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# Constructing the General Jensen-Cauchy Equations in Banach Space and Using Fixed Point Method to Establish Homomorphisms in Banach Algebras

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## Abstract

In this paper, we study to solve general Cauchy-Jensen additive mappings with $3k$-variables. First, we investigated the Cauchy-Jensen stability of the functional Equations (1.1), (1.2) and (1.3) in Banach-spaces and then I apply the fixed point method to establish homomorphisms on the Banach algebras.

## Subject Areas

Mathematics

## Keywords

Cauchy Additive Mapping, Jensen Additive Mapping, Cauchy-Jensen-Hyers-Ulam-Rassisa Stabilty, Fixed Point Method to Establish Homomorphisms in Banach Algebras

## 1. Introduction

Let $\mathbb{A}$ and $\mathbb{B}$ be a vector spaces on the same field $\mathbb{K}$, and $\phi : \mathbb{A} \to \mathbb{B}$. We use the notation $\|\cdot\|$ for all the norm on both $\mathbb{A}$ and $\mathbb{B}$. In this paper, we investisgate additive functional equations when $\mathbb{A}$ is a normed vector space and $\mathbb{B}$ is a Banach spaces.

In fact, when $\mathbb{A}$ is a normed vector space and $\mathbb{B}$ is a Banach spaces we solve and prove the general Cauchy-Jensen stability of forllowing additive functional equations.

$$k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)+k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)=\sum_{i=1}^{k}\phi(x_i)+2k\sum_{i=1}^{k}\phi(z_i) \quad (1)$$

$$k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)-k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)=\sum_{i=1}^{k}\phi(y_i) \quad (2)$$

$$2k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)=\sum_{i=1}^{k}\phi(x_i)+\sum_{i=1}^{k}\phi(y_i)+2k\sum_{i=1}^{k}\phi(z_i) \quad (3)$$

In 1940 Ulam in [1] raised the following question: under what conditions does there exist an additive mapping near an approximately additive mapping?

The Hyers [2] gave firts affirmative partial answer to the equation of Ulam in Banach spaces.

D. H. Hyers: Let $\mathbb{E},\mathbb{E}'$ to be two Banach spaces if $\varepsilon>0$ and $f:\mathbb{E}\to\mathbb{E}'$ be a mapping such that

$$\|f(x+y)-f(x)-f(y)\|\le\varepsilon$$

for all $x,y\in X$, then there exists a unique near an additive mapping $T:X\to Y$

$$\|f(x)-T(x)\|\le\varepsilon$$

for all $x\in X$

Next Th. M. Rassias: Consider $\mathbb{E},\mathbb{E}'$ to be two Banach spaces, and let $f:\mathbb{E}\to\mathbb{E}'$ be a mapping such that $f(tx)$ is continous in *t* for each fixed *x*. Assume that there exist $\theta>0$ and $p\in[0,1]$ such that

$$\|f(x+y)-f(x)-f(y)\|\le\varepsilon\left(\|x\|^p+\|y\|^p\right),\forall x,y\in\mathbb{E}.$$

then there exists a unique $\mathbb{R}$-linear $L:\mathbb{E}\to\mathbb{E}'$ satifies

$$\|f(x)-L(x)\|\le\frac{2\theta}{2-2^p}\|x\|^p,x\in\mathbb{E}.$$

Next in 1980 the topic of approximate homomsphisms and the stability of the equation of homomsphism, was studied by many mathematicians in the world. Găvruta [3] generalized the Hyers-Ulam-Rassias' result in the following form:

Let $(\mathbb{G},*)$ be a group Abelian and $\mathbb{E}$ a *Banach* space.

Denote by $\phi:\mathbb{G}\times\mathbb{G}\to[0,\infty)$ a function such that

$$\tilde{\phi}(x,y)=\sum_{n=0}^{\infty}k^{-n}\phi\left(k^n x,k^n y\right)<\infty$$

for all $x,y\in\mathbb{G}$. Suppose that $f:\mathbb{G}\to\mathbb{E}$ is a mapping satisfying

$$\|f(x+y)-f(x)-f(y)\|\le\varepsilon$$

for all $x,y\in\mathbb{G}$. Then there exists a unique additive mapping $T:\mathbb{G}\to\mathbf{E}$ such that

$$\|f(x)-T(x)\|\le\frac{1}{k}\tilde{\phi}(x,x)$$

and next

Jun-Lee: [4] Let $\phi:\mathbb{E}\setminus\{0\}\times\mathbb{E}\setminus\{0\}\to[0,\infty)$ a function such that

$$\tilde{\phi}(x,y)=\sum_{n=0}^{\infty}k^{-n}\phi\left(k^{n}x,k^{n}y\right)<\infty$$

for all $x,y\in\mathbb{E}\setminus\{0\}$ Suppose that $T:f:\mathbb{E}\to\mathbb{E}'$ is a mapping satisfying

$$\left\|f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right\|\leq\phi(x,y)$$

for all $x\in\mathbb{E}\setminus\{0\}$. Then there exists a unique additive mapping $T:f:\mathbb{E}\to\mathbb{E}'$ such that

$$\|f(x)-f(0)-T(x)\|\leq\frac{1}{3}\tilde{\phi}(x,-x)+\phi(-x,3x)$$

for all $x,y\in\mathbb{E}\setminus\{0\}$.

The stability problems for several functional equations have been extensively investigated by a number of authors and and there are many interesting results concerning this probem. Recently, the authors studied the classic Cauchy-Jensen stability for the following functional equations

$$f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x-y}{2}+z\right)=f(x)+2f(z) \tag{4}$$

$$f\left(\frac{x+y}{2}+z\right)-f\left(\frac{x-y}{2}+z\right)=f(y) \tag{5}$$

$$2f\left(\frac{x+y}{2}+z\right)=f(x)+f(y)+2f(z) \tag{6}$$

in Banach spaces. In this paper, we solve and proved the Hyers-Ulam stability for functional equations of the general form the following Equations (1.1), (1.2) and (1.3), *i.e.* the functional equations with $3k$-variables . Under suitable assumptions on spaces $\mathbb{A}$ and $\mathbb{B}$, we will prove that the mappings satisfy the functional Equations (1.1), (1.2) and (1.3). Thus, the results in this paper are generalization of those in [4] [5] [6] [7] for functional equations with $3k$-variables.

In the process of researching the solution for the cauchy-Jensen problem with a limited number of variables to overcome the above, so I came up with the cauchy-Jensen equation with a higher number of variables based on the works of world mathematicians. [1]-[4] [8]. Here, allow me to express my gratitude to mathematicians.

The construction of the general Cauchy-Jensen equation has great applications to help mathematicians when studying the solutions of Cauchy-Jensen equations on spaces where the number of variables is not limited and their existence solutions are also general solution. To create this work, I based on the ideas of Mathematicians in the world [1]-[31]. I would like to thank the Mathematicians. The paper is organized as follows:

In section preliminaries we remind some basic notations as Banach spaces, $\mathbb{R}$-linear mapping, Fixed point theory, Generalized metric theory and Solutions to Cauchy-Jensen Equations see [5] [6] [8]-[11].

**Section 3:** Constructing Lemma for Establishing Solutions to Cauchy-Jensen

Equations.

Constructing Lemma for Establishing Solutions to Cauchy-Jensen Equations. Note Here We assume that $\mathbb{A}$, $\mathbb{B}$ is a vector spaces.

**Section 4:** Establishing Solutions for general Cauchy-Jensen Equations.

Now, we first study the solutions of (1.1), (1.2) and (1.3). Note that for this equations, $\mathbb{A}$ is a vector space with norm $\|\cdot\|_{\mathbb{A}}$ and that $\mathbb{B}$ is a Banach space with norm $\|\cdot\|_{\mathbb{B}}$. Under this setting, we can show that the mappings satisfying (1.1), (1.2) and (1.3) is additive.

**Section 5:** Stability of homomorphisms in real Banach Algebras.

In this section, we use the fixed point method, to establish homomorphism on real Banach Algebra for Equation (1.1). Note that for this equations, $\mathbb{A}$ is a real Banach algebra with norm $\|\cdot\|_{\mathbb{A}}$ and that $\mathbb{B}$ is a real Banach with norm $\|\cdot\|_{\mathbb{B}}$.

## 2. Preliminaries

### 2.1. Banach Spaces

Let $\{x_n\}$ be a sequence in a normed space $\mathbb{X}$.

1) A sequence $\{x_n\}_{n=1}^{\infty}$ in a space $\mathbb{X}$ is a Cauchy sequence if the sequence $\{x_{n+1}-x_n\}_{n=1}^{\infty}$ converges to zero;

2) The sequence $\{x_n\}_{n=1}^{\infty}$ is said to be convergent if, there exists $x\in\mathbb{X}$ such that, for any $\varepsilon>0$, there is a positive integer $N$ such that

$$\|x_n-x\|\le\varepsilon,\forall n\ge N.$$

Then the point $x\in\mathbb{X}$ is called the limit of sequence $x_n$ and denoted by $\lim_{n\to\infty}x_n=x$;

3) If every sequence Cauchy in $\mathbb{X}$ converger, then the normed space $\mathbb{X}$ is called a Banach space.

### 2.2. $\mathbb{R}$-Linear Mapping

**Theorem 1.** Let $f:\mathbb{E}\to\mathbb{E}'$ be a mapping from anormed vector space $\mathbb{E}$ into a Banach spaces $\mathbb{E}'$ subject to the inequality

$$\|f(x+y)-f(x)-f(y)\|\le\varepsilon\left(\|x\|^p+\|y\|^p\right),\forall x,y\in\mathbb{E}. \tag{7}$$

where $\varepsilon$ and $p$ are constans with $\varepsilon>0$ and $p<1$. Then, the limit

$$L(x)=\lim_{n\to\infty}\frac{f(2^n x)}{2^n} \tag{8}$$

exists for all $x\in\mathbb{E}$ and $L:\mathbb{E}\to\mathbb{E}'$ is unique additive mapping which satifies

$$\|f(x)-L(x)\|\le\frac{2\varepsilon}{2-2^p}\|x\|^p,x\in\mathbb{E}. \tag{9}$$

for all $x\in\mathbb{E}$. Also, if each $x\in\mathbb{E}$ then function $f(tx)$ is continuous in $t\in\mathbb{R}$, then $L$ is $\mathbb{R}$-linear mapping.

**Theorem 2.** Let $\mathbb{X}$ be a real normed linear space and $\mathbb{Y}$ a real complete normed linear space. Assume that $f:\mathbb{X}\to\mathbb{Y}$ is an approximately additive mapping for which there exist constants $\theta \geq 0$ and $p \in \mathbb{R} - \{1\}$ such that $f$ satisfies the inequality

$$\left\| f(x+y) - f(x) - f(y) \right\| \leq \theta \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}}, \forall x, y \in \mathbb{X}, \tag{10}$$

Then, there exists a unique additive mapping $L : \mathbb{X} \to \mathbb{Y}$ satisfying

$$\left\| f(x) - L(x) \right\| \leq \frac{2\theta}{2 - 2^p} \|x\|^p, x \in \mathbb{E}. \tag{11}$$

for all $x \in \mathbb{X}$. If, in addition, $f : \mathbb{X} \to \mathbb{Y}$ is a mapping such that the transformation $t \to f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathbb{X}$, then $L$ is an $\mathbb{R}$-linear mapping.

### 2.3. Fixed Point Theory

**Theorem 3.** Let $(\mathbb{X}, d)$ be a complete generalized metric space and let $J : \mathbb{X} \to \mathbb{X}$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in \mathbb{X}$, either

$$d\left(J^n, J^{n+1}\right) = \infty$$

for all nonegative integers $n$ or there exists a positive integer $n_0$ such that

1) $d\left(J^n, J^{n+1}\right) < \infty$, $\forall n \geq n_0$;
2) The sequence $\left\{J^n x\right\}$ converges to a fixed point $y^*$ of $J$;
3) $y^*$ is the unique fixed point of $J$ in the set $\mathbb{Y} = \left\{y \in \mathbb{X} \mid d\left(J^n, J^{n+1}\right) < \infty\right\}$;
4) $d\left(y, y^*\right) \leq \frac{1}{1-l} d(y, Jy)$ $\forall y \in \mathbb{Y}$.

**Theorem 4.** Let $(\mathbb{X}, d)$ be a complete metric space and let $J : \mathbb{X} \to \mathbb{X}$ be a strictly contractive that is,

$$d(Jx, Jy) \leq Lf(x, y) \tag{12}$$

for some Lipschitz constant $L < 1$. Then,

1) the mapping $J$ has a unique fiexd point $x^* = Jx^*$;
2) the fixed point $x^*$ is globally attractive, that is

$$\lim_{n \to \infty} J^n x = x^* \tag{13}$$

for all starting point $x \in \mathbb{X}$;

3) one has the following estimation inequalities:

$$d\left(J^n x, x^*\right) \leq L^n d\left(x, x^*\right),$$

$$d\left(J^n x, x^*\right) \leq \frac{1}{1-L} d\left(J^n x, J^{n+1} x\right) \tag{14}$$

4) $d\left(x, x^*\right) \leq \frac{1}{1-L} d(x, Jx)$ for all nonnegatives n and all $x \in \mathbb{X}$.

### 2.4. Generalized Metric Theory

Let $X$ is a set. A function $d : \mathbb{X} \times \mathbb{X} \to [0, \infty)$ is called a generalized metric space on $\mathbb{X}$ if d staisfies the following:

1) $d(x,y)=0$ and only if $x=y$;
2) $d(x,y)=d(y,x)$ for all $x,y\in\mathbb{X}$;
3) $d(x,z)\le d(x,y)+d(y,z)$ for all $x,y,z\in\mathbb{X}$.

### 2.5. Solutions of the Equation

The functional equation

$$f(x+y)=f(x)+f(y)$$

is called the Cauchuy equation. In particular, every solution of the Cauchuy equation is said to be an additive Cauchy mapping.

The functional equation

$$f\left(\frac{x+y}{2}\right)=\frac{1}{2}f(x)+\frac{1}{2}f(y)$$

is called the Jensen additive equation. In particular, every solution of the Jensen equation is said to be Jensen additive mapping.

## 3. The Basis for Building Solutions for the Cauchy-Jensen. Equation

Note Here We assume that $\mathbb{A}$, $\mathbb{B}$ is a vector spaces

**Lemma 5.** Suppose that $\mathbb{A}$, $\mathbb{B}$ be vector space. It is shown if a mapping $\phi:\mathbb{A}\to\mathbb{B}$ satisfies

$$k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)+k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)=\sum_{i=1}^{k}\phi(x_i)+2k\sum_{i=1}^{k}\phi(z_i) \quad (15)$$

$$k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)-k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)=\sum_{i=1}^{k}\phi(y_i) \quad (16)$$

$$2k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)=\sum_{i=1}^{k}\phi(x_i)+\sum_{i=1}^{k}\phi(y_i)+2k\sum_{i=1}^{k}\phi(z_i) \quad (17)$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbb{A}$, then the mappings $\phi:\mathbb{A}\to\mathbb{B}$ is *Cauchy* additive.

*Proof.* Assume that $f:\mathbb{A}\to\mathbb{B}$ satisfies (15).

We replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(x,\cdots,x,x,\cdots,x,z,\cdots,0)$ in (15), we have

$$k\phi(x+z)+k\phi(z)=k\phi(x)+2k\phi(z)$$

for all $x,z\in\mathbf{X}$. So

$$\phi(x+z)=\phi(x)+\phi(z)$$

Hence $\phi:\mathbb{A}\to\mathbb{B}$ is *Cauchy* additive. We replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(x,\cdots,x,x,\cdots,x,z,\cdots,0)$ in (16), we have

$$k\phi(x+z)-k\phi(z)=k\phi(x)$$

for all $x,z\in\mathbb{A}$. So

$$\phi(x+z)=\phi(x)+\phi(z)$$

Hence $\phi:\mathbb{A}\to\mathbb{B}$ is Cauchy additive. Next We replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(x,\cdots,x,x,\cdots,x,z,\cdots,0)$ in (17), we have

$$2k\phi(x+z)=k\phi(x)+k\phi(x)+2k\phi(z)$$

for all $x,z\in\mathbb{A}$. So

$$\phi(x+z)=\phi(x)+\phi(z)$$

Hence $\phi:\mathbb{A}\to\mathbb{B}$ is *Cauchy* additive.

□

The mappings $\phi:\mathbb{A}\to\mathbb{B}$ given in the statement of lemma 3.1 are Cauchy-Jensen additive mappings.

We replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(x,\cdots,x,y,\cdots,y,0,\cdots,0)$ in (17), we get the Jensen additive mapping

$$2\phi\left(\frac{x+y}{2}\right)=\phi(x)+\phi(y)$$

and we replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(x,\cdots,x,x,\cdots,x,z,\cdots,0)$ in (17), in (17), we get the Cauchy additive mapping

$$\phi(x+z)=\phi(x)+\phi(z).$$

## 4. Establishing Solutions for General Cauchy-Jensen Equations

Now, we first study the solutions of (1.1), (1.2) and (1.3). Note that for this equations, $\mathbb{A}$ is a vector space with norm $\|\cdot\|_{\mathbb{A}}$ and that $\mathbb{B}$ is a Banach space with norm $\|\cdot\|_{\mathbb{B}}$. Under this setting, we can show that the mappings satisfying (1.1), (1.2) and (1.3) is additive.

**Theorem 6.** Suppose that $\phi:\mathbb{A}\to\mathbb{B}$ be a mapping. If there is a function $\varphi:\mathbb{A}^{3k}\to[0,\infty)$ such that satisfying

$$\left\|k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)+k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(x_i)-2k\sum_{i=1}^{k}\phi(z_i)\right\|_{\mathbb{B}} \le\varphi(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k) \quad (18)$$

and

$$\tilde{\varphi}(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k) =\sum_{j=1}^{\infty}2^j\varphi\left(\frac{x_1}{2^j},\cdots,\frac{x_k}{2^j},\frac{y_1}{2^j},\cdots,\frac{y_k}{2^j},\frac{z_1}{2^j},\cdots,\frac{z_k}{2^j}\right)<\infty \quad (19)$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbb{A}$

Then there exists a unique additive mapping $\psi:\mathbb{A}\to\mathbb{B}$ such that

$$\|\phi(x)-\psi(x)\|_{\mathbb{B}}\le\frac{1}{k}\tilde{\varphi}(x,\cdots,x,x,\cdots,x,x,\cdots,0) \quad (20)$$

for all $x\in\mathbf{X}$

*Proof.* We replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(x,\cdots,x,x,\cdots,x,x,\cdots,0)$ in (18), we have

$$\left\|k\phi(2x)-2k\phi(x)\right\|_{\mathbb{B}}\le\varphi(x,\cdots,x,x,\cdots,x,x,\cdots,0)\tag{21}$$

for all $x\in\mathbb{A}$. So

$$\left\|\phi(x)-2\phi\left(\frac{x}{2}\right)\right\|_{\mathbb{B}}\le\frac{1}{k}\varphi\left(\frac{x}{2},\cdots,\frac{x}{2},\frac{x}{2},\cdots,\frac{x}{2},\frac{x}{2},\cdots,0\right)$$

for all $x\in\mathbb{A}$. Hence

$$\left\|2^l\phi\left(\frac{x}{2^l}\right)-2^m\phi\left(\frac{x}{2^m}\right)\right\|_{\mathbb{B}}\le\frac{1}{k}\sum_{j=l+1}^{m}2^{j-1}\varphi\left(\frac{x}{2},\cdots,\frac{x}{2},\frac{x}{2},\cdots,\frac{x}{2},\frac{x}{2},\cdots,0\right)\tag{22}$$

for all nonnegative integers $m$ and $l$ with $m>l$ and for all $x\in\mathbb{A}$. It followns (19) and (22) that the sequence $\left\{2^n\phi\left(\frac{x}{2^n}\right)\right\}$ is a Cauchy sequence for all $x\in\mathbb{A}$. Sence $\mathbb{B}$ is complete, the sequence $\left\{2^n\phi\left(\frac{x}{2^n}\right)\right\}$ converges. So one can define the mapping $\psi:\mathbb{A}\to\mathbb{B}$ by

$$\psi(x)=\lim_{n\to\infty}2^n\phi\left(\frac{x}{2^n}\right)$$

for all $x\in\mathbb{A}$. By (19) and (18),

$$\begin{aligned}&\left\|k\psi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)+k\psi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\psi(x_i)-2k\sum_{i=1}^{k}\psi(x_i)\right\|_{\mathbb{B}}\\&=\lim_{n\to\infty}2^n\left\|k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2^{n+1}}+\sum_{i=1}^{k}\frac{z_i}{2^n}\right)+k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2^{n+1}}+\sum_{i=1}^{k}\frac{z_i}{2^n}\right)\right.\\&\quad\left.-\sum_{i=1}^{k}\phi\left(\frac{x_i}{2^n}\right)-2k\sum_{i=1}^{k}\phi\left(\frac{z_i}{2^n}\right)\right\|_{\mathbb{B}}\\&\le\lim_{n\to\infty}2^n\varphi\left(\frac{x_1}{2^n},\cdots,\frac{x_k}{2^n},\frac{y_1}{2^n},\cdots,\frac{y_k}{2^n},\frac{z_1}{2^n},\cdots,\frac{z_k}{2^n}\right)=0\end{aligned}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbb{A}$.

So

$$k\psi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)+k\psi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)=\sum_{i=1}^{k}\psi(x_i)+2k\sum_{i=1}^{k}\psi(z_i).$$

By Lemma 2.1, the mapping $\psi:\mathbb{A}\to\mathbb{B}$ is Cauchy additive mapping. Moreover, letting $l=0$ and passing to the limit $m\to\infty$ in (22), we get the inequality (20)

Now, let $\psi:\mathbb{A}\to\mathbb{B}$ be another generalized Cauchy-Jensen additive mapping satisfying (20). Then we have

$$\begin{aligned}\left\|\psi(x)-\psi'(x)\right\|_{\mathbb{B}}&=2^n\left\|\psi\left(\frac{x}{2^n}\right)-\psi'\left(\frac{x}{2^n}\right)\right\|_{\mathbb{B}}\\&\le2^n\left(\left\|\psi\left(\frac{x}{2^n}\right)-\phi\left(\frac{x}{2^n}\right)\right\|_{\mathbb{B}}+\left\|\psi'\left(\frac{x}{2^n}\right)-\phi\left(\frac{x}{2^n}\right)\right\|_{\mathbb{B}}\right)\\&\le2\frac{2^n}{k}\tilde{\varphi}\left(\frac{x}{2^n},\cdots,\frac{x}{2^n},\frac{x}{2^n},\cdots,\frac{x}{2^n},\frac{x}{2^n},\cdots,0\right)\end{aligned}\tag{23}$$

which tends to zero as $n\to\infty$ for all $x\in\mathbb{A}$. So we can conclude that $\psi(x)=\psi'(x)$ for all $x\in\mathbb{A}$. This proves the uniquence of $\psi'$.

□

**Corollary 1.** Supppose $p$ and $\theta$ be positive real numbers with $p>1$, and let $\phi:\mathbb{A}\to\mathbb{B}$ be a mapping such that

$$\left\|k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)+k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(x_i)-2k\sum_{i=1}^{k}\phi(z_i)\right\|_{\mathbb{B}}$$
$$\le\theta\left(\sum_{i=1}^{k}\|x_i\|_{\mathbb{A}}^{p}+\sum_{i=1}^{k}\|y_i\|_{\mathbb{A}}^{p}+\sum_{i=1}^{k}\|z_i\|_{\mathbb{A}}^{p}\right)$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbb{A}$. The there exists a unique additive mapping $\psi:\mathbb{A}\to\mathbb{B}$ such that

$$\|\phi(x)-\psi(x)\|_{\mathbb{B}}\le\left(2+\frac{1}{k}\right)\frac{\theta}{2^p-2}\|x\|_{\mathbb{A}}^{p}$$

for all $x\in\mathbb{A}$.

**Corollary 2.** Suppose $p_1,p_2,\cdots,p_k$ and $\theta$ be positive real numbers with $3p_1+2p_2+\cdots+2p_k>1$, and let $\phi:\mathbb{A}\to\mathbb{B}$ be a mapping such that

$$\left\|k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)+k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(x_i)-2k\sum_{i=1}^{k}\phi(z_i)\right\|_{\mathbb{B}}$$
$$\le\theta\prod_{i=1}^{k}\|x_i\|_{\mathbb{A}}^{p_i}\cdot\prod_{i=1}^{k}\|y_i\|_{\mathbb{A}}^{p_i}\cdot\|z_1\|_{\mathbb{A}}^{p_1}\cdot\left(1+\prod_{i=2}^{k}\|z_i\|_{\mathbb{A}}^{p_i}\right)$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbb{A}$. The there exists a unique additive mapping $\psi:\mathbb{A}\to\mathbb{B}$ such that

$$\|\phi(x)-\psi(x)\|_{\mathbb{B}}\le\frac{\theta}{k\left(2^{3p_1+2p_2+\cdots+2p_k}-2\right)}\|x\|_{\mathbb{A}}^{3p_1+2p_2+\cdots+2p_k}$$

for all $x\in\mathbb{A}$.

**Theorem 7.** Suppose $\phi:\mathbb{A}\to\mathbb{B}$ be a mapping. If there is a function $\varphi:\mathbb{A}^{3k}\to[0,\infty)$ such that satisfying

$$\left\|k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)+k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(x_i)-2k\sum_{i=1}^{k}\phi(z_i)\right\|_{\mathbb{B}}$$
$$\le\varphi(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k) \tag{24}$$

and

$$\tilde{\varphi}(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$$
$$=\sum_{j=1}^{\infty}\frac{1}{2^j}\varphi\left(2^jx_1,\cdots,2^jx_k,2^jy_1,\cdots,2^jy_k,2^jz_1,\cdots,2^jz_k\right)<\infty \tag{25}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbb{A}$

Then there exists a unique additive mapping $\psi:\mathbb{A}\to\mathbb{B}$ such that

$$\|\phi(x)-\psi(x)\|_{\mathbb{B}}\le\frac{1}{2k}\tilde{\varphi}(x,\cdots,x,x,\cdots,x,\cdots,x,\cdots,0) \tag{26}$$

for all $x\in\mathbb{A}$

*Proof.* We replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by

$(x,\cdots,x,x,\cdots,x,x,\cdots,0)$ in (24), we have

$$\left\|k\phi(2x)-2k\phi(x)\right\|_{\mathbb{B}} \le \varphi(x,\cdots,x,x,\cdots,x,x,\cdots,0) \tag{27}$$

for all $x\in\mathbb{A}$. So

$$\left\|\phi(x)-\frac{1}{2}\phi(2x)\right\|_{\mathbb{B}} \le \frac{1}{2k}\varphi(x,\cdots,x,x,\cdots,x,x,\cdots,0) \tag{28}$$

for all $x\in\mathbb{A}$. Hence

$$\begin{aligned}&\left\|\frac{1}{2^l}\phi(2^l x)-\frac{1}{2^m}\phi(2^m x)\right\|_{\mathbb{B}}\\ &\le \frac{1}{2k}\sum_{j=l+1}^{m}\frac{1}{2^{j-1}}\varphi(2^j x,\cdots,2^j x,2^j x,\cdots,2^j x,2^j x,\cdots,0)\end{aligned} \tag{29}$$

for all nonnegative integers $m$ and $l$ with $m>l$ and for all $x\in\mathbb{A}$. It followns (25) and (29) that the sequence $\left\{\frac{1}{2^n}\phi(2^n x)\right\}$ is a Cauchy sequence for all $x\in\mathbb{A}$. Sence $\mathbb{B}$ is complete, the sequence $\left\{\frac{1}{2^n}\phi(2^n x)\right\}$ converges. So one can define the mapping $\psi:\mathbb{A}\to\mathbb{B}$ by

$$\psi(x)=\lim_{n\to\infty}\frac{1}{2^n}f(2^n x)$$

for all $x\in\mathbb{A}$. By (25) and (z4),

$$\begin{aligned}&\left\|k\psi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)+k\psi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\psi(x_i)-2\sum_{i=1}^{k}\psi(z_i)\right\|_{\mathbb{B}}\\ &=\lim_{n\to\infty}\frac{1}{2^n}\left\|k\phi\left(2^n\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2}+\sum_{i=1}^{k}z_i\right)\right)+k\phi\left(2^n\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2}+\sum_{i=1}^{k}z_i\right)\right)\right.\\ &\quad\left.-\sum_{i=1}^{k}f(2^n x_i)-2k\sum_{i=1}^{k}f(2^n z_i)\right\|_{\mathbb{B}}\\ &\le \lim_{n\to\infty}\frac{1}{2^n}\varphi(2^n x_1,\cdots,2^n x_k,2^n y_1,\cdots,2^n y_k,2^n z_1,\cdots,2^n z_k)=0\end{aligned}$$

for all for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbb{A}$.

So

$$k\psi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)+k\psi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)=\sum_{i=1}^{k}\psi(x_i)+2\sum_{i=1}^{k}\psi(z_i)$$

By Lemma 2.1, the mapping $\psi:\mathbb{A}\to\mathbb{B}$ is Cauchy additive mapping. Moreover, letting $l=0$ and passing to the limit $m\to\infty$ in (29), we get the inequality (26)

Now, let $\psi':\mathbb{A}\to\mathbb{B}$ be another generalized Cauchy-Jensen additive mapping satisfying (26). Then we have

$$\begin{aligned}\left\|\psi(x)-\psi'(x)\right\|_{\mathbb{B}}&=\frac{1}{2^n}\left\|\psi(2^n x)-\psi'(2^n x)\right\|_{\mathbb{B}}\\ &\le\frac{1}{2^n}\left(\left\|\psi(2^n x)-\phi(2^n x)\right\|_{\mathbb{B}}+\left\|\psi'(2^n x)-\phi(2^n x)\right\|_{\mathbb{B}}\right)\\ &\le 2\frac{2^n}{2k}\tilde{\varphi}(2^n x,\cdots,2^n x,2^n x,\cdots,2^n x,2^n x,\cdots,0)\end{aligned} \tag{30}$$

which tends to zero as $n\to\infty$ for all $x\in\mathbb{A}$. So we can conclude that $\psi(x)=\psi'(x)$ for all $x\in\mathbb{A}$. This proves the uniquence of $\psi'$.

**Corollary 3.** Supppose $p$ and $\theta$ be positive real numbers with $p<1$, and let $\phi:\mathbb{A}\to\mathbb{B}$ be a mapping such that

□

$$\left\|k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)+k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(x_i)-2k\sum_{i=1}^{k}\phi(z_i)\right\|_{\mathbb{B}}$$
$$\leq\theta\left(\sum_{i=1}^{k}\|x_i\|_{\mathbb{A}}^{p}+\sum_{i=1}^{k}\|y_i\|_{\mathbb{A}}^{p}+\sum_{i=1}^{k}\|z_i\|_{\mathbb{A}}^{p}\right)$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbb{A}$. The there exists a unique additive mapping $\psi:\mathbb{A}\to\mathbb{B}$ such that

$$\|\phi(x)-\psi(x)\|_{\mathbb{B}}\leq\left(2+\frac{1}{2k}\right)\frac{\theta}{2-2^{p}}\|x\|_{\mathbb{A}}^{p}$$

for all $x\in\mathbb{A}$.

**Corollary 4.** Let $p_1,p_2,\cdots,p_k$ and $\theta$ be positive real numbers with $3p_1+2p_2+\cdots+2p_k<1$, and let $\phi:\mathbb{A}\to\mathbb{B}$ be a mapping such that

$$\left\|k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)+k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(x_i)-2k\sum_{i=1}^{k}\phi(z_i)\right\|_{\mathbb{B}}$$
$$\leq\theta\prod_{i=1}^{k}\|x_i\|_{\mathbb{A}}^{p_i}\cdot\prod_{i=1}^{k}\|y_i\|_{\mathbb{A}}^{p_i}\cdot\|z_1\|_{\mathbb{A}}^{p_1}\cdot\left(1+\prod_{i=2}^{k}\|z_i\|_{\mathbb{A}}^{p_i}\right)$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbb{A}$. The there exists a unique additive mapping $\psi:\mathbb{A}\to\mathbb{B}$ such that

$$\|\phi(x)-\psi(x)\|_{\mathbb{B}}\leq\frac{\theta}{2k\left(2-2^{3p_1+2p_2+\cdots+2p_k}\right)}\|x\|_{\mathbb{A}}^{3p_1+2p_2+\cdots+2p_k}$$

for all $x\in\mathbb{A}$.

**Theorem 8.** Let $\phi:\mathbb{A}\to\mathbb{B}$ be a mapping. If there is a function $\varphi:\mathbb{A}^{3k}\to[0,\infty)$ such that satisfying

$$\left\|k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)-k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(y_i)\right\|_{\mathbb{B}}$$
$$\leq\varphi(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k) \tag{31}$$

and

$$\tilde{\varphi}(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$$
$$=\sum_{j=1}^{\infty}2^{j}\varphi\left(\frac{x_1}{2^j},\cdots,\frac{x_k}{2^j},\frac{y_1}{2^j},\cdots,\frac{y_k}{2^j},\frac{z_1}{2^j},\cdots,\frac{z_k}{2^j}\right)<\infty \tag{32}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbb{A}$

Then there exists a unique additive mapping $\psi:\mathbb{A}\to\mathbb{B}$ such that

$$\|\phi(x)-\psi(x)\|\leq\frac{1}{k}\tilde{\varphi}(x,\cdots,x,x,\cdots,x,x,\cdots,0) \tag{33}$$

for all $x\in\mathbb{A}$

The rest of the proof is similar to the proof of Theorem 4.1.

**Theorem 9.** Suppose $\phi:\mathbb{A}\to\mathbb{B}$ be a mapping. If there is a function $\varphi:\mathbb{A}^{3k}\to[0,\infty)$ such that satisfying

$$\begin{aligned}&\left\|k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)-k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(y_i)\right\|_{\mathbb{B}}\\&\le\varphi(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)\end{aligned}\tag{34}$$

and

$$\begin{aligned}&\tilde{\varphi}(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)\\&=\sum_{j=1}^{\infty}\frac{1}{2^j}\varphi\left(2^jx_1,\cdots,2^jx_k,2^jy_1,\cdots,2^jy_k,2^jz_1,\cdots,2^jz_k\right)<\infty\end{aligned}\tag{35}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbb{A}$

Then there exists a unique additive mapping $\psi:\mathbb{A}\to\mathbb{B}$ such that

$$\|\phi(x)-\psi(x)\|\le\frac{1}{2k}\tilde{\phi}(x,\cdots,x,x,\cdots,x,x,\cdots,0)\tag{36}$$

for all $x\in\mathbb{A}$

The rest of the proof is similar to the proof of Theorem 4.1, Theorem 4.4.

**Theorem 10.** Suppose $\phi:\mathbb{A}\to\mathbb{B}$ be a mapping. If there is a function $\varphi:\mathbb{A}^{3k}\to[0,\infty)$ such that satisfying

$$\begin{aligned}&\left\|2k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(x_i)-\sum_{i=1}^{k}\phi(y_i)-2k\sum_{i=1}^{k}\phi(z_i)\right\|_{\mathbb{B}}\\&\le\varphi(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)\end{aligned}\tag{37}$$

and

$$\begin{aligned}&\tilde{\varphi}(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)\\&=\sum_{j=1}^{\infty}2^j\varphi\left(\frac{x_1}{2^j},\cdots,\frac{x_k}{2^j},\frac{y_1}{2^j},\cdots,\frac{y_k}{2^j},\frac{z_1}{2^j},\cdots,\frac{z_k}{2^j}\right)<\infty\end{aligned}\tag{38}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbb{A}$

Then there exists a unique additive mapping $\psi:\mathbb{A}\to\mathbb{B}$ such that.

$$\|\phi(x)-\psi(x)\|_{\mathbb{B}}\le\frac{1}{4k}\tilde{\varphi}(x,\cdots,x,x,\cdots,x,x,\cdots,0)\tag{39}$$

for all $x\in\mathbb{A}$

The rest of the proof is the same as in the proof of theorem 4.1.

**Corollary 5.** Suppose $p$ and $\theta$ be positive real numbers with $p>1$, and let $\phi:\mathbb{A}\to\mathbb{B}$ be a mapping such that

$$\begin{aligned}&\left\|2k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(x_i)-\sum_{i=1}^{k}\phi(y_i)-2k\sum_{i=1}^{k}\phi(z_i)\right\|_{\mathbb{B}}\\&\le\theta\left(\sum_{i=1}^{k}\|x_i\|_{\mathbb{A}}^p+\sum_{i=1}^{k}\|y_i\|_{\mathbb{A}}^p+\sum_{i=1}^{k}\|z_i\|_{\mathbb{A}}^p\right)\end{aligned}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbb{A}$. The there exists a unique additive mapping $\psi:\mathbb{A}\to\mathbb{B}$ such that

$$\left\|\phi(x)-\psi(x)\right\|_{\mathbb{B}} \le \left(\frac{1}{2}+\frac{1}{4k}\right)\frac{\theta(2k+1)}{2^{p+1}-2^2}\|x\|_{\mathbb{A}}^p$$

for all $x \in \mathbb{A}$.

**Corollary 6.** Suppose $p_1, p_2, \cdots, p_k$ and $\theta$ be positive real numbers with $3p_1 + 2p_2 + \cdots + 2p_k > 1$, and let $\phi : \mathbb{A} \to \mathbb{B}$ be a mapping such that

$$\begin{aligned}&\left\|2k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(x_i)-\sum_{i=1}^{k}\phi(y_i)-2k\sum_{i=1}^{k}\phi(z_i)\right\|_{\mathbb{B}}\\&\le \theta\prod_{i=1}^{k}\|x_i\|^{p_i}\cdot\prod_{i=1}^{k}\|y_i\|^{p_i}\cdot\|z_1\|^{p_1}\cdot\left(1+\prod_{i=1}^{k}\|z_k\|^{p_i}\right)\end{aligned}$$

for all $x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_k \in \mathbb{A}$. The there exists a unique additive mapping $\psi : \mathbf{X} \to \mathbf{Y}$ such that

$$\left\|\phi(x)-\psi(x)\right\| \le \frac{1}{4k}\cdot\frac{\theta}{2^{3p_1+2p_2+\cdots+2p_k+1}-2^2}\|x\|^{3p_1+2p_2+\cdots+2p_k}$$

for all $x \in \mathbf{X}$.

**Theorem 11.** Let $\phi : \mathbb{A} \to \mathbb{B}$ be a mapping. If there is a function $\varphi : \mathbb{A}^{3k} \to [0,\infty)$ such that satisfying

$$\begin{aligned}&\left\|2k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(x_i)-\sum_{i=1}^{k}\phi(y_i)-2k\sum_{i=1}^{k}\phi(z_i)\right\|_{\mathbb{B}}\\&\le \varphi(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)\end{aligned} \tag{40}$$

and

$$\begin{aligned}&\tilde{\varphi}(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)\\&=\sum_{j=1}^{\infty}\frac{1}{2^j}\varphi\left(2^jx_1,\cdots,2^jx_k,2^jy_1,\cdots,2^jy_k,2^jz_1,\cdots,2^jz_k\right)<\infty\end{aligned} \tag{41}$$

for all $x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_k \in \mathbb{A}$

Then there exists a unique additive mapping $\psi : \mathbb{A} \to \mathbb{B}$ such that

$$\left\|\phi(x)-\psi(x)\right\| \le \frac{1}{k}\tilde{\varphi}(x,\cdots,x,x,\cdots,x,x,\cdots,0) \tag{42}$$

for all $x \in \mathbb{A}$

*Proof.* We replacing $(x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_k)$ by $(x, \cdots, x, x, \cdots, x, x, \cdots, 0)$ in (40), we have

$$\left\|2k\phi(2x)-4k\phi(x)\right\| \le \varphi(x,\cdots,x,x,\cdots,x,x,\cdots,0) \tag{43}$$

for all $x \in \mathbb{A}$. So

$$\left\|\phi(x)-\frac{1}{2}\phi(2x)\right\| \le \frac{1}{4k}\varphi(x,\cdots,x,x,\cdots,x,x,\cdots,0)$$

for all $x \in \mathbb{A}$. The rest of the proof is the same as in the proof of theorem 4.1 and 4.4.

□

**Corollary 7.** Suppose $p$ and $\theta$ be positive real numbers with $p < 1$, and let $\phi : \mathbb{A} \to \mathbb{B}$ be a mapping such that

$$\left\| 2k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(x_i)-\sum_{i=1}^{k}\phi(y_i)-2k\sum_{i=1}^{k}\phi(z_i)\right\|_{\mathbb{B}}$$
$$\leq\theta\left(\sum_{i=1}^{k}\|x_i\|_{\mathbb{A}}^p+\sum_{i=1}^{k}\|y_i\|_{\mathbb{A}}^p+\sum_{i=1}^{k}\|z_i\|_{\mathbb{A}}^p\right)$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbb{A}$. The there exists a unique additive mapping $\psi:\mathbb{A}\to\mathbb{B}$ such that

$$\|\phi(x)-\psi(x)\|_{\mathbb{B}}\leq\left(\frac{1}{2}+\frac{1}{4k}\right)\frac{\theta}{2^2-2^{p+1}}\|x\|_{\mathbb{A}}^p$$

for all $x\in\mathbb{A}$.

**Corollary 8.** Suppose $p_1,p_2,\cdots,p_k$ and $\theta$ be positive real numbers with $3p_1+2p_2+\cdots+2p_k<1$, and let $\phi:\mathbb{A}\to\mathbb{B}$ be a mapping such that

$$\left\| 2k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(x_i)-\sum_{i=1}^{k}\phi(y_i)-2k\sum_{i=1}^{k}\phi(z_i)\right\|_{\mathbb{B}}$$
$$\leq\theta\prod_{i=1}^{k}\|x_i\|^{p_i}\cdot\prod_{i=1}^{k}\|y_i\|^{p_i}\cdot\|z_1\|^{p_1}\cdot\left(1+\prod_{i=1}^{k}\|z_k\|^{p_i}\right)$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbb{A}$. The there exists a unique additive mapping $\psi:\mathbf{X}\to\mathbf{Y}$ such that

$$\|\phi(x)-\psi(x)\|\leq\frac{1}{4k}\cdot\frac{\theta}{2^2-2^{3p_1+2p_2+\cdots+2p_k+1}}\|x\|^{3p_1+2p_2+\cdots+2p_k}$$

for all $x\in\mathbf{X}$.

## 5. Stability of Homomorphisms in Real Banach Algebras

In this section, we use the fixed point method, to establish homomorphism on real Banach Algebra for Equation (1.1). Note that for this equations, $\mathbb{A}$ is a real Banach algebra with norm $\|\cdot\|_{\mathbb{A}}$ and that $\mathbb{B}$ is a real Banach with norm $\|\cdot\|_{\mathbb{B}}$.

**Theorem 12.** Suppose that $\phi:\mathbb{A}\to\mathbb{B}$ be a mapping. If there is a function $\varphi:\mathbb{A}^{3k}\to[0,\infty)$ such that satisfying

$$\left\| k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)+k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(x_i)-2k\sum_{i=1}^{k}\phi(z_i)\right\|_{\mathbb{B}}\leq\varphi(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k),\quad(44)$$

$$\sum_{j=1}^{\infty}\frac{1}{2^j}\varphi\left(2^jx_1,\cdots,2^jx_k,2^jy_1,\cdots,2^jy_k,2^jz_1,\cdots,2^jz_k\right)<\infty\quad(45)$$

and

$$\left\|\phi\left(\prod_{i=1}^{k}x_i\prod_{i=1}^{k}y_i\right)-\prod_{i=1}^{k}\phi(x_i)\prod_{i=1}^{k}\phi(y_i)\right\|_{\mathbb{B}}\leq\varphi(x_1,\cdots,x_k,y_1,\cdots,y_k,0,\cdots,0),\quad(46)$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbb{A}$. If there exists an $M<1$ such that

$$\varphi(x,\cdots,x,x,\cdots,x,x,\cdots,x)\leq 2M\varphi\left(\frac{x}{2},\cdots,\frac{x}{2},\frac{x}{2},\cdots,\frac{x}{2},\frac{x}{2},\cdots,\frac{x}{2}\right)$$

for all $x \in \mathbb{A}$ and if $\phi(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathbb{A}$, then there exists a homomorphims $\psi : \mathbb{A} \to \mathbb{B}$ such that

$$\|\phi(x)-\psi(x)\|_{\mathbb{B}} \le \frac{1}{2-2M}\varphi(x,\cdots,x,x,\cdots,x,x,\cdots,x) \tag{47}$$

for all $x \in \mathbf{X}$

*Proof.* We consider the set

$$\mathbb{S} := \{h : \mathbb{A} \to \mathbb{B}\} \tag{48}$$

and introduce the generalized metric on $\mathbb{S}$:

$$d(g,h) := \inf\{\lambda \in \mathbb{R}_+ : \|g(x)-h(x)\| \le \lambda\varphi(x,\cdots,x,x,\cdots,x,x,\cdots,0), \forall x \in \mathbb{A}\}, \tag{49}$$

where, as usual, $\inf \phi = +\infty$. It easy to show that $(\mathbb{S},d)$ is complete see [16]. Now we consider the linear mapping $J : \mathbb{S} \to \mathbb{S}$ such that

$$Jg(x) := \frac{1}{2}g(2x) \tag{50}$$

for all $x \in \mathbb{A}$. By Theorem 2.3, we have

$$d(Jg,Jh) \le Md(g,h) \tag{51}$$

Let $g,h \in \mathbb{S}$

We replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(x,\cdots,x,x,\cdots,x,x,\cdots,0)$ in (18), we have

$$\left\|\phi(x)-\frac{1}{2}\phi(2x)\right\|_{\mathbb{B}} \le \frac{1}{2k}\varphi(x,\cdots,x,x,\cdots,x,x,\cdots,0) \tag{52}$$

for all $x \in \mathbb{A}$. Hence

$$d(\phi,J\phi) \le \frac{1}{2k}$$

By Theorem 2.1, there exists a mapping $\psi : \mathbb{A} \to \mathbb{B}$ satisfying the following:

1) $\psi$ is a fixed point of *J*, i.e.,

$$\psi(2x) = 2\psi(x) \tag{53}$$

for all $x \in \mathbf{X}$. The mapping $\psi$ is a unique fixed point *J* in the set

$$\mathbb{Q} = \{g \in \mathbb{S} : d(\phi,g) < \infty\}$$

This implies that $\psi$ is a unique mapping satisfying (53) such that there exists a $\lambda \in (0,\infty)$ satisfying

$$\|\phi(x)-\psi(x)\| \le \lambda\varphi(x,\cdots,x,x,\cdots,x,x,\cdots,0) \tag{54}$$

for all $x \in \mathbb{A}$ (2) $d(J^l\phi,\psi) \to 0$ as $l \to \infty$. This implies equality

$$\lim_{l\to\infty}\frac{1}{2^l}\phi(2^l x) = \psi(x) \tag{55}$$

for all $x \in \mathbb{A}$ (3) $d(\phi,\psi) \le \frac{1}{1-M}d(\phi,J\phi)$, which implies

$$d(\phi,\psi) \le \frac{1}{2-2M} \tag{56}$$

It follows (44), (45) and (55) that

$$\begin{aligned}&\left\|k\psi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)+k\psi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\psi(x_i)-2k\sum_{i=1}^{k}\psi(x_i)\right\|_{\mathbb{B}}\\&=\lim_{n\to\infty}\frac{1}{2^n}\left\|k\phi\left(2^n\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+2^n\sum_{i=1}^{k}z_i\right)+k\phi\left(2^n\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+2^n\sum_{i=1}^{k}z_i\right)\right.\\&\quad\left.-\sum_{i=1}^{k}\phi(2^n x_i)-2k\sum_{i=1}^{k}\phi(2^n z_i)\right\|_{\mathbb{B}}\\&\le\lim_{n\to\infty}\frac{1}{2^n}\varphi\left(2^n x_1,\cdots,2^n x_k,2^n y_1,\cdots,2^n y_k,2^n z_1,\cdots,2^n z_k\right)=0\end{aligned}\tag{57}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\in\mathbb{A}$.

So

$$k\psi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)+k\psi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)=\sum_{i=1}^{k}\psi(x_i)+2k\sum_{i=1}^{k}\psi(z_i)\tag{58}$$

By Lemma 2.1, the mapping $\psi:\mathbb{A}\to\mathbb{B}$ is Cauchy additive mapping. According to the theorem of Th.M. Rassias (see [8]) we infer that the mapping $\psi:\mathbb{A}\to\mathbb{B}$ is $\mathbb{R}$-linear. It forllows from (46).

$$\begin{aligned}&\left\|\varphi\left(\prod_{i=1}^{k}x_i\prod_{i=1}^{k}y_i\right)-\prod_{i=1}^{k}\varphi(x_i)\prod_{i=1}^{k}\varphi(y_i)\right\|_{\mathbb{B}}\\&=\lim_{n\to\infty}\frac{1}{2^{2kn}}\left\|\phi\left(2^{2kn}\prod_{i=1}^{k}x_i\prod_{i=1}^{k}y_i\right)-\prod_{i=1}^{k}\phi(2^{nk}x_i)\prod_{i=1}^{k}\phi(2^{kn}y_i)\right\|_{\mathbb{B}}\\&\le\lim_{n\to\infty}\frac{1}{2^{2kn}}\varphi\left(2^n x_1,\cdots,2^n x_k,2^n y_1,\cdots,2^n y_k,0,\cdots,0\right)\\&\le\lim_{n\to\infty}\frac{1}{2^n}\varphi\left(2^n x_1,\cdots,2^n x_k,2^n y_1,\cdots,2^n y_k,0,\cdots,0\right)=0,\end{aligned}\tag{59}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k\in\mathbb{A}$. So

$$\psi\left(\prod_{i=1}^{k}x_i\prod_{i=1}^{k}y_i\right)=\prod_{i=1}^{k}\psi(x_i)\cdot\prod_{i=1}^{k}\psi(x_i)\tag{60}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k\in\mathbb{A}$. Thus, $\psi:\mathbb{A}\to\mathbb{B}$ is a homomorphisms satisfying (46). □

**Theorem 13.** Suppose that $\phi:\mathbb{A}\to\mathbb{B}$ be a mapping. If there is a function $\varphi:\mathbb{A}^{3k}\to[0,\infty)$ such that satisfying

$$\begin{aligned}&\left\|k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)+k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(x_i)-2k\sum_{i=1}^{k}\phi(z_i)\right\|_{\mathbb{B}}\\&\le\varphi\left(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\right),\end{aligned}\tag{61}$$

$$\sum_{j=1}^{\infty}\frac{1}{2^{2kj}}\varphi\left(\frac{1}{2^j}x_1,\cdots,\frac{1}{2^j}x_k,\frac{1}{2^j}y_1,\cdots,\frac{1}{2^j}y_k,\frac{1}{2^j}z_1,\cdots,\frac{1}{2^j}z_k\right)<\infty\tag{62}$$

and

$$\left\|\phi\left(\prod_{i=1}^{k}x_i\prod_{i=1}^{k}y_i\right)-\prod_{i=1}^{k}\phi(x_i)\prod_{i=1}^{k}\phi(y_i)\right\|_{\mathbb{B}}\le\varphi\left(x_1,\cdots,x_k,y_1,\cdots,y_k,0,\cdots,0\right),\tag{63}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k \in \mathbb{A}$. If there exists an $M<1$ such that

$$\varphi(x,\cdots,x,x,\cdots,x,x,\cdots,x) \le \frac{1}{2} M \varphi(2x,\cdots,2x,2x,\cdots,2x,2x,\cdots,2x)$$

for all $x \in \mathbb{A}$ and if $\phi(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathbb{A}$, then there exists a homomorphims $\psi:\mathbb{A}\to\mathbb{B}$ such that

$$\|\phi(x)-\psi(x)\|_{\mathbb{B}} \le \frac{M}{2-2M}\varphi(x,\cdots,x,x,\cdots,x,x,\cdots,x) \tag{64}$$

for all $x \in \mathbf{X}$

*Proof.* Now we cosider the linear mapping $J:\mathbb{S}\to\mathbb{S}$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right) \tag{65}$$

for all $x \in \mathbb{A}$.

We replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(x,\cdots,x,x,\cdots,x,x,\cdots,0)$ in (61), we have

$$\begin{aligned}\left\|\phi(x)-2\phi\left(\frac{1}{2}x\right)\right\|_{\mathbb{B}} &\le \frac{1}{k}\varphi\left(\frac{x}{2},\cdots,\frac{x}{2},\frac{x}{2},\cdots,\frac{x}{2},\frac{x}{2},\cdots,0\right)\\ &\le \frac{M}{2k}\varphi(x,\cdots,x,x,\cdots,x,x,\cdots,0)\end{aligned} \tag{66}$$

for all $x \in \mathbb{A}$. Hence

$$d(\phi, J\phi) \le \frac{M}{2k}$$

The complete proof is similar to Theorem 5.2.

□

From the theorems we have the consequences:

**Corollary 9.** Supppose $p<1$ and $\theta$ be nonnegative real numbers, and let $\phi:\mathbb{A}\to\mathbb{B}$ be a mapping such that

$$\begin{aligned}&\left\|k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)+k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(x_i)-2k\sum_{i=1}^{k}\phi(z_i)\right\|_{\mathbb{B}}\\ &\le \theta\left(\sum_{i=1}^{k}\|x_i\|_{\mathbb{A}}^p+\sum_{i=1}^{k}\|y_i\|_{\mathbb{A}}^p+\sum_{i=1}^{k}\|z_i\|_{\mathbb{A}}^p\right)\end{aligned} \tag{67}$$

$$\left\|\phi\left(\prod_{i=1}^{k}x_i\prod_{i=1}^{k}y_i\right)-\prod_{i=1}^{k}\phi(x_i)\prod_{i=1}^{k}\phi(y_i)\right\|_{\mathbb{B}} \le \theta\left(\sum_{i=1}^{k}\|x_i\|_{\mathbb{A}}^p+\sum_{i=1}^{k}\|y_i\|_{\mathbb{A}}^p+\sum_{i=1}^{k}\|z_i\|_{\mathbb{A}}^p\right) \tag{68}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k \in \mathbb{A}$.

If $\phi(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathbb{A}$, then there exists a homomorphims $\psi:\mathbb{A}\to\mathbb{B}$ such that

$$\|\phi(x)-\psi(x)\|_{\mathbb{B}} \le \left(2+\frac{1}{k}\right)\frac{\theta}{2-2^p}\|x\|_{\mathbb{A}}^p \tag{69}$$

for all $x \in \mathbb{A}$.

**Corollary 10.** Supppose $p>2$ and $\theta$ be nonnegative real numbers, and let $\phi:\mathbb{A}\to\mathbb{B}$ be a mapping such that

$$\left\| k\phi\left(\sum_{i=1}^{k}\frac{x_i+y_i}{2k}+\sum_{i=1}^{k}z_i\right)+k\phi\left(\sum_{i=1}^{k}\frac{x_i-y_i}{2k}+\sum_{i=1}^{k}z_i\right)-\sum_{i=1}^{k}\phi(x_i)-2k\sum_{i=1}^{k}\phi(z_i)\right\|_{\mathbb{B}} \\ \le \theta\left(\sum_{i=1}^{k}\|x_i\|_{\mathbb{A}}^p+\sum_{i=1}^{k}\|y_i\|_{\mathbb{A}}^p+\sum_{i=1}^{k}\|z_i\|_{\mathbb{A}}^p\right) \tag{70}$$

$$\left\|\phi\left(\prod_{i=1}^{k}x_i\prod_{i=1}^{k}y_i\right)-\prod_{i=1}^{k}\phi(x_i)\prod_{i=1}^{k}\phi(y_i)\right\|_{\mathbb{B}} \le \theta\left(\sum_{i=1}^{k}\|x_i\|_{\mathbb{A}}^p+\sum_{i=1}^{k}\|y_i\|_{\mathbb{A}}^p+\sum_{i=1}^{k}\|z_i\|_{\mathbb{A}}^p\right) \tag{71}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k \in \mathbb{A}$.

If $\phi(tx)$ is continuous in $t\in\mathbb{R}$ for each fixed $x\in\mathbb{A}$, then there exists a homomorphims $\psi:\mathbb{A}\to\mathbb{B}$ such that

$$\|\phi(x)-\psi(x)\|_{\mathbb{B}} \le \left(2+\frac{1}{k}\right)\frac{\theta}{2^p-2}\|x\|_{\mathbb{A}}^p \tag{72}$$

for all $x\in\mathbb{A}$.

## 6. Conclusion

In this paper, I have built a general Cauchy-Jensen equation to improve the classical Cauchy-Jensen equation when we build a general solution for the equation on space with an arbitrary number of variables.

## Conflicts of Interest

The author declares no conflicts of interest.

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