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Indefinite Cross Divisions of Vectors in Natural Space

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Abstract

In this paper, in order to solve the problem that cross product has no corresponding division in natural space, indefinite cross divisions are firstly introduced as the inverse operations of cross products, which solve the problem from another angle. Then a lot of basic properties of indefinite cross divisions are obtained, such as the Conversion Formulas between left and right indefinite cross quotients, and linear operation properties, where some are expected and some are special. Especially, the geometric expressions of indefinite cross divisions are presented so that their structures are provided. Finally, some important coordinate formulas and corresponding examples on indefinite cross divisions are presented.

Subject Areas

Vector Analysis, Analytic Geometry, Mechanics

Keywords

Cross Product, Indefinite Cross Division, Indefinite Cross Quotient, Left Indefinite Cross Division, Vector Quotient

1. Introduction

Although the cross product of two vectors in natural space is widely used in the geometry, mechanics, computer graphics etc. [1] [2] [3] [4] [5], that is even extended to $\mathbb{C}^3$ [6] and to (2^n-1)-dimensional vector spaces [7] [8] [9], we still feel something imperfect, since it does not have corresponding division. When we face cross product equation $\boldsymbol{a}\times\boldsymbol{b}=\boldsymbol{c}$, naturally hope $\boldsymbol{a}=\dfrac{\boldsymbol{c}}{\boldsymbol{b}}$ holds. Almost everyone spends some time to consider the divisions of vectors when learning cross products, and unfortunately obtains the result: Generally speaking, the division of two vectors on cross products does not exist. As a result, there are no papers which successfully study the divisions of vectors on cross products.

By profoundly researching cross products of vectors, we find that we might ignore something important such as angle. For instance, some books [10] [11] use coordinates of vectors to directly define cross product as

$$\boldsymbol{a}\times\boldsymbol{b}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix} \tag{1}$$

or

$$\boldsymbol{a}\times\boldsymbol{b}=\left\{a_2b_3-a_3b_2,-a_1b_3+a_3b_1,a_1b_2-a_2b_1\right\} \tag{2}$$

where $\boldsymbol{a}=\{a_1,a_2,a_3\}$ and $\boldsymbol{b}=\{b_1,b_2,b_3\}$, and $\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}$ are an orthonormal basis. Though the two definitions above are correct and useful, there are no angles appearing on the face. It is easy to make people ignore the role of angles between vectors when computing cross products. Fortunately some books [12] [13] [14] stressed angles, like the following definition:

Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two vectors, and $\theta(0\le\theta\le\pi)$ be the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. The **cross product** (also called **vector product**) of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is the vector, denoted by $\boldsymbol{a}\times\boldsymbol{b}$, whose magnitude is

$$|\boldsymbol{a}\times\boldsymbol{b}|=|\boldsymbol{a}||\boldsymbol{b}|\sin\theta \tag{3}$$

and whose direction is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$, having the same direction as the translation of a right-handed screw due to a rotation from $\boldsymbol{a}$ to $\boldsymbol{b}$ (See Figure 1).

This definition tells us that, when we compute $\boldsymbol{a}\times\boldsymbol{b}$, we actually know not only $\boldsymbol{a}$ and $\boldsymbol{b}$ but also the angle between them, which actually play very important roles. However, we do not know the angle condition when we inversely want to obtain $\boldsymbol{a}$ from $\boldsymbol{c}(=\boldsymbol{a}\times\boldsymbol{b})$ and $\boldsymbol{b}$. We find that we can inversely obtain the exact $\boldsymbol{a}$ from $\boldsymbol{c}$ and $\boldsymbol{b}$ by adding the angle condition.

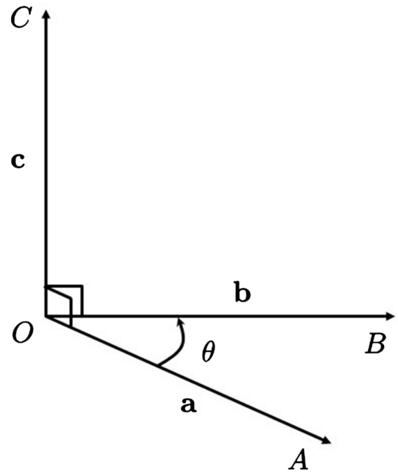

Figure 1. Cross product.

When we actually pay more attention to angles, we then successfully establish the theory of indefinite cross divisions, as the inverse operations of cross products. This paper is divided into 6 sections: In Section 2, the definitions of indefinite cross divisions are introduced, and some basic properties are presented. In Section 3, the some basic operations of indefinite cross divisions are discussed. In Section 4, the geometric expressions of indefinite cross divisions are provided, and their structures with real parameters are presented. In Section 5, the coordinate formulas and corresponding examples on indefinite cross divisions are simply presented after studying the structures.

2. Indefinite Cross Divisions

In this section, we will present the definitions of indefinite cross divisions when the angles are not zero and π. And some notations and basic properties are proposed.

Definition 2.1. *Let* $\boldsymbol{c},\boldsymbol{b}$ *be two vectors with* $\boldsymbol{b} \neq \boldsymbol{0}$ *and* $\boldsymbol{c} \perp \boldsymbol{b}$, *and let* $\theta \in (0, \pi)$ *be an angle. The vector, denoted by* $\frac{\boldsymbol{c}}{{}_{\theta}\boldsymbol{b}}$ $\left(\frac{\boldsymbol{c}}{\boldsymbol{b}_{\theta}}\right)$, *is called the* ***left*** (***right***) ***indefinite cross division*** *of two vectors* $\boldsymbol{c}$ *and* $\boldsymbol{b}$, *simply* ***left*** (***right***) ***cross division***, *if its magnitude is defined as*

$$\left|\frac{\boldsymbol{c}}{{}_{\theta}\boldsymbol{b}}\right| = \left|\frac{\boldsymbol{c}}{\boldsymbol{b}_{\theta}}\right| = \frac{|\boldsymbol{c}|}{|\boldsymbol{b}|\sin\theta} \tag{4}$$

and its direction is perpendicular to $\boldsymbol{c}$ such that $\frac{\boldsymbol{c}}{{}_{\theta}\boldsymbol{b}} \times \boldsymbol{b} = \boldsymbol{c}$ $\left(\boldsymbol{b} \times \frac{\boldsymbol{c}}{\boldsymbol{b}_{\theta}} = \boldsymbol{c}\right)$ (See Figure 2).

More specifically, the direction of $\frac{\boldsymbol{c}}{{}_{\theta}\boldsymbol{b}}$ $\left(\frac{\boldsymbol{c}}{\boldsymbol{b}_{\theta}}\right)$ is determined by the following 3 steps:

Step 1. Let O be any point in the space, and make $\overrightarrow{OC} = \boldsymbol{c}$, $\overrightarrow{OB} = \boldsymbol{b}$.

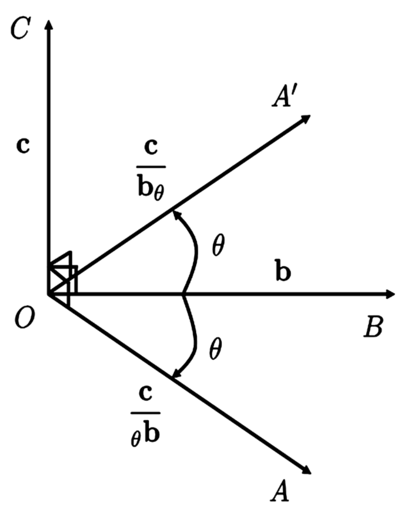

Figure 2. Cross division.

Step 2. Extend the left (right) hand, satisfying five fingers are on the plane *BOC*, and the thumb is perpendicular to the other 4 fingers, and point the thumb in the direction of $\overrightarrow{OC}$ and the other four fingers in the direction of $\overrightarrow{OB}$.

Step 3. The left (right) hand rotates angle θ around vector $\overrightarrow{OC}$.

Then, the direction pointed by the four fingers is that direction of $\dfrac{\boldsymbol{c}}{{}_{\theta}\boldsymbol{b}}$ $\left(\dfrac{\boldsymbol{c}}{\boldsymbol{b}_{\theta}}\right)$ (See Figure 2).

Definition 2.2. *The* θ *in notation* $\dfrac{\boldsymbol{c}}{{}_{\theta}\boldsymbol{b}}$ $\left(\dfrac{\boldsymbol{c}}{\boldsymbol{b}_{\theta}}\right)$ *is called an* ***indefinite angle parameter****, simply* ***angle parameter****. The left* (*right*) *indefinite cross division, of course, can be called the* ***left*** (***right***) ***indefinite cross quotient****. The left and right indefinite cross divisions are collectively called the* ***indefinite cross divisions****, simply* ***cross divisions****.*

When $\boldsymbol{c}=\mathbf{0}$ and $\boldsymbol{b}\neq\mathbf{0}$ and $\theta=0$ or π, the problem becomes very simple since $\dfrac{\boldsymbol{c}}{{}_{\theta}\boldsymbol{b}}$ and $\dfrac{\boldsymbol{c}}{\boldsymbol{b}_{\theta}}$ are parallel to $\boldsymbol{b}$ so that they can be easily dealt with the simple form of $\lambda\boldsymbol{b}, \lambda\in\mathbb{R}$ in Section 4. In light of these considerations, without special statement, as we meet the notations $\dfrac{\boldsymbol{c}}{{}_{\theta}\boldsymbol{b}}$ and $\dfrac{\boldsymbol{c}}{\boldsymbol{b}_{\theta}}$, we always suppose $\theta\in(0,\pi)$, $\boldsymbol{b}\neq\mathbf{0}$ and $\boldsymbol{c}\perp\boldsymbol{b}$. We do not repeat later.

From Definition 2.1, $\dfrac{\boldsymbol{c}}{{}_{\theta}\boldsymbol{b}}$ and $\dfrac{\boldsymbol{c}}{\boldsymbol{b}_{\theta}}$ have the following simple results:

(1.1) $\dfrac{\mathbf{0}}{{}_{\theta}\boldsymbol{b}}=\dfrac{\mathbf{0}}{\boldsymbol{b}_{\theta}}=\mathbf{0}$ for any $\theta\in(0,\pi)$.

(1.2) $\boldsymbol{c}\perp\dfrac{\boldsymbol{c}}{{}_{\theta}\boldsymbol{b}}$, $\boldsymbol{c}\perp\dfrac{\boldsymbol{c}}{\boldsymbol{b}_{\theta}}$.

(1.3) $\dfrac{\boldsymbol{c}}{{}_{\theta}\boldsymbol{b}}$, $\boldsymbol{b}$, $\boldsymbol{c}$ obey the right-handed rule, and $\boldsymbol{b}$, $\dfrac{\boldsymbol{c}}{\boldsymbol{b}_{\theta}}$, $\boldsymbol{c}$, of course, also.

(1.4) $\boldsymbol{b}\times\dfrac{\boldsymbol{c}}{{}_{\theta}\boldsymbol{b}}=\dfrac{\boldsymbol{c}}{\boldsymbol{b}_{\theta}}\times\boldsymbol{b}=-\boldsymbol{c}$.

(1.5) $\dfrac{\boldsymbol{c}}{{}_{\frac{\pi}{2}}\boldsymbol{b}}=-\dfrac{\boldsymbol{c}}{\boldsymbol{b}_{\frac{\pi}{2}}}$ and $\left|\dfrac{\boldsymbol{c}}{{}_{\frac{\pi}{2}}\boldsymbol{b}}\right|=\left|\dfrac{\boldsymbol{c}}{\boldsymbol{b}_{\frac{\pi}{2}}}\right|=\dfrac{|\boldsymbol{c}|}{|\boldsymbol{b}|}$.

Indefinite cross divisions have the following two important properties:

(1.6) $\dfrac{\boldsymbol{c}}{{}_{\theta}\boldsymbol{b}}=\dfrac{-\boldsymbol{c}}{\boldsymbol{b}_{\theta}}$, $\dfrac{\boldsymbol{c}}{\boldsymbol{b}_{\theta}}=\dfrac{-\boldsymbol{c}}{{}_{\theta}\boldsymbol{b}}$. (Conversion Formulas)

(1.7) $-\dfrac{\boldsymbol{c}}{{}_{\theta}\boldsymbol{b}}=\dfrac{\boldsymbol{c}}{{}_{\theta}(-\boldsymbol{b})}$, $-\dfrac{\boldsymbol{c}}{\boldsymbol{b}_{\theta}}=\dfrac{\boldsymbol{c}}{(-\boldsymbol{b})_{\theta}}$. (Inverse Formulas)

Of course, they have some other properties such as

(1.8) $\dfrac{-\boldsymbol{c}}{{}_{\theta}\boldsymbol{b}}=\dfrac{\boldsymbol{c}}{{}_{\pi-\theta}(-\boldsymbol{b})}$, $\dfrac{-\boldsymbol{c}}{\boldsymbol{b}_{\theta}}=\dfrac{\boldsymbol{c}}{(-\boldsymbol{b})_{\pi-\theta}}$.

(Angle Formulas)

(1.6), (1.7) and (1.8) can be easily understood by Figure 3. Note that, Definition 2.1 ensures, for any $\theta \in (0, \pi)$, $\frac{\boldsymbol{c}}{{}_{\theta}\boldsymbol{b}}$ ($\frac{\boldsymbol{c}}{\boldsymbol{b}_{\theta}}$) is a vector such that $\frac{\boldsymbol{c}}{{}_{\theta}\boldsymbol{b}} \times \boldsymbol{b} = \boldsymbol{c}$ ($\boldsymbol{b} \times \frac{\boldsymbol{c}}{\boldsymbol{b}_{\theta}} = \boldsymbol{c}$). Conversely, we have

Theorem 2.1. *If there is a vector* $\boldsymbol{a}$ ($\boldsymbol{b}$) *such that* $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{c}$ *and* $\boldsymbol{c} \neq \boldsymbol{0}$, *then there is the unique* $\theta \in (0, \pi)$ *such that* $\frac{\boldsymbol{c}}{{}_{\theta}\boldsymbol{b}} = \boldsymbol{a}$ ($\frac{\boldsymbol{c}}{\boldsymbol{a}_{\theta}} = \boldsymbol{b}$).

Proof. Since $\boldsymbol{c} \neq \boldsymbol{0}$, the angle $\angle(\boldsymbol{a}, \boldsymbol{b})$ between $\boldsymbol{a}$ and $\boldsymbol{b}$ is in $(0, \pi)$. Let $\theta = \angle(\boldsymbol{a}, \boldsymbol{b})$. According to the definition of left (right) indefinite cross division, $\frac{\boldsymbol{c}}{{}_{\theta}\boldsymbol{b}}$ and $\boldsymbol{a}$ ($\frac{\boldsymbol{c}}{\boldsymbol{a}_{\theta}}$ and $\boldsymbol{b}$) have the same direction. Since $|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}|\sin\theta = |\boldsymbol{c}|$, they have the same magnitude $\frac{|\boldsymbol{c}|}{|\boldsymbol{b}|\sin\theta}$ ($\frac{|\boldsymbol{c}|}{|\boldsymbol{a}|\sin\theta}$). Thus $\frac{\boldsymbol{c}}{{}_{\theta}\boldsymbol{b}} = \boldsymbol{a}$ ($\frac{\boldsymbol{c}}{\boldsymbol{a}_{\theta}} = \boldsymbol{b}$). Since θ specifies the direction, θ is unique.

Similarly, we have $\frac{\boldsymbol{c}}{\boldsymbol{a}_{\theta}} = \boldsymbol{b}$ with the unique θ. □

The above theorem implies

Corollary 2.1. *If* $\boldsymbol{c} \neq \boldsymbol{0}$, *then*

$$\{\boldsymbol{a} \mid \boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{c}\} = \left\{\frac{\boldsymbol{c}}{{}_{\theta}\boldsymbol{b}} \mid \theta \in (0, \pi)\right\};$$

$$\{\boldsymbol{b} \mid \boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{c}\} = \left\{\frac{\boldsymbol{c}}{\boldsymbol{a}_{\theta}} \mid \theta \in (0, \pi)\right\}.$$

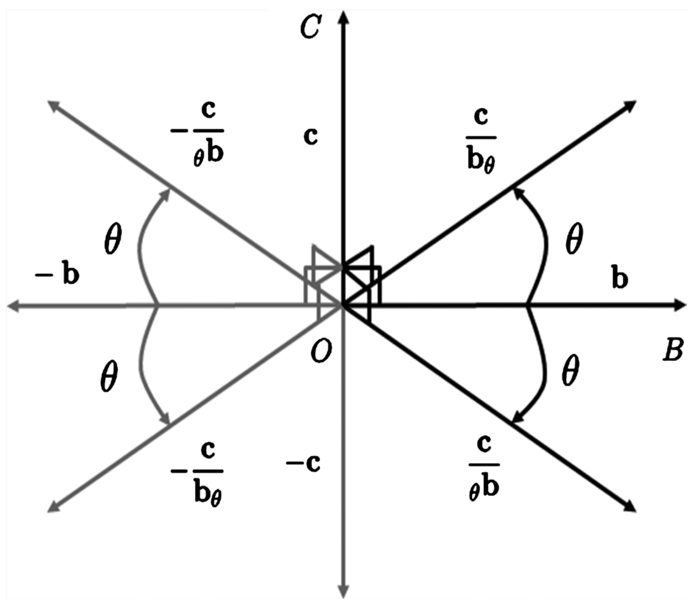

Figure 3. Conversion.

Denote $V_L(\boldsymbol{c},\boldsymbol{b})=\{\boldsymbol{u}\mid\boldsymbol{u}\times\boldsymbol{b}=\boldsymbol{c}\}$ and $V_R(\boldsymbol{c},\boldsymbol{b})=\{\boldsymbol{v}\mid\boldsymbol{b}\times\boldsymbol{v}=\boldsymbol{c}\}$. We have

Theorem 2.2. *Let* $\boldsymbol{u}_1,\boldsymbol{u}_2,\cdots,\boldsymbol{u}_n\in V_L(\boldsymbol{c},\boldsymbol{b})$ *and* $\boldsymbol{v}_1,\boldsymbol{v}_2,\cdots,\boldsymbol{v}_n\in V_R(\boldsymbol{c},\boldsymbol{b})$. *Let* $\lambda_1,\lambda_2,\cdots,\lambda_n$ *be n real numbers such that* $\sum_{i=1}^n\lambda_i=1$. *Then,* $\sum_{i=1}^n\lambda_i\boldsymbol{u}_i\in V_L(\boldsymbol{c},\boldsymbol{b})$ *and* $\sum_{i=1}^n\lambda_i\boldsymbol{v}_i\in V_R(\boldsymbol{c},\boldsymbol{b})$.

Proof. $\left(\sum_{i=1}^n\lambda_i\boldsymbol{u}_i\right)\times\boldsymbol{b}=\sum_{i=1}^n\lambda_i(\boldsymbol{u}_i\times\boldsymbol{b})=\sum_{i=1}^n\lambda_i\boldsymbol{c}=\boldsymbol{c}$, and

$$\boldsymbol{b}\times\left(\sum_{i=1}^n\lambda_i\boldsymbol{v}_i\right)=\sum_{i=1}^n\lambda_i(\boldsymbol{b}\times\boldsymbol{v}_i)=\sum_{i=1}^n\lambda_i\boldsymbol{c}=\boldsymbol{c}.\ \square$$

After we have indefinite cross divisions, it is easy to find the general solution of the following cross product vector equation:

$$\boldsymbol{x}\times\boldsymbol{a}+\boldsymbol{y}\times\boldsymbol{b}=\boldsymbol{c} \tag{5}$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are given such that at least one is not zero, and $\boldsymbol{x}$ and $\boldsymbol{y}$ are two unknown vectors. Then, how to get the solution of the above equation? Assume $\boldsymbol{a}\neq\boldsymbol{0}$. Let $\boldsymbol{y}$ take an vector $\boldsymbol{d}$, we have equation

$$\boldsymbol{x}\times\boldsymbol{a}=\boldsymbol{c}-\boldsymbol{d}\times\boldsymbol{b}. \tag{6}$$

Then we get the general solution of Equation (5) is

$$\begin{cases}\boldsymbol{x}=\dfrac{\boldsymbol{c}-\boldsymbol{d}\times\boldsymbol{b}}{{}_\theta\boldsymbol{a}}, & \theta\in(0,\pi),\\ \boldsymbol{y}=\boldsymbol{d}, & \boldsymbol{d}\text{ is an arbitrary vector.}\end{cases}$$

Similarly, when $\boldsymbol{b}\neq\boldsymbol{0}$, the general solution of Equation (5) is

$$\begin{cases}\boldsymbol{x}=\boldsymbol{d}, & \boldsymbol{d}\text{ is an arbitrary vector,}\\ \boldsymbol{y}=\dfrac{\boldsymbol{c}-\boldsymbol{d}\times\boldsymbol{a}}{{}_\theta\boldsymbol{b}}, & \theta\in(0,\pi).\end{cases}$$

How to understand indefinite cross division? Cross product is like derivation, and indefinite cross division is like indefinite integral, where angle parameter is like arbitrary constant in indefinite integral.

3. Operations

In this section, we will further discuss the rules of multiplications of scalars and cross divisions. Because a cross division involves three factors (a numerator vector and a denominator vector and an angle parameter θ), the multiplications become very interesting. For the symmetry of left and right indefinite cross divisions, we only prove the properties about left cross divisions.

Theorem 3.1. For $\lambda\neq0$, $\lambda\dfrac{\boldsymbol{c}}{{}_\theta\boldsymbol{b}}=\dfrac{\boldsymbol{c}}{{}_\theta\left(\frac{1}{\lambda}\boldsymbol{b}\right)}$, $\lambda\dfrac{\boldsymbol{c}}{\boldsymbol{b}_\theta}=\dfrac{\boldsymbol{c}}{\left(\frac{1}{\lambda}\boldsymbol{b}\right)_\theta}$.

Proof. When $\lambda>0$, $\lambda\dfrac{\boldsymbol{c}}{{}_\theta\boldsymbol{b}}=\dfrac{\boldsymbol{c}}{{}_\theta\left(\frac{1}{\lambda}\boldsymbol{b}\right)}$ is obvious.

When $\lambda<0$, by Inverse Formulas (1.7), we have

$$\lambda\frac{\boldsymbol{c}}{_\theta\boldsymbol{b}}=-|\lambda|\frac{\boldsymbol{c}}{_\theta\boldsymbol{b}}=-\frac{\boldsymbol{c}}{_\theta\left(\frac{1}{|\lambda|}\boldsymbol{b}\right)}=\frac{\boldsymbol{c}}{_\theta\left(-\frac{1}{|\lambda|}\boldsymbol{b}\right)}=\frac{\boldsymbol{c}}{_\theta\left(\frac{1}{\lambda}\boldsymbol{b}\right)}. \square$$

Theorem 3.2. (1) For $\lambda\ge 0$, $\lambda\frac{\boldsymbol{c}}{_\theta\boldsymbol{b}}=\frac{\lambda\boldsymbol{c}}{_\theta\boldsymbol{b}}$, $\lambda\frac{\boldsymbol{c}}{\boldsymbol{b}_\theta}=\frac{\lambda\boldsymbol{c}}{\boldsymbol{b}_\theta}$;

(2) for $\lambda<0$, $\lambda\frac{\boldsymbol{c}}{_\theta\boldsymbol{b}}=\frac{\lambda\boldsymbol{c}}{(-\boldsymbol{b})_\theta}$, $\lambda\frac{\boldsymbol{c}}{\boldsymbol{b}_\theta}=\frac{\lambda\boldsymbol{c}}{_\theta(-\boldsymbol{b})}$.

Proof. (1) Obvious.

(2) For $\lambda<0$, by Theorem 0.3 and Conversion Formulas (1.6),

$$\lambda\frac{\boldsymbol{c}}{_\theta\boldsymbol{b}}=-|\lambda|\frac{\boldsymbol{c}}{_\theta\boldsymbol{b}}=\frac{|\lambda|\boldsymbol{c}}{_\theta(-\boldsymbol{b})}=\frac{-|\lambda|\boldsymbol{c}}{(-\boldsymbol{b})_\theta}=\frac{\lambda\boldsymbol{c}}{(-\boldsymbol{b})_\theta}. \square$$

Corollary 3.1. (1) *For* $\lambda>0$, $\frac{\boldsymbol{c}}{_\theta\left(\frac{1}{\lambda}\boldsymbol{b}\right)}=\frac{\lambda\boldsymbol{c}}{_\theta\boldsymbol{b}}$, $\frac{\boldsymbol{c}}{\left(\frac{1}{\lambda}\boldsymbol{b}\right)_\theta}=\frac{\lambda\boldsymbol{c}}{\boldsymbol{b}_\theta}$*;*

(2) for $\lambda<0$, $\frac{\boldsymbol{c}}{_\theta\left(\frac{1}{\lambda}\boldsymbol{b}\right)}=\frac{\lambda\boldsymbol{c}}{(-\boldsymbol{b})_\theta}$, $\frac{\boldsymbol{c}}{\left(\frac{1}{\lambda}\boldsymbol{b}\right)_\theta}=\frac{\lambda\boldsymbol{c}}{_\theta(-\boldsymbol{b})}$.

Proof. Obvious by Theorem 3.1 and Theorem 3.2. □

Corollary 3.3. (1) *For* $\lambda>0$, $\frac{\boldsymbol{c}}{_\theta\boldsymbol{b}}=\frac{\lambda\boldsymbol{c}}{_\theta(\lambda\boldsymbol{b})}$, $\frac{\boldsymbol{c}}{\boldsymbol{b}_\theta}=\frac{\lambda\boldsymbol{c}}{(\lambda\boldsymbol{b})_\theta}$*;*

(2) for $\lambda<0$, $\frac{\boldsymbol{c}}{_\theta\boldsymbol{b}}=\frac{\lambda\boldsymbol{c}}{(-\lambda\boldsymbol{b})_\theta}$, $\frac{\boldsymbol{c}}{\boldsymbol{b}_\theta}=\frac{\lambda\boldsymbol{c}}{_\theta(-\lambda\boldsymbol{b})}$.

Proof. (1) If $\lambda>0$, then $\frac{\boldsymbol{c}}{_\theta\boldsymbol{b}}=\left(\frac{1}{\lambda}\times\lambda\right)\frac{\boldsymbol{c}}{_\theta\boldsymbol{b}}=\frac{1}{\lambda}\frac{\lambda\boldsymbol{c}}{_\theta\boldsymbol{b}}=\frac{\lambda\boldsymbol{c}}{_\theta(\lambda\boldsymbol{b})}$.

(2) If $\lambda<0$, then $\frac{\boldsymbol{c}}{_\theta\boldsymbol{b}}=\frac{-\lambda}{-\lambda}\frac{\boldsymbol{c}}{_\theta\boldsymbol{b}}=\frac{-\lambda\boldsymbol{c}}{_\theta(-\lambda\boldsymbol{b})}=\frac{\lambda\boldsymbol{c}}{(-\lambda\boldsymbol{b})_\theta}$. □

Theorem 3.3. *If two nonzero vectors* $\boldsymbol{c}_1$ *and* $\boldsymbol{c}_2$ *have the same direction, then*

(1) $\frac{\boldsymbol{c}_1+\boldsymbol{c}_2}{_\theta\boldsymbol{b}}=\frac{\boldsymbol{c}_1}{_\theta\boldsymbol{b}}+\frac{\boldsymbol{c}_2}{_\theta\boldsymbol{b}}$; (2) $\frac{\boldsymbol{c}_1+\boldsymbol{c}_2}{\boldsymbol{b}_\theta}=\frac{\boldsymbol{c}_1}{\boldsymbol{b}_\theta}+\frac{\boldsymbol{c}_2}{\boldsymbol{b}_\theta}$.

Proof. (1)

There is a real number $\lambda>0$ satisfying $\boldsymbol{c}_2=\lambda\boldsymbol{c}_1$, since two nonzero vectors $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ have the same direction. Thus,

$$\frac{\boldsymbol{c}_1}{_\theta\boldsymbol{b}}+\frac{\boldsymbol{c}_2}{_\theta\boldsymbol{b}}=\frac{\boldsymbol{c}_1}{_\theta\boldsymbol{b}}+\frac{\lambda\boldsymbol{c}_1}{_\theta\boldsymbol{b}}=\frac{\boldsymbol{c}_1}{_\theta\boldsymbol{b}}+\lambda\frac{\boldsymbol{c}_1}{_\theta\boldsymbol{b}}=(1+\lambda)\frac{\boldsymbol{c}_1}{_\theta\boldsymbol{b}}=\frac{(1+\lambda)\boldsymbol{c}_1}{_\theta\boldsymbol{b}}=\frac{\boldsymbol{c}_1+\lambda\boldsymbol{c}_1}{_\theta\boldsymbol{b}}=\frac{\boldsymbol{c}_1+\boldsymbol{c}_2}{_\theta\boldsymbol{b}}. \square$$

Note that, if $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ have opposite directions, then the results do not hold when $\theta\ne\frac{\pi}{2}$. In fact, simply suppose that $\boldsymbol{c}_1+\boldsymbol{c}_2$ and $\boldsymbol{c}_1$ have the same direction. Then $\frac{\boldsymbol{c}_1+\boldsymbol{c}_2}{_\theta\boldsymbol{b}}$ and $\frac{\boldsymbol{c}_1}{_\theta\boldsymbol{b}}$ have the same direction which is quite different with that of $\frac{\boldsymbol{c}_1}{_\theta\boldsymbol{b}}+\frac{\boldsymbol{c}_2}{_\theta\boldsymbol{b}}$. Not to mention, for general $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$.

When $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ have opposite directions and $\theta = \frac{\pi}{2}$, the equations hold. In fact, at this time, it is enough to recognize the directions of $\frac{\boldsymbol{c}_1}{{}_{\frac{\pi}{2}}\boldsymbol{b}}$ and $\frac{\boldsymbol{c}_2}{{}_{\frac{\pi}{2}}\boldsymbol{b}}$ are opposite. The angle $\frac{\pi}{2}$ is special and important, which even results in

Theorem 3.4. $\frac{\boldsymbol{c}}{{}_{\frac{\pi}{2}}\boldsymbol{b}} = \frac{\boldsymbol{b} \times \boldsymbol{c}}{|\boldsymbol{b}|^2}$ *and* $\frac{\boldsymbol{c}}{\boldsymbol{b}_{\frac{\pi}{2}}} = \frac{\boldsymbol{c} \times \boldsymbol{b}}{|\boldsymbol{b}|^2}$.

Proof. If $\boldsymbol{c} = \boldsymbol{0}$, the results hold evidently. Assume that $\boldsymbol{c} \neq \boldsymbol{0}$. Since $\frac{\boldsymbol{c}}{{}_{\frac{\pi}{2}}\boldsymbol{b}} \times \boldsymbol{b} = \boldsymbol{c}$, $\boldsymbol{b} \times \boldsymbol{c}$ and $\frac{\boldsymbol{c}}{{}_{\frac{\pi}{2}}\boldsymbol{b}}$ have the same direction. Thus, there is a real number $\lambda > 0$, satisfying $\boldsymbol{b} \times \boldsymbol{c} = \lambda \frac{\boldsymbol{c}}{{}_{\frac{\pi}{2}}\boldsymbol{b}}$. Therefore

$$\lambda \left| \frac{\boldsymbol{c}}{{}_{\frac{\pi}{2}}\boldsymbol{b}} \right| = |\boldsymbol{b} \times \boldsymbol{c}| = |\boldsymbol{b}||\boldsymbol{c}| = |\boldsymbol{b}| \left| \frac{\boldsymbol{c}}{{}_{\frac{\pi}{2}}\boldsymbol{b}} \times \boldsymbol{b} \right| = |\boldsymbol{b}| \left| \frac{\boldsymbol{c}}{{}_{\frac{\pi}{2}}\boldsymbol{b}} \right| |\boldsymbol{b}| = \left| \frac{\boldsymbol{c}}{{}_{\frac{\pi}{2}}\boldsymbol{b}} \right| |\boldsymbol{b}|^2 \Rightarrow \lambda = |\boldsymbol{b}|^2 \Rightarrow$$

$\frac{\boldsymbol{c}}{{}_{\frac{\pi}{2}}\boldsymbol{b}} = \frac{\boldsymbol{b} \times \boldsymbol{c}}{|\boldsymbol{b}|^2}$. Similarly, we have $\frac{\boldsymbol{c}}{\boldsymbol{b}_{\frac{\pi}{2}}} = \frac{\boldsymbol{c} \times \boldsymbol{b}}{|\boldsymbol{b}|^2}$. □

4. Structure of Cross Divisions

In this section, in order to conveniently study the structures of cross divisions, we always suppose that $\boldsymbol{c}$ and $\boldsymbol{b}$ are two vectors with $\boldsymbol{b} \neq \boldsymbol{0}$ and $\boldsymbol{c} \perp \boldsymbol{b}$, and $\theta_0 \in (0, \pi)$ is a given angle. We firstly present the following geometric expressions of indefinite cross divisions:

Theorem 4.1. In Figure 4, let O be a point in the natural space, and take

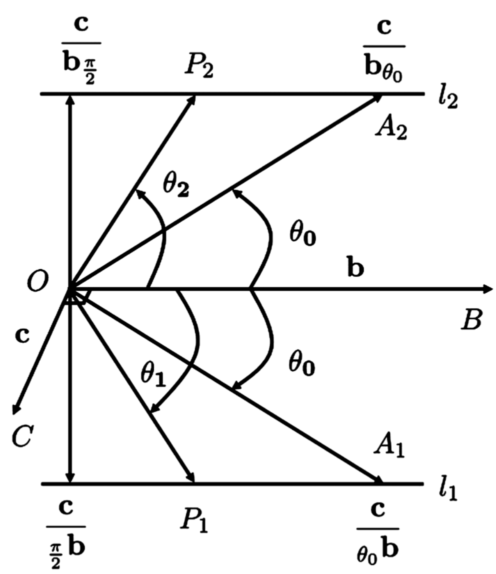

Figure 4. Structure.

$\overrightarrow{OB}=\boldsymbol{b}$, $\overrightarrow{OC}=\boldsymbol{c}$, $\overrightarrow{OA_1}=\dfrac{\boldsymbol{c}}{{}_{\theta_0}\boldsymbol{b}}$, $\overrightarrow{OA_2}=\dfrac{\boldsymbol{c}}{\boldsymbol{b}_{\theta_0}}$. For $i=1,2$, through A_i, draw a straight line l_i parallel to vector $\overrightarrow{OB}$. Then

(1) point P_1 is on the line l_1 if and only if there exists a $\theta_1\in(0,\pi)$ such that $\overrightarrow{OP_1}=\dfrac{\boldsymbol{c}}{{}_{\theta_1}\boldsymbol{b}}$;

(2) point P_2 is on the line l_2 if and only if there exists a $\theta_2\in(0,\pi)$ such that $\overrightarrow{OP_2}=\dfrac{\boldsymbol{c}}{\boldsymbol{b}_{\theta_2}}$.

Proof. (1) In fact, when P_1 is on the line l_1,

$$\overrightarrow{OP_1}\times\boldsymbol{b}=\left(\overrightarrow{OA_1}+\overrightarrow{A_1P_1}\right)\times\boldsymbol{b}=\overrightarrow{OA_1}\times\boldsymbol{b}+\overrightarrow{A_1P_1}\times\boldsymbol{b}=\boldsymbol{c}.$$

Thus, there exists a $\theta_1\in(0,\pi)$ such that $\overrightarrow{OP_1}=\dfrac{\boldsymbol{c}}{{}_{\theta_1}\boldsymbol{b}}$ according to Theorem 2.1.

Conversely, if there exists a $\theta_1\in(0,\pi)$ such that $\overrightarrow{OP_1}=\dfrac{\boldsymbol{c}}{{}_{\theta_1}\boldsymbol{b}}$. We have

$$\overrightarrow{P_1A_1}\times\boldsymbol{b}=\left(\overrightarrow{OA_1}-\overrightarrow{OP_1}\right)\times\boldsymbol{b}=\overrightarrow{OA_1}\times\boldsymbol{b}-\overrightarrow{OP_1}\times\boldsymbol{b}=\boldsymbol{c}-\boldsymbol{c}=\boldsymbol{0},$$

which implies P_1 is on the line l_1.

(2) Similarly.

Corollary 4.1. *The point sets* $\left\{P\mid\overrightarrow{OP}=\dfrac{\boldsymbol{c}}{{}_{\theta}\boldsymbol{b}},\theta\in(0,\pi)\right\}$ *and* $\left\{P\mid\overrightarrow{OP}=\dfrac{\boldsymbol{c}}{\boldsymbol{b}_{\theta}},\theta\in(0,\pi)\right\}$ *form two parallel lines, whose distance is* $\dfrac{2|\boldsymbol{c}|}{|\boldsymbol{b}|}$.

Proof. Obvious by Attribute (1.5) and Theorem 4.1. □

Corollary 4.2. For any $\theta\in(0,\pi)$, there is a real number λ such that

$$\frac{\boldsymbol{c}}{{}_{\theta}\boldsymbol{b}}=\frac{\boldsymbol{c}}{{}_{\theta_0}\boldsymbol{b}}+\lambda\boldsymbol{b}\text{ and }\frac{\boldsymbol{c}}{\boldsymbol{b}_{\theta}}=\frac{\boldsymbol{c}}{\boldsymbol{b}_{\theta_0}}+\lambda\boldsymbol{b}. \tag{7}$$

Proof. Obvious by Theorem 4.1. □

Especially, when $\theta_0=\dfrac{\pi}{2}$, we have

Corollary 4.3. *For any* $\theta\in(0,\pi)$, *there is a real number* λ *such that*

$$\frac{\boldsymbol{c}}{{}_{\theta}\boldsymbol{b}}=\frac{\boldsymbol{c}}{{}_{\frac{\pi}{2}}\boldsymbol{b}}+\lambda\boldsymbol{b}\text{ and }\frac{\boldsymbol{c}}{\boldsymbol{b}_{\theta}}=\frac{\boldsymbol{c}}{\boldsymbol{b}_{\frac{\pi}{2}}}+\lambda\boldsymbol{b} \tag{8}$$

where $\lambda=\dfrac{|\boldsymbol{c}|\cot\theta}{|\boldsymbol{b}|^2}$.

Proof. From **Figure 4**, we have $\cot\theta=\dfrac{\lambda|\boldsymbol{b}|}{\left|\dfrac{\boldsymbol{c}}{{}_{\frac{\pi}{2}}\boldsymbol{b}}\right|}=\dfrac{\lambda|\boldsymbol{b}|}{\dfrac{|\boldsymbol{c}|}{|\boldsymbol{b}|}}=\dfrac{\lambda|\boldsymbol{b}|^2}{|\boldsymbol{c}|}$

$$\Rightarrow \lambda = \frac{|\boldsymbol{c}|\cot\theta}{|\boldsymbol{b}|^2}. \square$$

Corollary 4.4. *For any* $\theta_1 < \theta_2$, *there is a real number* $\lambda > 0$ *such that*

$$\frac{\boldsymbol{c}}{{}_{\theta_1}\boldsymbol{b}} - \frac{\boldsymbol{c}}{{}_{\theta_2}\boldsymbol{b}} = \frac{\boldsymbol{c}}{\boldsymbol{b}_{\theta_1}} - \frac{\boldsymbol{c}}{\boldsymbol{b}_{\theta_2}} = \lambda\boldsymbol{b}. \quad (9)$$

Proof. Obvious. $\square$

(7) and (8) show the relations between indefinite cross divisions and λb with a fixed angle. They successfully put angle parameter into real parameter and can be also regard as the definitions of indefinite cross divisions. In **Figure 4**, we can find that, when θ goes to 0 or π, the vectors $\frac{\boldsymbol{c}}{{}_{\theta}\boldsymbol{b}}$ and $\frac{\boldsymbol{c}}{\boldsymbol{b}_{\theta}}$ are all closing to the straight line *OB*. When $\theta = 0$ or π, we have $\boldsymbol{c} = \boldsymbol{0}$ and $\frac{\boldsymbol{0}}{{}_{\theta_0}\boldsymbol{b}} = \frac{\boldsymbol{0}}{\boldsymbol{b}_{\theta_0}} = \boldsymbol{0}$ and $\frac{\boldsymbol{0}}{{}_{\theta}\boldsymbol{b}} = \frac{\boldsymbol{0}}{\boldsymbol{b}_{\theta}} = \lambda\boldsymbol{b}$ where $\lambda \in \mathbb{R}$ is determined by other conditions.

Thus, for $\theta = 0$ or π, we can present a supplementary definition of indefinite cross divisions to complete our theory.

Definition 4.1. Let $\boldsymbol{b}$ be a nonzero vector. For $\theta = 0$ or π, $\frac{\boldsymbol{0}}{{}_{\theta}\boldsymbol{b}} = \frac{\boldsymbol{0}}{\boldsymbol{b}_{\theta}} \triangleq \lambda\boldsymbol{b}$, where $\lambda \in R$ is called the **real parameter**.

Theorem 4.2. *For* $\theta = 0$ *or* π, *if there is a real number* μ *such that* $\frac{\boldsymbol{0}}{{}_{\theta}\boldsymbol{b}} \cdot \boldsymbol{b} = \mu$, *then*

$$\frac{\boldsymbol{0}}{{}_{\theta}\boldsymbol{b}} = \frac{\boldsymbol{0}}{\boldsymbol{b}_{\theta}} = \frac{\mu\boldsymbol{b}}{|\boldsymbol{b}|^2}$$

Proof. According to the above definition, there is a real number λ such that $\frac{\boldsymbol{0}}{{}_{\theta}\boldsymbol{b}} = \lambda\boldsymbol{b}$. We have, $\frac{\boldsymbol{0}}{{}_{\theta}\boldsymbol{b}} \cdot \boldsymbol{b} = \mu \Leftrightarrow (\lambda\boldsymbol{b}) \cdot \boldsymbol{b} = \mu \Leftrightarrow \lambda|\boldsymbol{b}|^2 = \mu \Leftrightarrow \lambda = \frac{\mu}{|\boldsymbol{b}|^2}$. $\square$

Combining Corollary 4.3 and Theorem 4.2 and $\frac{\boldsymbol{0}}{{}_{\frac{\pi}{2}}\boldsymbol{b}} = \frac{\boldsymbol{0}}{\boldsymbol{b}_{\frac{\pi}{2}}} = \boldsymbol{0}$, we can provide two unified forms by

Corollary 4.5. *For any* $\theta \in [0, \pi]$, *there is a real number* λ *such that*

$$\frac{\boldsymbol{c}}{{}_{\theta}\boldsymbol{b}} = \frac{\boldsymbol{c}}{{}_{\frac{\pi}{2}}\boldsymbol{b}} + \lambda\boldsymbol{b} \text{ and}$$
$$\frac{\boldsymbol{c}}{\boldsymbol{b}_{\theta}} = \frac{\boldsymbol{c}}{\boldsymbol{b}_{\frac{\pi}{2}}} + \lambda\boldsymbol{b} \quad (10)$$

where $\lambda = \begin{cases} \dfrac{|\boldsymbol{c}|\cot\theta}{|\boldsymbol{b}|^2}, & \theta \in (0, \pi) \\ \dfrac{\frac{\boldsymbol{0}}{{}_{\theta}\boldsymbol{b}} \cdot \boldsymbol{b}}{|\boldsymbol{b}|^2}, & \theta = 0, \pi \end{cases}$ if $\dfrac{\boldsymbol{0}}{{}_{\theta}\boldsymbol{b}} \cdot \boldsymbol{b}$ is known.

The general solution to Equation (5) can be expressed by an indefinite real number and a fixed angle such as $\frac{\pi}{2}$. When $\boldsymbol{a} \neq \boldsymbol{0}$, the general solution of Equation (5) is

$$\begin{cases} \boldsymbol{x} = \dfrac{\boldsymbol{c} - \boldsymbol{d} \times \boldsymbol{b}}{{}_{\frac{\pi}{2}}\boldsymbol{a}} + \lambda \boldsymbol{a}, & \lambda \in R, \\ \boldsymbol{y} = \boldsymbol{d}, & \boldsymbol{d} \text{ is an arbitrary vector.} \end{cases}$$

When $\boldsymbol{b} \neq \boldsymbol{0}$, the general solution of Equation (5) is

$$\begin{cases} \boldsymbol{x} = \boldsymbol{d}, & \boldsymbol{d} \text{ is an arbitrary vector,} \\ \boldsymbol{y} = \dfrac{\boldsymbol{c} - \boldsymbol{d} \times \boldsymbol{a}}{{}_{\frac{\pi}{2}}\boldsymbol{b}} + \lambda \boldsymbol{b}, & \lambda \in R. \end{cases}$$

Let θ_0 be a fixed angle and $\boldsymbol{b}$, $\boldsymbol{c}$ be two arbitrary vectors with $\boldsymbol{b} \neq \boldsymbol{0}$. Denote $G_{\theta_0} = \left\{ \dfrac{\boldsymbol{c}}{{}_{\theta_0}\boldsymbol{b}} + \lambda \boldsymbol{b} \mid \lambda \in R \right\}$ and $B^{\lambda} = \dfrac{\boldsymbol{c}}{{}_{\theta_0}\boldsymbol{b}} + \lambda \boldsymbol{b}$. Then G_{θ_0} is an Abelian group with the binary mapping $*$: $B^{\lambda} * B^{\mu} = B^{\lambda+\mu}$.

5. Coordinates of Cross Divisions

In this section, we just consider the coordinate formulas of indefinite cross divisions in some rectangular coordinate system. We firstly study the case of $\theta = \frac{\pi}{2}$.

Theorem 5.1. In a rectangular coordinate system $\{O; \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$, let $\boldsymbol{b} = \{X_2, Y_2, Z_2\} \neq \boldsymbol{0}$, $\boldsymbol{c} = \{X_3, Y_3, Z_3\}$. Suppose $\dfrac{\boldsymbol{c}}{{}_{\frac{\pi}{2}}\boldsymbol{b}} = \{X_1, Y_1, Z_1\}$. Then

$$\begin{cases} X_1 = \dfrac{Y_2 Z_3 - Y_3 Z_2}{X_2^2 + Y_2^2 + Z_2^2} \\ Y_1 = \dfrac{X_3 Z_2 - X_2 Z_3}{X_2^2 + Y_2^2 + Z_2^2} \\ Z_1 = \dfrac{X_2 Y_3 - X_3 Y_2}{X_2^2 + Y_2^2 + Z_2^2} \end{cases} \tag{11}$$

Proof. According to Theorem 3.4, we have

$$\frac{\boldsymbol{c}}{{}_{\frac{\pi}{2}}\boldsymbol{b}} = \frac{\boldsymbol{b} \times \boldsymbol{c}}{|\boldsymbol{b}|^2} = \frac{\{Y_2 Z_3 - Y_3 Z_2, X_3 Z_2 - X_2 Z_3, X_2 Y_3 - X_3 Y_2\}}{X_2^2 + Y_2^2 + Z_2^2}. \square$$

Theorem 5.2. *In a rectangular coordinate system* $\{O; \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$, *let*

$\boldsymbol{b}=\{X_2,Y_2,Z_2\}\neq\boldsymbol{0}$, $\boldsymbol{c}=\{X_3,Y_3,Z_3\}$, *and* $\theta\in(0,\pi)$. *Suppose* $\frac{\boldsymbol{c}}{{}_{\theta}\boldsymbol{b}}=\{X_1,Y_1,Z_1\}$. *Then*

$$\begin{cases} X_1=\dfrac{Y_2Z_3-Y_3Z_2+X_2\cot\theta\sqrt{X_3^2+Y_3^2+Z_3^2}}{X_2^2+Y_2^2+Z_2^2} \\ Y_1=\dfrac{X_3Z_2-X_2Z_3+Y_2\cot\theta\sqrt{X_3^2+Y_3^2+Z_3^2}}{X_2^2+Y_2^2+Z_2^2} \\ Z_1=\dfrac{X_2Y_3-X_3Y_2+Z_2\cot\theta\sqrt{X_3^2+Y_3^2+Z_3^2}}{X_2^2+Y_2^2+Z_2^2} \end{cases} \tag{12}$$

Proof. According to Corollary 4.3,

$$\frac{\boldsymbol{c}}{{}_{\theta}\boldsymbol{b}}=\frac{\boldsymbol{c}}{{}_{\frac{\pi}{2}}\boldsymbol{b}}+\lambda\boldsymbol{b}=\frac{\boldsymbol{b}\times\boldsymbol{c}}{|\boldsymbol{b}|^2}+\frac{|\boldsymbol{c}|\cot\theta}{|\boldsymbol{b}|^2}\boldsymbol{b}=\frac{\boldsymbol{b}\times\boldsymbol{c}+\boldsymbol{b}|\boldsymbol{c}|\cot\theta}{|\boldsymbol{b}|^2}.$$

Then we obtain the formula (12) by using the coordinates of $\boldsymbol{b}$ and $\boldsymbol{c}$. □

The formula (12), of course, can be obtained from other ways. For example, from the following three equations:

$$\left|\frac{\boldsymbol{c}}{{}_{\theta}\boldsymbol{b}}\right|=\frac{|\boldsymbol{c}|}{|\boldsymbol{b}|\sin\theta} \text{ and } \frac{\boldsymbol{c}}{{}_{\theta}\boldsymbol{b}}\times\boldsymbol{b}=\boldsymbol{c} \text{ and } \boldsymbol{c}\cdot\boldsymbol{b}=0.$$

In fact, substituting $\frac{\boldsymbol{c}}{{}_{\theta}\boldsymbol{b}}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ by their coordinates, by very complicated coordinate calculation, we can also derive the formula (12).

Theorem 5.3. *In a rectangular coordinate system* $\{O;\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}\}$, *let* $\mu\in R$, $\boldsymbol{b}=\{X_2,Y_2,Z_2\}\neq\boldsymbol{0}$. *For* $\theta=0$ *or* π, *suppose* $\frac{\boldsymbol{0}}{{}_{\theta}\boldsymbol{b}}=\{X_1,Y_1,Z_1\}$. *If* $\frac{\boldsymbol{0}}{{}_{\theta}\boldsymbol{b}}\cdot\boldsymbol{b}=\mu$, *then*

$$\begin{cases} X_1=\dfrac{\mu X_2}{X_2^2+Y_2^2+Z_2^2} \\ Y_1=\dfrac{\mu Y_2}{X_2^2+Y_2^2+Z_2^2} \\ Z_1=\dfrac{\mu Z_2}{X_2^2+Y_2^2+Z_2^2} \end{cases} \tag{13}$$

Proof. If $\theta=0$ or π, from Theorem 4.2, $\frac{\boldsymbol{0}}{{}_{\theta}\boldsymbol{b}}=\frac{\mu\boldsymbol{b}}{|\boldsymbol{b}|^2}$, we get Equation (13) by substituting $\boldsymbol{b}$ by its coordinates. □

Similarly, for right indefinite cross division, we have

Theorem 5.4. *In a rectangular coordinate system* $\{O;\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}\}$, *let* $\boldsymbol{a}=\{X_1,Y_1,Z_1\}$, $\boldsymbol{c}=\{X_3,Y_3,Z_3\}$. *Suppose* $\frac{\boldsymbol{c}}{\boldsymbol{a}_{\frac{\pi}{2}}}=\{X_2,Y_2,Z_2\}$. *Then*

$$\begin{cases} X_2 = \dfrac{Y_3 Z_1 - Y_1 Z_3}{X_1^2 + Y_1^2 + Z_1^2} \\ Y_2 = \dfrac{X_1 Z_3 - X_3 Z_1}{X_1^2 + Y_1^2 + Z_1^2} \\ Z_2 = \dfrac{X_3 Y_1 - X_1 Y_3}{X_1^2 + Y_1^2 + Z_1^2} \end{cases} \tag{14}$$

Theorem 5.5. *In a rectangular coordinate system* $\{O; \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$, *let* $\boldsymbol{a} = \{X_1, Y_1, Z_1\}$, $\boldsymbol{c} = \{X_3, Y_3, Z_3\}$, *and* $\theta \in (0, \pi)$. *Suppose* $\dfrac{\boldsymbol{c}}{\boldsymbol{a}_\theta} = \{X_2, Y_2, Z_2\}$. *Then*

$$\begin{cases} X_2 = \dfrac{Y_3 Z_1 - Y_1 Z_3 + X_1 \cot\theta \sqrt{X_3^2 + Y_3^2 + Z_3^2}}{X_1^2 + Y_1^2 + Z_1^2} \\ Y_2 = \dfrac{X_1 Z_3 - X_3 Z_1 + Y_1 \cot\theta \sqrt{X_3^2 + Y_3^2 + Z_3^2}}{X_1^2 + Y_1^2 + Z_1^2} \\ Z_2 = \dfrac{X_3 Y_1 - X_1 Y_3 + Z_1 \cot\theta \sqrt{X_3^2 + Y_3^2 + Z_3^2}}{X_1^2 + Y_1^2 + Z_1^2} \end{cases} \tag{15}$$

Theorem 5.6. *In a rectangular coordinate system* $\{O; \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$, *let* $\mu \in R$, $\boldsymbol{a} = \{X_1, Y_1, Z_1\}$. *For* $\theta = 0$ *or* π, *suppose* $\dfrac{\boldsymbol{0}}{\boldsymbol{a}_\theta} = \{X_2, Y_2, Z_2\}$. *If* $\boldsymbol{a} \cdot \dfrac{\boldsymbol{0}}{\boldsymbol{a}_\theta} = \mu$, *then*

$$\begin{cases} X_2 = \dfrac{\mu X_1}{X_1^2 + Y_1^2 + Z_1^2} \\ Y_2 = \dfrac{\mu Y_1}{X_1^2 + Y_1^2 + Z_1^2} \\ Z_2 = \dfrac{\mu Z_1}{X_1^2 + Y_1^2 + Z_1^2} \end{cases} \tag{16}$$

Now we consider how to use our theory. Though we can give some application examples of cross divisions in different fields, such as in physics, here we just give two very simple examples to show how to use our coordinate formulas and to test our theory by the way.

Example 5.1. *Given two vectors* $\boldsymbol{a} = \{X_1, Y_1, Z_1\} = \{1, 2, 3\}$ *and* $\boldsymbol{b} = \{X_2, Y_2, Z_2\} = \{2, 1, -2\}$, *then their cross product*

$\boldsymbol{a} \times \boldsymbol{b} \triangleq \boldsymbol{c} = \{X_3, Y_3, Z_3\} = \{Y_1 Z_2 - Y_2 Z_1, X_2 Z_1 - X_1 Z_2, X_1 Y_2 - X_2 Y_1\} = \{-7, 8, -3\}$.

Furthermore we have $|\boldsymbol{a}| = \sqrt{X_1^2 + Y_1^2 + Z_1^2} = \sqrt{14}$, $|\boldsymbol{b}| = \sqrt{X_2^2 + Y_2^2 + Z_2^2} = 3$, $|\boldsymbol{c}| = \sqrt{X_3^2 + Y_3^2 + Z_3^2} = \sqrt{122}$. *Since* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *are known, the angle* θ *between them is determined by* $\cos\theta = \dfrac{\boldsymbol{ab}}{|\boldsymbol{a}||\boldsymbol{b}|} = \dfrac{-2}{3\sqrt{14}}$. *By the way, we have*

$\sin\theta = \sqrt{1 - \cos^2\theta} = \dfrac{\sqrt{122}}{3\sqrt{14}}$, $\cot\theta = \dfrac{-2}{\sqrt{122}}$, *and*

$\cot\theta \sqrt{X_3^2 + Y_3^2 + Z_3^2} = \cot\theta |\boldsymbol{c}| = \dfrac{-2}{\sqrt{122}} \times \sqrt{122} = -2$.

It is no doubt that $\boldsymbol{c} \perp \boldsymbol{b}$ and $\boldsymbol{b} \neq \boldsymbol{0}$ and $\theta \in (0, \pi)$. Thus, from Formula (12), we have the coordinates of $\frac{\boldsymbol{c}}{{}_{\theta}\boldsymbol{b}}$:

$$X_1 = \frac{Y_2 Z_3 - Y_3 Z_2 + X_2 \cot\theta \sqrt{X_3^2 + Y_3^2 + Z_3^2}}{X_2^2 + Y_2^2 + Z_2^2}$$
$$= \frac{1}{9}\left[1 \times (-3) - (-2) \times 8 + 2 \times (-2)\right] = 1;$$

$$Y_1 = \frac{X_3 Z_2 - X_2 Z_3 + Y_2 \cot\theta \sqrt{X_3^2 + Y_3^2 + Z_3^2}}{X_2^2 + Y_2^2 + Z_2^2}$$
$$= \frac{1}{9}\left[-2 \times (-7) - 2 \times (-3) + 1 \times (-2)\right] = 2;$$

$$Z_1 = \frac{X_2 Y_3 - X_3 Y_2 + Z_2 \cot\theta \sqrt{X_3^2 + Y_3^2 + Z_3^2}}{X_2^2 + Y_2^2 + Z_2^2}$$
$$= \frac{1}{9}\left[2 \times 8 - (-7) \times 1 + (-2) \times (-2)\right] = 3.$$

It is readily seen that $\frac{\boldsymbol{c}}{{}_{\theta}\boldsymbol{b}}$ is really equal to $\boldsymbol{a}$.

Similarly, since $\boldsymbol{c} \perp \boldsymbol{a}$ and $\boldsymbol{a} \neq \boldsymbol{0}$ and $\theta \in (0, \pi)$, by Formula (15), we can obtain the coordinates of $\frac{\boldsymbol{c}}{\boldsymbol{a}_{\theta}}$.

$$X_2 = \frac{Y_3 Z_1 - Y_1 Z_3 + X_1 \cot\theta \sqrt{X_3^2 + Y_3^2 + Z_3^2}}{X_1^2 + Y_1^2 + Z_1^2}$$
$$= \frac{1}{14}\left[8 \times 3 - 2 \times (-3) + 1 \times (-2)\right] = 2;$$

$$Y_2 = \frac{X_1 Z_3 - X_3 Z_1 + Y_1 \cot\theta \sqrt{X_3^2 + Y_3^2 + Z_3^2}}{X_1^2 + Y_1^2 + Z_1^2}$$
$$= \frac{1}{14}\left[1 \times (-3) - (-7) \times 3 + 2 \times (-2)\right] = 1;$$

$$Z_2 = \frac{X_3 Y_1 - X_1 Y_3 + Z_1 \cot\theta \sqrt{X_3^2 + Y_3^2 + Z_3^2}}{X_1^2 + Y_1^2 + Z_1^2}$$
$$= \frac{1}{14}\left[(-7) \times 2 - 1 \times 8 + 3 \times (-2)\right] = -2.$$

It is also seen that $\frac{\boldsymbol{c}}{\boldsymbol{a}_{\theta}}$ is exactly equal to $\boldsymbol{b}$.

Example 5.2. *Given two vectors* $\boldsymbol{a} = \{X_1, Y_1, Z_1\} = \{3, 2\sqrt{2}, 2\}$ *and* $\boldsymbol{b} = \{X_2, Y_2, Z_2\} = \{1.5, \sqrt{2}, 1\}$. *Since* $\boldsymbol{a} = 2\boldsymbol{b}$, *their cross product* $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{0}$ *and* $\theta = 0$. *Furthermore, their dot product*
$c = \boldsymbol{a} \cdot \boldsymbol{b} = 3 \times 1.5 + 2\sqrt{2} \times \sqrt{2} + 2 \times 1 = 10.5$, *and*
$|\boldsymbol{a}|^2 = X_1^2 + Y_1^2 + Z_1^2 = 9 + 8 + 4 = 21$, *and*
$|\boldsymbol{b}|^2 = X_2^2 + Y_2^2 + Z_2^2 = 2.25 + 2 + 1 = 5.25$. *If we regard* $\boldsymbol{a} \cdot \boldsymbol{b} = 10.5$ *as known,*

then $\mu = 10.5$.

Thus, from Formula (13), we have the coordinates of $\frac{\mathbf{0}}{{}_0\boldsymbol{b}}$:

$$X_1 = \frac{\mu X_2}{X_2^2 + Y_2^2 + Z_2^2} = \frac{1.5\mu}{5.25} = \frac{1.5 \times 10.5}{5.25} = 3;$$

$$Y_1 = \frac{\mu Y_2}{X_2^2 + Y_2^2 + Z_2^2} = \frac{\sqrt{2}\mu}{5.25} = \frac{\sqrt{2} \times 10.5}{5.25} = \frac{\sqrt{2} \times 10.5}{5.25} = 2\sqrt{2};$$

$$Z_1 = \frac{\mu Z_3}{X_2^2 + Y_2^2 + Z_2^2} = \frac{\mu}{5.25} = \frac{10.5}{5.25} = 2.$$

It is readily seen that $\frac{\mathbf{0}}{{}_0\boldsymbol{b}}$ is exactly equal to $\boldsymbol{a}$.

Similarly, we have the coordinates of $\frac{\mathbf{0}}{\boldsymbol{a}_0}$:

$$X_2 = \frac{\mu X_1}{X_1^2 + Y_1^2 + Z_1^2} = \frac{3\mu}{21} = \frac{\mu}{7} = \frac{10.5}{7} = 1.5;$$

$$Y_2 = \frac{\mu Y_1}{X_1^2 + Y_1^2 + Z_1^2} = \frac{2\sqrt{2}\mu}{21} = \frac{2\sqrt{2} \times 10.5}{21} = \sqrt{2};$$

$$Z_2 = \frac{\mu Z_1}{X_1^2 + Y_1^2 + Z_1^2} = \frac{2\mu}{21} = \frac{2 \times 10.5}{21} = 1.$$

It is also seen that $\frac{\mathbf{0}}{\boldsymbol{a}_0}$ is fully equal to $\boldsymbol{b}$.

When we want to find $\boldsymbol{a}$ from $\mathbf{0}$ and $\boldsymbol{b}$ such that $\boldsymbol{b} \neq \mathbf{0}$ and $\frac{\mathbf{0}}{{}_0\boldsymbol{a}} \cdot \boldsymbol{b} = \mu$. At this time, $\boldsymbol{b}$ is known, so we can regard $\frac{\mathbf{0}}{{}_0\boldsymbol{a}} \cdot \boldsymbol{b} = \mu = -5$ as known if we need, that results in $\frac{\mathbf{0}}{{}_0\boldsymbol{b}} = \left\{\frac{1.5 \times (-5)}{5.25}, \frac{\sqrt{2} \times (-5)}{5.25}, \frac{1 \times (-5)}{5.25}\right\} = \left\{-\frac{10}{7}, -\frac{20\sqrt{2}}{21}, -\frac{20}{21}\right\}$, and $\frac{\mathbf{0}}{\boldsymbol{a}_0} = \left\{-\frac{20}{7}, -\frac{40\sqrt{2}}{21}, -\frac{40}{21}\right\}$.

It is seen that, if we know the angle (≠0 or π) or dot product when the angle is 0 or π, we can inversely find the unique expected indefinite cross quotients $\frac{\mathbf{0}}{{}_\theta\boldsymbol{b}}$ and $\frac{\mathbf{0}}{\boldsymbol{a}_\theta}$ accurately. Furthermore, our formulas can tell that according to our new needs, we can quickly get other new vectors to fit our new needs from $\boldsymbol{b}$ and $\boldsymbol{c}$ by changing angle parameter θ.

6. Conclusions

This paper has solved the problem that cross product has no corresponding division by introducing the indefinite cross divisions. When we know two vectors $\boldsymbol{c}$ and $\boldsymbol{b}$ such that $\boldsymbol{c} \perp \boldsymbol{b}$ and $\boldsymbol{b} \neq \mathbf{0}$, according to our theory, we can inversely obtain two vectors $\frac{\boldsymbol{c}}{{}_{\theta}\boldsymbol{b}}$ and $\frac{\boldsymbol{c}}{\boldsymbol{b}_{\theta}}$ such that $\frac{\boldsymbol{c}}{{}_{\theta}\boldsymbol{b}}\times\boldsymbol{b}=\boldsymbol{c}$ and $\boldsymbol{b}\times\frac{\boldsymbol{c}}{\boldsymbol{b}_{\theta}}=\boldsymbol{c}$ where θ is an angle parameter. Furthermore, we can design indefinite cross quotients by adjusting angle parameter to fit new situation in the application. If we know the coordinates of $\boldsymbol{c}$ and $\boldsymbol{b}$, the coordinate formulas (11)-(16) can help us to get the coordinates of $\frac{\boldsymbol{c}}{{}_{\theta}\boldsymbol{b}}$ and $\frac{\boldsymbol{c}}{\boldsymbol{b}_{\theta}}$. It is worth mentioning that Corollary 4.5 not only puts angle parameter into real parameter but also presents two unified expressions:

$$\frac{\boldsymbol{c}}{{}_{\theta}\boldsymbol{b}}=\frac{\boldsymbol{c}}{{}_{\frac{\pi}{2}}\boldsymbol{b}}+\lambda\boldsymbol{b} \text{ and } \frac{\boldsymbol{c}}{\boldsymbol{b}_{\theta}}=\frac{\boldsymbol{c}}{\boldsymbol{b}_{\frac{\pi}{2}}}+\lambda\boldsymbol{b},$$

which avoids concerning the angle is in $(0,\pi)$ or not so that let us solve some problems easily. When meeting the equations of cross products in practical applications, the indefinite cross divisions can help us obtain the solutions to the equations of cross products. Our theory of indefinite cross divisions makes cross product theory more perfect.

The relation between indefinite cross divisions and cross products likes that between indefinite integrals and derivatives.

Conflicts of Interest

The authors declare no conflicts of interest.

References

[1] Alikhani, R. and Bahrami, F. (2019) Fuzzy Partial Differential Equations under the Cross Product of Fuzzy Numbers. *Information Sciences*, **494**, 80-99. https://doi.org/10.1016/j.ins.2019.04.030

[2] Hausner, M. (1998) A Vector Space Approach to Geometry. Dover Publications, New York.

[3] McDavid, A.W. and McMullen, C.D. (2006) Generalizing Cross Products and Maxwell's Equations to Universal Extra Dimensions. arXiv.org

[4] Song, Y.L. (2017) A K-Homological Approach to the Quantization Commutes with Reduction Problem. *Journal of Geometry and Physics*, **112**, 29-44. https://doi.org/10.1016/j.geomphys.2016.08.017

[5] Springer, C.E. (2012) Tensor and Vector Analysis: With Applications to Differential Geometry. Dover Publications, Mineola, New York.

[6] Gross, J., Trenkler, G. and Troschke, S.-O. (1999) The Vector Cross Product in $\mathbb{C}^3$. *International Journal of Mathematical Education in Science and Technology*, **30**, 549-555. https://doi.org/10.1080/002073999287815

[7] Gonano, C.A. and Zich, R.E. (2014) Cross Product in N Dimensions: The Double-wedge Product. Polytechnic University of Milan, Italy.

[8] Massey, W.S. (1983) Cross Products of Vectors in Higher Dimensional Euclidean Spaces. *The American Mathematical Monthly*, **90**, 697-701. https://doi.org/10.1080/00029890.1983.11971316

[9] Silagadze, Z.K. (2002) Multi-Dimensional Vector Product. *Journal of Physics A: Mathematical and General*, **35**, 4949. https://doi.org/10.1088/0305-4470/35/23/310

[10] Galbis, A. and Maestre, M. (2012) Vector Analysis versus Vector Calculus. Springer New York Dordrecht Heidelberg, London. https://doi.org/10.1007/978-1-4614-2200-6

[11] Lengyel, E. (2012) Mathematics for 3D Game Programming and Computer Graphics. Third Edition, Course Technology, Boston.

[12] Lv, L.G. and Xu, Z.D. (2019) Analytic Geometry. Fifth Edition, Higher Education Press, Beijing.

[13] Vince, J. (2008) Geometric Algebra for Computer Graphics. Springer-Verlag London Limited, London. https://doi.org/10.1007/978-1-84628-997-2

[14] Weatherburn, C.E., Came, M.A. and Sydney, D.S. (1921) Elementry Vector Analysis with Application to Geometry and Physics. G. Bell and Sons, Ltd., London.