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Generalized Stability of Functional Inequalities with 3*k*-Variables Associated for Jordan-von Neumann-Type Additive Functional Equation

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Abstract

In this paper, we study to solve the Cauchy, Jensen and Cauchy-Jensen additive function inequalities with 3*k*-variables related to Jordan-von Neumann type in the spirit of the Rassias stability approach for approximate homomorphisms in Banach space. These are the main results of this paper.

Subject Areas

Mathematics

Keywords

Normed Spaces, Banach Space, Stability Jordan-von Neumann-Type Additive Functional Equation, Cauchy, Jensen and Cauchy-Jensen Additive Function Inequalities

1. Introduction

Let $\mathbf{X}$ and $\mathbf{Y}$ be normed spaces on the same field $\mathbb{K}$, and $f:\mathbf{X}\to\mathbf{Y}$ be a mapping. We use the notation $\|\cdot\|_{\mathbf{X}}$ ($\|\cdot\|_{\mathbf{Y}}$) for corresponding the norms on $\mathbf{X}$ and $\mathbf{Y}$. In this paper, we investigate additive functional inequalities associated with Jordan-von Neumann type additive functional equation when $\mathbf{X}$ is a normed space with norm $\|\cdot\|_{\mathbf{X}}$ and that $\mathbf{Y}$ is a *Banach* space with norm $\|\cdot\|_{\mathbf{Y}}$.

In fact, when $\mathbf{X}$ is a normed space with norm $\|\cdot\|_{\mathbf{X}}$ and that $\mathbf{Y}$ is a *Banach* space with norm $\|\cdot\|_{\mathbf{Y}}$ we solve and prove the Hyers-Ulam-Rassias type stability of following additive functional inequalities.

$$\left\|\sum_{j=1}^{k} f(x_j)+\sum_{j=1}^{k} f(y_j)+\sum_{j=1}^{k} f(z_j)\right\|_{\mathbf{Y}} \le \left\|2kf\left(\frac{\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}y_j+\sum_{j=1}^{k}z_j}{2k}\right)\right\|_{\mathbf{Y}}, \quad (1)$$

and

$$\left\| \sum_{j=1}^{k} f\left(x_j\right) + \sum_{j=1}^{k} f\left(y_j\right) + \sum_{j=1}^{k} f\left(z_j\right) \right\|_{\mathbf{Y}} \leq \left\| f\left(\sum_{j=1}^{k} x_j + \sum_{j=1}^{k} y_j + \sum_{j=1}^{k} z_j \right) \right\|_{\mathbf{Y}}, \tag{2}$$

final

$$\left\| \sum_{j=1}^{k} f\left(x_j\right) + \sum_{j=1}^{k} f\left(y_j\right) + 2k\sum_{j=1}^{k} f\left(z_j\right) \right\|_{\mathbf{Y}} \leq \left\| 2kf\left(\frac{\sum_{j=1}^{k} x_j + \sum_{j=1}^{k} y_j}{2k} + \sum_{j=1}^{k} z_j \right) \right\|_{\mathbf{Y}}. \tag{3}$$

The study of the stability of generalized additive functional inequalities associated with Jordan-von Neumann type additive functional equational originated from a question of S.M. Ulam [1], concerning the stability of group homomorphisms.

Let $(\mathbf{G},*)$ be a group and let $(\mathbf{G}',\circ,d)$ be a metric group with metric $d(\cdot,\cdot)$. Geven $\varepsilon > 0$, does there exist a $\delta > 0$ such that if $f:\mathbf{G} \to \mathbf{G}'$ satisfies

$$d\left(f\left(x * y\right), f\left(x\right) \circ f\left(y\right)\right) < \delta, \forall x \in \mathbf{G}$$

then there is a homomorphism $h:\mathbf{G} \to \mathbf{G}'$ with

$$d\left(f\left(x\right), h\left(x\right)\right) < \varepsilon, \forall x \in \mathbf{G}$$

The concept of stability for a functional equation arises when we replace functional equation with an inequality that acts as a perturbation of the equation. Thus the stability question of functional equations is how the solutions of the inequality differ from those of the given function equation.

Hyers gave a first affirmative answer to the question of Ulam as follows: In 1941 D. H. Hyers [2] Let $\varepsilon \geq 0$ and let $f:\mathbf{E}_1 \to \mathbf{E}_2$ be a mapping between *Banach* space such that

$$\left\| f\left(x+y\right) - f\left(x\right) - f\left(y\right) \right\| \leq \varepsilon, \tag{4}$$

for all $x, y \in \mathbf{E}_1$ and some $\varepsilon \geq 0$. It was shown that the limit

$$T\left(x\right) = \lim_{n\to\infty} \frac{f\left(2^n x\right)}{2^n} \tag{5}$$

exists for all $x \in \mathbf{E}_1$ and that $T:\mathbf{E}_1 \to \mathbf{E}_2$ is that unique additive mapping satisfying

$$\left\| f\left(x\right) - T\left(x\right) \right\| \leq \varepsilon, \forall x \in \mathbf{E}_1. \tag{6}$$

Next in 1978 Th. M. Rassias [3] provided a generalization of Hyers' Theorem which allows the Cauchy difference to be unbounded:

Consider $\mathbf{E}, \mathbf{E}'$ to be two Banach spaces, and let $f:\mathbf{E} \to \mathbf{E}'$ be a mapping such that $f(tx)$ is continuous in *t* for each fixed *x*. Assume that there exist $\theta \geq 0$ and $p \in [0,1)$, $\varepsilon > 0$ such that

$$\left\| f\left(x+y\right) - f\left(x\right) - f\left(y\right) \right\| \leq \varepsilon\left(\|x\|^p + \|y\|^p \right), \forall x, y \in \mathbb{E}. \tag{7}$$

where ε and *p* is constants with $\varepsilon > 0$ and $p < 1$. Then the limit

$$T(x)=\lim_{n\to\infty}\frac{f(2^n x)}{2^n} \tag{8}$$

there exists a unique linear $L:\mathbf{E}\to\mathbf{E}'$ satifies

$$\|f(x)-L(x)\|\le\frac{2\varepsilon}{2-2^p}\|x\|^p, x\in\mathbf{E}. \tag{9}$$

If $p<0$, then inequality (7) holds for $x,y\neq 0$ and (9) for $x\neq 0$.

We notice that in Rassias' functional inequality (7) Mathematicians around the world such as [4] [5] as well as Rassias have asserted that the inequality (7) no longer holds true when $p=1$ from the assertion that gave rise to the idea to generalize the generalized functional equation Hyers-Ulam more specifically.

Thus, to replace the non-existent condition mentioned above, Mathematician Rassias [3] has given the following specific conditions: $\|x\|^p+\|y\|^p$ by $\|x\|^p\|y\|^p$ for $p,q\in\mathbb{R}$ with $p+q\neq 1$.

For all $x\in\mathbf{E}$. Găvruta [6] provided a further generalization of Rassias' theorem. During the last two decades, a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers-Ulam stability to a number of functional equations and mappings.

Afterward Gilány [7] showed that is if satisfies the functional inequality

$$\left\|2f(x)+2f(y)-f\left(xy^{-1}\right)\right\|\le\|f(xy)\| \tag{10}$$

Then *f* satisfies the Jordan-von Neumann functional equation

$$2f(x)+2f(y)=f(xy)+f\left(xy^{-1}\right) \tag{11}$$

Then, mathematicians in the world proved to extend the functional inequality (11) as [7] [8] [9]. In addition, mathematicians have developed the achievements of their predecessors who have built mathematical models from advanced to modern mathematics, especially functional equations applied on function spaces to Unlocking means connecting with other Maths [3]-[34]. Recently, the authors studied the Hyers-Ulam-Rassias type stability for the following functional inequalities (see [30] [31] [33])

$$\|f(x)+f(y)+f(z)\|\le\left\|k\left(f\left(\frac{x+y+z}{k}\right)\right)\right\|,|k|<|3|, \tag{12}$$

$$\|f(x_1)+f(x_2)+\cdots+f(x_n)\|\le\left\|kf\left(\frac{x_1+x_2+\cdots+x_n}{k}\right)\right\|,|n|>|k|, \tag{13}$$

$$\left\|\sum_{j=1}^n f(x_j)+\sum_{j=1}^n f\left(\frac{x_{n+j}}{n}\right)\right\|\le\left\|kf\left(\frac{\sum_{j=1}^n x_j}{k}+\frac{\sum_{j=1}^n x_{n+j}}{n\cdot k}\right)\right\|,|n|>|k|. \tag{14}$$

in Banach spaces.

In this paper, we solve and proved the Hyers-Ulam-Rassias type stability for functional inequality (1). (2) and (3) are the functional inequalities with 3*k*-variables. Under suitable assumptions on spaces $\mathbf{X}$ and $\mathbf{Y}$, we will prove that the mappings satisfy the functional inequality (1). (2) and (3). Thus, the results in this paper are generalization of those in [21] [30] [31] [33] for functional inequality with $3k$-variables.

The paper is organized as follows:

In the section preliminary, we remind some basic notations such as solutions to the inequalities.

Section 3: The basis for building solutions for functional inequalities related to the type of Jordan-von Neumann additive functional equations.

Section 4: Establishing solutions to functional inequality (1) related to the type of Jensen additive functional equation.

Section 5: Establishing solutions to functional inequality (2) related to the type of Cauchy additive functional equation.

Section 6: Establishing solutions to functional inequality (3) related to the type of Cauchy-Jensen additive functional equation.

2. Preliminaries

Solutions to the Inequalities

The functional equation

$$f(x+y)=f(x)+f(y)$$

is called the Cauchy equation. In particular, every solution of the Cauchy equation is said to be an *additive mapping*.

The functional equation

$$f\left(\frac{x+y}{2}\right)=\frac{1}{2}f(x)+\frac{1}{2}f(y)$$

is called the Jensen equation. In particular, every solution of the Jensen equation is said to be a Jensen *additive mapping*.

The functional equation

$$2f\left(\frac{x+y}{2}+z\right)=f(x)+f(y)+2f(z)$$

is called the Cauchy-Jensen equation. In particular, every solution of the Cauchy-Jensen equation is said to be a Jensen-Cauchy *additive mapping*.

3. The Basis for Building Solutions for Functional Inequalities Related to the Type of Jordan-von Neumann Additive Functional Equations

The basis for building solutions for functional inequalities related to the type of Jordan-von Neumann additive functional equations. Now, we first study the solutions of (1), (2) and (3). Note that for this inequality, $\mathbf{X}$ is a normed space with norm $\|\cdot\|_{\mathbf{X}}$ and that $\mathbf{Y}$ is a *Banach* space with norm $\|\cdot\|_{\mathbf{Y}}$. Under this setting, we can show that the mappings satisfying (1), (2) and (3) are additive.

Here we assume that G is a $3k$-divisible abelian group.

Proposition 1. Suppose $f:\mathbf{X}\to\mathbf{Y}$ be a mapping such that

$$\left\|\sum_{j=1}^{k} f\left(x_j\right)+\sum_{j=1}^{k} f\left(y_j\right)+\sum_{j=1}^{k} f\left(z_j\right)\right\|_{\mathbf{Y}} \leq\left\|2kf\left(\frac{\sum_{j=1}^{k} x_j+\sum_{j=1}^{k} y_j+\sum_{j=1}^{k} z_j}{2k}\right)\right\|_{\mathbf{Y}} \tag{15}$$

for all $x_j, y_n, z_j \in \mathbf{X}$ for all $j=1 \to n$ then f is additive.

Proof. Assume that $f:\mathbf{X}\to\mathbf{Y}$ satisfies (15).

We replaced $\left(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\right)$ by $\left(0,\cdots,0,0,\cdots,0,0,\cdots,0\right)$ in (15), we have $f\left(0\right)=0$.

Next, we replaced $\left(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\right)$ by $\left(x,0,\cdots,0,-x,0,\cdots,0,0,\cdots,0\right)$ in (15), we have

$$\left\|f\left(x\right)+f\left(-x\right)\right\|_{\mathbf{Y}} \leq \left\|2nf\left(0\right)\right\|_{\mathbf{Y}} \tag{16}$$

for all $x\in\mathbf{X}$.

Hence $f\left(x\right)=-f\left(-x\right)$, $\forall x\in\mathbf{X}$.

Next, we replaced $\left(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\right)$ by $\left(x,0,\cdots,0,y,0,\cdots,0,-x-y,\cdots,0\right)$ in (15), we have

$$\left\|f\left(x\right)+f\left(y\right)-f\left(x+y\right)\right\|_{Y}=\left\|f\left(x\right)+f\left(y\right)+f\left(-x-y\right)\right\|_{Y}\leq\left\|2nf\left(0\right)\right\|_{Y}=0 \tag{17}$$

for all $x,y\in X$. It follows that $f\left(x+y\right)=f\left(x\right)+f\left(y\right)$. This completes the proof. □

Proposition 2. Suppose $f:\mathbf{X}\to\mathbf{Y}$ be a mapping such that

$$\left\|\sum_{j=1}^{n} f\left(x_j\right)+\sum_{j=1}^{n} f\left(y_j\right)+\sum_{j=1}^{n} f\left(z_j\right)\right\|_{\mathbf{Y}} \leq\left\|f\left(\sum_{j=1}^{n} x_j+\sum_{j=1}^{n} y_j+\sum_{j=1}^{n} z_j\right)\right\|_{\mathbf{Y}} \tag{18}$$

for all $x_j, y_j, z_j \in \mathbf{X}$ for all $j=1 \to n$ then f is additive.

Proof. Assume that $f:X\to Y$ satisfies (18).

We replaced $\left(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\right)$ by $\left(0,\cdots,0,0,\cdots,0,0,\cdots,0\right)$ in (18), we have

$$f\left(0\right)=0.$$

Next, we replaced $\left(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\right)$ by $\left(x,\cdots,0,-x,\cdots,0,0,\cdots,0\right)$ in (18), we have

$$\left\|f\left(x\right)+f\left(-x\right)\right\|_{\mathbf{Y}} \leq \left\|f\left(0\right)\right\|_{\mathbf{Y}} \tag{19}$$

for all $x\in\mathbf{X}$.

Hence $f\left(x\right)=-f\left(-x\right)$, $\forall x\in\mathbf{X}$.

Next, we replaced $\left(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\right)$ by $\left(x,0,\cdots,0,y,0,\cdots,0,-x-y,\cdots,0\right)$ in (18), we have

$$\left\|f\left(x\right)+f\left(y\right)-f\left(x+y\right)\right\|_{\mathbf{Y}}=\left\|f\left(x\right)+f\left(y\right)+f\left(-x-y\right)\right\|_{Y}\leq\left\|f\left(0\right)\right\|_{\mathbf{Y}}=0 \tag{20}$$

for all $x,y\in\mathbf{X}$. It follows that $f\left(x+y\right)=f\left(x\right)+f\left(y\right)$ This completes the proof.

Proposition 3. Suppose $f:\mathbf{G}\to\mathbf{Y}$ be a mapping such that

$$\left\|\sum_{j=1}^{n} f\left(x_j\right)+\sum_{j=1}^{n} f\left(y_j\right)+2n\sum_{j=1}^{n} f\left(z_j\right)\right\|_{\mathbf{Y}} \leq\left\|2nf\left(\frac{\sum_{j=1}^{n} x_j+\sum_{j=1}^{n} y_j}{2n}+\sum_{j=1}^{n} z_j\right)\right\|_{\mathbf{Y}} \tag{21}$$

for all $x_j, y_j, z_j \in \mathbf{X}$ for all $j = 1 \to n$ then *f* is additive.

Proof. Assume that $f : \mathbf{G} \to \mathbf{Y}$ satisfies (21).

We replaced $(x_1, \cdots, x_n, y_1, \cdots, y_n, z_1, \cdots, z_n)$ by $(0, \cdots, 0, 0, \cdots, 0, 0, \cdots, 0)$ in (21), we have we get

$$\left\| \left(2n^2 + 2n\right) f(0) \right\| \le \left\| 2nf(0) \right\|_{\mathbf{Y}} \tag{22}$$

So $f(0) = 0$

Next, we replaced $(x_1, \cdots, x_n, y_1, \cdots, y_n, z_1, \cdots, z_n)$ by $(x, 0, \cdots, 0, -x, 0, \cdots, 0, 0, \cdots, 0)$ in (21), we have

$$\left\| f(x) + f(-x) \right\|_Y \le \left\| 2nf(0) \right\|_{\mathbf{Y}} \tag{23}$$

for all $x \in X$.

Hence $f(x) = -f(-x)$, $\forall x \in \mathbf{X}$.

Next, we replaced $(x_1, \cdots, x_n, y_1, \cdots, y_n, z_1, \cdots, z_n)$ by $(-2nz, 0, \cdots, 0, 0, \cdots, 0, z, 0, \cdots, 0)$ in (21), we have

$$\left\| f(-2nz) + 2nf(z) \right\|_{\mathbf{Y}} \le \left\| 2nf(0) \right\|_{\mathbf{Y}} \tag{24}$$

for all $x \in \mathbf{X}$.

Thus $f(2nz) = 2nf(z)$, $\forall z \in \mathbf{G}$.

Next, we replaced $(x_1, \cdots, x_n, y_1, \cdots, y_n, z_1, \cdots, z_n)$ by $\left(x_1, \cdots, x_n, y_1, \cdots, y_n, -\frac{x_1 + y_1}{2n}, \cdots, -\frac{x_n + y_n}{2n}\right)$ in (21), we have

$$\begin{aligned} &\left\| \sum_{j=1}^{n} f(x_j) + \sum_{j=1}^{n} f(y_j) - \sum_{j=1}^{n} f(x_j + y_j) \right\|_{\mathbf{Y}} \\ &= \left\| \sum_{j=1}^{n} f(x_j) + \sum_{j=1}^{n} f(y_j) + 2n \sum_{j=1}^{n} f\left(-\frac{x_j + y_j}{2n}\right) \right\|_{\mathbf{Y}} \\ &\le \left\| 2nf(0) \right\|_{\mathbf{Y}} \end{aligned} \tag{25}$$

$$\forall x_1, \cdots, x_k, y_1, \cdots, y_k, -\frac{x_1 + y_1}{2n}, \cdots, -\frac{x_k + y_k}{2n} \in \mathbf{G}.$$

Thus

$$\sum_{j=1}^{n} f(x_j) + \sum_{j=1}^{n} f(y_j) - \sum_{j=1}^{n} f(x_j + y_j) = 0 \tag{26}$$

Next put $x = x_j, y = y_j$ for all $j = 1 \to n$ in (26), we have

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in \mathbf{G}$. It follows that *f* is an additive mapping and the proof is complete. □

4. Establishing Solutions to Functional Inequality (1) Related to the Type of Jensen Additive Functional Equation

Now, we first study the solutions of (1). Note that for this inequality, $\mathbf{X}$ is a normed space with norm $\|\cdot\|_{\mathbf{X}}$ and that $\mathbf{Y}$ is a *Banach* space with norm $\|\cdot\|_{\mathbf{Y}}$. Under this setting, we can show that the mappings satisfying (1) are Jensen additive. These results are given in the following.

Theorem 4. Suppose $q>1$, θ be non-negative real and $f:\mathbf{X}\to\mathbf{Y}$ be an odd mapping such that

$$\begin{aligned}&\left\|\sum_{j=1}^{k} f\left(x_j\right)+\sum_{j=1}^{k} f\left(y_j\right)+\sum_{j=1}^{k} f\left(z_j\right)\right\|_{\mathbf{Y}}\\ &\le\left\|2kf\left(\frac{\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}y_j+\sum_{j=1}^{k}z_j}{2k}\right)\right\|_{\mathbf{Y}}+\theta\left(\sum_{j=1}^{k}\left\|x_j\right\|_{\mathbf{X}}^{q}+\sum_{j=1}^{k}\left\|y_j\right\|_{\mathbf{X}}^{q}+\sum_{j=1}^{k}\left\|z_j\right\|_{\mathbf{X}}^{q}\right)\end{aligned} \tag{27}$$

for all $x_j, y_j, z_j\in\mathbf{X}$ for all $j=1\to n$.

Then there exists a unique additive mapping $H:\mathbf{X}\to\mathbf{Y}$ such that

$$\left\|\frac{f(x)+f(-x)}{2k}-H(x)\right\|_{Y}\le\frac{(2k)^q+2k}{(2k)^q-2k}\theta\|x\|_X^q. \tag{28}$$

for all $x\in\mathbf{X}$.

Proof. Assume that $f:\mathbf{X}\to\mathbf{Y}$ satisfies (27).

We replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(x,\cdots,x,x,\cdots,x,-2kx,0,\cdots,0)$ in (27), we have

$$\left\|2kf(x)+f(-2kx)\right\|_{\mathbf{Y}}\le\left((2k)^q+2k\right)\theta\|x\|_{\mathbf{X}}^q \tag{29}$$

for all $x\in\mathbf{X}$. Replacing x by $-x$ in (29), we get

$$\left\|2kf(-x)+f(2kx)\right\|_{\mathbf{Y}}\le\left((2k)^q+2k\right)\theta\|x\|_{\mathbf{X}}^q \tag{30}$$

for all $x\in\mathbf{X}$. It follows from (29) and (30) that

$$\begin{aligned}&\left\|2k\left(f(x)+f(-x)\right)-\left(f(2kx)+f(-2kx)\right)\right\|_{\mathbf{Y}}\\ &\le\left\|2kf(x)+f(-2kx)\right\|_{\mathbf{Y}}+\left\|2kf(-x)+f(2kx)\right\|_{\mathbf{Y}}\\ &\le 2\left((2k)^q+2k\right)\theta\|x\|_{\mathbf{X}}^q\end{aligned} \tag{31}$$

for all $x\in\mathbf{X}$. Let $Q(x)=\dfrac{f(x)+f(-x)}{2k}$. From (31) we have

$$\left\|2kQ(x)-Q(2kx)\right\|\le\left((2k)^q+2k\right)\frac{\theta}{k}\|x\|_{\mathbf{X}}^q \tag{32}$$

for all $x\in\mathbf{X}$. So

$$\left\|Q(x)-2kQ\left(\frac{x}{2k}\right)\right\|\le\frac{(2k)^q+2k}{(2k)^q}\frac{\theta}{k}\|x\|_{\mathbf{X}}^q \tag{33}$$

Hence we have

$$\begin{aligned}&\left\|(2k)^l Q\left(\frac{x}{(2k)^l}\right)-(2k)^m Q\left(\frac{x}{(2k)^m}\right)\right\|_Y\\ &\le\sum_{j=l}^{m-1}\left\|(2k)^j Q\left(\frac{x}{(2k)^j}\right)-(2k)^{j+1} Q\left(\frac{x}{(2k)^{j+1}}\right)\right\|_Y\\ &\le\frac{(2k)^q+2k}{(2k)^q}\frac{\theta}{k}\sum_{j=l}^{m-1}\frac{(2k)^j}{(2k)^{qj}}\theta\|x\|_{\mathbf{X}}^q.\end{aligned} \tag{34}$$

for all nongnegative m and l with $m > l$, $\forall x \in \mathbf{X}$. It follows from (34) that the sequence $\left\{(2k)^n f\left(\frac{x}{(2k)^n}\right)\right\}$ is a cauchy sequence for all $x \in \mathbf{X}$. Since $\mathbf{Y}$ is a Banach space, the sequence $\left\{(2k)^n f\left(\frac{x}{(2k)^n}\right)\right\}$ coverges.

So one can define the mapping $H : \mathbf{X} \to \mathbf{Y}$ by

$$H(x) := \lim_{n\to\infty} (2k)^n f\left(\frac{x}{(2k)^n}\right)$$

for all $x \in \mathbf{X}$.

$$\begin{aligned}
&\left\|\sum_{j=1}^{k} H(x_j) + \sum_{j=1}^{k} H(y_j) + \sum_{j=1}^{k} H(z_j)\right\|_{\mathbf{Y}} \\
&= \lim_{n\to\infty} (2k)^n \left\|\sum_{j=1}^{k} Q\left(\frac{x_j}{(2k)^n}\right) + \sum_{j=1}^{k} Q\left(\frac{y_j}{(2k)^n}\right) + \sum_{j=1}^{k} Q\left\|\left(\frac{z_j}{(2k)^n}\right)\right._{\mathbf{Y}} \\
&= \lim_{n\to\infty} \frac{(2k)^n}{2k} \left\|\sum_{j=1}^{k} f\left(\frac{x_j}{(2k)^n}\right) + \sum_{j=1}^{k} f\left(\frac{y_j}{(2k)^n}\right) + \sum_{j=1}^{k} f\left(\frac{z_j}{(2k)^n}\right)\right. \\
&\quad \left. - \sum_{j=1}^{k} f\left(-\frac{x_j}{(2k)^n}\right) - \sum_{j=1}^{k} f\left(-\frac{y_j}{(2k)^n}\right) - \sum_{j=1}^{k} f\left(-\frac{z_j}{(2k)^n}\right)\right\|_{\mathbf{Y}} \\
&\le \lim_{n\to\infty} \frac{(2k)^n}{2k} \left\| 2kf\left(\frac{\sum_{j=1}^{k} x_j + \sum_{j=1}^{k} y_j + \sum_{j=1}^{k} z_j}{(2k)^{n+1}}\right)\right. \\
&\quad \left. - 2kf\left(\frac{\sum_{j=1}^{k} x_j + \sum_{j=1}^{k} y_j + \sum_{j=1}^{k} z_j}{(-2k)^{n+1}}\right)\right\|_{\mathbf{Y}} \\
&\quad + \lim_{n\to\infty} \frac{(2k)^n}{2k} \theta\left(\sum_{j=1}^{k} \|x_j\|_{\mathbf{X}}^q + \sum_{j=1}^{k} \|y_j\|_{\mathbf{X}}^q + \sum_{j=1}^{k} \|z_j\|_{\mathbf{X}}^q\right) \\
&= \left\| 2kH\left(\frac{\sum_{j=1}^{k} x_j + \sum_{j=1}^{k} y_j + \sum_{j=1}^{k} z_j}{2k}\right)\right\|_{\mathbf{Y}}
\end{aligned} \tag{35}$$

for all $x_j, y_j, z_j \in \mathbf{X}$ for all $j = 1 \to k$. So

$$\left\|\sum_{j=1}^{k} H(x_j) + \sum_{j=1}^{k} H(y_j) + \sum_{j=1}^{k} H(z_j)\right\|_{\mathbf{Y}} \le \left\| 2kH\left(\frac{\sum_{j=1}^{k} x_j + \sum_{j=1}^{k} y_j + \sum_{j=1}^{k} z_j}{2k}\right)\right\|_{\mathbf{Y}} \tag{36}$$

for all $x_j, y_j, z_j \in \mathbf{X}$ for all $j = 1 \to n$. By Proposition 3.1, the mapping $H : \mathbf{X} \to \mathbf{Y}$ is additive. Now, let $T : \mathbf{X} \to \mathbf{Y}$ be another additive mapping satisfy (28) then we have

$$\|H(x) - T(x)\|_{\mathbf{Y}} = (2k)^n \left\| h\left(\frac{x}{(2k)^n}\right) - T\left(\frac{x}{(2k)^n}\right)\right\|_{\mathbf{Y}}$$

$$\le (2k)^n \left(\left\| h\left(\frac{x}{(2k)^n}\right) - f\left(\frac{x}{(2k)^n}\right) \right\|_{\mathbf{Y}} + \left\| T\left(\frac{x}{(2k)^n}\right) - f\left(\frac{x}{(2k)^n}\right) \right\|_{\mathbf{Y}} \right)$$
$$\le \frac{2\left((2k)^n + 2k\right)}{(2k)^n - 2k} \cdot \frac{\theta}{k} \cdot \frac{(2k)^n}{(2k)^{nq}} \|x\|_{\mathbf{X}}^q . \tag{37}$$

which tends to zero as $q \to \infty$ for all $x \in \mathbf{X}$. So we can conclude that $H(x) = T(x)$ for all $x \in \mathbf{X}$. This proves the uniqueness of *H*. Thus the mapping $H : \mathbf{X} \to \mathbf{Y}$ is additive mapping satisfying (28). ☐

Theorem 5. Suppose $q < 1$, θ be positive real numbers and $f : \mathbf{X} \to \mathbf{Y}$ be a mapping such that

$$\left\| \sum_{j=1}^k f(x_j) + \sum_{j=1}^k f(y_j) + \sum_{j=1}^k f(z_j) \right\|_{\mathbf{Y}}$$
$$\le \left\| 2kf\left(\frac{\sum_{j=1}^k x_j + \sum_{j=1}^k y_j + \sum_{j=1}^k z_j}{2k} \right) \right\|_{\mathbf{Y}} \tag{38}$$
$$+ \theta \left(\sum_{j=1}^k \|x_j\|_{\mathbf{X}}^q + \sum_{j=1}^k \|y_j\|_{\mathbf{X}}^q + \sum_{j=1}^k \|z_j\|_{\mathbf{X}}^q \right)$$

for all $x_j, y_j, z_j \in \mathbf{X}$ for all $j = 1 \to n$.

Then there exists a unique additive mapping $H : \mathbf{X} \to \mathbf{Y}$ such that

$$\left\| \frac{f(x) + f(-x)}{2k} - H(x) \right\|_Y \le \frac{2k + (2k)^q}{2k - (2k)^q} \theta \|x\|_X^q . \tag{39}$$

for all $x \in \mathbf{X}$.

The rest of the Proof is similar to the Proof of Theorem 4.

Theorem 6. Suppose $q > p^{-1}$ with $p \ge 3$, θ be non-negative real and $f : \mathbf{X} \to \mathbf{Y}$ be a mapping such that

$$\left\| \sum_{j=1}^k f(x_j) + \sum_{j=1}^k f(y_j) + \sum_{j=1}^k f(z_j) \right\|_{\mathbf{Y}}$$
$$\le \left\| 2kf\left(\frac{\sum_{j=1}^k x_j + \sum_{j=1}^k y_j + \sum_{j=1}^k z_j}{2k} \right) \right\|_{\mathbf{Y}} \tag{40}$$
$$+ \theta \prod_{j=1}^k \|x_j\|_{\mathbf{X}}^q \cdot \prod_{j=1}^k \|y_j\|_{\mathbf{X}}^q \cdot \|z_1\|_{\mathbf{X}}^{kq} \cdot \left(1 + \prod_{j=2}^k \|z_j\|_{\mathbf{X}}^q \right)$$

for all $x_j, y_j, z_j \in \mathbf{X}$ for all $j = 1 \to n$.

Then there exists a unique additive mapping $H : \mathbf{X} \to \mathbf{Y}$ such that

$$\left\| \frac{f(x) + f(-x)}{2k} - H(x) \right\|_Y \le \frac{(2k)^q}{(2k)^{3kq} - 2k} \theta \|x\|_X^{3kq} . \tag{41}$$

for all $x \in \mathbf{X}$.

Proof. Assume that $f : \mathbf{X} \to \mathbf{Y}$ satisfies (40).

We replacing $(x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_k)$ by

$\left(x,\cdots,x,x,\cdots,x,-2kx,0,\cdots,0\right)$ in (40), we have

$$\left\|2kf\left(x\right)+f\left(-2kx\right)\right\|_{\mathbf{Y}}\leq\left|2k\right|^{kq}\theta\left\|x\right\|_{\mathbf{X}}^{3kq} \tag{42}$$

for all $x\in\mathbf{X}$. Replacing x by $-x$ in (42), we get

$$\left\|2kf\left(-x\right)+f\left(2kx\right)\right\|_{\mathbf{Y}}\leq\left|2k\right|^{kq}\theta\left\|x\right\|_{\mathbf{X}}^{3kq} \tag{43}$$

for all $x\in\mathbf{X}$. It follows from (42) and (43) that

$$\begin{aligned}&\left\|2k\left(f\left(x\right)+f\left(-x\right)\right)-\left(f\left(2kx\right)+f\left(-2kx\right)\right)\right\|_{\mathbf{Y}}\\&\leq\left\|2kf\left(x\right)+f\left(-2kx\right)\right\|_{\mathbf{Y}}+\left\|2kf\left(-x\right)+f\left(2kx\right)\right\|_{\mathbf{Y}}\\&\leq2\left|2k\right|^{kq}\theta\left\|x\right\|_{\mathbf{X}}^{3kq}\end{aligned} \tag{44}$$

for all $x\in\mathbf{X}$. Let $Q\left(x\right)=\dfrac{f\left(x\right)+f\left(-x\right)}{2k}$. From (31) we have

$$\left\|2kQ\left(x\right)-Q\left(2kx\right)\right\|\leq\left|2k\right|^{kq}\frac{\theta}{k}\left\|x\right\|_{\mathbf{X}}^{3kq} \tag{45}$$

for all $x\in\mathbf{X}$. So

$$\left\|Q\left(x\right)-2kQ\left(\frac{x}{2k}\right)\right\|\leq\frac{\left|2k\right|^{kq}}{\left|2k\right|^{3kq}}\frac{\theta}{k}\left\|x\right\|_{\mathbf{X}}^{3kq} \tag{46}$$

Hence we have

$$\begin{aligned}&\left\|\left(2k\right)^{l}Q\left(\frac{x}{\left(2k\right)^{l}}\right)-\left(2k\right)^{m}Q\left(\frac{x}{\left(2k\right)^{m}}\right)\right\|_{Y}\\&\leq\sum_{j=l}^{m-1}\left\|\left(2k\right)^{j}Q\left(\frac{x}{\left(2k\right)^{j}}\right)-\left(2k\right)^{j+1}Q\left(\frac{x}{\left(2k\right)^{j+1}}\right)\right\|_{Y}\\&\leq\frac{\left|2k\right|^{kq}}{\left|2k\right|^{3kq}}\frac{\theta}{k}\sum_{j=l}^{m-1}\frac{\left(2k\right)^{j}}{\left(2k\right)^{3kqj}}\theta\left\|x\right\|_{\mathbf{X}}^{3kq}.\end{aligned} \tag{47}$$

for all nongnegative m and l with $m>l$, $\forall x\in\mathbf{X}$. It follows from (47) that the sequence $\left\{\left(2k\right)^{n}Q\left(\frac{x}{\left(2k\right)^{n}}\right)\right\}$ is a Cauchy sequence for all $x\in\mathbf{X}$. Since $\mathbf{Y}$ is a Banach space, the sequence $\left\{\left(2k\right)^{n}Q\left(\frac{x}{\left(2k\right)^{n}}\right)\right\}$ converges.

So one can define the mapping $H:\mathbf{X}\to\mathbf{Y}$ by

$$H\left(x\right):=\lim_{n\to\infty}\left(2k\right)^{n}Q\left(\frac{x}{\left(2k\right)^{n}}\right)$$

for all $x\in\mathbf{X}$. Moreover, letting $l=0$ and passing the *limit* $m\to\infty$ in (47), we have (41). The rest of the Proof is similar to the Proof of Theorem 4. □

Theorem 7. Suppose $q<p^{-1}$ with $p\geq3$, θ be non-negative real and

$f:\mathbf{X}\to\mathbf{Y}$ be a mapping such that

$$\left\|\sum_{j=1}^{k} f\left(x_j\right)+\sum_{j=1}^{k} f\left(y_j\right)+\sum_{j=1}^{k} f\left(z_j\right)\right\|_{\mathbf{Y}} \leq \left\|2kf\left(\frac{\sum_{j=1}^{k} x_j+\sum_{j=1}^{k} y_j+\sum_{j=1}^{k} z_j}{2k}\right)\right\|_{\mathbf{Y}} + \theta\prod_{j=1}^{k}\left\|x_j\right\|_{\mathbf{X}}^{q}\cdot\prod_{j=1}^{k}\left\|y_j\right\|_{\mathbf{X}}^{q}\cdot\left\|z_1\right\|_{\mathbf{X}}^{kq}\cdot\left(1+\prod_{j=2}^{k}\left\|z_j\right\|_{\mathbf{X}}^{q}\right) \quad (48)$$

for all $x_j, y_j, z_j\in\mathbf{X}$ for all $j=1\to n$.

Then there exists a unique additive mapping $H:\mathbf{X}\to\mathbf{Y}$ such that

$$\left\|\frac{f(x)+f(-x)}{2k}-H(x)\right\|_{Y} \leq \frac{(2k)^{q}}{2k-(2k)^{3kq}}\theta\|x\|_{X}^{3kq}. \quad (49)$$

for all $x\in\mathbf{X}$.

Proof. Assume that $f:\mathbf{X}\to\mathbf{Y}$ satisfies (40).

We replacing $\left(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\right)$ by $\left(x,\cdots,x,x,\cdots,x,-2kx,0,\cdots,0\right)$ in (40), we have

$$\left\|2kf(x)+f(-2kx)\right\|_{\mathbf{Y}} \leq |2k|^{kq}\theta\|x\|_{\mathbf{X}}^{3kq} \quad (50)$$

for all $x\in\mathbf{X}$. Replacing *x* by -*x* in (50), we get

$$\left\|2kf(-x)+f(2kx)\right\|_{\mathbf{Y}} \leq |2k|^{kq}\theta\|x\|_{\mathbf{X}}^{3kq} \quad (51)$$

for all $x\in\mathbf{X}$. It follows from (50) and (51) that

$$\begin{aligned}&\left\|2k\left(f(x)+f(-x)\right)-\left(f(2kx)+f(-2kx)\right)\right\|_{\mathbf{Y}}\\ &\leq\left\|2kf(x)+f(-2kx)\right\|_{\mathbf{Y}}+\left\|2kf(-x)+f(2kx)\right\|_{\mathbf{Y}}\\ &\leq 2|2k|^{kq}\theta\|x\|_{\mathbf{X}}^{3kq}\end{aligned} \quad (52)$$

for all $x\in\mathbf{X}$. Let $Q(x)=\dfrac{f(x)+f(-x)}{2k}$. From (52) we have

$$\left\|2kQ(x)-Q(2kx)\right\|_{\mathbf{Y}} \leq |2k|^{kq}\frac{\theta}{k}\|x\|_{\mathbf{X}}^{3kq} \quad (53)$$

for all $x\in\mathbf{X}$. So

$$\left\|Q(x)-\frac{1}{2k}Q(2kx)\right\|_{\mathbf{Y}} \leq |2k|^{kq}\frac{\theta}{2k^2}\|x\|_{\mathbf{X}}^{3kq} \quad (54)$$

Hence we have

$$\begin{aligned}&\left\|\frac{1}{(2k)^{l}}Q\left((2k)^{l}x\right)-\frac{1}{(2k)^{m}}Q\left((2k)^{m}x\right)\right\|_{\mathbf{Y}}\\ &\leq\sum_{j=l}^{m-1}\left\|\frac{1}{(2k)^{j}}Q\left((2k)^{j}x\right)-\frac{1}{(2k)^{j+1}}Q\left((2k)^{j+1}x\right)\right\|_{\mathbf{Y}}\\ &\leq|2k|^{kq}\frac{\theta}{2k^2}\sum_{j=l}^{m-1}\frac{(2k)^{3kqj}}{(2k)^{j}}\theta\|x\|_{\mathbf{X}}^{3kq}.\end{aligned} \quad (55)$$

for all nongnegative *m* and *l* with $m > l$, $\forall x \in \mathbf{X}$. It follows from (55) that the sequence $\left\{ \frac{1}{(2k)^n} Q\left((2k)^n x\right) \right\}$ is a Cauchy sequence for all $x \in \mathbf{X}$. Since $\mathbf{Y}$ is a Banach space, the sequence $\left\{ \frac{1}{(2k)^n} Q\left((2k)^n x\right) \right\}$ converges.

So one can define the mapping $H : \mathbf{X} \to \mathbf{Y}$ by

$$H(x) := \lim_{n\to\infty} \frac{1}{(2k)^n} Q\left((2k)^n x\right)$$

for all $x \in \mathbf{X}$. Moreover, letting $l = 0$ and passing the *limit* $m \to \infty$ in (55), we have (49). The rest of the Proof is similar to the Proof of Theorem 4. □

5. Establishing Solutions to Functional Inequality (2) Related to the Type of Cauchy Additive Functional Equation

Now, we first study the solutions of (2). Note that for this inequality, $\mathbf{X}$ is a normed space with norm $\|\cdot\|_{\mathbf{X}}$ and that $\mathbf{Y}$ is a *Banach* space with norm $\|\cdot\|_{\mathbf{Y}}$. Under this setting, we can show that the mappings satisfying (2) are Cauchy additive. These results are given in the following.

Theorem 8. Suppose $q > 1$, θ be non-negative real and $f : \mathbf{X} \to \mathbf{Y}$ be an odd mapping such that

$$\begin{aligned} &\left\| \sum_{j=1}^{k} f(x_j) + \sum_{j=1}^{k} f(y_j) + \sum_{j=1}^{k} f(z_j) \right\|_{\mathbf{Y}} \\ &\le \left\| f\left(\sum_{j=1}^{k} x_j + \sum_{j=1}^{k} y_j + \sum_{j=1}^{k} z_j \right) \right\|_{\mathbf{Y}} + \theta \left(\sum_{j=1}^{k} \|x_j\|_{\mathbf{X}}^q + \sum_{j=1}^{k} \|y_j\|_{\mathbf{X}}^q + \sum_{j=1}^{k} \|z_j\|_{\mathbf{X}}^q \right) \end{aligned} \tag{56}$$

for all $x_j, y_j, z_j \in \mathbf{X}$ for all $j = 1 \to n$.

Then there exists a unique additive mapping $H : \mathbf{X} \to \mathbf{Y}$ such that

$$\left\| \frac{f(x) + f(-x)}{2k} - H(x) \right\|_{\mathbf{Y}} \le \frac{(2k)^q + 2k}{(2k)^q - 2k} \theta \|x\|_{\mathbf{X}}^q. \tag{57}$$

for all $x \in \mathbf{X}$.

Proof. Assume that $f : \mathbf{X} \to \mathbf{Y}$ satisfies (56).

We replacing $(x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_k)$ by $(x, \cdots, x, x, \cdots, x, -2kx, 0, \cdots, 0)$ in (56), we have

$$\left\| 2kf(x) + f(-2kx) \right\|_{\mathbf{Y}} \le \left((2k)^q + 2k\right) \theta \|x\|_{\mathbf{X}}^q \tag{58}$$

for all $x \in \mathbf{X}$. Replacing *x* by *-x* in (58), we get

$$\left\| 2kf(-x) + f(2kx) \right\|_{\mathbf{Y}} \le \left((2k)^q + 2k\right) \theta \|x\|_{\mathbf{X}}^q \tag{59}$$

for all $x \in \mathbf{X}$. It follows from (58) and (59) that

$$\begin{aligned} &\left\| 2k\left(f(x) + f(-x)\right) - \left(f(2kx) + f(-2kx)\right) \right\|_{\mathbf{Y}} \\ &\le \left\| 2kf(x) + f(-2kx) \right\|_{\mathbf{Y}} + \left\| 2kf(-x) + f(2kx) \right\|_{\mathbf{Y}} \\ &\le 2\left((2k)^q + 2k\right) \theta \|x\|_{\mathbf{X}}^q \end{aligned} \tag{60}$$

for all $x \in \mathbf{X}$. Let $Q(x) = \frac{f(x)+f(-x)}{2k}$. From (60) we have

$$\|2kQ(x) - Q(2kx)\| \le \left((2k)^q + 2k\right)\frac{\theta}{k}\|x\|_{\mathbf{X}}^q \tag{61}$$

for all $x \in \mathbf{X}$. The rest of the Proof is similar to the Proof of Theorem 4.

Theorem 9. Suppose $q < 1$, θ be positive real numbers and $f : \mathbf{X} \to \mathbf{Y}$ be a mapping such that

$$\begin{aligned} &\left\|\sum_{j=1}^{k} f(x_j) + \sum_{j=1}^{k} f(y_j) + \sum_{j=1}^{k} f(z_j)\right\|_{\mathbf{Y}} \\ &\le \left\|f\left(\sum_{j=1}^{k} x_j + \sum_{j=1}^{k} y_j + \sum_{j=1}^{k} z_j\right)\right\|_{\mathbf{Y}} + \theta\left(\sum_{j=1}^{k}\|x_j\|_{\mathbf{X}}^q + \sum_{j=1}^{k}\|y_j\|_{\mathbf{X}}^q + \sum_{j=1}^{k}\|z_j\|_{\mathbf{X}}^q\right) \end{aligned} \tag{62}$$

for all $x_j, y_j, z_j \in \mathbf{X}$ for all $j = 1 \to n$.

Then there exists a unique additive mapping $H : \mathbf{X} \to \mathbf{Y}$ such that

$$\left\|\frac{f(x)+f(-x)}{2k} - H(x)\right\|_Y \le \frac{2k + (2k)^q}{2k - (2k)^q}\theta\|x\|_X^q. \tag{63}$$

for all $x \in \mathbf{X}$.

The rest of the Proof is similar to the Proof of Theorems 4 and 5.

Theorem 10. Suppose $q > p^{-1}$ with $p \ge 3$, θ be non-negative real and $f : \mathbf{X} \to \mathbf{Y}$ be a mapping such that

$$\begin{aligned} &\left\|\sum_{j=1}^{k} f(x_j) + \sum_{j=1}^{k} f(y_j) + \sum_{j=1}^{k} f(z_j)\right\|_{\mathbf{Y}} \\ &\le \left\|f\left(\sum_{j=1}^{k} x_j + \sum_{j=1}^{k} y_j + \sum_{j=1}^{k} z_j\right)\right\|_{\mathbf{Y}} + \theta\prod_{j=1}^{k}\|x_j\|_{\mathbf{X}}^q \cdot \prod_{j=1}^{k}\|y_j\|_{\mathbf{X}}^q \cdot \|z_1\|_{\mathbf{X}}^{kq} \cdot \left(1 + \prod_{j=1}^{k}\|z_j\|_{\mathbf{X}}^q\right) \end{aligned} \tag{64}$$

for all $x_j, y_j, z_j \in \mathbf{X}$ for all $j = 1 \to n$.

Then there exists a unique additive mapping $H : \mathbf{X} \to \mathbf{Y}$ such that

$$\left\|\frac{f(x)+f(-x)}{2k} - H(x)\right\|_Y \le \frac{(2k)^q}{(2k)^{3kq} - 2k}\theta\|x\|_X^{3kq}. \tag{65}$$

for all $x \in \mathbf{X}$.

Proof. Assume that $f : \mathbf{X} \to \mathbf{Y}$ satisfies (64).

We replaced $(x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_k)$ by $(x, \cdots, x, x, \cdots, x, -2kx, 0, \cdots, 0)$ in (64), we have

$$\|2kf(x) + f(-2kx)\|_{\mathbf{Y}} \le |2k|^{kq}\theta\|x\|_{\mathbf{X}}^{3kq} \tag{66}$$

for all $x \in \mathbf{X}$. we have

$$\|2kQ(x) - Q(2kx)\| \le |2k|^{kq}\frac{\theta}{k}\|x\|_{\mathbf{X}}^{3kq} \tag{67}$$

for all $x \in \mathbf{X}$. The rest of the Proof is similar to the Proof of Theorems 4 and 6. □

Theorem 11. Suppose $q < p^{-1}$ with $p \ge 3$, θ be non-negative real and

$f:\mathbf{X}\to\mathbf{Y}$ be a mapping such that

$$\left\|\sum_{j=1}^{k} f\left(x_j\right)+\sum_{j=1}^{k} f\left(y_j\right)+\sum_{j=1}^{k} f\left(z_j\right)\right\|_{\mathbf{Y}} \le \left\|2kf\left(\frac{\sum_{j=1}^{k} x_j+\sum_{j=1}^{k} y_j+\sum_{j=1}^{k} z_j}{2k}\right)\right\|_{\mathbf{Y}} + \theta\prod_{j=1}^{k}\left\|x_j\right\|_{\mathbf{X}}^{q}\cdot\prod_{j=1}^{k}\left\|y_j\right\|_{\mathbf{X}}^{q}\cdot\left\|z_1\right\|_{\mathbf{X}}^{kq}\cdot\left(1+\prod_{j=1}^{k}\left\|z_j\right\|_{\mathbf{X}}^{q}\right) \quad (68)$$

for all $x_j, y_j, z_j\in\mathbf{X}$ for all $j=1\to n$.

Then there exists a unique additive mapping $H:\mathbf{X}\to\mathbf{Y}$ such that

$$\left\|\frac{f(x)+f(-x)}{2k}-H(x)\right\|_{\mathbf{Y}} \le \frac{(2k)^q}{2k-(2k)^{3kq}}\theta\|x\|_{\mathbf{X}}^{3kq}. \quad (69)$$

for all $x\in\mathbf{X}$.

The rest of the Proof is similar to the Proof of Theorems 4 and 7.

6. Establishing Solutions to Functional Inequality (3) Related to the Type of Cauchy-Jensen Additive Functional Equation

Now, we first study the solutions of (3). Note that for this inequality, $\mathbf{X}$ is a normed space with norm $\|\cdot\|_{\mathbf{X}}$ and that $\mathbf{Y}$ is a *Banach* space with norm $\|\cdot\|_{\mathbf{Y}}$. Under this setting, we can show that the mappings satisfying (3) are Cauchy-Jensen additive. These results are given in the following.

Theorem 12. Suppose $q>1$, θ be non-negative real, $f(0)=0$ and $f:\mathbf{X}\to\mathbf{Y}$ be a mapping such that

$$\left\|\sum_{j=1}^{k} f\left(x_j\right)+\sum_{j=1}^{k} f\left(y_j\right)+\sum_{j=1}^{k} f\left(z_j\right)\right\|_{\mathbf{Y}} \le \left\|2kf\left(\frac{\sum_{j=1}^{k} x_j+\sum_{j=1}^{k} y_j}{2k}+\sum_{j=1}^{k} z_j\right)\right\|_{\mathbf{Y}} + \theta\left(\sum_{j=1}^{k}\left\|x_j\right\|_{\mathbf{X}}^{q}+\sum_{j=1}^{k}\left\|y_j\right\|_{\mathbf{X}}^{q}+\sum_{j=1}^{k}\left\|z_j\right\|_{\mathbf{X}}^{q}\right) \quad (70)$$

for all $x_j, y_j, z_j\in\mathbf{X}$ for all $j=1\to n$.

Then there exists a unique additive mapping $H:\mathbf{X}\to\mathbf{Y}$ such that

$$\left\|\frac{f(x)+f(-x)}{2k}-H(x)\right\|_{Y} \le \frac{(2k)^q+1}{(2k)^q-2k}\theta\|x\|_{X}^{q}. \quad (71)$$

for all $x\in\mathbf{X}$.

Proof. Assume that $f:\mathbf{X}\to\mathbf{Y}$ satisfies (70).

We replacing $\left(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\right)$ by $\left(2kx,0,\cdots,0,0,\cdots,0,-x,0,\cdots,0\right)$ in (70), we have

$$\left\|f(2kx)+2kf(-x)\right\|_{\mathbf{Y}} \le \left((2k)^q+1\right)\theta\|x\|_{\mathbf{X}}^{q} \quad (72)$$

for all $x\in\mathbf{X}$. Replacing *x* by -*x* in (72), we get

$$\left\| f(-2kx) + 2kf(x) \right\|_{\mathbf{Y}} \le \left((2k)^q + 1 \right) \theta \left\| x \right\|_{\mathbf{X}}^q \tag{73}$$

for all $x \in \mathbf{X}$. It follows from (72) and (73) that

$$\begin{aligned} &\left\| 2k\left(f(x) + f(-x) \right) - \left(f(2kx) + f(-2kx) \right) \right\|_{\mathbf{Y}} \\ &\le \left\| 2kf(x) + f(-2kx) \right\|_{\mathbf{Y}} + \left\| 2kf(-x) + f(2kx) \right\|_{\mathbf{Y}} \\ &\le 2\left((2k)^q + 1 \right) \theta \left\| x \right\|_{\mathbf{X}}^q \end{aligned} \tag{74}$$

for all $x \in \mathbf{X}$. Let $Q(x) = \frac{f(x) + f(-x)}{2k}$. From (74) we have

$$\left\| 2kQ(x) - Q(2kx) \right\| \le \left((2k)^q + 1 \right) \frac{\theta}{k} \left\| x \right\|_{\mathbf{X}}^q \tag{75}$$

for all $x \in \mathbf{X}$. So

$$\left\| Q(x) - 2kQ\left(\frac{x}{2k} \right) \right\| \le \frac{(2k)^q + 1}{(2k)^q} \frac{\theta}{k} \left\| x \right\|_{\mathbf{X}}^q \tag{76}$$

The rest of the Proof is similar to the Proof of Theorem 4. □

Theorem 13. Suppose $q < 1$, θ be positive real numbers and $f : \mathbf{X} \to \mathbf{Y}$ be a mapping such that

$$\begin{aligned} &\left\| \sum_{j=1}^{k} f(x_j) + \sum_{j=1}^{k} f(y_j) + \sum_{j=1}^{k} f(z_j) \right\|_{\mathbf{Y}} \\ &\le \left\| 2kf\left(\frac{\sum_{j=1}^{k} x_j + \sum_{j=1}^{k} y_j}{2k} + \sum_{j=1}^{k} z_j \right) \right\|_{\mathbf{Y}} + \theta \left(\sum_{j=1}^{k} \left\| x_j \right\|_{\mathbf{X}}^q + \sum_{j=1}^{k} \left\| y_j \right\|_{\mathbf{X}}^q + \sum_{j=1}^{k} \left\| z_j \right\|_{\mathbf{X}}^q \right) \end{aligned} \tag{77}$$

for all $x_j, y_j, z_j \in \mathbf{X}$ for all $j = 1 \to n$

$$\left\| \frac{f(x) + f(-x)}{2k} - H(x) \right\|_{Y} \le \frac{2k + (2k)^q}{2k - (2k)^q} \theta \left\| x \right\|_{X}^q . \tag{78}$$

for all $x \in \mathbf{X}$.

The rest of the proof is similar to the proof of Theorems 4 and 5.

7. Conclusion

In this paper, I have given three general functional inequalities and I have shown that their solutions are determined on normalized spaces and take values in Banach spaces.

Conflicts of Interest

The author declares no conflicts of interest.

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