t))$$

with $\mathcal{P}_V = \left\{P \in C([0,1], X) : P(0) = 0, P(1) = u_V\right\}$.

Step 1: $\|u_n\|_H$ is bounded. Next we show that $\|u_n\|_H$ is bounded. Since $\|u_n\|_H \to \infty$. Now, for any $n \in \mathbb{N}$, let $v_n = \frac{u_n}{\|u_n\|_H}$. Then we get $v_n \rightharpoonup v$ weakly in X and $v_n(x) \to v(x) \neq 0$ a.e. $x \in \mathbb{R}^3$. Since $v(x) = 0$ a.e. $x \in \mathbb{R}^3$. By (2.10), let $\varepsilon = \frac{4s-3}{24} > 0$, there exists $L_2 > 0$ such that for $x \in \mathbb{R}^3$ and $u \geq 0$, there holds

$$\left|\frac{1}{4}f(u)u - F(u)\right| \leq \frac{4s-3}{24}\left(|u|^2 + |u|^{2_s^*}\right) + L_2|u|^{s_0}. \tag{3.6}$$

Thanks to the Young's inequality, we have

$$|u|^{s_0} = |u|^{\frac{2(2_s^*-s_0)}{2_s^*-2}}|u|^{\frac{2_s^*(s_0-2)}{2_s^*-2}} \leq \frac{2_s^*-s_0}{2_s^*-2}\frac{1}{\varepsilon^{\frac{2_s^*-2}{2_s^*-s_0}}}|u|^2 + \frac{s_0-2}{2_s^*-2}\varepsilon^{\frac{2_s^*-2}{s_0-2}}|u|^{2_s^*}, \tag{3.7}$$

where $s_0 \in (2, 2_s^*)$. On the basis of choosing $\varepsilon > 0$ small in (3.7), we get that there exists $L_3 > 0$ such that

$$\frac{1}{4}f(u)u - F(u) + \frac{4s-3}{12}|u|^{2_s^*} \geq -L_3|u|^2, \quad \text{for } |x| \leq \tilde{R}_0 \text{ and } u \geq 0. \tag{3.8}$$

By Lemma 2.1, we have that $\frac{1}{4}f(u)u - F(u) \geq 0$ for $|x| \geq \tilde{R}_0$ and $u \geq 0$. Next, by (3.8) and Lemma 2.3, we get

$$\begin{aligned}\frac{\mathcal{G}(u_n)}{\|u_n\|_H^2} &= \frac{1}{\|u_n\|_H^2}\left(\mathcal{G}(u_n) - \frac{1}{4}(\mathcal{G}'(u_n), u_n)\right) \\ &\geq \frac{1}{4} - L_3\int_{|x|\leq\tilde{R}_0}\frac{|u_n|^2}{\|u_n\|_H^2}\mathrm{d}x \\ &= \frac{1}{4} - L_3\int_{|x|\leq\tilde{R}_0}|v_n|^2\,\mathrm{d}x \to \frac{1}{4}\end{aligned} \tag{3.9}$$

as $n\to+\infty$. By $\|u_n\|_H\to\infty$ and $\mathcal{G}(u_n)\to c_V$, we get a contradiction.

As a result, $v(x)\neq 0$. By $(\mathcal{G}'(u_n),u_n^-)=o_n(1)$, we have $\|u_n^-\|_H=o_n(1)$. So $u_n^-(x)\to 0$ a.e. $x\in\mathbb{R}^3$, from which we get that $v^-(x)=0$ a.e. $x\in\mathbb{R}^3$. Might as well set $v(x)\geq 0$. Let Ω is an open bounded subset of $\mathbb{R}^3$ defined as

$$\Omega=\left\{x\in\mathbb{R}^3:v(x)>0\right\}.$$

And we can see that the measure of Ω is positive. By $v_n(x)=\dfrac{u_n(x)}{\|u_n\|_H}\to v(x)\neq 0$ and $\|u_n\|_H\to\infty$, for $x\in\Omega$, we get that $u_n(x)\to+\infty$ as $n\to\infty$. Obviously, $\lim\limits_{n\to\infty}\dfrac{F(u_n(x))+\frac{1}{2_s^*}(u_n^+(x))^{2_s^*}}{|u_n(x)|^4}|v_n(x)|^4=+\infty$ for $x\in\Omega$, from which we have

$$\lim_{n\to\infty}\int_\Omega\frac{F(u_n)+\frac{1}{2_s^*}(u_n^+)^{2_s^*}}{|u_n|^4}|v_n|^4\,\mathrm{d}x=+\infty. \tag{3.10}$$

By (2.10), we set that $|F(u)|\leq\dfrac{4s-3}{12}\left(|u|^2+|u|^{2_s^*}\right)+C_\varepsilon|u|^{s_0}$ for $x\in\mathbb{R}^3$, $u\geq 0$ and $C_\varepsilon>0$. On the basis of choosing ε small in (3.7), we get that there exists $L_0>0$ such that $F(u)+\dfrac{1}{2_s^*}|u|^{2_s^*}\geq -L_0|u|^2$ for $|x|\leq\tilde{R}_0$ and $u\geq 0$. By $F(u)+\dfrac{1}{2_s^*}|u|^{2_s^*}\geq 0$ for $|x|\geq\tilde{R}_0$ and $u\geq 0$, we get

$$\int_{\mathbb{R}^3\setminus\Omega}\frac{F(u_n)+\frac{1}{2_s^*}(u_n^+)^{2_s^*}}{\|u_n\|_H^4}\mathrm{d}x\geq-\frac{L_0\int_{\mathbb{R}^3}|u_n|^2\,\mathrm{d}x}{\|u_n\|_H^4}. \tag{3.11}$$

Together with (2.2), we get that

$$\lim_{n\to\infty}\int_{\mathbb{R}^3\setminus\Omega}\frac{F(u_n)+\frac{1}{2_s^*}(u_n^+)^{2_s^*}}{\|u_n\|_H^4}\mathrm{d}x\geq 0. \tag{3.12}$$

By (3.10) and (3.12), we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}\frac{F(u_n)+\frac{1}{2_s^*}(u_n^+)^{2_s^*}}{\|u_n\|_H^4}\mathrm{d}x=+\infty. \tag{3.13}$$

However, by the embedding $X\hookrightarrow H^s(\mathbb{R}^3)$ is continuous, $\mathcal{G}(u_n)\to c_V>0$ and Lemma 2.3, we get

$$\begin{aligned}&\lim_{n\to\infty}\int_{\mathbb{R}^3}\frac{\mathcal{G}(u_n)+F(u_n)+\frac{1}{2_s^*}(u_n^+)^{2_s^*}}{\|u_n\|_H^4}\mathrm{d}x\\&\leq\lim_{n\to\infty}\frac{1}{2}\int_{\mathbb{R}^3}\frac{\|u_n\|_H^2+\omega C_3\|u_n\|_{H^s}^4}{\|u_n\|_H^4}\mathrm{d}x\leq\lim_{n\to\infty}\frac{C\left(\|u_n\|_H^2+\|u_n\|_H^4\right)}{\|u_n\|_H^4}\mathrm{d}x\leq C,\end{aligned}$$

which contradict (3.13).

Step 2: $c_V < \tau_\infty$. Let $\sigma = (1,0,0)$. By (2.10) and Lemma 2.3, we get that for $\varepsilon \in \left(0, \frac{1}{2_s^*}\right)$, there exists $C > 0$ such that

$$\begin{aligned}\mathcal{G}\left(tw_\infty(x-R\sigma)\right) \le & \frac{t^2}{2}\int_{\mathbb{R}^3}\left(|\nabla w_\infty|^2 + V_M|w_\infty|^2\right)\mathrm{d}x + \frac{\omega C_3 t^4}{2}\|w_\infty\|_{H^s}^4 \\ & + \varepsilon\int_{\mathbb{R}^3}\left(t^2|w_\infty|^2 + t^{2_s^*}|w_\infty|^{2_s^*}\right)\mathrm{d}x + Ct^{s_0}\int_{\mathbb{R}^3}|w_\infty|^{s_0}\,\mathrm{d}x \\ & -\frac{t^{2_s^*}}{2_s^*}\int_{\mathbb{R}^3}|w_\infty|^{2_s^*}\,\mathrm{d}x.\end{aligned} \tag{3.14}$$

So there exist a small t_1 and a large t_2 such that $0 < t_1 < 1 < t_2$ independent of $R > 0$ satisfying

$$\sup_{t\in[0,t_1]\cup[t_2,+\infty)} \mathcal{G}\left(tw_\infty(x-R\sigma)\right) < \tau_\infty. \tag{3.15}$$

Observe that

$$\mathcal{G}(tu) = \mathcal{G}_\infty(tu) + \frac{t^2}{2}\int_{\mathbb{R}^3}\left(V(x)-V_\infty\right)u^2\mathrm{d}x. \tag{3.16}$$

Choose $\delta \in \left(0, 1-\frac{h_0}{2\sqrt{V_\infty}}\right)$. By Lemma 3.2, there exists $C_\delta > 0$ such that

$$w_\infty(x-R\sigma) \le C_\delta \mathrm{e}^{-(1-\delta)\sqrt{V_\infty}|x-R\sigma|}, \quad x\in\mathbb{R}^3. \tag{3.17}$$

From absolute value inequality, we get $|R||\sigma| - |x| \le |x-R\sigma|$, by ($V_2$) and (3.17), set $\tilde{R} = \max\{R_1, R_0\}$, we get that

$$\begin{aligned}&\int_{\mathbb{R}^3}\left(V(x)-V_\infty\right)\left|w_\infty(x-R\sigma)\right|^2\mathrm{d}x \\ &= \int_{|x|\le\tilde{R}}\left(V(x)-V_\infty\right)\left|w_\infty(x-R\sigma)\right|^2\mathrm{d}x + \int_{|x|\ge\tilde{R}}\left(V(x)-V_\infty\right)\left|w_\infty(x-R\sigma)\right|^2\mathrm{d}x \\ &\le 2V_M\int_{|x|\le\tilde{R}} C_\delta^2 \mathrm{e}^{-2(1-\delta)\sqrt{V_\infty}|x-R\sigma|}\mathrm{d}x + \int_{|x|\ge\tilde{R}} C_v \mathrm{e}^{-h_0|x|} C_\delta^2 \mathrm{e}^{-2(1-\delta)\sqrt{V_\infty}|x-R\sigma|}\mathrm{d}x \\ &\le 2V_M C_\delta^2 \mathrm{e}^{-2(1-\delta)\sqrt{V_\infty}R}\int_{|x|\le\tilde{R}}\mathrm{e}^{2(1-\delta)\sqrt{V_\infty}|x|}\mathrm{d}x + C_v C_\delta^2\mathrm{e}^{-2(1-\delta)\sqrt{V_\infty}R}\int_{|x|\ge\tilde{R}}\mathrm{e}^{\left[2(1-\delta)\sqrt{V_\infty}-h_0\right]|x|}\mathrm{d}x \\ &\le C\mathrm{e}^{-2(1-\delta)\sqrt{V_\infty}R}.\end{aligned} \tag{3.18}$$

Then, set $\mathcal{I}(t)$ for $t\in(0,\infty)$, defined as

$$\begin{aligned}\mathcal{I}(t) = & \frac{1}{2}t^2\iint_{\mathbb{R}^6}\frac{\left(w_\infty(x)-w_\infty(y)\right)^2}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^3}V_\infty w_\infty^2\mathrm{d}x - \frac{1}{2}t^4\int_{\mathbb{R}^3}\omega\phi_{w_\infty}w_\infty^2\mathrm{d}x \\ & -\frac{t^{2_s^*}}{2_s^*}\int_{\mathbb{R}^3}|w_\infty|^{2_s^*}\,\mathrm{d}x - \int_{\mathbb{R}^3}F(tw_\infty)\mathrm{d}x.\end{aligned}$$

And

$$\mathcal{I}'(t) = t\|\omega_\infty\|^2 - 2t^3\int_{\mathbb{R}^3}\omega\phi_{w_\infty}w_\infty^2\mathrm{d}x + t^{2_s^*-1}\int_{\mathbb{R}^3}|w_\infty|^{2_s^*}\,\mathrm{d}x - t^3\int_{\mathbb{R}^3}\frac{f(tw_\infty)}{(tw_\infty)^3}w_\infty^4\mathrm{d}x. \tag{3.19}$$

By (f_3), we get that $\frac{f(tw_\infty)}{(tw_\infty)^3}$ is increasing for $t>0$, we deduce that $\mathcal{I}(t)$

has a unique critical point which is its maximum value. By $\mathcal{G}'_\infty(w_\infty)=0$, the critical point is reached, *i.e.* $\mathcal{I}'(1)=0$, this critical point should be achieved. So $\sup_{t\ge0}\mathcal{I}(t)=\mathcal{I}(1)=\tau_\infty$. By (3.16), (3.18) and Lemma 2.3, we get

$$\begin{aligned}&\sup_{t\in[t_1,t_2]}\mathcal{G}\left(tw_\infty(x-R\sigma)\right)\\&\le\sup_{t\ge0}\mathcal{I}(t)+\sup_{t\in[t_1,t_2]}\left[\frac{1}{2}t^4\int_{\mathbb{R}^3}\omega\phi_{w_\infty}w_\infty^2\mathrm{d}x-\frac{1}{2}\int_{\mathbb{R}^3}\omega\phi_{tw_\infty}(tw_\infty)^2\,\mathrm{d}x\right]\\&+\frac{t_1^2}{2}Ce^{-2(1-\delta)\sqrt{V_\infty}R}\\&\le\tau_\infty+\frac{\omega C_3t_2^4}{2}\|w_\infty\|_{H^s}^4+\frac{t_1^2}{2}Ce^{-2(1-\delta)\sqrt{V_\infty}R}.\end{aligned}\tag{3.20}$$

By $0<h_0<2(1-\delta)\sqrt{V_\infty}$, we set $R>\tilde{R}$, there holds $\sup_{t\in[t_1,t_2]}\mathcal{G}\left(tw_\infty(x-R\gamma)\right)\le\tau_\infty+\frac{\omega C_3t_2^4}{2}\|w_\infty\|_{H^s}^4-Ce^{-a_0R}$. Then there exists $\omega_0>0$ such that for $\omega\in(0,\omega_0)$

$$\sup_{t\in[t_1,t_2]}\mathcal{G}\left(tw_\infty(x-R_0\gamma)\right)<\tau_\infty.\tag{3.21}$$

Combining with (3.15) and (3.21), we get that $\sup_{t\ge0}\mathcal{G}\left(w_\infty(x-R_0\gamma)\right)<\tau_\infty$ for $\omega\in(0,\omega_0)$ and $R>\tilde{R}$. By the definition of c_V, we proved that $c_V<\tau_\infty$.

Now, by step 1 $\|u_n\|_H$ is bounded, $\mathcal{G}(u_n)\to c_V<\tau_\infty<\frac{s}{3}S^{\frac{3}{2s}}$ and $\mathcal{G}'(u_n)\to 0$, without loss of generality, we may assume that $u_n\ge0$ a.e. and $u_n\rightharpoonup u\ne0$ weakly in *X*. Because if $u=0$, $u_n\rightharpoonup0$ weakly in *X*. If $\lim_{n\to\infty}\sup_{y\in\mathbb{R}^3}\int_{B_2(y)}|u_n|^2\,\mathrm{d}x=0$, by the Lions Lemma, we derive that $u_n\to0$ in $L^t(\mathbb{R}^3)$, where $t\in(2,2_s^*)$. Similar to the principle of Lemma 3.1, we launch a contradiction. We're not going to prove it here. There exists $\zeta>0$ such that $\lim_{n\to\infty}\sup_{y\in\mathbb{R}^3}\int_{B_2(y)}|u_n|^2\,\mathrm{d}x\ge\zeta>0$, so we deduce that there exists $y_n\in\mathbb{R}^3$ with $|y_n|\to\infty$ satisfying $\tilde{v}_n=u_n(.+y_n)\rightharpoonup\tilde{v}\ne0$ weakly in *X*. By $u_n\rightharpoonup0$ weakly in *X* and $\lim_{|x|\to\infty}V(x)=V_\infty$, we get $\mathcal{G}_\infty(u_n)=c_V+o_n(1)$ and $\mathcal{G}_\infty(u_n)=o_n(1)$. Therefore,

$$\mathcal{G}_\infty(\tilde{v}_n)=c_V+o_n(1),\quad\mathcal{G}'_\infty(\tilde{v}_n)=o_n(1).\tag{3.22}$$

From $\tilde{v}_n\rightharpoonup\tilde{v}$ weakly in *X* and Lemma 2.2, we have $\mathcal{G}'_\infty(\tilde{v})=0$. So $\mathcal{G}_\infty(\tilde{v})\ge\tau_\infty$. By (3.22), we get

$$\begin{aligned}c_V&=\mathcal{G}_\infty(\tilde{v}_n)-\frac{1}{4}\left(\mathcal{G}'_\infty(\tilde{v}_n),\tilde{v}_n\right)+o_n(1)\\&=\frac{1}{4}\|\tilde{v}_n\|_H^2+\frac{1}{4}\int_{\mathbb{R}^3}\phi_{\tilde{v}_n}^2\tilde{v}_n^2\mathrm{d}x+\int_{\mathbb{R}^3}\left(\frac{1}{4}f(\tilde{v}_n)\tilde{v}_n-F(\tilde{v}_n)\right)\mathrm{d}x\\&\quad+\frac{4s-3}{12}\int_{\mathbb{R}^3}|\tilde{v}_n|^{2_s^*}\,\mathrm{d}x+o_n(1).\end{aligned}\tag{3.23}$$

So let's take the limit of both sides, by Fatou's lemma, we get

$$\begin{aligned}c_V&\ge\frac{1}{4}\|\tilde{v}\|_H^2+\frac{1}{4}\int_{\mathbb{R}^3}\phi_{\tilde{v}}^2\tilde{v}^2\mathrm{d}x+\int_{\mathbb{R}^3}\left(\frac{1}{4}f(\tilde{v})\tilde{v}-F(\tilde{v})\right)\mathrm{d}x+\frac{4s-3}{12}\int_{\mathbb{R}^3}|\tilde{v}|^{2_s^*}\,\mathrm{d}x\\&=\mathcal{G}_\infty(\tilde{v})-\frac{1}{4}\left(\mathcal{G}'_\infty(\tilde{v}),\tilde{v}\right)=\mathcal{G}_\infty(\tilde{v})\ge\tau_\infty.\end{aligned}\tag{3.24}$$

This contradiction with Step 2 $c_V < \tau_\infty$. Therefore, $u_n \rightharpoonup u_V \neq 0$ weakly in *X*. By $\mathcal{G}'(u_n) \to 0$ and Lemma 2.2, we get $\mathcal{G}'(u_V) = 0$.

Conflicts of Interest

The author declares no conflicts of interest.

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