Open Access Library Journal
2022, Volume 9, e7970
ISSN Online: 2333-9721
ISSN Print: 2333-9705

Optimal Control of Assets Allocation on a Defined Contribution Pension Plan

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How to cite this paper: Keganneng, O. and Basimanebotlhe, O. (2022) Optimal Control of Assets Allocation on a Defined Contribution Pension Plan. *Open Access Library Journal*, **9**: e7970.
https://doi.org/10.4236/oalib.1107970

Received: May 23, 2022
Accepted: June 27, 2022
Published: June 30, 2022

Abstract

This paper investigates the optimal control of asset allocation on a defined contribution pension plan. In our model, the plan member is allowed to invest in a risk-free asset (bank account), a risky asset (stock) and an inflation-linked bond. The dynamics of the wealth in our model take into account a certain proportion of the client's salary paid as the contribution towards the pension fund. By applying the Hamilton-Jacobi-Bellman equation we find the explicit solutions for the CARA and CRRA utility functions. This helps us to calculate the investment strategies associated with the stock and inflation-linked bond. Finally, a numerical simulation is presented to illustrate the behaviour of the model.

Subject Areas

Mathematics

Keywords

Asset Allocations, Defined Contribution, Defined Benefit Pension Fund, Stochastic Salary, Brownian Motion, Utility Function, Power Utility, Optimal Portfolio, Stochastic Optimal Control, Hamilton-Jacobi-Bellman Equation

1. Introduction

It is very imperative to assess the performance of pension fund managers or investment managers in relation to the large amounts of funds that pension managers are handling, and the scale of competition this market is facing. From the investor's point of view, the need is even more significant when one looks at the increasing volatility that the financial market is subjected to.

A number of authors have dealt with the problem of managing the financial resources of a pensioner after retirement, which arises from the fact that whole life annuities are felt by policyholders to be "poor value for money" and have investigated the other alternatives available to a retiree [1]. [2] studies how to optimally manage a pension fund by taking positions in a money market account, stock and an inflation-linked bond, while financing investments through a continuous stochastic income stream such as the plan member's contributions. The martingale method is used in order to compute an analytic expression for the optimal strategy and express it in terms of observable market variables. [3] compares the purchase of an index-linked annuity with two alternatives: level annuity and income withdrawal option. [4] proposes a strategy for the post-retirement period where the pensioner's consumption exactly matches what a level annuity purchased at retirement would pay and the pensioner invests the remaining part. [5] considers investment decisions in the decumulation phase of a DC plan by means of stochastic optimal control, choosing between equities and annuities, and finds that complete annuitization eventually occurs. [6], in an attempt to answer the question of why people do not buy annuities, compares the expected present value of payments from an annuity (calculated with a proper term structure of interest rates) with the amount of premium charged, and compare the expected utility from the annuity payments with the expected utility from an optimal consumption path, if there were no annuities in the market. [7] considers the adoption of a drawdown option assuming a fixed amount withdrawn every year and investment of the remaining fund in one risky asset. They calculate exactly the eventual probability of ruin and then approximate the probability that ruin occurs before the random time of death, comparing their approximations with the frequency of ruin found via Monte Carlo simulations. [8] proposes a product for the post-retirement period (the second paper being a more detailed exposition of the product) in which pensioners have flexibility in the way that they invest the fund and withdraw money, as in the drawdown option, with the difference that mortality credits are given to the survivors and there is no bequest at death. [9] uses expected utility to compare immediate annuitization at retirement with two kinds of drawdown plans (one approach has survival bonuses but no bequest—as in the proposal by [10]. [10] considers the probability that the pensioner outlives her own assets when taking the income drawdown option, withdrawing exactly the amount of money that an immediate level annuity bought at retirement would provide. [11] finds the optimal mix (constant over time) between a fixed immediate annuity and a variable immediate annuity, with different mortality assumptions, via the maximization of expected utility, and then compares it with the optimal mix found in the accumulation phase of a DC scheme. [12] lists nine alternatives to immediate and complete annuitization at retirement: three kinds of annuity, three kinds of income drawdown and three kinds of combinations of the two: he proposes criteria for the design of a post-retirement product and analyses all of the choices with reference to these criteria. [13] deals with the pension fund management issue and focuses on defined-contribution plans where a guarantee is given on the benefits, and the guarantee depends on the level of the stochastic interest rate when the employee retires. It is particularly shown that the optimal composition of this kind of pension fund can be divided into three different parts: a loan that amounts to the present value of the contributions, a contingent claim delivering the guarantee and a hedging fund. [14] solves in a closed form the problem of a pension fund maximizing the expected CRRA (Constant Relative Risk Aversion), a utility of its surplus till the stochastic death time of a representative agent. A unique asset allocation problem for both accumulation and decumulation phases is considered. The optimal investment in the risky assets must decrease during the first phase and increase during the second one. It is not optimal to manage the two phases separately, and outsourcing of allocation decisions should be avoided in both phases. Optimal asset allocation by a pension fund is analyzed which maximizes the expected utility of its surplus at the death time of a member. This surplus is defined as the difference between the total managed wealth and the retrospective mathematical reserve. [15] investigates asset-allocation strategies open to members of defined-contribution pension plans with a model that incorporates asset, salary (labor-income) and interest-rate risk. [16] derives a formula for the optimal investment allocation (derived from a dynamic programming approach) in a defined contribution (DC) pension scheme whose fund is invested in n assets. The particular case of two assets is analyzed and the investment allocation and the downside risk faced by the retiring member of the DC scheme are studied, where optimal investment strategies have been adopted. [17] uses the stochastic dynamic programming approach to investigate the optimal asset allocation for a defined-contribution pension plan with downside protection under stochastic inflation. [18] investigates a pension's trustees responsible for choosing long investment advice and whose actuary is normally required to advise the trustees and the employees. [13] considers portfolio selection problem of a member of a defined contribution pension plan in a hidden Markov-modulated economy modulated by a continuous time, finite state, hidden Markov chain whose states represent different hidden states of the underlying economy. [19] considers a stochastic model for a defined contribution pension fund in continuous time. They focus on the portfolio problem of a fund manager who wants to maximize the expected utility of his terminal wealth in a complete financial market with a stochastic interest rate. [20] analyzes optimal investment policies for pension funds of a defined benefit (DB) type.

Most of the studies left a literature gap as their models focused on investors investing in only one risky asset (being the stock), and so our study aimed to fill this gap by introducing another risky asset (being an inflation-linked bond). Our study has some similarities with the work of [21], but there are some differences anyway. The difference between our work and that of [21] is in the number of assets that the fund manager trades on. In [21] the fund manager traded only one risky asset (stock) whereas in our model the fund manager trades in both stock and bond. Therefore [21] only solved for the optimal investment strategy for the stock while our model solved for the optimal investment strategy for both the stock and bond. The other thing that differentiates our work from that of [21] is that [21] used only the CARA utility function while our model made use of both the CARA and CRRA utility functions, which is considered as the function whereby the manager aspires to maximize his rewards. Furthermore, we presented the numerical application chapter while [21] did not present it.

The main objective of the present study is to find the optimal investment strategies for a defined contribution (DC) pension with stochastic salary. This work is concerned with maximizing the optimal investment strategy for DC Pension with a stochastic salary. The present research attempts to:

1) Study optimal control of assets allocation on a defined contribution pension plan.

2) Apply the Hamilton-Bellman equation to find the explicit solutions for the CARA and CRRA utility function.

3) Obtain the optimal investment strategies (cash, inflation-linked bond and the stock) for the three investments under the exponential utility function.

2. Model Formulation

2.1. The Financial Market

We assume that the instantaneous nominal interest rate $r(t)$ is stochastic and follows an Ornstein-Uhlenbeck process (see [22]). Then, under the historical probability measure $\mathbb{P}$, the process $r(t)$ satisfies the following stochastic differential equation

$$\mathrm{d}r(t) = b(a - r(t))\mathrm{d}t + \sigma_r \mathrm{d}W_1(t), \quad r(0) = r_0, \tag{1}$$

where $W_1(t)$ is a standard Brownian motion and a, b and σ are strictly positive constants.

2.1.1. Riskless Asset

The riskless asset or bank account is given by the following SDE

$$\frac{\mathrm{d}B_A(t)}{B_A(t)} = r(t)\mathrm{d}t, \quad B_A(0) = B_{A0}. \tag{2}$$

2.1.2. Stock

Based on the work of [1], a stock with price $S(t)$ is given by the following stochastic differential equation

$$\frac{\mathrm{d}S(t)}{S(t)} = (r(t) + \kappa\eta\lambda_r\sigma_S)\mathrm{d}t + \sigma_S\sqrt{\eta(t)}\mathrm{d}W_1(t), \quad S(0) = S_0. \tag{3}$$

where κ, η, λ_r and σ_S are positive constants see [1].

2.1.3. Inflation Rate

We define the inflation rate process as:

$$\frac{\mathrm{d}P(t)}{P(t)} = i\mathrm{d}t + \sigma_P \mathrm{d}W_2(t), \quad P(0) = P_0. \tag{4}$$

The constant i is the expected rate of inflation and σ_P represents its volatility. $W_2(t)$ is another Brownian motion under the physical measure, which generates uncertainty in the price level and is independent of $W_1(t)$. The correlation between the two Brownian motions is given by $\mathrm{d}W_1\mathrm{d}W_2 = 0$.

2.1.4. Bond

The inflation-linked bond is given by the SDE

$$\frac{\mathrm{d}I(t)}{I(t)} = \left(r(t) + \psi\sigma_r\lambda_r\kappa\sqrt{\eta} + \sigma_P\lambda_P\right)\mathrm{d}t + \psi\sigma_r\mathrm{d}W_1(t) + \sigma_P\mathrm{d}W_2(t). \tag{5}$$

2.1.5. Salary

The salary of the client is given by the following SDE

$$\frac{\mathrm{d}L(t)}{L(t)} = \left(\varpi_L(t) + \gamma(t)\right)\mathrm{d}t + \rho_{L,1}(t)\mathrm{d}W_1(t) + \rho_{L,2}(t)\mathrm{d}W_2(t), \quad L(0) = L_0, \tag{6}$$

where the instantaneous mean of the salary $\mu_L(t) = \varpi_L(t) + \gamma(t)$ has two components, where the former $\varpi_L(t)$ is caused by inflation and the latter $\gamma(t)$ is caused by other factors like economy growth. On the other hand $\rho_{L,1}(t)$ and $\rho_{L,2}(t)$ are two volatility factors that determine the effect of stock and inflation on salary.

2.1.6. Contribution

Each employee contributes a constant proportion of his salary at the rate $c(t)$, then the process of contribution $\Gamma(t)$ is given by

$$\Gamma(t) = c(t)L(t)\mathrm{d}t. \tag{7}$$

2.1.7. Wealth

The dynamics of the pension wealth are given by

$$\begin{aligned}\mathrm{d}V(t) = (1-\pi_1-\pi_2)V(t)\frac{\mathrm{d}B_A(t)}{B_A(t)} + \pi_1 V(t)\frac{\mathrm{d}S(t)}{S(t)} \\ + \pi_2 V(t)\frac{\mathrm{d}I(t)}{I(t)} + \Gamma(t), \quad V(0) = V_0,\end{aligned} \tag{8}$$

where $V(t)$ is the wealth of pension fund at time $t \in [0,1]$ and π_1 and π_2 respectively represent the proportion of the pension fund invested in the Stock and Bond. The proportion of the pension fund invested in the riskless asset is given by $1-\pi_1-\pi_2$.

Substituting (2), (3), (5) and (7) above into (8) we get

$$\begin{aligned}\mathrm{d}V(t) = V(t)\Big(\Big(r(t) + \pi_1\lambda_r\sigma_S\kappa\eta + \pi_2\left(\psi\sigma_r\lambda_r\kappa\sqrt{\eta} + \sigma_P\lambda_P\right)\Big)\mathrm{d}t \\ + \left(\pi_1\sigma_S\sqrt{\eta(t)} + \pi_2\psi\sigma_r\right)\mathrm{d}W_1 + \pi_2\sigma_P\mathrm{d}W_2\Big) + c(t)L(t)\mathrm{d}t, \; V(0) = V_0.\end{aligned} \tag{9}$$

2.1.8. Real Wealth

Now taking $X = \frac{V}{P}$ to be the real wealth process and then applying chain rule to it by Ito *i.e.*

$$\mathrm{d}X = \mathrm{d}\left(\frac{V}{P}\right) = \frac{V}{P}\left(\frac{\mathrm{d}V}{V} - \frac{\mathrm{d}P}{P} - \frac{\mathrm{d}V}{V}\frac{\mathrm{d}P}{P} + \left(\frac{\mathrm{d}P}{P}\right)^2\right). \tag{10}$$

Substituting (4) and (9) into (10) above we get

$$\begin{aligned}\mathrm{d}X(t) = X(t)\Big(\big(r(t) + \pi_1\lambda_r\sigma_S\kappa\eta - i + \sigma_P^2 + \pi_2\left(\psi\sigma_r\lambda_r\kappa\sqrt{\eta} + \sigma_P\lambda_P - \sigma_P^2\right)\big)\mathrm{d}t \\ + \left(\pi_1\sigma_S\sqrt{\eta(t)} + \pi_2\psi\sigma_r\right)\mathrm{d}W_1 + \left(\pi_2\sigma_P - \sigma_P\right)\mathrm{d}W_2\Big) + c(t)\mathrm{d}t.\end{aligned} \tag{11}$$

2.1.9. The HJB Equation

Let $H:[0,T]\times\mathbb{R}\mapsto\mathbb{R}$ denote a value function. Then $H(\cdot)$ (see [22]) is given by

$$H(t,x) = \sup_{u\in A(t,x)} J(t,x,u(t)). \tag{12}$$

The HJB equation associated with (11) is given by

$$\sup\{\mathrm{d}H\} = \sup\left\{\frac{\partial H}{\partial t}\mathrm{d}t + \frac{\partial H}{\partial X}\mathrm{d}t + \frac{1}{2}\frac{\partial^2 H}{\partial X^2}(\mathrm{d}X)^2\right\}, \tag{13}$$

which simplifies to

$$\begin{aligned}H(t,x) = \sup\Big\{&H_t + x\big(r(t) + \pi_1\lambda_r\sigma_S\kappa\eta - i + \sigma_P^2 + \pi_2\big(\psi\sigma_r\lambda_r\kappa\sqrt{\eta} \\ &+ \sigma_P\lambda_P - \sigma_P^2\big)\big)H_x + c + \frac{1}{2}x^2\big(\pi_1^2\sigma_S^2\eta + \pi_2^2\psi^2\sigma_r^2 \\ &+ 2\pi_1\pi_2\sigma_S\sigma_r\psi\sqrt{\eta} + \pi_2^2\sigma_P^2 + \sigma_P^2 - 2\sigma_P^2\pi_2\big)H_{xx}\Big\} \\ =\ &0.\end{aligned} \tag{14}$$

Differentiating (14) with respect to π_1 and π_2 we get the first order maximizing conditions for the optimal strategies π_1^* and π_2^* below:

$$x\sigma_S\lambda_r\kappa\eta H_x + x^2\sigma_S^2\eta\pi_1^*H_{xx} + x^2\sigma_S\sigma_r\psi\sqrt{\eta}\pi_2^*H_{xx} = 0 \tag{15}$$

$$\begin{aligned}&\left(x\sigma_P\lambda_P + x\psi\sigma_r\lambda_r\kappa\sqrt{\eta} - x\sigma_P^2\right)H_x + \big(x^2\psi^2\sigma_r^2\pi_2^* + x^2\sigma_S\sigma_r\psi\sqrt{\eta}\pi_1^* \\ &+ x^2\sigma_P^2\pi_2^* - x^2\sigma_P^2\big)H_{xx} = 0.\end{aligned} \tag{16}$$

(15) and (16) simplify to

$$\sigma_S\lambda_r\kappa\eta H_x + x\sigma_S^2\eta\pi_1^*H_{xx} + x\sigma_S\sigma_r\psi\sqrt{\eta}\pi_2^*H_{xx} = 0 \tag{17}$$

$$\begin{aligned}&\left(\sigma_P\lambda_P + \psi\sigma_r\lambda_r\kappa\sqrt{\eta} - \sigma_P^2\right)H_x + \big(x\psi^2\sigma_r^2\pi_2^* + x\sigma_S\sigma_r\psi\sqrt{\eta}\pi_1^* \\ &+ x\sigma_P^2\pi_2^* - x\sigma_P^2\big)H_{xx} = 0,\end{aligned} \tag{18}$$

for $x \neq 0$.

We then solve (17) and (18) for π_1^* and π_2^* to obtain

$$\pi_2^* = \frac{H_x}{xH_{xx}} - \frac{\lambda_P}{x\sigma_P}\frac{H_x}{H_{xx}} + 1 \tag{19}$$

and

$$\pi_1^* = -\frac{\sigma_r \psi \sqrt{\eta}}{x\sigma_S \eta}\frac{H_x}{H_{xx}} + \frac{\lambda_P \sigma_r \psi \sqrt{\eta}}{x\sigma_P \sigma_S \eta}\frac{H_x}{H_{xx}} - \frac{\sigma_r \psi \sqrt{\eta}}{\sigma_S \eta} - \frac{\kappa \lambda_r}{x\sigma_S}\frac{H_x}{H_{xx}}. \tag{20}$$

Substituting (19) and (20) into (14) above we get

$$\begin{aligned} H(t,x) &= H_t + \left(xr(t) - xi + x\sigma_P \lambda_P + c\right)H_x \\ &\quad - \frac{1}{2}\frac{H_x^2}{H_{xx}}\left(\eta \lambda_r^2 \kappa^2 + \sigma_P^2 + \lambda_P^2 + 2\lambda_P \sigma_P\right), \end{aligned} \tag{21}$$

a nonlinear second order partial differential equation difficult to solve in this form.

3. Explicit Solution for the CARA and CRRA Utility Functions

The section demonstrates the explicit solutions for the CARA utility functions via power transformation technique and variable change method. It also illustrates the explicit solution for the CRRA utility function.

3.1. Explicit Solution for the CARA Utility Function

Assuming that the plan member takes an exponential utility function:

$$U(x) = -\frac{1}{q}\mathrm{e}^{-qx}; \ (q > 0) \tag{22}$$

The absolute risk aversion, q, of a decision maker with utility described in (22) is constant and (22) is a CARA utility. We conjecture a solution to (21) with the following form

$$H(t,x) = -\frac{1}{q}\mathrm{e}^{-q(a(t)x + g(t))} \tag{23}$$

with the boundary conditions given by $a(T) = 1$ and $g(T) = 0$.

3.2. Solution in CRRA Utility

The Constant relative risk aversion of an investor is given by:

$$H(t,x) = \frac{b(t)\left(x - a(t)\right)^{\gamma}}{\gamma} \tag{24}$$

where the degree of risk aversion parameter $\gamma \in (-\infty, 0) \cup (0, 1)$ with initial conditions $a(N+T) = 0$ and $b(N+T) = 0$.

4. Main Results

The main results are obtained using the utility functions given in (23) and (24) above. They are presented in the following theorems:

Theorem 1. Suppose the utility function is given as in (22). Then the optimal investment strategies for the bond and stock are respectively given as:

$$\pi_2^* = -\frac{\mathrm{e}^{-\varsigma(T-t)}}{xq} + \frac{\lambda_P \mathrm{e}^{-\varsigma(T-t)}}{q} + 1 \tag{25}$$

and

$$\pi_1^* = \frac{\sigma_r \psi \sqrt{\eta} e^{-\varsigma(T-t)}}{q x \sigma_S \eta} - \frac{\lambda_P \sigma_r \psi \sqrt{\eta} e^{-\varsigma(T-t)}}{q x \sigma_P \sigma_S \eta} + \frac{k \lambda_r e^{-\varsigma(T-t)}}{x \sigma_S q} \tag{26}$$

Proof. See Appendix A.

Theorem 2. Suppose the utility function is given as in (24). Then the optimal investment strategies for the bond and stock are respectively given as:

$$\pi_2^* = \frac{x - \frac{2xi - x\sigma_P \lambda_P - 2c}{r(t) + \sigma_P \lambda_P} + \frac{-2xi + x\sigma_P \lambda_P + 2c}{r(t) + \sigma_P \lambda_P} e^{(-r(t) - \sigma_P \lambda_P)(N+T-t)}}{x(\gamma - 1)} - \frac{\lambda_P}{x\sigma_P} \frac{x - \frac{2xi - x\sigma_P \lambda_P - 2c}{r(t) + \sigma_P \lambda_P} + \frac{-2xi + x\sigma_P \lambda_P + 2c}{r(t) + \sigma_P \lambda_P} e^{(-r(t) - \sigma_P \lambda_P)(N+T-t)}}{\gamma - 1} + 1 \tag{27}$$

and

$$\begin{aligned} \pi_1^* = & -\frac{\sigma_r \psi \sqrt{\eta}}{x \sigma_S \eta} \frac{x - \frac{2xi - x\sigma_P \lambda_P - 2c}{r(t) + \sigma_P \lambda_P} + \frac{-2xi + x\sigma_P \lambda_P + 2c}{r(t) + \sigma_P \lambda_P} e^{(-r(t) - \sigma_P \lambda_P)(N+T-t)}}{\gamma - 1} \\ & + \frac{\lambda_P \sigma_r \psi \sqrt{\eta}}{x \sigma_P \sigma_S \eta} \frac{x - \frac{2xi - x\sigma_P \lambda_P - 2c}{r(t) + \sigma_P \lambda_P} + \frac{-2xi + x\sigma_P \lambda_P + 2c}{r(t) + \sigma_P \lambda_P} e^{(-r(t) - \sigma_P \lambda_P)(N+T-t)}}{\gamma - 1} \\ & - \frac{\sigma_r \psi \sqrt{\eta}}{\sigma_S \eta} - \frac{\kappa \lambda_r}{x \sigma_S} \frac{x - a(t)}{\gamma - 1} \end{aligned} \tag{28}$$

Proof. See Appendix B.

5. Numerical Application

In this chapter, we illustrate the relationship between the optimal investment strategies and the parameters in our model. The graphs were done using MATLAB program. The parameter values are as follows unless otherwise stated:

$$T = 40; t = 0; i = 0.8; \sigma_P = 0.005; \sigma_r = 0.6; \sigma_s = 0.6;$$
$$\lambda_P = 0.2; \lambda_r = 0.4; \psi = 0.2; q = 20; \eta = 3; x = 1; \kappa = 500.$$

The figures show the gesture of optimal investment proportions from initial time to retirement. The parameters are given by the previous paragraph. Figures 1-3 give us the trends how the optimal investment strategies in the three assets change with time. As time runs, the optimal amount invested in cash and bond decrease as shown in Figure 1 and Figure 3. However, the optimal strategy in stock increases as time goes by, as indicated in 2. Results suggest that the pension fund manager maintains diversifying the portfolio by investing more in stock since the optimal investment strategies in stock increase with time. This shows that as time goes on, the investors are told to more position in stock and shorter position in cash and bond. Figures 1-3 also show the different degrees or measures of risk aversion for different optimal investment strategies. Figure 1 and Figure 3 show that as the degree of the risk preference increases so do the investment strategies in bond and cash respectively. Figure 2 indicates that the higher the degree of risk preference, the lower the investment strategy for the stock.

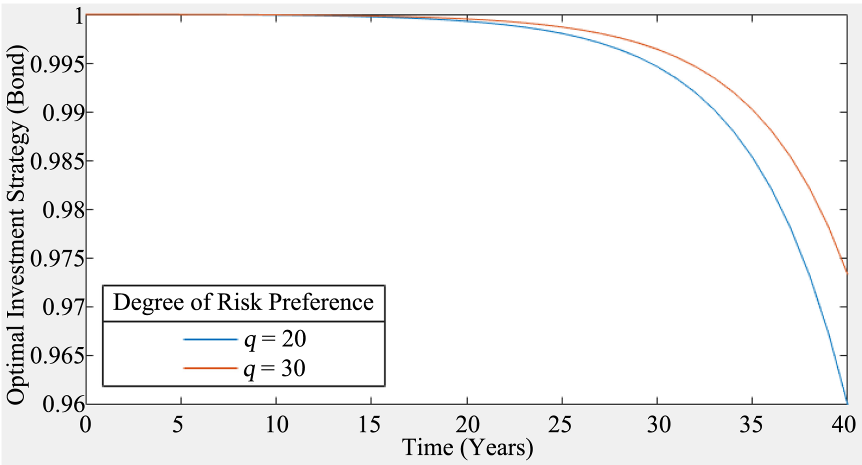

Figure 1. Investment strategies (Bond) vs time for CARA.

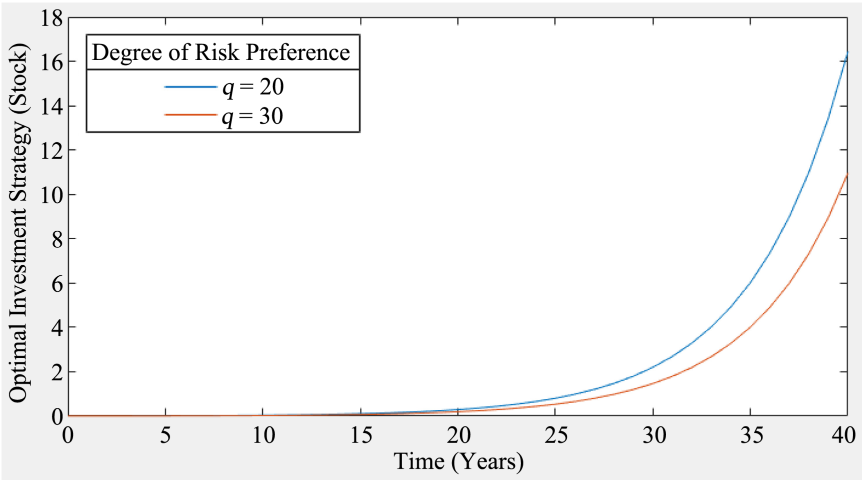

Figure 2. Investment strategies (Stock) vs time for CARA.

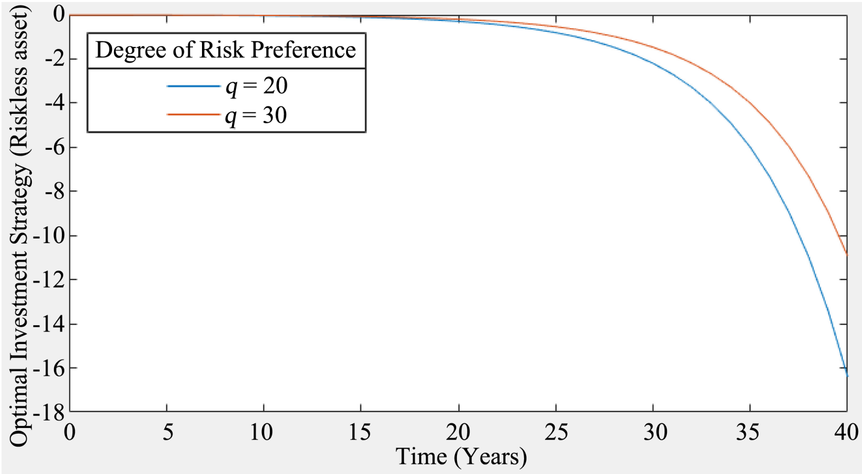

Figure 3. Investment strategies (Riskless asset) vs time for CARA.

Figures 4-6 show the gesture of optimal investment proportions from initial time to retirement using the CRRA technique. The parameters are stated as follows:

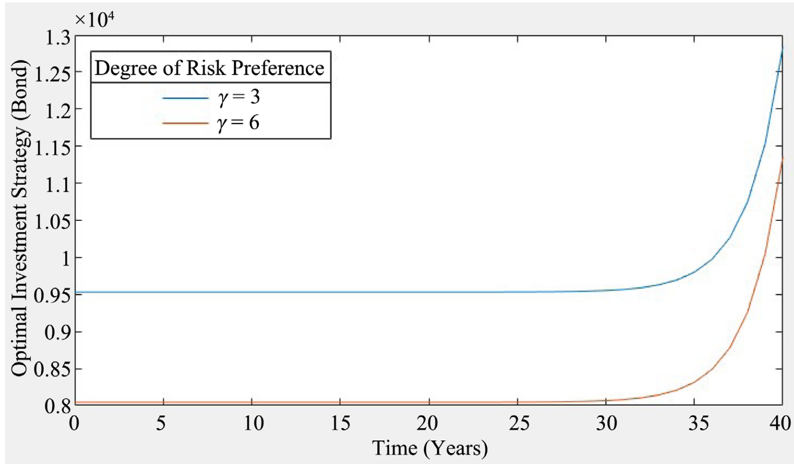

Figure 4. Investment strategies (Bond) vs time for CRRA.

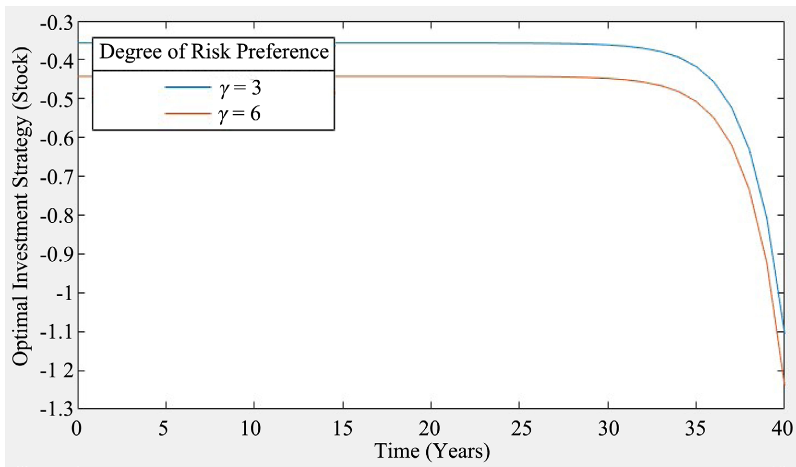

Figure 5. Investment strategies (Stock) vs time for CRRA.

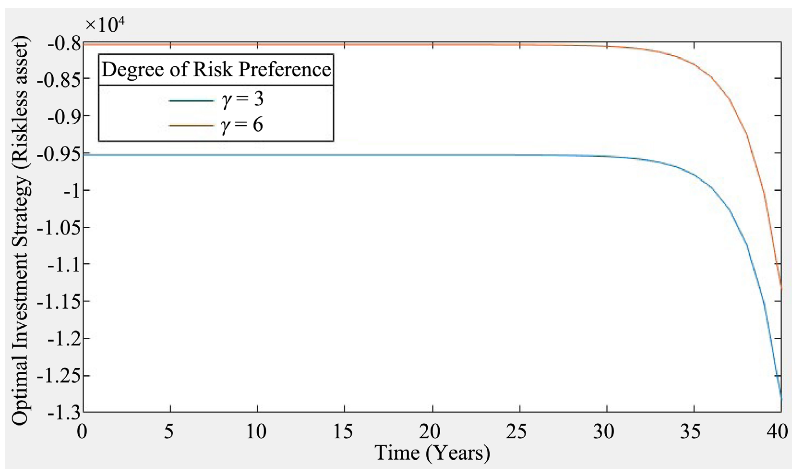

Figure 6. Investment strategies (Riskless asset) vs time for CRRA.

$\kappa = 1500; i = 0.03; x = 0.5; \sigma_P = 0.04; \lambda_P = 0.05; \lambda_r = 0.03; a = 0.005; \varphi = -0.3;$
$r = 0.5; N = 0.025; c = 0.49; T = 40; \gamma = 0.0003; \sigma_S = 0.03; \sigma_r = 0.5; \eta = 0.0002.$

The above figures show the gesture of optimal investment proportions from initial time to retirement. The parameters are given by the previous paragraph. Figures 4-6 above give us the trends how the optimal investment strategies in the three assets change with time. As time runs, the optimal amount invested in cash and bond increase as shown in Figure 4 above. However, the optimal strategy in stock and cash decrease as time goes by, as indicated in Figure 5 and Figure 6 respectively. Results suggest that the pension fund manager maintains diversifying the portfolio by investing more in bond since the optimal investment strategies in bond increase with time. This shows that as time goes on, the investors are told to more position in bond and shorter position in cash and stock. Figures 4-6 also show the different degrees or measures of risk aversion (risk aversion parameter) for different optimal investment strategies. Figure 4 and Figure 5 indicate that the higher the degree of risk preference the lower the investment strategy for the bond and stock respectively. Figure 6 shows that as the degree of the risk aversion increases so does the investment strategy invested in cash.

6. Conclusion

We studied optimal control of asset allocation on a defined contribution pension plan. We considered three different investment strategies which include cash, inflation-linked bond and the stock. We obtained the optimal investment strategies for the three investments under the exponential utility function. The dynamics of the wealth in our model considered a certain proportion of the client's salary paid as the contribution towards the pension fund. By applying the Hamilton-Bellman equation we found the explicit solutions for the CARA and CRRA utility functions. This helped us to calculate the investment strategies associated with the stock and inflation-linked bond. Finally, a numerical simulation was presented to illustrate the behaviour of the model.

Acknowledgements

I would like to thank Dr. Othusitse Basimanebotlhe for his guidance through each stage of the write-up of this article. I extended gratitude to the members of JAMP for their valuable comments and support.

Conflicts of Interest

The authors declare no conflicts of interest.

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Appendix A

Proof. Taking the partial derivatives of (23) we obtain:

$$H_t = -q\left(a_t x + g_t\right)H$$

$$H_x = -qa\left(t\right)H$$

$$H_{xx} = q^2 a\left(t\right)^2 H$$

where a_t and g_t denote derivatives of $a\left(t\right)$ and $g\left(t\right)$ respectively.

Introducing these derivatives in (21) above we get:

$$\left(qa_t + qa - qai + qa\sigma_P\lambda_P\right)x + qg_t + qac + \frac{1}{2}\left(\eta\lambda_r^2\kappa^2 + \left(\sigma_P + \lambda_P\right)^2\right) = 0 \quad (29)$$

splitting (29) into two equations we get:

$$\left(qa_t + qa - qai + qa\sigma_P\lambda_P\right)x = 0 \quad (30)$$

and

$$qg_t + qac + \frac{1}{2}\left(\eta\lambda_r^2\kappa^2 + \left(\sigma_P + \lambda_P\right)^2\right) = 0 \quad (31)$$

Simplifying (30) and (31) respectively we get:

$$\left(a_t + a - ai + a\sigma_P\lambda_P\right) = 0 \quad (32)$$

and

$$g_t + ac + \frac{1}{2q}\left(n\lambda_r^2\kappa^2 + \left(\sigma_P + \lambda_P\right)^2\right) = 0 \quad (33)$$

Solving (32) and (33) we get:

$$a\left(t\right) = \mathrm{e}^{\varsigma(T-t)} \quad (34)$$

where $\varsigma = 1 - i + \sigma_P\lambda_P$.

And

$$g\left(t\right) = \rho t \quad (35)$$

where $\rho = ac + \dfrac{\eta\lambda_r^2\kappa^2 + \left(\sigma_P + \lambda_P\right)^2}{2q}$

$$\frac{H_x}{H_{xx}} = -\frac{\mathrm{e}^{-\varsigma(T-t)}}{q} \quad (36)$$

where $\varsigma = 1 - i + \sigma_P\lambda_P$

Substituting (36) into (19) and (20) we obtain the optimal investment strategies for the inflation-linked bond and stock respectively as:

$$\pi_2^* = -\frac{\mathrm{e}^{-\varsigma(T-t)}}{xq} + \frac{\lambda_P\mathrm{e}^{-\varsigma(T-t)}}{q} + 1 \quad (37)$$

and

$$\pi_1^* = \frac{\sigma_r\psi\sqrt{\eta}\mathrm{e}^{-\varsigma(T-t)}}{qx\sigma_S\eta} - \frac{\lambda_P\sigma_r\psi\sqrt{\eta}\mathrm{e}^{-\varsigma(T-t)}}{qx\sigma_P\sigma_S\eta} + \frac{k\lambda_r\mathrm{e}^{-\varsigma(T-t)}}{x\sigma_S q} \quad (38)$$

Appendix B

Proof. Taking the partial derivatives of (24) we obtain:

$$H_t = \frac{b'(t)(x-a(t))^{\gamma}}{\gamma} - b(t)(x-a(t))^{\gamma-1} a'(t) \tag{39}$$

$$H_x = b(t)(x-a(t))^{\gamma-1} \tag{40}$$

$$H_{xx} = b(t)(\gamma-1)(x-a(t))^{\gamma-2} \tag{41}$$

substituting the above into (21) we get

$$\begin{aligned} &\frac{b'(t)(x-a(t))^{\gamma}}{\gamma} - b(t)(x-a(t))^{\gamma-1} a'(t) + xr(t)b(t)(x-a(t))^{\gamma-1} \\ &- xib(t)(x-a(t))^{\gamma-1} + x\sigma_P\lambda_P b(t)(x-a(t))^{\gamma-1} + cb(t)(x-a(t))^{\gamma-1} \\ &-\frac{1}{2}\frac{b(t)(x-a(t))^{\gamma}}{\gamma-1}\left(\eta\lambda_r^2\kappa^2 + \sigma_P^2 + \lambda_P^2 + 2\lambda_P\sigma_P\right) = 0 \end{aligned} \tag{42}$$

splitting (42) into terms containing $(x-a(t))^{\gamma}$ and $(x-a(t))^{\gamma-1}$ respectively gives:

$$\begin{aligned} &\frac{b'(t)(x-a(t))^{\gamma}}{\gamma} + r(t)b(t)(x-a(t))^{\gamma} + \sigma_P\lambda_P b(t)(x-a(t))^{\gamma} \\ &-\frac{1}{2}\frac{b(t)(x-a(t))^{\gamma}}{\gamma-1}\left(\eta\lambda_r^2\kappa^2 + \sigma_P^2 + \lambda_P^2 + 2\lambda_P\sigma_P\right) = 0 \end{aligned} \tag{43}$$

and

$$\begin{aligned} &-b(t)(x-a(t))^{\gamma-1} a'(t) + (r(t)a(t) - xi)b(t)(x-a(t))^{\gamma-1} \\ &+(\sigma_P\lambda_P a(t) + c)b(t)(x-a(t))^{\gamma-1} - xib(t)(x-a(t))^{\gamma-1} \\ &+x\sigma_P\lambda_P b(t)(x-a(t))^{\gamma-1} + cb(t)(x-a(t))^{\gamma-1} = 0 \end{aligned} \tag{44}$$

simplifying (43) and (44) we get

$$\frac{b'(t)}{\gamma} + r(t)b(t) + \sigma_P\lambda_P b(t) - \frac{1}{2}\frac{b(t)}{\gamma-1}\left(\eta\lambda_r^2\kappa^2 + \sigma_P^2 + \lambda_P^2 + 2\lambda_P\sigma_P\right) = 0 \tag{45}$$

and

$$-a'(t) + r(t)a(t) + \sigma_P\lambda_P a(t) - 2xi + x\sigma_P\lambda_P + 2c = 0 \tag{46}$$

from (45) we have:

$$b'(t) + \left(\gamma(r(t) + \sigma_P\lambda_P) - \frac{1}{2}\frac{\gamma}{\gamma-1}\left(\eta\lambda_r^2\kappa^2 + \sigma_P^2 + \lambda_P^2 + 2\lambda_P\sigma_P\right)\right)b(t) = 0 \tag{47}$$

let

$$B = \gamma(r(t) + \sigma_P\lambda_P) - \frac{1}{2}\frac{\gamma}{\gamma-1}\left(\eta\lambda_r^2\kappa^2 + \sigma_P^2 + \lambda_P^2 + 2\lambda_P\sigma_P\right) \tag{48}$$

then (47) becomes

$$b'(t) + Bb(t) = 0 \tag{49}$$

solving (49) using integrating factor (*IF*) by the below steps results in:

$$IF = p(t) = e^{\int B dt} = e^{Bt}$$

$$b'(t)e^{Bt} + Be^{Bt}b(t) = 0 \tag{50}$$

$$d\left[b(t)e^{Bt}\right] = 0$$

integrating both sides gives

$$b(t) = 0 + Ce^{-Bt}$$

using initial conditions gives

$$1 = Ce^{-B(N+T)}$$

and so

$$b(t) = e^{B(N+T-t)} \tag{51}$$

substituting (48) back into (51) gives

$$b(t) = e^{\left(\gamma\left(r(t)+\sigma_P\lambda_P\right)-\frac{1}{2}\frac{\gamma}{\gamma-1}\left(\eta\lambda_r^2\kappa^2+\sigma_P^2+\lambda_P^2+2\lambda_P\sigma_P\right)\right)(N+T-t)} \tag{52}$$

then (46) is solved as:

$$a'(t) - \left(r(t) + \sigma_P\lambda_P\right)a(t) = -2xi + x\sigma_P\lambda_P + 2c \tag{53}$$

let

$$A = r(t) + \sigma_P\lambda_P \text{ and } K = -2xi + x\sigma_P\lambda_P + 2c \tag{54}$$

then

$$a'(t) - Aa(t) = K \tag{55}$$

multiplying the Equation (55) above by integrating factor $p(t) = e^{\int -A dt} = e^{-At}$ gives:

$$a'(t)e^{-At} - Aa(t)e^{-At} = Ke^{-At} \tag{56}$$

$$d\left[a(t)e^{-At}\right] = Ke^{-At}$$

integrating both sides gives

$$a(t)e^{-At} = \frac{K}{-A}e^{-At} + C \tag{57}$$

$$a(t) = \frac{-K}{A} + Ce^{At} \tag{58}$$

using initial conditions $a(N+T) = 0$ yields:

$$0 = \frac{-K}{A} + Ce^{A(N+T)} \tag{59}$$

and so:

$$a(t) = \frac{-K}{A} + \frac{K}{A}e^{-A(N+T-t)} \tag{60}$$

Now substituting (54) into (60) above gives:

$$a(t)=\frac{2xi-x\sigma_P\lambda_P-2c}{r(t)+\sigma_P\lambda_P}+\frac{-2xi+x\sigma_P\lambda_P+2c}{r(t)+\sigma_P\lambda_P}\mathrm{e}^{(-r(t)-\sigma_P\lambda_P)(N+T-t)} \tag{61}$$

and therefore we have

$$\frac{H_x}{H_{xx}}=\frac{x-a(t)}{\gamma-1} \tag{62}$$

substituting (62) into (19) and (20) above gives

$$\pi_2^*=\frac{x-a(t)}{x(\gamma-1)}-\frac{\lambda_P}{x\sigma_P}\frac{x-a(t)}{\gamma-1}+1 \tag{63}$$

and

$$\pi_1^*=-\frac{\sigma_r\psi\sqrt{\eta}}{x\sigma_S\eta}\frac{x-a(t)}{\gamma-1}+\frac{\lambda_P\sigma_r\psi\sqrt{\eta}}{x\sigma_P\sigma_S\eta}\frac{x-a(t)}{\gamma-1}-\frac{\sigma_r\psi\sqrt{\eta}}{\sigma_S\eta}-\frac{\kappa\lambda_r}{x\sigma_S}\frac{x-a(t)}{\gamma-1} \tag{64}$$

now substituting (61) into (63) and (64) gives optimal investment strategies for the bond and stock respectively as:

$$\begin{aligned}\pi_2^*=&\frac{x-\frac{2xi-x\sigma_P\lambda_P-2c}{r(t)+\sigma_P\lambda_P}+\frac{-2xi+x\sigma_P\lambda_P+2c}{r(t)+\sigma_P\lambda_P}\mathrm{e}^{(-r(t)-\sigma_P\lambda_P)(N+T-t)}}{x(\gamma-1)}\\&-\frac{\lambda_P}{x\sigma_P}\frac{x-\frac{2xi-x\sigma_P\lambda_P-2c}{r(t)+\sigma_P\lambda_P}+\frac{-2xi+x\sigma_P\lambda_P+2c}{r(t)+\sigma_P\lambda_P}\mathrm{e}^{(-r(t)-\sigma_P\lambda_P)(N+T-t)}}{\gamma-1}+1\end{aligned} \tag{65}$$

and

$$\begin{aligned}\pi_1^*=&-\frac{\sigma_r\psi\sqrt{\eta}}{x\sigma_S\eta}\frac{x-\frac{2xi-x\sigma_P\lambda_P-2c}{r(t)+\sigma_P\lambda_P}+\frac{-2xi+x\sigma_P\lambda_P+2c}{r(t)+\sigma_P\lambda_P}\mathrm{e}^{(-r(t)-\sigma_P\lambda_P)(N+T-t)}}{\gamma-1}\\&+\frac{\lambda_P\sigma_r\psi\sqrt{\eta}}{x\sigma_P\sigma_S\eta}\frac{x-\frac{2xi-x\sigma_P\lambda_P-2c}{r(t)+\sigma_P\lambda_P}+\frac{-2xi+x\sigma_P\lambda_P+2c}{r(t)+\sigma_P\lambda_P}\mathrm{e}^{(-r(t)-\sigma_P\lambda_P)(N+T-t)}}{\gamma-1}\\&-\frac{\sigma_r\psi\sqrt{\eta}}{\sigma_S\eta}-\frac{\kappa\lambda_r}{x\sigma_S}\frac{x-a(t)}{\gamma-1}\end{aligned} \tag{66}$$