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Generalized Hyers-Ulam-Rassias Type Stability Additive α-Functional Inequalities with $3k$-Variable in Complex Banach Spaces

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Abstract

In this paper we study to solve two-additive α-functional inequality with 3k-variables and their Hyers-Ulam-Rassias type stability. It is investigated in complex Banach spaces. These are the main results of this paper.

Subject Areas

Mathematics

Keywords

Additive β-Functional Equation, Additive β-Functional Inequality, Complex Banach Space, Hyers-Ulam-Rassisa Stability

Mathematics Subject Classification

Primary 4610, 4710, 39B62, 39B72, 39B52

1. Introduction

Let $\mathbf{X}$ and $\mathbf{Y}$ be normed spaces on the same field $\mathbb{K}$, and $f:\mathbf{X}\to\mathbf{Y}$. We use the notation $\|\cdot\|$ for all the norms on both $\mathbf{X}$ and $\mathbf{Y}$. In this paper, we investigate some additive α-functional inequality when $\mathbf{X}$ is a real or complex normed space and $\mathbf{Y}$ is a complex Banach space.

In fact, when $\mathbf{X}$ is a real or complex normed space and $\mathbf{Y}$ is a complex Banach space, we solve and prove the Hyers-Ulam stability of following additive α-functional inequality.

$$\left\| f\left((m+1)\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j \right)-\sum_{j=1}^{k} f\left(m\frac{x_j+y_j}{2k}-z_j \right)-\sum_{j=1}^{k} f\left(\frac{x_j+y_j}{2k} \right) \right\|_{\mathbf{Y}} \tag{1}$$
$$\left\| \alpha\left(f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j \right)-\sum_{j=1}^{k} f\left(\frac{x_j+y_j}{2k} \right)-\sum_{j=1}^{k} f\left(z_j \right) \right) \right\|_{\mathbf{Y}}$$

and when we change the role of the function inequality (1), we continue to prove the following function inequality

$$\left\| f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j \right)-\sum_{j=1}^{k} f\left(\frac{x_j+y_j}{2k} \right)-\sum_{j=1}^{k} f\left(z_j \right) \right\|_{\mathbf{Y}}$$
$$\le \left\| \alpha\left(f\left((m+1)\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j \right)-\sum_{j=1}^{k}\left(m\frac{x_j+y_j}{2k}-z_j \right)-\sum_{j=1}^{k} f\left(\frac{x_j+y_j}{2k} \right) \right) \right\|_{\mathbf{Y}} \tag{2}$$

So (1) and (2) are equivalent propositions.

Where α is a fixed complex number with $|\alpha|<1$ and m be a fixed integer with $m>1$.

The Hyers-Ulam stability was first investigated for functional equation of Ulam in [1] concerning the stability of group homomorphisms.

The functional equation

$$f(x+y)=f(x)+f(y)$$

is called the Cauchy equation. In particular, every solution of the Cauchy equation is said to be an additive mapping.

The Hyers [2] gave first affirmative partial answer to the equation of Ulam in Banach spaces. After that, Hyers' Theorem was generalized by Aoki [3] additive mappings and by Rassias [4] for linear mappings considering an unbouned Cauchy diffrence. Ageneralization of the Rassias theorem was obtained by Găvruta [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

The Hyers-Ulam stability for functional inequalities has been investigated such as in [5] [6] [7]. Gilány showed that if it satisfies the functional inequality

$$\|2f(x)+2f(y)-f(x-y)\| \le \|f(x+y)\| \tag{3}$$

Then f satisfies the Jordan-von Newman functional equation

$$2f(x)+2f(y)=f(x+y)+f(x-y) \tag{4}$$

Gilányi [5] and Fechner [8] proved the Hyers-Ulam stability of the functional inequality (3).

Next Choonkil Park [9] proved the Hyers-Ulam stability of additive β-functional inequalities. Recently, the author has studied the addition inequalities of mathematicians in the world as [5] [8] [10]-[24] and I have introduced two general additive function inequalities (1) and (2) based on the (β_1,β_2) -function inequality result, see [25]. When inserting the parameter m this is the opening for modern functional equations. That is, it demonstrates the superiority of the field of functional equations and is also a bright horizon for the special development of functional equations. So in this paper, we solve and proved the Hyers-Ulam stability for two α-functional inequalities (1)-(2), *i.e.* the α-functional inequalities with $3k$-variables. Under suitable assumptions on spaces $\mathbf{X}$ and $\mathbf{Y}$, we will prove that the mappings satisfying the α-functional inequatilies (1) or (2). Thus, the results in this paper are generalization of those in [7] [9] [17] [25] [26] [27] for α-functional inequatilies with $3k$-variables. The paper is organized as followns: In section preliminarier we remind a basic property such as We only redefine the solution definition of the equation of the additive function.

Notice here that we make the general assumption that: $\mathbf{G}$ be a k-divisible abelian group.

Section 3: is devoted to prove the Hyers-Ulam stability of the addive α-functional inequalities (1) when $\mathbf{X}$ is a real or complex normed space and $\mathbf{Y}$ complex Banach space.

Section 4: is devoted to prove the Hyers-Ulam stability of the addive α-functional inequalities (2) when $\mathbf{X}$ is a real or complex normed space and $\mathbf{Y}$ complex Banach space.

2. Preliminaries

Solutions of the Inequalities

The functional equation

$$f(x+y)=f(x)+f(y)$$

is called the cauchuy equation. In particular, every solution of the cauchuy equation is said to be an additive mapping.

3. Establish the Solution of the Additive α-Function Inequalities

Now, we first study the solutions of (1). Note that for these inequalities, $\mathbf{G}$ be a k-divisible abelian group, $\mathbf{X}$ is a real or complex normed space and $\mathbf{Y}$ is a complex Banach spaces. Under this setting, we can show that the mapping satisfying (1.1) is additive. These results are give in the following.

Lemma 1. Let $m\in\mathbb{N}$ and a mapping $f:\mathbf{G}\to\mathbf{Y}$ satilies

$$\begin{aligned}&\left\|f\left((m+1)\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)-\sum_{j=1}^{k}f\left(m\frac{x_j+y_j}{2k}-z_j\right)-\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)\right\|_{\mathbf{Y}}\\&\leq\left\|\alpha\left(f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)-\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f(z_j)\right)\right\|_{\mathbf{Y}}\end{aligned}\tag{5}$$

for all $x_j,y_j,z_j\in\mathbf{G}$ for $j=1\to n$, then $f:\mathbf{G}\to\mathbf{Y}$ is additive

Proof. Assume that $f:\mathbf{G}\to\mathbf{Y}$ satisfies (5).

We replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(0,\cdots,0,0,\cdots,0,0,\cdots,0)$ in (5), we have

$$\|(2k-1)f(0)\|_{\mathbf{Y}}\leq\|\alpha(2k-1)f(0)\|_{\mathbf{Y}}\leq 0$$

therefore

$$\left(\left|2k-1\right|-\left|\alpha\left(2k-1\right)\right|\right)\left\|f\left(0\right)\right\|_{\mathbf{Y}}\leq 0$$

So $f\left(0\right)=0$.

Replacing $\left(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\right)$ by $\left(x_1,\cdots,x_k,y_1,\cdots,y_k,m\cdot\frac{x_1+y_1}{2k}-v_1,\cdots,m\cdot\frac{x_k+y_k}{2k}-v_k\right)$ in (5), we have

$$\begin{aligned}&\left\|f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}v_j\right)-\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(v_j\right)\right\|_{\mathbf{Y}}\\&\leq\left\|\alpha\left(f\left((m+1)\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}v_j\right)-\sum_{j=1}^{k}f\left(m\frac{x_j+y_j}{2k}-v_j\right)-\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)\right)\right\|_{\mathbf{Y}}\end{aligned}\tag{6}$$

for all $x_1,\cdots,x_k,y_1,\cdots,y_k,\frac{x_1+y_1}{2k}-v_1,\cdots,\frac{x_k+y_k}{2k}-v_k\in\mathbf{G}$. From (5) and (6) we infer that

$$\begin{aligned}&\left\|f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)-\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(z_j\right)\right\|_{\mathbf{Y}}\\&\leq\left\|\alpha\left(f\left((m+1)\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)-\sum_{j=1}^{k}f\left(m\frac{x_j+y_j}{2k}-z_j\right)-\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)\right)\right\|_{\mathbf{Y}}\\&\leq\left\|\alpha^2\left(f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)-\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(z_j\right)\right)\right\|_{\mathbf{Y}}\end{aligned}\tag{7}$$

and so

$$f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)=\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)+\sum_{j=1}^{k}f\left(z_j\right)$$

for all $x_j,y_j,z_j\in\mathbf{G}$ for $j=1\to n$, as we expected.

Theorem 2. Let $r>1,m\in\mathbb{Z},m>1$, θ be nonngative real number, and let $f:\mathbf{X}\to\mathbf{Y}$ be a mapping such that

$$\begin{aligned}&\left\|f\left((m+1)\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)-\sum_{j=1}^{k}f\left(m\frac{x_j+y_j}{2k}-z_j\right)-\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)\right\|_{\mathbf{Y}}\\&\leq\left\|\alpha\left(f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)-\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(z_j\right)\right)\right\|_{\mathbf{Y}}\\&\quad+\theta\left(\sum_{j=1}^{k}\left\|x_j\right\|_{\mathbf{X}}^{r}+\sum_{j=1}^{k}\left\|y_j\right\|_{\mathbf{X}}^{r}+\sum_{j=1}^{k}\left\|z_j\right\|_{\mathbf{X}}^{r}\right)\end{aligned}\tag{8}$$

for all $x_j,y_j,z_j\in X$ for all $j=1\to n$. Then there exists a unique additive mapping $\phi:\mathbf{X}\to\mathbf{Y}$ such that

$$\left\|f\left(x\right)-h\left(x\right)\right\|_{\mathbf{Y}}\leq\frac{\sum_{q=1}^{m-1}\left(q^r+2k^r\right)}{\left(1-\left|\alpha\right|\right)\left(m^r-m\right)}\theta\left\|x\right\|_{\mathbf{X}}^{r}.\tag{9}$$

for all $x\in\mathbf{X}$.

Proof. Assume that $f:\mathbf{X}\to\mathbf{Y}$ satisfies (8).

Replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(0,\cdots,0,0,\cdots,0,0,\cdots,0)$ in (8), we have

$$\|(2k-1)f(0)\|_{\mathbf{Y}}\le\|\alpha(2k-1)f(0)\|_{\mathbf{Y}}\le 0$$

therefore

$$(|2k-1|-|\alpha(2k-1)|)\|f(0)\|_{\mathbf{Y}}\le 0$$

So $f(0)=0$.

Next we:

Replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(kx,0,\cdots,0,kx,0,\cdots,0,0,\cdots,0)$ in (8), we get

$$\|f((m+1)x)-f(mx)-f(x)\|_{\mathbf{Y}}\le 2k^r\theta\|x\|_{\mathbf{X}}^r \tag{10}$$

for all $x\in\mathbf{X}$. Thus for $q\in\mathbb{N}$.

We replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(kx,0,\cdots,0,kx,0,\cdots,0,qx,0,\cdots,0)$ in (8), we have

$$\begin{aligned}&\|f((m-q+1)x)-f((m-q)x)-f(x)\|_{\mathbf{Y}}\\&\le\|\alpha(f((q+1)x)-f(qx)-f(x))\|_{\mathbf{Y}}+\theta(2k^r+q^r)\|x\|_{\mathbf{Y}}^r\end{aligned} \tag{11}$$

for all $x\in\mathbf{X}$.

For (10) and (11)

$$\begin{aligned}&\sum_{q=1}^{m-1}\|f((m-q+1)x)-f((m-q)x)-f(x)\|_{\mathbf{Y}}\\&\le\sum_{q=1}^{m-1}\|\alpha(f((q+1)x)-f(qx)-f(x))\|_{\mathbf{Y}}+\theta\left(\sum_{q=1}^{m-1}(2k^r+q^r)\|x\|_{\mathbf{X}}^r\right)\end{aligned} \tag{12}$$

for all $x\in\mathbf{X}$.

From (11) and (12) and triangle inequality, we have

$$\begin{aligned}&(1-|\alpha|)\|f(mx)-mf(x)\|_{\mathbf{Y}}\\&=(1-|\alpha|)\sum_{q=1}^{m-1}\|f((q+1)x)-f(qx)-f(x)\|_{\mathbf{Y}}\\&\le\sum_{q=1}^{m-1}(1-|\alpha|)\|f((q+1)x)-f(qx)-f(x)\|_{\mathbf{Y}}\\&\le\sum_{q=1}^{m-1}\|f((q+1)x)-f(qx)-f(x)\|_{\mathbf{Y}}-\sum_{q=1}^{m-1}\|\alpha(f((q+1)x)-f(qx)-f(x))\|_{\mathbf{Y}}\\&\le\theta\left(\sum_{q=1}^{m-1}(2k^r+q^r)\|x\|_{\mathbf{X}}^r\right)\end{aligned} \tag{13}$$

for all $x\in\mathbf{X}$. from

$$\begin{aligned}&\sum_{q=1}^{m-1}\|f((m-q+1)x)-f((m-q)x)-f(x)\|_{\mathbf{Y}}\\&=\sum_{q=1}^{m-1}\|f((q+1)x)-f(qx)-f(x)\|_{\mathbf{Y}}\end{aligned}$$

Since $|\alpha|<1$, the mapping f satisfies the inequalities

$$\|f(mx)-mf(x)\|_{\mathbf{Y}} \le \frac{\theta\left(\sum_{q=1}^{m-1}\left(2k^r+q^r\right)\|x\|_{\mathbf{X}}^r\right)}{1-|\alpha|}$$

for all $x\in\mathbf{X}$.

Therefore

$$\left\|f(x)-mf\left(\frac{x}{m}\right)\right\|_{\mathbf{Y}} \le \frac{\theta\left(\sum_{q=1}^{m-1}\left(2k^r+q^r\right)\|x\|_{\mathbf{X}}^r\right)}{(1-|\alpha|)m^r} \tag{14}$$

for all $x\in X$. So

$$\begin{aligned}\left\|m^l f\left(\frac{x}{m^n}\right)-m^p f\left(\frac{x}{m^h}\right)\right\|_{\mathbf{Y}} &\le \sum_{j=l}^{p-1}\left\|m^j f\left(\frac{x}{m^j}\right)-m^{j+1} f\left(\frac{x}{m^{j+1}}\right)\right\|_{\mathbf{Y}} \\ &\le \frac{\theta\left(\sum_{q=1}^{m-1}\left(2k^r+q^r\right)\right)}{(1-|\alpha|)m^r}\sum_{j=l}^{p-1}\frac{m^j}{m^{rj}}\|x\|_{\mathbf{X}}^r\end{aligned} \tag{15}$$

for all nonnegative integers p,l with $p>l$ and all $x\in\mathbf{X}$. It follows from (15) that the sequence $\left\{m^n f\left(\frac{x}{m^n}\right)\right\}$ is a cauchy sequence for all $x\in\mathbf{X}$. Since $\mathbf{Y}$ is complete, the sequence $\left\{m^n f\left(\frac{x}{m^n}\right)\right\}$ coverges.

So one can define the mapping $\phi:\mathbf{X}\to\mathbf{Y}$ by $\phi(x):=\lim_{n\to\infty} m^n f\left(\frac{x}{m^n}\right)$ for all $x\in\mathbf{X}$. Moreover, letting $l=0$ and passing the limit $m\to\infty$ in (15), we get (9).

It follows from (8) that

$$\begin{aligned}&\left\|\phi\left((m+1)\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)-\sum_{j=1}^{k}\phi\left(m\frac{x_j+y_j}{2k}-z_j\right)-\sum_{j=1}^{k}\phi\left(\frac{x_j+y_j}{2k}\right)\right\|_{\mathbf{Y}} \\ &=\lim_{n\to\infty} m^n\left\|f\left(\frac{m+1}{m^n}\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\frac{1}{m^n}\sum_{j=1}^{k}z_j\right)-\sum_{j=1}^{k}f\left(\frac{m}{m^n}\frac{x_j+y_j}{2k}-\frac{1}{m^n}z_j\right)\right. \\ &\quad\left.-\sum_{j=1}^{k}f\left(\frac{1}{m^n}\frac{x_j+y_j}{2k}\right)\right\|_{\mathbf{Y}} \\ &\le \lim_{n\to\infty} m^n\left\|\alpha\left(f\left(\frac{1}{m^n}\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\frac{1}{m^n}\sum_{j=1}^{k}z_j\right)-\sum_{j=1}^{k}f\left(\frac{1}{m^n}\frac{x_j+y_j}{2k}\right)\right.\right. \\ &\quad\left.\left.-\sum_{j=1}^{k}f\left(\frac{1}{m^n}z_j\right)\right)\right\|_{\mathbf{Y}}+\lim_{n\to\infty}\frac{m^n}{m^{nr}}\theta\left(\sum_{j=1}^{k}\|x_j\|_{\mathbf{X}}^r+\sum_{j=1}^{k}\|y_j\|_{\mathbf{X}}^r+\sum_{j=1}^{k}\|z_j\|_{\mathbf{X}}^r\right) \\ &\le|\alpha|\left\|\phi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)-\sum_{j=1}^{k}\phi\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}\phi(z_j)\right\|_{\mathbf{Y}}\end{aligned} \tag{16}$$

for all $x_j,y_j,z_j\in X$ for all $j=1\to n$.

$$\begin{aligned}&\left\|\phi\left((m+1)\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)-\sum_{j=1}^{k}\phi\left(m\frac{x_j+y_j}{2k}-z_j\right)-\sum_{j=1}^{k}\phi\left(\frac{x_j+y_j}{2k}\right)\right\|_{\mathbf{Y}} \\ &\le|\alpha|\left\|\phi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)-\sum_{j=1}^{k}\phi\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}\phi(z_j)\right\|_{\mathbf{Y}}\end{aligned}$$

for all $x_j, y_j, z_j \in X$ for all $j = 1 \to n$. So by lemma 21 it follows that the mapping $\phi : \mathbf{X} \to \mathbf{Y}$ is additive. Now we need to prove uniqueness, suppose $\phi' : \mathbf{X} \to \mathbf{Y}$ is also an additive mapping that satisfies (9). Then we have

$$\begin{aligned} &\left\|\phi(x)-\phi'(x)\right\|_{\mathbf{Y}} \\ &= m^n \left\|\phi\left(\frac{x}{m^n}\right)-\phi'\left(\frac{x}{m^n}\right)\right\|_{\mathbf{Y}} \\ &\le m^n \left(\left\|\phi\left(\frac{x}{m^n}\right)-f\left(\frac{x}{m^n}\right)\right\|_{\mathbf{Y}}+\left\|\phi'\left(\frac{x}{m^n}\right)-f\left(\frac{x}{m^n}\right)\right\|_{\mathbf{Y}}\right) \\ &\le \frac{2\cdot m^n \cdot \sum_{q=1}^{m-1}\left(q^r+2k^r\right)}{\left(1-|\alpha|\right)m^{nr}\left(m^r-m\right)}\theta\|x\|_{\mathbf{X}}^r \end{aligned} \tag{17}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $\phi(x) = \phi'(x)$ for all $x \in \mathbf{X}$. This proves thus the mapping $\phi : \mathbf{X} \to \mathbf{Y}$ is a unique mapping satisfying (9) as we expected.

Theorem 3. Let $r > 1, m \in \mathbb{Z}, m > 1$, θ be nonngative real number, and let $f : \mathbf{X} \to \mathbf{Y}$ be a mapping such that

$$\begin{aligned} &\left\|f\left((m+1)\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)-\sum_{j=1}^{k}f\left(m\frac{x_j+y_j}{2k}-z_j\right)-\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)\right\|_{\mathbf{Y}} \\ &\le \left\|\alpha\left(f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)-\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(z_j\right)\right)\right\|_{\mathbf{Y}} \\ &\quad +\theta\left(\sum_{j=1}^{k}\left\|x_j\right\|_{\mathbf{X}}^r+\sum_{j=1}^{k}\left\|y_j\right\|_{\mathbf{X}}^r+\sum_{j=1}^{k}\left\|z_j\right\|_{\mathbf{X}}^r\right) \end{aligned} \tag{18}$$

for all $x_j, y_j, z_j \in X$ for all $j = 1 \to n$. Then there exists a unique mapping $\phi : \mathbf{X} \to \mathbf{Y}$ such that

$$\left\|f(x)-\phi(x)\right\|_{\mathbf{Y}} \le \frac{m^n \cdot \sum_{q=1}^{m-1}\left(q^r+2k^r\right)}{\left(1-|\alpha|\right)\left(m-m^r\right)}\theta\|x\|_{\mathbf{X}}^r. \tag{19}$$

for all $x \in \mathbf{X}$.

The rest of the proof is similar to the proof of Theorem 2.2.

4. Establish the Solution of the Additive α-Function Inequalities

Next, we study the solutions of (2). Note that for these inequalities, when $\mathbb{X}$ be a real or complete normed space and $\mathbb{Y}$ complex Banach space. Now, we study the solutions of (2). Note that for these inequalities, $\mathbf{G}$ be a k-divisible abelian group, $\mathbf{X}$ is a real or complex normed space and $\mathbf{Y}$ is complex Banach spaces. Under this setting, we can show that the mapping satisfying (2) is additive. These results are give in the following.

Lemma 4. Let $m \in \mathbb{N}$ and a mapping $f : \mathbf{G} \to \mathbf{Y}$ satilies

$$\left\|f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)-\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(z_j\right)\right\|_{\mathbf{Y}}$$

$$\le \left\| \alpha \left(f\left((m+1)\sum_{j=1}^{k}\frac{x_j+y_j}{2k} - \sum_{j=1}^{k} z_j \right) - \sum_{j=1}^{k} f\left(m\frac{x_j+y_j}{2k} - z_j \right) - \sum_{j=1}^{k} f\left(\frac{x_j+y_j}{2k} \right) \right) \right\|_{\mathbf{Y}} \tag{20}$$

for all $x_j, y_j, z_j \in \mathbf{X}$ for $j = 1 \to n$, then $f : \mathbf{X} \to \mathbf{Y}$ is additive.

Proof. Assume that $f : \mathbf{G} \to \mathbf{Y}$ satisfies (20).

Replacing $(x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_k)$ by $(0, \cdots, 0, 0, \cdots, 0, 0, \cdots, 0)$ in (20), we have

$$\left\| (2k-1) f(0) \right\|_{\mathbf{Y}} \le \left\| (2k-1)\alpha f(0) \right\|_{\mathbf{Y}} \le 0$$

therefore

$$\left(|2k-1| - |\alpha(2k-1)| \right) \left\| f(0) \right\|_{\mathbf{Y}} \le 0$$

So $f(0) = 0$.

Replacing $(x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_k)$ by $\left(x_1, \cdots, x_k, y_1, \cdots, y_k, m \cdot \frac{x_1+y_1}{2k} - v_1, \cdots, m \cdot \frac{x_k+y_k}{2k} - v_k \right)$ in (20), we have

$$\begin{aligned} &\left\| f\left((m+1)\sum_{j=1}^{k}\frac{x_j+y_j}{2k} - \sum_{j=1}^{k} v_j \right) - \sum_{j=1}^{k} f\left(m\frac{x_j+y_j}{2k} - v_j \right) - \sum_{j=1}^{k} f\left(\frac{x_j+y_j}{2k} \right) \right\|_{\mathbf{Y}} \\ &\le \left\| \alpha \left(f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k} + \sum_{j=1}^{k} v_j \right) - \sum_{j=1}^{k} f\left(\frac{x_j+y_j}{2k} \right) - \sum_{j=1}^{k} f(v_j) \right) \right\|_{\mathbf{Y}} \end{aligned} \tag{21}$$

for all $x_1, \cdots, x_k, y_1, \cdots, y_k, \frac{x_1+y_1}{2k} - v_1, \cdots, \frac{x_k+y_k}{2k} - v_k \in \mathbf{G}$. From (20) and (21) we infer that

$$\begin{aligned} &\left\| f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k} + \sum_{j=1}^{k} v_j \right) - \sum_{j=1}^{k} f\left(\frac{x_j+y_j}{2k} \right) - \sum_{j=1}^{k} f(v_j) \right\|_{\mathbf{Y}} \\ &\le \left\| \alpha \left(f\left((m+1)\sum_{j=1}^{k}\frac{x_j+y_j}{2k} - \sum_{j=1}^{k} v_j \right) - \sum_{j=1}^{k} f\left(m\frac{x_j+y_j}{2k} - v_j \right) - \sum_{j=1}^{k} f\left(\frac{x_j+y_j}{2k} \right) \right) \right\|_{\mathbf{Y}} \\ &\le \left\| \alpha^2 \left(f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k} + \sum_{j=1}^{k} v_j \right) - \sum_{j=1}^{k} f\left(\frac{x_j+y_j}{2k} \right) - \sum_{j=1}^{k} f(v_j) \right) \right\|_{\mathbf{Y}} \end{aligned} \tag{22}$$

and so

$$f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k} + \sum_{j=1}^{k} z_j \right) = \sum_{j=1}^{k} f\left(\frac{x_j+y_j}{2k} \right) + \sum_{j=1}^{k} f(z_j)$$

for all $x_j, y_j, z_j \in \mathbf{G}$ for $j = 1 \to n$, as we expected.

Theorem 5. *Let* $r > 1, m \in \mathbb{Z}, m > 1$, θ *be nonngative real number, and let* $f : \mathbf{X} \to \mathbf{Y}$ *be a mapping such that*

$$\begin{aligned} &\left\| f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k} + \sum_{j=1}^{k} z_j \right) - \sum_{j=1}^{k} f\left(\frac{x_j+y_j}{2k} \right) - \sum_{j=1}^{k} f(z_j) \right\|_{\mathbf{Y}} \\ &\le \left\| \alpha \left(f\left((m+1)\sum_{j=1}^{k}\frac{x_j+y_j}{2k} - \sum_{j=1}^{k} z_j \right) - \sum_{j=1}^{k} f\left(m\frac{x_j+y_j}{2k} - z_j \right) - \sum_{j=1}^{k} f\left(\frac{x_j+y_j}{2k} \right) \right) \right\|_{\mathbf{Y}} \end{aligned}$$

$$+\theta\left(\sum_{j=1}^{k}\left\|x_j\right\|^r+\sum_{j=1}^{k}\left\|y_j\right\|^r+\sum_{j=1}^{k}\left\|z_j\right\|^r\right) \tag{23}$$

for all $x_j, y_j, z_j \in X$ for all $j=1\to n$. Then there exists a unique mapping $\phi:\mathbf{X}\to\mathbf{Y}$ such that

$$\left\|f(x)-h(x)\right\|_{\mathbf{Y}} \le \frac{\sum_{q=1}^{m-1}\left(q^r+2k^r\right)}{\left(1-|\alpha|\right)\left(m-m^r\right)}\theta\|x\|_{\mathbf{X}}^r. \tag{24}$$

for all $x\in\mathbf{X}$.

Proof. Assume that $f:\mathbf{X}\to\mathbf{Y}$ satisfies (23).

Replacing $\left(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\right)$ by $\left(0,\cdots,0,0,\cdots,0,0,\cdots,0\right)$ in (23), we have

$$\left\|2kf(0)\right\| \le \left\|\alpha(2k-1)f(0)\right\|_{\mathbf{Y}} \le 0$$

therefore

$$\left(|2k-1|-\left|\alpha(2k-1)\right|\right)\left\|f(0)\right\|_{\mathbf{Y}} \le 0$$

So $f(0)=0$.

Next we:

Replacing $\left(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\right)$ by $\left(kx,0,\cdots,0,kx,0,\cdots,0,0,\cdots,0\right)$ in (23), we get

$$\left\|f\left((m+1)x\right)-f(mx)-f(x)\right\|_{\mathbf{Y}} \le 2k^r\theta\|x\|_{\mathbf{X}}^r \tag{25}$$

for all $x\in\mathbf{X}$. Thus for $q\in\mathbb{N}$.

We replacing $\left(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\right)$ by $\left(kx,0,\cdots,0,kx,0,\cdots,0,qx,0,\cdots,0\right)$ in (23), we have

$$\begin{aligned}&\left\|f\left((m-q+1)x\right)-f\left((m-q)x\right)-f(x)\right\|_{\mathbf{Y}}\\ &\le\left\|\alpha\left(f\left((q+1)x\right)-f(qx)-f(x)\right)\right\|+\theta\left(2k^r+q^r\right)\|x\|_{\mathbf{Y}}^r\end{aligned} \tag{26}$$

for all $x\in\mathbf{X}$.

For (25) and (26)

$$\begin{aligned}&\sum_{q=1}^{m-1}\left\|f\left((m-q+1)x\right)-f\left((m-q)x\right)-f(x)\right\|_{\mathbf{Y}}\\ &\le\sum_{q=1}^{m-1}\left\|\alpha\left(f\left((q+1)x\right)-f(qx)-f(x)\right)\right\|_{\mathbf{Y}}+\theta\left(\sum_{q=1}^{m-1}\left(2k^r+q^r\right)\|x\|^r\right)\end{aligned} \tag{27}$$

for all $x\in\mathbf{X}$.

From (26) and (27) and triangle inequality, we have

$$\begin{aligned}&\left(1-|\alpha|\right)\left\|f(mx)-mf(x)\right\|_{\mathbf{Y}}\\ &=\left(1-|\alpha|\right)\sum_{q=1}^{m-1}\left\|f\left((q+1)x\right)-f(qx)-f(x)\right\|_{\mathbf{Y}}\\ &\le\sum_{q=1}^{m-1}\left(1-|\alpha|\right)\left\|f\left((q+1)x\right)-f(qx)-f(x)\right\|_{\mathbf{Y}}\end{aligned}$$

$$\le \sum_{q=1}^{m-1}\left\|f\left((q+1)x\right)-f(qx)-f(x)\right\|-\sum_{q=1}^{m-1}\left\|\alpha\left(f\left((q+1)x\right)-f(qx)-f(x)\right)\right\|_{\mathbf{Y}}$$
$$\le \theta\left(\sum_{q=1}^{m-1}\left(2k^r+q^r\right)\|x\|_{\mathbf{X}}^r\right) \tag{28}$$

for all $x \in \mathbf{X}$. from

$$\sum_{q=1}^{m-1}\left\|f\left((m-q+1)x\right)-f\left((m-q)x\right)-f(x)\right\|_{\mathbf{Y}}$$
$$=\sum_{q=1}^{m-1}\left\|f\left((q+1)x\right)-f(qx)-f(x)\right\|_{\mathbf{Y}}$$

Since $|\alpha|<1$, the mapping f satisfies the inequalities

$$\left\|f(mx)-mf(x)\right\|_{\mathbf{Y}} \le \frac{\theta\left(\sum_{q=1}^{m-1}\left(2k^r+q^r\right)\|x\|_{\mathbf{X}}^r\right)}{1-|\alpha|}$$

for all $x \in \mathbf{X}$.

Therefore

$$\left\|f(x)-mf\left(\frac{x}{m}\right)\right\|_{\mathbf{Y}} \le \frac{\theta\left(\sum_{q=1}^{m-1}\left(2k^r+q^r\right)\|x\|_{\mathbf{X}}^r\right)}{\left(1-|\alpha|\right)m^r} \tag{29}$$

for all $x \in X$. So

$$\left\|m^l f\left(\frac{x}{m^n}\right)-m^p f\left(\frac{x}{m^h}\right)\right\|_{\mathbf{Y}} \le \sum_{j=l}^{p-1}\left\|m^j f\left(\frac{x}{m^j}\right)-m^{j+1} f\left(\frac{x}{m^{j+1}}\right)\right\|_{\mathbf{Y}}$$
$$\le \frac{\theta\left(\sum_{q=1}^{m-1}\left(2k^r+q^r\right)\right)}{\left(1-|\alpha|\right)m^r}\sum_{j=l}^{p-1}\frac{m^j}{m^{rj}}\|x\|_{\mathbf{X}}^r \tag{30}$$

for all nonnegative integers p,l with $p>l$ and all $x \in \mathbf{X}$. It follows from (30) that the sequence $\left\{m^n f\left(\frac{x}{m^n}\right)\right\}$ is a Cauchy sequence for all $x \in \mathbf{X}$. Since $\mathbf{Y}$ is complete, the sequence $\left\{m^n f\left(\frac{x}{m^n}\right)\right\}$ coverges.

So one can define the mapping $\phi:\mathbf{X}\to\mathbf{Y}$ by $\phi(x):=\lim_{n\to\infty} m^n f\left(\frac{x}{m^n}\right)$ for all $x \in \mathbf{X}$. Moreover, letting $l=0$ and passing the limit $m\to\infty$ in (30), we get (24).

It follows from (23) that

$$\left\|\phi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)-\sum_{j=1}^{k}\phi\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}\phi\left(z_j\right)\right\|_{\mathbf{Y}}$$
$$=\lim_{n\to\infty} m^n\left\|f\left(\frac{1}{m^n}\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\frac{1}{m^n}\sum_{j=1}^{k}z_j\right)-\sum_{j=1}^{k}f\left(\frac{1}{m^n}\sum_{j=1}^{k}\frac{x_j+y_j}{2k}\right)\right.$$
$$\left.-\sum_{j=1}^{k}f\left(\frac{1}{m^n}z_j\right)\right\|_{\mathbf{Y}}+\lim_{n\to\infty}\frac{m^n}{m^{nr}}\theta\left(\sum_{j=1}^{k}\|x_j\|_{\mathbf{X}}^r+\sum_{j=1}^{k}\|y_j\|_{\mathbf{X}}^r+\sum_{j=1}^{k}\|z_j\|_{\mathbf{X}}^r\right)$$
$$\le \lim_{n\to\infty} m^n|\alpha|\left\|f\left(\frac{m+1}{m^n}\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\frac{1}{m^n}\sum_{j=1}^{k}z_j\right)\right.$$

$$-\sum_{j=1}^{k} f\left(\frac{m}{m^n}\left(\frac{x_j+y_j}{2k}\right)-\frac{1}{m^n}z_j\right)-\sum_{j=1}^{k} f\left(\frac{1}{m^n}z_j\right)\Bigg\|_{\mathbf{Y}}$$
$$\le|\alpha|\left\|\phi\left((m+1)\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)-\sum_{j=1}^{k}\phi\left(m\frac{x_j+y_j}{2k}-z_j\right)-\sum_{j=1}^{k}\phi\left(z_j\right)\right\|_{\mathbf{Y}} \tag{31}$$

for all $x_j, y_j, z_j \in X$ for all $j=1 \to n$. So

$$\left\|\phi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)-\sum_{j=1}^{k}\phi\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}\phi\left(z_j\right)\right\|_{\mathbf{Y}}$$
$$\le|\alpha|\left\|\phi\left((m+1)\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)-\sum_{j=1}^{k}\phi\left(m\frac{x_j+y_j}{2k}-z_j\right)-\sum_{j=1}^{k}\phi\left(\frac{x_j+y_j}{2k}\right)\right\|_{\mathbf{Y}}$$

for all $x_j, y_j, z_j \in X$ for all $j=1 \to n$. So by lemma 4.1 it follows that the mapping $\phi:\mathbf{X}\to\mathbf{Y}$ is additive. Now we need to prove uniqueness, suppose $\phi':\mathbf{X}\to\mathbf{Y}$ is also an additive mapping that satisfies (24). Then we have

$$\begin{aligned}\left\|\phi(x)-\phi'(x)\right\| &= m^n\left\|\phi\left(\frac{x}{m^n}\right)-\phi'\left(\frac{x}{m^n}\right)\right\| \\ &\le m^n\left(\left\|\phi\left(\frac{x}{m^n}\right)-f\left(\frac{x}{m^n}\right)\right\|+\left\|\phi'\left(\frac{x}{m^n}\right)-f\left(\frac{x}{m^n}\right)\right\|\right) \\ &\le \frac{2\cdot m^n\cdot\sum_{q=1}^{m-1}\left(q^r+2k^r\right)}{\left(1-|\alpha|\right)m^{nr}\left(m^r-m\right)}\theta\|x\|^r\end{aligned} \tag{32}$$

which tends to zero as $n\to\infty$ for all $x\in X$. So we can conclude that $\phi(x)=\phi'(x)$ for all $x\in\mathbf{X}$.This proves thus the mapping $\phi:\mathbf{X}\to\mathbf{Y}$ is a unique mapping satisfying (24) as we expected.

5. Conclusion

In this article, I have solved two problems posed as establishing the solution of the additive α-function inequality (1) and (2) in complex Banach spaces with $3k$ variable. So when I develop this result, I rely on the inequality (β_1,β_2) -function.

Conflicts of Interest

The author declares no conflicts of interest.

References

[1] ULam, S.M. (1960) A Collection of Mathematical Problems. Vol. 8, Interscience Publishers, New York.

[2] Hyers, D.H. (1941) On the Stability of the Functional Equation. *Proceedings of the National Academy of the United States of America*, **27**, 222-224. https://doi.org/10.1073/pnas.27.4.222

[3] Aoki, T. (1950) On the Stability of the Linear Transformation in Banach Space. *Journal of the Mathematical Society of Japan*, **2**, 64-66. https://doi.org/10.2969/jmsj/00210064

[4] Rassias, T.M. (1978) On the Stability of the Linear Mapping in Banach Space. *Pro-*

ceedings of the American Mathematical Society, **27**, 297-300.
https://doi.org/10.1090/S0002-9939-1978-0507327-1

[5] Găvruta, P. (1994) A Generalization of the Hyers-Ulam-Rassias Stability of Approximately Additive Mappings. *Journal of Mathematical Analysis and Applications*, **184**, 431-436. https://doi.org/10.1006/jmaa.1994.1211

[6] Gilányi, A. (2002) On a Problem by K. Nikodem. *Mathematical Inequalities & Applications*, **5**, 707-710.

[7] Prager, W. and Schwaiger, J. (2013) A System of Two Inhomogeneous Linear Functional Equations. *Acta Mathematica Hungarica*, **140**, 377-406.
https://doi.org/10.1007/s10474-013-0315-y

[8] Fechner, W. (2006) Stability of a Functional Inequlities Associated with the Jordan-Von Neumann Functional Equation. *Aequationes Mathematicae*, **71**, 149-161.
https://doi.org/10.1007/s00010-005-2775-9

[9] Park, C. (2014) Additive β-Functional Inequalities. *Journal of Nonlinear Sciences and Applications*, **7**, 296-310. https://doi.org/10.22436/jnsa.007.05.02

[10] Park, C. (2015) Additive ρ-Functional Inequalities and Equations. *Journal of Mathematical Inequalities*, **9**, 17-26.

[11] Park, C. (2015) Additive ρ-Functional Inequalities in Non-Archimedean Normed Spaces. *Journal of Mathematical Inequalities*, **9**, 397-407.

[12] Skof, F. (1983) Propriet locali e approssimazione di operatori. *Rendiconti del Seminario Matematico e Fisico di Milano*, **53**, 113-129.
https://doi.org/10.1007/BF02924890

[13] Fechner, W. (2010) On Some Functional Inequalities Related to the Logarithmic Mean. *Acta Mathematica Hungarica*, **128**, 36-45.
https://doi.org/10.1007/s10474-010-9153-3

[14] Cadariu, L. and Radu, V. (2003) Fixed Points and the Stability of Jensen's Functional Equation. *Journal of Inequalities in Pure and Applied Mathematics*, **4**, Article No. 4.

[15] Diaz, J. and Margolis, B. (1968) A Fixed Point Theorem of the Alternative for Contractions on a Generalized Complete Metric Space. *Bulletin of the American Mathematical Society*, **74**, 305-309. https://doi.org/10.1090/S0002-9904-1968-11933-0

[16] Lee, J.R., Park, C. and Shin, D.Y. (2014) Additive and Quadratic Functional in Equalities in Non-Archimedean Normed Spaces. *International Journal of Mathematical Analysis*, **8**, 1233-1247. https://doi.org/10.12988/ijma.2014.44113

[17] Yun, S. and Shin, D.Y. (2017) Stability of an Additive (p_1, p_2)-Functional Inequality in Banach Spaces. *The Pure and Applied Mathematics*, **24**, 21-31.
https://doi.org/10.7468/jksmeb.2017.24.1.21

[18] Mihet, D. and Radu, V. (2008) On the Stability of the Additive Cauchy Functional Equation in Random Normed Spaces. *Journal of Mathematical Analysis and Applications*, **343**, 567-572. https://doi.org/10.1016/j.jmaa.2008.01.100

[19] Bahyrycz, A. and Piszczek, M. (2014) Hyers Stability of the Jensen Function Equation. *Acta Mathematica Hungarica*, **142**, 353-365.
https://doi.org/10.1007/s10474-013-0347-3

[20] Balcerowski, M. (2013) On the Functional Equations Related to a Problem of Z Boros and Z. Dróczy. *Acta Mathematica Hungarica*, **138**, 329-340.
https://doi.org/10.1007/s10474-012-0278-4

[21] Gilányi, A. (2002) Eine zur parallelogrammleichung *äquivalente* ungleichung. *Aeq-*

uations, **5**, 707-710.*

[22] Qarawani, M. (2012) Hyers-Ulam Stability of a Generalized Second-Order Nonlinear Differential Equation. *Applied Mathematics*, **3**, 1857-1861. https://doi.org/10.4236/am.2012.312252 https://www.scirp.org/journal/am/

[23] Park, C., Cho, Y. and Han, M. (2007) Functional Inequalities Associated with Jordan-Von Newman-Type Additive Functional Equations. *Journal of Inequalities and Applications*, **2007**, Article No. 41820. https://doi.org/10.1155/2007/41820

[24] Rätz, J. (2003) On Inequalities Assosciated with the Jordan-Von Neumann Functional Equation. *Aequationes Matheaticae*, **66**, 191-200. https://doi.org/10.1007/s00010-003-2684-8

[25] Van An, L.Y. (2022) Generalized Hyers-Ulam-Rassisa Stabilityof an Additive (1; 2)-Functional Inequalities with nVariables in Complex Banach. *Open Access Library Journal*, **9**, e9183. https://doi.org/10.4236/oalib.1109183

[26] Van An, L.Y. (2019) Hyers-Ulam Stability of Functional Inequalities with Three Variable in Banach Spaces and Non-Archemdean Banach Spaces. *International Journal of Mathematical Analysis*, **13**, 519-537. https://doi.org/10.12988/ijma.2019.9954

[27] Van An, L.Y. (2020) Hyers-Ulam Stability of β-Functional Inequalities with Three Variable in Non-Archemdean Banach Spaces and Complex Banach. *International Journal of Mathematical Analysis*, **14**, 219-239. https://doi.org/10.12988/ijma.2020.91169