**Open Access Library Journal**
2022, Volume 9, e8837
ISSN Online: 2333-9721
ISSN Print: 2333-9705

# Ground State Solutions for *p*-Fractional Choquard-Kirchhoff Equations Involving Electromagnetic Fields and Critical Nonlinearity

**Xueqian Yan**

School of Mathematics, Liaoning Normal University, Dalian, China
Email: yanxq526@163.com

**How to cite this paper:** Yan, X.Q. (2022) Ground State Solutions for *p*-Fractional Choquard-Kirchhoff Equations Involving Electromagnetic Fields and Critical Nonlinearity. *Open Access Library Journal*, **9**: e8837.
https://doi.org/10.4236/oalib.1108837

**Received:** May 4, 2022
**Accepted:** June 12, 2022
**Published:** June 15, 2022



## Abstract

This paper is concerned with the existence of ground state solutions for *p*-fractional Choquard-Kirchhoff equations involving electromagnetic fields and critical nonlinearity. Under assumptions on the nonlinear term, by applying the method of Nehari manifold, we obtain that the equation possesses a ground state solution.

## Subject Areas

Functional Analysis

## Keywords

Fractional Choquard-Kirchhoff Equation, Magnetic Fractional *p*-Laplacian, Critical Growth

## 1. Introduction

We consider the existence of ground state solutions for following *p*-fractional Choquard-Kirchhoff equations with electromagnetic fields and critical growth

$$\begin{aligned}&\left(a+b\left[u\right]_{s,A}^{p}\right)\left(-\Delta\right)_{p,A}^{s}u+V\left(x\right)\left|u\right|^{p-2}u\\&=\mu h\left(x,\left|u\right|^{2}\right)u+\left[I_{\sigma}\left(x\right)*\left|u\right|^{p_{\sigma,s}^{*}}\right]\left|u\right|^{p_{\sigma,s}^{*}-2}u,\ x\in\mathbb{R}^{N},\end{aligned}\tag{1.1}$$

where $a,b>0$, $\mu>0$, $I_{\sigma}\left(x\right)=\left|x\right|^{-\sigma}$ is the Riesz potential. $\left(-\Delta\right)_{p,A}^{s}$ denotes the *p*-fractional magnetic operator with $0<s<1$, $2\le p<N/s$, $0<\sigma<2ps$ and $p_{\sigma,s}^{*}=\dfrac{p\left(2N-\sigma\right)}{2\left(N-ps\right)}$. $A\in C\left(\mathbb{R}^{N},\mathbb{R}^{N}\right)$ and $V\in C\left(\mathbb{R}^{N},\mathbb{R}\right)$ are the electric

and magnetic potentials, respectively. $h$ is a continuous function satisfying some conditions.

When $p=2$, the fractional magnetic Laplacian $(-\Delta)_A^s$, up to normalization constants, which is defined on smooth functions $u$ as

$$(-\Delta)_A^s u(x) = 2\lim_{\eta\to 0}\int_{\mathbb{R}^N\setminus B_\eta(x)} \frac{u(x)-\mathrm{e}^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y)}{|x-y|^{N+2s}}\mathrm{d}y, \quad x\in\mathbb{R}^N. \tag{1.2}$$

Here, $B_\eta(x)$ denotes the ball of $\mathbb{R}^N$ centered at $x\in\mathbb{R}^N$ and of radius $\eta>0$. This operator was defined by d'Avenia and Squassina [1], and it can be considered as the fractional counterpart of the magnetic Laplacian

$$(-\Delta)_A u := \left(\frac{1}{i}\nabla - A\right)^2 u = -\Delta u - \frac{2}{i}A(x)\cdot\nabla u + |A(x)|^2 u - \frac{1}{i}u\mathrm{div}(A(x)), \tag{1.3}$$

which plays a fundamental role in quantum mechanics in the description of the dynamics of the particle in a non-relativistic setting. In this context, the curl of $A$ represents magnetic field acting on a charged particle. Motivated by this fact, many authors dealt with the existence of nontrivial solutions of the Schrödinger equations with magnetic fields.

For more details on fractional magnetic operators, we refer to d'Avenia and Squassina [1], and for the physical background, we can refer to previous studies [2] and [3]. This paper was inspired by previous works concerning the magnetic Schrödinger equations. Next, let us mention some enlightening works related to the problem (1.1). Recently, a great attention has been devoted to the study of the following fractional magnetic Schrödinger equation

$$\varepsilon^{2s}(-\Delta)_A^s u + V(x)u = f\left(x,|u|^2\right)u, \quad x\in\mathbb{R}^N. \tag{1.4}$$

For instance, Ambrosio and d'Avenia established with the existence and multiplicity of solutions to (1.4) for small $\varepsilon>0$, when $f$ has a subcritical growth and the potential $V$ satisfies some global conditions, by applying variational methods and Ljusternick-Schnirelmann theory in [4]. By employing the fractional version of the concentration compactness principle and variational methods, Liang *et al.*, in [5], studied the existence and multiplicity of solutions for the fractional Schrödinger-Kirchhoff equations with external magnetic operator and critical nonlinearity

$$\begin{cases} \varepsilon^{2s}M\left([u]_{s,A_\varepsilon}^2\right)(-\Delta)_{A_\varepsilon}^s u + V(x)u = |u|^{2_s^*-2}u + h\left(x,|u|^2\right)u, & x\in\mathbb{R}^N, \\ u(x)\to 0, \quad \text{as } |x|\to\infty. \end{cases} \tag{1.5}$$

Others related fractional Schrödinger-Kirchhoff equations can be seen in [6]-[11]. Moreover, as mentioned above, if the magnetic field $A\equiv 0$, the operator $(-\Delta)_{p,A}^s$ can be reduced to the $p$-fractional Laplacian operator $(-\Delta)_p^s$, up to normalization constants, which is defined as

$$(-\Delta)_p^s u(x) = 2\lim_{\eta\to 0}\int_{\mathbb{R}^N\setminus B_\eta(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}\mathrm{d}y, \quad x\in\mathbb{R}^N, \tag{1.6}$$

where $B_\eta(x) := \{y\in\mathbb{R}^N : |x-y|<\eta\}$. There are also some interesting results

that are obtained by using some different approaches under various hypotheses on the potential and the nonlinearity. Xiang *et al.* [12] obtained weak solutions for the following Kirchhoff type problem involving the fractional $p$-Laplacian by using the mountain pass theorem

$$\begin{cases} M\left(\iint_{\mathbb{R}^{2N}} |u(x)-u(y)|^p K(x-y)\mathrm{d}x\mathrm{d}y\right) L_K^p u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.7}$$

Iannizzotto *et al.* [13] studied a class of quasilinear nonlocal problems involving the fractional $p$-Laplacian and obtained the existence and multiplicity of solutions by Morse theory. In [14], the authors investigated the existence of weak solutions for a perturbed nonlinear elliptic equation driven by the fractional $p$-Laplacian operator by variational methods.

For the Choquard equation, we refer to [15], Shen *et al.* considered the following Choquard equation, and proved that the existence of ground states for it by variational methods

$$(-\Delta)^s u + u = \left(|x|^{-\mu} * F(u)\right) f(u). \tag{1.8}$$

And in [16], by applying the variational methods, Ma and Zhang obtained the existence and multiplicity of weak solutions, considering the following fractional Choquard equation with critical nonlinearity

$$(-\Delta)^{\frac{\alpha}{2}} u + \left(\lambda V(x) - \beta\right) u = \left[|x|^{-\mu} * |u|^{2_\mu^*}\right] |u|^{2_\mu^*-2} u, \quad x \in \mathbb{R}^N. \tag{1.9}$$

And Li *et al.* [17] obtained a ground state solution for fractional Choquard equation involving upper critical exponent. For others related, we can see [18]-[24]. It is worth mentioning that Li *et al.*, in [25], established that the following fractional equation has a ground state solution by the Nehari methods, when $\lambda$ is quite large

$$(-\Delta)^s u + V(x) u = |u|^{2_s^*-2} u + \lambda f(x,u), \quad x \in \mathbb{R}^N. \tag{1.10}$$

We borrowed some brilliant ideas from them, while the structure of Choquard-Kirchhoff equations and appearance of the magnetic fields, such that our results are different from theirs and extend their results in some degree.

Inspired by the above works, in this paper, we focus our attention on the existence of ground state solutions to (1.1). To our best knowledge, there are a few results in the literature to study the $p$-fractional Choquard-Kirchhoff equations with electromagnetic fields and critical growth. Some difficulties arise when dealing with this problem, the main difficulty origins from the strongly nonlocality in the sense that the leading operator takes care of the behavior of the solutions in the whole space. Indeed, the appearance of the magnetic fields and the existence of criticality also bring additional difficulties into the study of our problem, such as the effects of the magnetic fields on the linear spectral sets and on the structure of solutions, and the possible interactions between the magnetic fields and the linear potentials. Therefore, we need to take more considerations

to overcome the difficulties induced by these new traits.

The main goal of this paper is to investigate the existence of ground state solutions for the problem (1.1), when $\mu > 0$ is sufficiently large, $A \in C\left(\mathbb{R}^N, \mathbb{R}^N\right)$, under assumptions ($V_1$) - ($V_2$) on the potential $V$ and $h$ is a superlinear but subcritical function satisfying the following conditions. Let $K$ be the class of functions $k \in L^\infty\left(\mathbb{R}^N\right)$ such that for every $\delta > 0$, the set $\left\{x \in \mathbb{R}^N : |k(x)| \geq \delta\right\}$ has a finite Lebesgue measure. We shall assume that $V$ satisfies

($V_1$) $V \in L^\infty\left(\mathbb{R}^N\right)$ and $V_0 = \inf_{x \in \mathbb{R}^N} V(x) > 0$.

($V_2$) There exists a function $V_\alpha \in L^\infty\left(\mathbb{R}^N\right)$, which is 1-periodic in $x_i\ (i = 1, \cdots, N)$, such that $V(x) - V_\alpha(x) \in K$ and $V(x) \leq V_\alpha(x)$ for all $x \in \mathbb{R}^N$.

And $h$ satisfies the assumptions:

($h_1$) $h \in C\left(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}\right)$ and there exists $p < q < p_s^*$ such that

$$|h(x,t)| \leq C\left(1 + |t|^{\frac{q-2}{2}}\right)$$

for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, where $C$ is a positive constant.

($h_2$) $h(x,t) = o(1)$ uniformly in $x \in \mathbb{R}^N$ as $|t| \to 0$.

($h_3$) $th(x,t) - pH(x,t) \geq \omega th(x,\omega t) - pH(x,\omega t)$ for all $x \in \mathbb{R}^N \times \mathbb{R}$ and $\omega \in [0,1]$, where $H(x,t) = \int_0^t h(x,\tau)\mathrm{d}\tau$.

($h_4$) $h(x,t)t > 0$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R} \setminus \{0\}$.

($h_5$) There exists a function $h_\alpha \in C\left(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}\right)$, which is 1-periodic in $x_i\ (i = 1, \cdots, N)$, such that

1) $|h_\alpha(x,t)| \leq |h(x,t)|$, $\forall (x,t) \in \mathbb{R}^N \times \mathbb{R}$;

2) $|h_\alpha(x,t) - h(x,t)| \leq |k(x)|\left(1 + |t|^{\frac{q-2}{2}}\right)$, $\forall (x,t) \in \mathbb{R}^N \times \mathbb{R}$, where $k \in K$ and $q$ is given by ($h_1$);

3) $th_\alpha(x,t) - pH_\alpha(x,t) \geq \omega th_\alpha(x,\omega t) - pH_\alpha(x,\omega t)$ for all $x \in \mathbb{R}^N \times \mathbb{R}$, and $\omega \in [0,1]$, where $H_\alpha(x,t) = \int_0^t h_\alpha(x,\tau)\mathrm{d}\tau$;

4) $h_\alpha(x,t)t \geq 0$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$.

The main result of this paper can be summarized as follows:

**Theorem 1.1.** *Let* $0 < s < 1$, $2 \leq p < N/s$, $0 < \sigma < 2ps$. *Assume that* $A \in C\left(\mathbb{R}^N, \mathbb{R}^N\right)$, *$V$ satisfies* ($V_1$) - ($V_2$) *and $h$ satisfies* ($h_1$) - ($h_5$). *Then there exists* $\mu^* > 0$ *such that for each* $\mu > \mu^*$, *problem* (1.1) *possesses a positive ground state solution.*

## 2. Preliminaries

Let $0 < s < 1$, $2 \leq p < N/s$. The magnetic Gagliardo seminorm is defined by

$$[u]_{s,A} = \left( \iint_{\mathbb{R}^{2N}} \frac{\left| u(x) - \mathrm{e}^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)} u(y) \right|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{p}},$$

and $W_A^{s,p}\left(\mathbb{R}^N,\mathbb{C}\right)$ is denoted by

$$W_A^{s,p}\left(\mathbb{R}^N,\mathbb{C}\right)=\left\{u\in L^p\left(\mathbb{R}^N,\mathbb{C}\right):[u]_{s,A}^p<+\infty\right\},$$

endowed with the norm

$$\|u\|_{W_A^{s,p}\left(\mathbb{R}^N,\mathbb{C}\right)}=\left([u]_{s,A}^p+|u|_p^p\right)^{\frac{1}{p}}.$$

In view of the $V$, the subspace of $W_A^{s,p}\left(\mathbb{R}^N,\mathbb{C}\right)$ is defined by

$$X=\left\{u\in W_A^{s,p}\left(\mathbb{R}^N,\mathbb{C}\right):\int_{\mathbb{R}^N}V(x)|u|^p\,\mathrm{d}x<+\infty\right\},$$

where the norm

$$\|u\|_{s,A}=\left(a[u]_{s,A}^p+\int_{\mathbb{R}^N}V(x)|u|^p\,\mathrm{d}x\right)^{\frac{1}{p}}.$$

On account of $(V_1)$ - $(V_2)$, we know that the norms $\|u\|_{s,A}$ and

$$\|u\|_{s,\alpha,A}=\left(a[u]_{s,A}^p+\int_{\mathbb{R}^N}V_\alpha(x)|u|^p\,\mathrm{d}x\right)^{\frac{1}{p}}$$

are equivalent. In addition, the best constant of Hardy-Littlehood-Sobolev inequality is

$$S=\inf_{u\in W^{s,p}\left(\mathbb{R}^N\right)\setminus\{0\}}\frac{[u]_s^p}{\left(\int_{\mathbb{R}^N}\left[I_\sigma(x)*|u|^{p_{\sigma,s}^*}\right]|u|^{p_{\sigma,s}^*}\,\mathrm{d}x\right)^{\frac{p}{2p_{\sigma,s}^*}}},\tag{2.1}$$

where $[u]_s^p$ is Gagliardo seminorm defined in $W^{s,p}\left(\mathbb{R}^N\right)$. We will show the existence of ground solutions of (1.1) by searching for the critical points of energy functional associated to (1.1)

$$J(u)=\frac{1}{p}\int_{\mathbb{R}^N}\left(a[u]_{s,A}^p+V(x)|u|^p\right)\mathrm{d}x+\frac{b}{2p}[u]_{s,A}^{2p}-\frac{\mu}{2}\int_{\mathbb{R}^N}H\left(x,|u|^2\right)\mathrm{d}x$$
$$-\frac{1}{2p_{\sigma,s}^*}\int_{\mathbb{R}^N}\left[I_\sigma(x)*|u|^{p_{\sigma,s}^*}\right]|u|^{p_{\sigma,s}^*}\,\mathrm{d}x.$$

The Nehari manifolds can be defined on $X$ as follows:

$$\mathcal{N}:=\left\{u\in X\setminus\{0\}:\left\langle J'(u),u\right\rangle=0\right\}$$

and

$$\mathcal{N}_\alpha:=\left\{u\in X\setminus\{0\}:\left\langle J_\alpha'(u),u\right\rangle=0\right\},$$

where

$$J_\alpha(u)=\frac{1}{p}\int_{\mathbb{R}^N}\left(a[u]_{s,A}^p+V(x)|u|^p\right)\mathrm{d}x+\frac{b}{2p}[u]_{s,A}^{2p}-\frac{\mu}{2}\int_{\mathbb{R}^N}H_\alpha\left(x,|u|^2\right)\mathrm{d}x$$
$$-\frac{1}{2p_{\sigma,s}^*}\int_{\mathbb{R}^N}\left[I_\sigma(x)*|u|^{p_{\sigma,s}^*}\right]|u|^{p_{\sigma,s}^*}\,\mathrm{d}x.$$

Now we give the definition of weak solutions for problem (1.1).

**Lemma 2.1.** (*Diamagnetic inequality*) *For every* $u\in W_A^{s,p}\left(\mathbb{R}^N,\mathbb{C}\right)$, *it holds* $|u|\in W^{s,p}\left(\mathbb{R}^N\right)$. *More precisely*,

$$\left\|\left|u\right|\right\|_{W^{s,p}\left(\mathbb{R}^N\right)} \le \left\|u\right\|_{W_A^{s,p}\left(\mathbb{R}^N,\mathbb{C}\right)}, \quad \text{for each } u \in W_A^{s,p}\left(\mathbb{R}^N,\mathbb{C}\right).$$

**Proof.** It follows from Pointwise Diamagnetic inequality in [1] that

$$\left|\left|u(x)\right|-\left|u(y)\right|\right| \le \left|u(x)-\mathrm{e}^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y)\right|,$$

which implies the conclusion holds.

**Lemma 2.2.** (*Magnetic Sobolev embeddings*) *Let* $s\in(0,1)$ *and* $N>ps$, $p_s^*=\dfrac{Np}{N-ps}$, *then the embedding* $W_A^{s,p}\left(\mathbb{R}^N,\mathbb{C}\right)\hookrightarrow L^r\left(\mathbb{R}^N,\mathbb{C}\right)$ *is continuous for* $r\in\left[p,p_s^*\right]$ *and is locally compact for* $r\in\left[p,p_s^*\right)$.

**Proof.** In view of Theorem 6.7 in [24], we know that the embedding $W^{s,p}\left(\mathbb{R}^N\right)\hookrightarrow L^r\left(\mathbb{R}^N\right)$ is continuous for $r\in\left[p,p_s^*\right]$, that is, there exists a constant $C_0$ such that

$$\left|u\right|_{L^r\left(\mathbb{R}^N\right)} \le C_0\left\|u\right\|_{W^{s,p}\left(\mathbb{R}^N\right)},$$

and similar to the argument of Lemma 3.3 in [1], since Pointwise Diamagnetic inequality, we have

$$\left(\int_{\mathbb{R}^N}\frac{\left|u(x)-u(y)\right|^p}{\left|x-y\right|^{N+ps}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{p}} \le C\left(\int_{\mathbb{R}^N}\frac{\left|u(x)-\mathrm{e}^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y)\right|^p}{\left|x-y\right|^{N+ps}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{p}}.$$

Consequently,

$$\left\|u\right\|_{L^{p_s^*}\left(\mathbb{R}^N\right)} \le C\left(\int_{\mathbb{R}^N}\frac{\left|u(x)-\mathrm{e}^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y)\right|^p}{\left|x-y\right|^{N+ps}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{p}}, \text{ for all } u\in W_A^{s,p}\left(\mathbb{R}^N,\mathbb{C}\right).$$

Then, by interpolation the assertion immediately follows. For the compact embedding, note that the embedding $W_A^{s,p}\left(\mathbb{R}^N,\mathbb{C}\right)\hookrightarrow W^{s,p}\left(\mathbb{R}^N\right)$ is continuous, the assertion follows by the Corollary 7.2 [26].

**Lemma 2.3.** [27] *Let* $r,t>1$ *and* $0<\sigma<N$ *with* $\dfrac{1}{r}+\dfrac{1}{t}+\dfrac{N-\sigma}{N}=2$. *Assume that* $f_1\in L^r\left(\mathbb{R}^N\right)$ *and* $f_2\in L^t\left(\mathbb{R}^N\right)$. *Then there exists a sharp constant* $C_{N,\sigma,r,t}$ *independent of* $f_1$ *and* $f_2$ *such that*

$$\left|\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{f_1(x)\,f_2(y)}{\left|x-y\right|^{N-\sigma}}\mathrm{d}x\mathrm{d}y\right| \le C_{N,\sigma,r,t}\left|f_1\right|_r\left|f_2\right|_t.$$

## 3. Proof of Main Results

**Lemma 3.1.** *For each* $\mu>0$, $u\in X\setminus\{0\}$, *we have*

1) *Set* $\Phi(t)=J(tu)$. *Then there exists a unique* $t_u>0$ *such that*

$$\Phi(t_u)=\max_{t\geq 0}\Phi(t),$$

$\Phi'(t)>0$ *for* $0<t<t_u$ *and* $\Phi'(t)<0$ *for* $t_u<t$. *Moreover,* $tu\in\mathcal{N}$ *if and only if* $t=t_u$.

2) *Set* $\Psi(t)=J_\alpha(tv)$. *Then there exists a unique* $t_v>0$ *such that*

$$\Psi(t_v)=\max_{t\geq 0}\Psi(t),$$

$\Psi_v'(t)>0$ *for* $0<t<t_v$ *and* $\Psi_v'(t)<0$ *for* $t_v<t$. *Moreover,* $tv\in\mathcal{N}_\alpha$ *if and only if* $t=t_v$.

**Proof.** 1) For any $\varepsilon>0$, by ($h_1$) and ($h_2$), there exists $C_\varepsilon>0$ such that

$$\left|h(x,u)\right|\leq\varepsilon+C_\varepsilon|u|^{q-2} \tag{3.1}$$

and

$$\left|H(x,u)\right|\leq\varepsilon|u|+\frac{C_\varepsilon}{q}|u|^q. \tag{3.2}$$

In view of Lemma 2.2 and Lemma 2.3, we get

$$\left|\int_{\mathbb{R}^N}\left[I_\sigma(x)*|u|^{2^*_{\sigma,s}}\right]|u|^{2^*_{\sigma,s}}\,\mathrm{d}x\right|\leq C_1|u|_{p_s^*}^{2p^*_{\sigma,s}}\leq C_2\|u\|_{s,A}^{2p^*_{\sigma,s}}. \tag{3.3}$$

Hence, for small $\varepsilon>0$ and $t>0$, it follows from (3.2) and (3.3) that

$$\begin{aligned}\Phi(t)&=J(tu)\\&=\frac{1}{p}t^p\|u\|_{s,A}^p+\frac{bt^{2p}}{2p}[u]_{s,A}^{2p}-\frac{\mu}{2}\int_{\mathbb{R}^N}H\left(x,|tu|^2\right)\mathrm{d}x\\&\quad-\frac{1}{2p^*_{\sigma,s}}t^{2p^*_{\sigma,s}}\int_{\mathbb{R}^N}\left[I_\sigma(x)*|u|^{p^*_{\sigma,s}}\right]|u|^{p^*_{\sigma,s}}\,\mathrm{d}x\\&\geq\frac{1}{p}t^p\|u\|_{s,A}^2-\frac{\mu}{2}\left(C_3\varepsilon t^2\|u\|_{s,A}^2+C_4C_\varepsilon t^q\|u\|_{s,A}^q\right)-C_1t^{2p^*_{\sigma,s}}|u|_{p_s^*}^{2p^*_{\sigma,s}}\\&>0,\end{aligned}$$

and due to (3.1), we have

$$\begin{aligned}\Phi'(t)&=\left\langle J'(tu),u\right\rangle\\&=t^{p-1}\|u\|_{s,A}^p+bt^{2p-1}[u]_{s,A}^{2p}-\mu\int_{\mathbb{R}^N}h\left(x,|tu|^2\right)tu^2\mathrm{d}x\\&\quad-t^{2p^*_{\sigma,s}-1}\int_{\mathbb{R}^N}\left[I_\sigma(x)*|u|^{p^*_{\sigma,s}}\right]|u|^{p^*_{\sigma,s}}\,\mathrm{d}x\\&\geq t^{p-1}\|u\|_{s,A}^p-\mu\left(C_3\varepsilon t\|u\|_{s,A}^2+C_4C_\varepsilon t^{q-1}\|u\|_{s,A}^q\right)-C_1t^{2p^*_{\sigma,s}-1}|u|_{p_s^*}^{2p^*_{\sigma,s}}\\&>0.\end{aligned}$$

Furthermore, by means of ($h_4$), we obtain that

$$\begin{aligned}\Phi(t)&=J(tu)\\&\leq\frac{1}{p}t^p\|u\|_{s,A}^p+\frac{b}{2p}t^{2p}[u]_{s,A}^{2p}\\&\quad-\frac{1}{2p^*_{\sigma,s}}t^{2p^*_{\sigma,s}}\int_{\mathbb{R}^N}\left[I_\sigma(x)*|u|^{p^*_{\sigma,s}}\right]|u|^{p^*_{\sigma,s}}\,\mathrm{d}x\\&\to-\infty,\end{aligned}$$

as $t \to +\infty$. Therefore, $\Phi(t)$ has a maximum and then there exists $t_u > 0$ such that $\Phi'(t_u) = 0$ and $\Phi'(t) > 0$ for $0 < t < t_u$. We claim that $\Phi'(t) \neq 0$ for all $t > t_u$. Indeed, if the conclusion is false, then, from the above arguments, there exists a $t_u < t_\omega < +\infty$ such that $\Phi'(t_\omega) > 0$ and $\Phi(t_u) \geq \Phi(t_\omega)$. Nevertheless, ($h_3$) implies that

$$\begin{aligned}
\Phi(t_\omega) &= \Phi(t_\omega) - \frac{1}{2p}\left\langle \Phi'(t_\omega), t_\omega \right\rangle \\
&= \frac{1}{2p} t_\omega^p \|u\|_{s,A}^p + \frac{\mu}{2p}\int_{\mathbb{R}^N} \left[ h\left(x, |t_\omega u|^2\right)|t_\omega u|^2 - pH\left(x, |t_\omega u|^2\right)\right] \mathrm{d}x \\
&\quad + \left(\frac{1}{2p} - \frac{1}{2p^*_{\sigma,s}}\right) t_\omega^{2p^*_{\sigma,s}} \int_{\mathbb{R}^N} \left[ I_\sigma(x) * |u|^{p^*_{\sigma,s}} \right] |u|^{p^*_{\sigma,s}} \mathrm{d}x \\
&\geq \frac{1}{2p} t_u^p \|u\|_{s,A}^p + \frac{\mu}{2p}\int_{\mathbb{R}^N} \left[ h\left(x, |t_u u|^2\right)|t_u u|^2 - pH\left(x, |t_u u|^2\right)\right] \mathrm{d}x \\
&\quad + \left(\frac{1}{2p} - \frac{1}{2p^*_{\sigma,s}}\right) t_u^{2p^*_{\sigma,s}} \int_{\mathbb{R}^N} \left[ I_\sigma(x) * |u|^{p^*_{\sigma,s}} \right] |u|^{p^*_{\sigma,s}} \mathrm{d}x \\
&= \Phi(t_u) - \frac{1}{2p}\left\langle \Phi'(t_u), t_u \right\rangle = \Phi(t_u),
\end{aligned}$$

which is a contradiction. Thereupon, the one conclusion of (1) has been proved, we can obtain the other one by the fact that $\Phi'(t) = t^{-1}\left\langle J'(tu), tu \right\rangle$. This completes the proof of (1).

2) Similar to the proof of (1), we can obtain that (2) holds.

**Lemma 3.2.** *For each* $\mu > 0$, *we have the following results.*

1) *There exists* $t_\delta > 0$ *such that* $t_u \geq t_\delta$ *for each*

$$u \in S_1 := \left\{ u \in X : \|u\|_{s,A} = 1 \right\}.$$

*Moreover, for each compact subset* $\Lambda \subset S_1$, *there exsits* $C_\Lambda > 0$ *such that* $t_u \leq C_\Lambda$ *for all* $u \in \Lambda$.

2) *There exists* $\rho > 0$ *such that*

$$c_\mu = \inf_{u \in \mathcal{N}} J(u) \geq \inf_{u \in S_\rho} J(u) > 0,$$

*where* $S_\rho := \left\{ u \in X : \|u\|_{s,A} = \rho \right\}$.

**Proof.** 1) For $u \in S_1$, owing to Lemma 3.1 (1), there exists $t_u > 0$ such that $t_u u \in \mathcal{N}$. Also, by (3.1) and (3.3), we have

$$\begin{aligned}
0 &= \left\langle J'(t_u u), t_u u \right\rangle \\
&= t_u^p \|u\|_{s,A}^p + b t_u^{2p} [u]_{s,A}^{2p} - \mu \int_{\mathbb{R}^N} h\left(x, |t_u u|^2\right)|t_u u|^2 \mathrm{d}x \\
&\quad - t_u^{2p^*_{\sigma,s}} \int_{\mathbb{R}^N} \left[ I_\sigma(x) * |u|^{p^*_{\sigma,s}} \right] |u|^{p^*_{\sigma,s}} \mathrm{d}x \\
&\geq t_u^p \|u\|_{s,A}^p - \mu\left( C_3 \varepsilon t_u^2 \|u\|_{s,A}^2 + C_4 C_\varepsilon t_u^q \|u\|_{s,A}^q \right) - C_2 t_u^{2p^*_{\sigma,s}} \|u\|_{s,A}^{2p^*_{\sigma,s}} \\
&\geq C_5 t_u^p - \mu C_4 C_\varepsilon t_u^q - C_2 t_u^{2p^*_{\sigma,s}},
\end{aligned}$$

which implies that there exists $t_\delta > 0$ such that $t_u \geq t_\delta$ for all $u \in S_1$. Assume that there exists $\{u_n\} \subset \Lambda \subset S_1$ such that $t_n = t_{u_n} \to +\infty$ as $n \to \infty$. Since $\Lambda$

is compact, there exists $u \in \Lambda$ such that $u_n \to u$ in $X$. Set

$$\beta(u) = \frac{1}{2p^*_{\sigma,s}} \int_{\mathbb{R}^N} \left[ I_\sigma(x) * |u|^{p^*_{\sigma,s}} \right] |u|^{p^*_{\sigma,s}} \, \mathrm{d}x, \quad \forall u \in X.$$

Indeed, we have

$$\beta(t_n u_n) \geq C_6 t_n^{2p^*_{\sigma,s}} \|u_n\|^{2p^*_{\sigma,s}}_{s,A}. \tag{3.4}$$

It follows from (3.4) and ($h_4$) that

$$\begin{aligned} J(t_n u_n) &\leq \frac{1}{p} t_n^p \|u_n\|^p_{s,A} + \frac{b}{2p} t_n^{2p} [u_n]^{2p}_{s,A} - \beta(t_n u_n) \\ &\leq \frac{1}{p} t_n^p \|u_n\|^p_{s,A} + \frac{b}{2p} t_n^{2p} [u_n]^{2p}_{s,A} - C_6 t_n^{2p^*_{\sigma,s}} \|u_n\|^{2p^*_{\sigma,s}}_{s,A} \\ &\to -\infty \end{aligned}$$

as $n \to \infty$. However, by ($h_3$), we have

$$\begin{aligned} J(t_n u_n) &= J(t_n u_n) - \frac{1}{2p} \langle J'(t_n u_n), t_n u_n \rangle \\ &= \frac{1}{2p} t_n^p \|u_n\|^p_{s,A} + \frac{\mu}{2p} \int_{\mathbb{R}^N} \left[ h\left(x, t_n^2 |u_n|^2\right) t_n^2 |u_n|^2 - pH\left(x, t_n^2 |u_n|^2\right) \right] \mathrm{d}x \\ &\quad + \left( \frac{1}{2p} - \frac{1}{2p^*_{\sigma,s}} \right) t_n^{2p^*_{\sigma,s}} \int_{\mathbb{R}^N} \left[ I_\sigma(x) * |u_n|^{p^*_{\sigma,s}} \right] |u_n|^{p^*_{\sigma,s}} \, \mathrm{d}x \\ &\geq 0, \end{aligned}$$

a contradiction. Hence the conclusion holds.

2) For $u \in S_\rho$, and small $\varepsilon > 0$, it follows from (3.2) and (3.3) that

$$\begin{aligned} J(u) &\geq \frac{1}{p} \|u\|^p_{s,A} - \mu \left( C_3 \varepsilon \|u\|^2_{s,A} + C_4 C_\varepsilon \|u\|^q_{s,A} \right) - C_2 \|u\|^{2p^*_{\sigma,s}}_{s,A} \\ &\geq C_7 \|u\|^2_{s,A} = C_7 \rho^2 > 0 \end{aligned}$$

for small $\rho > 0$. Furthermore, for each $u \in \mathcal{N}$, there exists $t_\varepsilon > 0$ such that $t_\varepsilon u \in S_\rho$. Then we have

$$0 < C_6 \rho^2 \leq \inf_{u \in S_\rho} J(u) \leq J(t_\varepsilon u) \leq \max_{t>0} J(tu) = J(u),$$

which implies that

$$c_\mu = \inf_{u \in \mathcal{N}} J(u) \geq \inf_{u \in S_\rho} J(u) > 0.$$

The proof is completed.

It follows from [28] that we have the following lemma.

**Lemma 3.3.** *The mapping $I : S_1 \to \mathcal{N}$ is a homeomorphism between $S_1$ and $\mathcal{N}$, and the inverse of $I$ is given by $I^{-1}(u) = \frac{u}{\|u\|_{s,A}}$. Considering the functional $\phi_\mu : S_1 \to \mathbb{R}$ given by*

$$\phi_\mu(w) = J(I(w)),$$

*then the lemma follows.*

**Lemma 3.4.** 1) *If* $\{w_n\}$ *is a Palais-Smale sequence for* $\phi_\mu$, *then* $\{I(w_n)\}$ *is a Palais-Smale sequence for J. If* $\{u_n\} \subset \mathcal{N}$ *is a bounded Palais-Smale sequence for J, then* $\{I^{-1}(u_n)\}$ *is a Palais-Smale sequence for* $\phi_\mu$.

2) $w \in S_1$ *is a critical point of* $\phi_\mu$ *if and only if* $I(w)$ *is a nontrivial critical point of J. Moreover, the corresponding values of* $\phi_\mu$ *and J coincide and* $\inf_{S_1} \phi_\mu = \inf_{\mathcal{N}} J$.

3) *A minimizer of J on* $\mathcal{N}$ *is a ground state solution of* (1.1).

Similar to the argument of Lemma 2.6 in [8], the results as follows

**Lemma 3.5.** *If* $\{u_n\} \subset X$ *satisfies* $u_n \rightharpoonup 0$ *in X and* $\varphi_n \in X$ *is bounded. Then*

$$\int_{\mathbb{R}^N} \left[V(x) - V_\alpha(x)\right] u_n \overline{\varphi_n} \mathrm{d}x \to 0 \tag{3.5}$$

*and*

$$\int_{\mathbb{R}^N} \left[h\left(x, |u_n|^2\right) - h_\alpha\left(x, |u_n|^2\right)\right] u_n \overline{\varphi_n} \mathrm{d}x \to 0, \tag{3.6}$$

*also*

$$\int_{\mathbb{R}^N} \left[H\left(x, |u_n|^2\right) - H_\alpha\left(x, |u_n|^2\right)\right] \mathrm{d}x \to 0. \tag{3.7}$$

**Lemma 3.6.** *There exists* $\mu^* > 0$ *such that*

$$0 < c_\mu < \frac{2ps - \sigma}{2p(2N - \sigma)} S^{\frac{p(2N-\sigma)}{2N(p-1)+p(2s-\sigma)}} \quad \text{for all} \quad \mu > \mu^*.$$

**Proof.** Assume that the conclusion is not true. Then there exists a sequence $\mu_n$ with $\mu_n \to +\infty$ such that $c_{\mu_n} \geq \frac{2ps - \sigma}{2p(2N - \sigma)} S^{\frac{p(2N-\sigma)}{2N(p-1)+p(2s-\sigma)}}$. Take $u \in X \setminus \{0\}$, by Lemma 3.1 (1), there exists a unique $t_{\mu_n} > 0$ such that $\max_{t>0} J(tu) = J(t_{\mu_n} u)$. Since ($h_4$), we have

$$\begin{aligned}
&t_{\mu_n}^p \|u\|_{s,A}^p + b t_{\mu_n}^{2p} [u]_{s,A}^{2p} \\
&= t_{\mu_n}^{2p^*_{\sigma,s}} \int_{\mathbb{R}^N} \left[I_\sigma(x) * |u|^{p^*_{\sigma,s}}\right] |u|^{p^*_{\sigma,s}} \mathrm{d}x + \mu_n \int_{\mathbb{R}^N} h\left(x, |t_{\mu_n} u|^2\right) |t_{\mu_n} u|^2 \mathrm{d}x \\
&\geq t_{\mu_n}^{2p^*_{\sigma,s}} \int_{\mathbb{R}^N} \left[I_\sigma(x) * |u|^{p^*_{\sigma,s}}\right] |u|^{p^*_{\sigma,s}} \mathrm{d}x,
\end{aligned}$$

which means that $\{t_{\mu_n}\}$ is bounded. Therefore, up to a subsequence, and there exists $t_\kappa \geq 0$ such that $t_{\mu_n} \to t_\kappa$ as $n \to \infty$. Suppose $t_\kappa > 0$. In view of ($h_4$), one has

$$\lim_{n\to\infty} \left[\mu_n \int_{\mathbb{R}^N} h\left(x, |t_{\mu_n} u|^2\right) |t_{\mu_n} u|^2 \mathrm{d}x + t_{\mu_n}^{2p^*_{\sigma,s}} \int_{\mathbb{R}^N} \left[I_\sigma(x) * |u|^{p^*_{\sigma,s}}\right] |u|^{p^*_{\sigma,s}} \mathrm{d}x\right] = +\infty.$$

However, we know that

$$t_{\mu_n}^p \|u\|_{s,A}^p \to t_\kappa^p \|u\|_{s,A}^p,$$

which is a contradiction. Thereupon, we get $t_\kappa = 0$. And it follows from ($h_4$) that

$$\max_{t>0} J(tu) = J(t_{\mu_n} u) \leq \frac{1}{p} t_{\mu_n}^p \|u\|_{s,A}^2 - \frac{1}{2p^*_{\sigma,s}} t_{\mu_n}^{2p^*_{\sigma,s}} \int_{\mathbb{R}^N} \left[I_\sigma(x) * |u|^{p^*_{\sigma,s}}\right] |u|^{p^*_{\sigma,s}} \mathrm{d}x \to 0,$$

as $n \to \infty$. Hence,

$$0 < \frac{2ps-\sigma}{2p(2N-\sigma)} S^{\frac{p(2N-\sigma)}{2N(p-1)+p(2s-\sigma)}} \le \inf_{u \in X\setminus\{0\}} \sup_{t>0} J(tu) \le \max_{t>0} J(tu) \to 0,$$

a contradiction. As a result, there exists $\mu^* > 0$ such that

$0 < c_\mu < \frac{2ps-\sigma}{2p(2N-\sigma)} S^{\frac{p(2N-\sigma)}{2N(p-1)+p(2s-\sigma)}}$ for all $\mu > \mu^*$. The proof is completed.

*Proof of Theorem 1.1.* In virtue of Lemma 3.4 (3), we know that $c_\mu$ is achieved. For $\mu > \mu^*$. let $\{w_n\} \subset S_1$ be a minimizing sequence satisfying

$$\phi_\mu(w_n) \to c_\mu = \inf_{S_1} \phi_\mu.$$

Thanks to the Ekeland variational principle, we assume that $\phi'_\mu(w_n) \to 0$ in $X'$. Set $u_n = I(w_n) \in \mathcal{N}$. By Lemma 3.4 (1), we have

$$J(u_n) = \phi_\mu(w_n) \to c_\mu,$$

and $J'(u_n) \to 0$ in $X'$. Thus, by virtue of ($h_3$), we get

$$\begin{aligned} c_\mu + o_n(1)\|u_n\|_{s,A} &= J(u_n) - \frac{1}{2p}\langle J'(u_n), u_n\rangle \\ &= \frac{1}{2p}\|u_n\|_{s,A}^p + \frac{\mu}{2p}\int_{\mathbb{R}^N}\left[h\left(x,|u_n|^2\right)|u_n|^2 - pH\left(x,|u_n|^2\right)\right]\mathrm{d}x \\ &\quad + \left(\frac{1}{2p} - \frac{1}{2p^*_{\sigma,s}}\right)\int_{\mathbb{R}^N}\left[I_\sigma(x) * |u_n|^{p^*_{\sigma,s}}\right]|u_n|^{p^*_{\sigma,s}}\,\mathrm{d}x \\ &\ge \frac{1}{2p}\|u_n\|_{s,A}^p, \end{aligned}$$

which implies $\{u_n\}$ is bounded in $X$. Hence, there exists a subsequence, still denoted by $\{u_n\}$, and $u \in X$. Then we have

$$\begin{aligned} &u_n \rightharpoonup u \text{ in } X, \\ &u_n \to u \text{ in } L^q_{\mathrm{loc}}\left(\mathbb{R}^N, \mathbb{C}\right) \text{ for } p \le q < p^*_s. \end{aligned}$$

Thereupon, $J'(u) = 0$. The next, we prove it by case.

If $u \neq 0$. we know that $u \in \mathcal{N}$ and $c_\mu \le J(u)$. It follows from Fatou's Lemma, the weakly lower semi-continuity of the norm and ($h_3$) that

$$\begin{aligned} c_\mu &= \lim_{n\to\infty}\left[J(u_n) - \frac{1}{2p}\langle J'(u_n), u_n\rangle\right] \\ &\ge \frac{1}{2p}\liminf_{n\to\infty}\|u_n\|_{s,A}^p + \frac{\mu}{2p}\liminf_{n\to\infty}\int_{\mathbb{R}^N}\left[h\left(x,|u_n|^2\right)|u_n|^2 - pH\left(x,|u_n|^2\right)\right]\mathrm{d}x \\ &\quad + \liminf_{n\to\infty}\left(\frac{1}{2p} - \frac{1}{2p^*_{\sigma,s}}\right)\int_{\mathbb{R}^N}\left[I_\sigma(x) * |u_n|^{p^*_{\sigma,s}}\right]|u_n|^{p^*_{\sigma,s}}\,\mathrm{d}x \\ &\ge \frac{1}{2p}\|u\|_{s,A}^p + \frac{\mu}{2p}\int_{\mathbb{R}^N}\left[h\left(x,|u|^2\right)|u|^2 - pH\left(x,|u|^2\right)\right]\mathrm{d}x \\ &\quad + \left(\frac{1}{2p} - \frac{1}{2p^*_{\sigma,s}}\right)\int_{\mathbb{R}^N}\left[I_\sigma(x) * |u|^{p^*_{\sigma,s}}\right]|u|^{p^*_{\sigma,s}}\,\mathrm{d}x \\ &= J(u) - \frac{1}{2p}\langle J'(u), u\rangle = J(u) \ge c_\mu. \end{aligned}$$

Consequently, we get $J(u)=c_\mu$.

In the following, we consider the case for $u=0$. On account of the concentration-compactness principle by Lions, we know that two cases may happen:

1): Vanishing, that is, $\lim_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_1(y)}|u_n(x)|^2\,\mathrm{d}x=0$.

2): Nonvanishing, that is, there exists a sequence $\{y_n\}\subset\mathbb{R}^N$ and a constant $d>0$ such that

$$\liminf_{n\to\infty}\int_{B_1(y_n)}|u_n(x)|^2\,\mathrm{d}x\geq d. \tag{3.8}$$

Assume that (1) occurs. In view of Lemma 1.21 in [29], we get $|u_n|\to 0$ in $L^q(\mathbb{R}^N)$ for $p<q<p_s^*$. Thus, by means of (3.1) and (3.2), we have

$$\int_{\mathbb{R}^N}H\left(x,|u_n|^2\right)\mathrm{d}x\to 0 \text{ and } \int_{\mathbb{R}^N}h\left(x,|u_n|^2\right)|u_n|^2\,\mathrm{d}x\to 0. \tag{3.9}$$

Consequently,

$$o_n(1)=\|u_n\|_{s,A}^p+b[u_n]_{s,A}^{2p}-\int_{\mathbb{R}^N}\left[I_\sigma(x)*|u_n|^{p_{\sigma,s}^*}\right]|u_n|^{p_{\sigma,s}^*}\,\mathrm{d}x.$$

Suppose $\|u_n\|_{s,A}^p+b[u_n]_{s,A}^{2p}\to m$. Then $\int_{\mathbb{R}^N}\left[I_\sigma(x)*|u_n|^{p_{\sigma,s}^*}\right]|u_n|^{p_{\sigma,s}^*}\,\mathrm{d}x\to m$. If $m>0$, in virtue of (2.1), we get

$$S\left(\int_{\mathbb{R}^N}\left[I_\sigma(x)*|u_n|^{p_{\sigma,s}^*}\right]|u_n|^{p_{\sigma,s}^*}\,\mathrm{d}x\right)^{\frac{1}{2p_{\sigma,s}^*}}\leq[u_n]_{s,A}^p\leq\|u_n\|_{s,A}^p\leq\|u_n\|_{s,A}^p+b[u_n]_{s,A}^{2p}.$$

Hence, $Sm^{\frac{2(N-ps)}{p(2N-\sigma)}}\leq m$. Then we have either $m=0$ or $m\geq S^{\frac{p(2N-\sigma)}{2N(p-1)+p(2s-\sigma)}}$.

If $m=0$, we have $c_\mu=0$, which contradicts with Lemma 3.6. When $m\geq S^{\frac{p(2N-\sigma)}{2N(p-1)+p(2s-\sigma)}}$, it follows from $J(u_n)\to c_\mu$ and (3.9) that

$$\begin{aligned}c_\mu&=\lim_{n\to\infty}\left[\frac{1}{p}\|u_n\|_{s,A}^p+\frac{b}{2p}[u_n]_{s,A}^{2p}-\frac{\mu}{2}\int_{\mathbb{R}^N}H\left(x,|u_n|^2\right)\mathrm{d}x\right.\\&\quad\left.-\frac{1}{2p_{\sigma,s}^*}\int_{\mathbb{R}^N}\left[I_\sigma(x)*|u_n|^{p_{\sigma,s}^*}\right]|u_n|^{p_{\sigma,s}^*}\,\mathrm{d}x\right]\\&\geq\frac{m}{2p}-\frac{m}{2p_{\sigma,s}^*}=\frac{2ps-\sigma}{2p(2N-\sigma)}m\\&\geq\frac{2ps-\sigma}{2p(2N-\sigma)}S^{\frac{p(2N-\sigma)}{2N(p-1)+p(2s-\sigma)}},\end{aligned}$$

which also contradicts with Lemma 3.6. Therefore, nonvanishing occurs. Without loss of generality, we may suppose $y_n\in\mathbb{Z}^N$. Let $\tilde{u}_n(\cdot)=u_n(\cdot+y_n)$. Up to a subsequence, then there exists $\tilde{u}\in X$ such that $\tilde{u}_n\rightharpoonup\tilde{u}$ in $X$, $\tilde{u}_n\to\tilde{u}$ in $L_{\mathrm{loc}}^q(\mathbb{R}^N,\mathbb{C})$ for $p\leq q<p_s^*$, and $u_n\to u$ a.e. on $\mathbb{R}^N$. Due to (3.8), we know that $\tilde{u}\neq 0$.

The next job is to prove that $J_\alpha'(\tilde{u})=0$. For all $\varphi\in X$, set $\varphi_n(\cdot)=\varphi_n(\cdot-y_n)$. Owing to Lemma 3.5, we know that

$$\int_{\mathbb{R}^N}\left[V(x)-V_\alpha(x)\right]u_n\overline{\varphi_n}\mathrm{d}x\to 0$$

and

$$\int_{\mathbb{R}^N}\left[h\left(x,|u_n|^2\right)-h_\alpha\left(x,|u_n|^2\right)\right]u_n\overline{\varphi_n}\mathrm{d}x\to 0.$$

Hence we have

$$\begin{aligned}&\left\langle J'(u_n),\varphi_n\right\rangle-\left\langle J'_\alpha(u_n),\varphi_n\right\rangle\\&=\mathcal{R}\left\{\int_{\mathbb{R}^N}\left[V(x)-V_\alpha(x)\right]u_n\overline{\varphi_n}\mathrm{d}x-\mu\int_{\mathbb{R}^N}\left[h\left(x,|u_n|^2\right)-h_\alpha\left(x,|u_n|^2\right)\right]u_n\overline{\varphi_n}\mathrm{d}x\right\}\\&\to 0.\end{aligned}$$

Consequently, $\left\langle J'_\alpha(u_n),\varphi_n\right\rangle\to 0$. In addition, it follows from the periodicity of $V_\alpha$ and $h_\alpha$ with regard to the variable $x$ and $y_n\in\mathbb{Z}^N$ that

$$\left\langle J'_\alpha(\tilde{u}_n),\varphi\right\rangle=\left\langle J'_\alpha(u_n),\varphi_n\right\rangle,$$

which means that $\left\langle J'_\alpha(\tilde{u}_n),\varphi\right\rangle\to 0$. Therefore, as previous arguments we can conclude that $J'_\alpha(\tilde{u})=0$.

What follows is to prove $J_\alpha(\tilde{u})\le c_\mu$. In fact, it follows from the boundedness of $\left\{\|u_n\|_{s,A}\right\}$ and Lemma 3.5 that

$$\int_{\mathbb{R}^N}\left[V(x)-V_\alpha(x)\right]|u_n|^2\,\mathrm{d}x\to 0$$

and

$$\int_{\mathbb{R}^N}\left[h\left(x,|u_n|^2\right)-h_\alpha\left(x,|u_n|^2\right)\right]|u_n|^2\,\mathrm{d}x\to 0,$$

also

$$\int_{\mathbb{R}^N}\left[H\left(x,|u_n|^2\right)-H_\alpha\left(x,|u_n|^2\right)\right]\mathrm{d}x\to 0.$$

Thereupon,

$$\int_{\mathbb{R}^N}\left[h\left(x,|u_n|^2\right)-H\left(x,|u_n|^2\right)\right]\mathrm{d}x=\int_{\mathbb{R}^N}\left[h_\alpha\left(x,|u_n|^2\right)-H_\alpha\left(x,|u_n|^2\right)\right]\mathrm{d}x+o_n(1).$$

By the periodicity of $V_\alpha$ and $h_\alpha$ in the variable $x$ again, (3) in ($h_5$), and it follows from the weakly lower semi-continuity of the norm and Fatou's Lemma that

$$\begin{aligned}c_\mu&=\lim_{n\to\infty}\left[J(u_n)-\frac{1}{2p}\left\langle J'(u_n),u_n\right\rangle\right]\\&\ge\frac{1}{2p}\liminf_{n\to\infty}\|u_n\|_{s,A}^p+\frac{\mu}{2p}\liminf_{n\to\infty}\int_{\mathbb{R}^N}\left[h\left(x,|u_n|^2\right)|u_n|^2-pH\left(x,|u_n|^2\right)\right]\mathrm{d}x\\&\quad+\liminf_{n\to\infty}\left(\frac{1}{2p}-\frac{1}{2p^*_{\sigma,s}}\right)\int_{\mathbb{R}^N}\left[I_\sigma(x)*|u_n|^{p^*_{\sigma,s}}\right]|u_n|^{p^*_{\sigma,s}}\,\mathrm{d}x\\&=\frac{1}{2p}\liminf_{n\to\infty}\|u_n\|_{s,\alpha,A}^p+\frac{\mu}{2p}\liminf_{n\to\infty}\int_{\mathbb{R}^N}\left[h_\alpha\left(x,|u_n|^2\right)|u_n|^2-pH_\alpha\left(x,|u_n|^2\right)\right]\mathrm{d}x\\&\quad+\liminf_{n\to\infty}\left(\frac{1}{2p}-\frac{1}{2p^*_{\sigma,s}}\right)\int_{\mathbb{R}^N}\left[I_\sigma(x)*|u_n|^{p^*_{\sigma,s}}\right]|u_n|^{p^*_{\sigma,s}}\,\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2p}\liminf_{n\to\infty}\|\tilde{u}_n\|^p_{s,\alpha,A} + \frac{\mu}{2p}\liminf_{n\to\infty}\int_{\mathbb{R}^N}\left[h_\alpha\left(x,|\tilde{u}_n|^2\right)|\tilde{u}_n|^2 - pH_\alpha\left(x,|\tilde{u}_n|^2\right)\right]\mathrm{d}x \\
&\quad + \liminf_{n\to\infty}\left(\frac{1}{2p}-\frac{1}{2p^*_{\sigma,s}}\right)\int_{\mathbb{R}^N}\left[I_\sigma(x)*|\tilde{u}_n|^{p^*_{\sigma,s}}\right]|\tilde{u}_n|^{p^*_{\sigma,s}}\,\mathrm{d}x \\
&\geq \frac{1}{2p}\|\tilde{u}\|^p_{s,\alpha,A} + \frac{\mu}{2p}\int_{\mathbb{R}^N}\left[h_\alpha\left(x,|\tilde{u}|^2\right)|\tilde{u}|^2 - pH_\alpha\left(x,|\tilde{u}|^2\right)\right]\mathrm{d}x \\
&\quad + \left(\frac{1}{2p}-\frac{1}{2p^*_{\sigma,s}}\right)\int_{\mathbb{R}^N}\left[I_\sigma(x)*|\tilde{u}|^{p^*_{\sigma,s}}\right]|\tilde{u}|^{p^*_{\sigma,s}}\,\mathrm{d}x \\
&= J_\alpha(\tilde{u}) - \frac{1}{2p}\left\langle J'_\alpha(\tilde{u}),\tilde{u}\right\rangle = J_\alpha(\tilde{u}).
\end{aligned}
$$

Finally, we argue that $\max_{t>0} J_\alpha(t\tilde{u}) = J_\alpha(\tilde{u})$. In virtue of $\tilde{u}\neq 0$ and $J'_\alpha(\tilde{u})=0$, we get $\tilde{u}\in\mathcal{N}_\alpha$. Therefore, we can deduce that the conclusion holds from Lemma 3.1 (2). It follows from $\tilde{u}\neq 0$ and Lemma 3.1 (1) that there exists $t_{\tilde{u}}>0$ such that $t_{\tilde{u}}\tilde{u}\in\mathcal{N}$. Then, we have

$$
c_\mu = \inf_{\mathcal{N}} J \leq J\left(t_{\tilde{u}}\tilde{u}\right) \leq J_\alpha\left(t_{\tilde{u}}\tilde{u}\right) \leq \max_{t>0} J_\alpha\left(t\tilde{u}\right) = J_\alpha\left(\tilde{u}\right) \leq c_\mu,
$$

which means that $J\left(t_{\tilde{u}}\tilde{u}\right)=c_\mu$.

In summary, $c_\mu$ is achieved. Moreover, by Lemma 3.4 (3), the corresponding minimizer is a ground state solution of (1.1). Then, we complete the proof of Theorem 1.1.

## Conflicts of Interest

The author declares no conflicts of interest.

## References


[1] D'Avenia, P. and Squassina, M. (2018) Ground States for Fractional Magnetic Operators. *ESAIM*, **24**, 1-24. https://doi.org/10.1051/cocv/2016071

[2] Ichinose, T. (1993) On Essential Selfadjointness of the Weyl Quantized Relativistic Hamiltonian. *Forum Mathematicum*, **5**, 539-560. https://doi.org/10.1515/form.1993.5.539

[3] Ichinose, T. and Tamura, H. (1986) Imaginary-Time Path Integral for a Relativistic Spinless Particle in an Electromagnetic Field. *Communications in Mathematical Physics*, **105**, 239-257. https://doi.org/10.1007/BF01211101

[4] Ambrosio, V. and D 'Avenia, P. (2018) Nonlinear Fractional Magnetic Schrödinger Equation: Existence and Multiplicity. *Journal of Differential Equations*, **264**, 3336-3368. https://doi.org/10.1016/j.jde.2017.11.021

[5] Liang, S., Repovs, D. and Zhang, B. (2018) On the Fractional Schrödinger-Kirchhoff Equations with Electromagnetic Fields and Critical Nonlinearity. *Computers and Mathematics with Applications*, **75**, 1778-1794. https://doi.org/10.1016/j.camwa.2017.11.033

[6] Xiang, M., Zhang, B. and Guo, X. (2015) Infinitely Many Solutions for a Fractional Kirchhoff Type Problem via Fountain Theorem. *Nonlinear Analysis*, **120**, 299-313. https://doi.org/10.1016/j.na.2015.03.015

[7] Guo, Y. and Nie. J. (2015) Existence and Multiplicity of Nontrivial Solutions for *P*-Laplacian Schrödinger-Kirchhoff-Type Equations. *Journal of Mathematical*

*Analysis and Applications*, **428**, 1054-1069.
https://doi.org/10.1016/j.jmaa.2015.03.064

[8] Li, Q., Wang, W., Teng, K. and Wu, X. (2020) Ground States for Fractional Schrödinger Equations with Electromagnetic Fields and Critical Growth. *Acta Mathematica Scientia*, **40**, 59-74. https://doi.org/10.1007/s10473-020-0105-0

[9] Wang, L., Han, T. and Wang, J.X. (2021) Infinitely Many Solutions for Schrödinger-Choquard-Kirchhoff Equations Involving the Fractional *p*-Laplacian. *Acta Mathematica Sinica*, *English Series*, **37**, 315-332.
https://doi.org/10.1007/s10114-021-0125-z

[10] Ambrosio, V. (2019) Multiplicity and Concentration of Solutions for a Fractional Kirchhoff Equation with Magnetic Field and Critical Growth. *Annales Henri Poincaré*, **20**, 2717-2766. https://doi.org/10.1007/s00023-019-00803-5

[11] Xiang, M., Rădulescu, V.D. and Zhang. B. (2018) A Critical Fractional Choquard-Kirchhoff Problem with Magnetic Field. *Communications in Contemporary Mathematics*, **21**, Article ID: 1850004. https://doi.org/10.1142/S0219199718500049

[12] Ferrara. M., Zhang. B. and Xiang, M. (2015) Existence of Solutions for Kirchhoff Type Problem Involving the Non-Local Fractional P-Laplacian. *Journal of Mathematical Analysis and Applications*, **424**, 1021-1041.
https://doi.org/10.1016/j.jmaa.2014.11.055

[13] Iannizzotto, A., Liu, S., Perera, K. and Squassina, M. (2014) Existence Results for Fractional P-Laplacian Problems via Morse Theory. *Advances in Calculus of Variations*, **9**, 101-125. https://doi.org/10.1515/acv-2014-0024

[14] Radulescu, V.D., Xiang, M. and Zhang, B. (2016). Existence of Solutions for Perturbed Fractional *p*-Laplacian Equations. *Journal of Differential Equations*, **260**, 1392-1413. https://doi.org/10.1016/j.jde.2015.09.028

[15] Shen, Z., Gao, F. and Yang, M. (2016) Ground States for Nonlinear Fractional Choquard Equations with General Nonlinearities. *Mathematical Methods in the Appliedences*, **39**, 4082-4098. https://doi.org/10.1002/mma.3849

[16] Ma, P. and Zhang, J. (2017) Existence and Multiplicity of Solutions for Fractional Choquard Equations. *Nonlinear Analysis*, **164**, 100-117.
https://doi.org/10.1016/j.na.2017.07.011

[17] Li, Q., Teng, K. and Zhang, J. (2020) Ground State Solutions for Fractional Choquard Equations Involving upper Critical Exponent. *Nonlinear Analysis*, **197**, Article ID: 111846. https://doi.org/10.1016/j.na.2020.111846

[18] Wang, F. and Xiang, M. (2017) Multiplicity of Solutions for a Class of Fractional Choquard-Kirchhoff Equations Involving Critical Nonlinearity. *Analysis and Mathematical Physics*, **9**, 1-16. https://doi.org/10.1007/s13324-017-0174-8

[19] Guo, Y.H., Sun, H.R. and Cui, N. (2021) Existence and Multiplicity Results for the Fractional Magnetic Schrödinger Equations with Critical Growth. *Journal of Mathematical Physics*, **62**, Article ID: 061503. https://doi.org/10.1063/5.0041372

[20] Xiang, M., Pucci, P., Squassina, M. and Zhang, B. (2017) Nonlocal Schrödinger-Kirchhoff Equations with External Magnetic Field. *Discrete and Continuous Dynamical Systems*, **37**, 1631-1649. https://doi.org/10.3934/dcds.2017067

[21] Pucci, P., Xiang, M. and Zhang, B. (2017) Existence Results for Schrödinger-Choquard-Kirchhoff Equations Involving the Fractional *P*-Laplacian. *Advances in Calculus of Variations*, **12**, 253-275. https://doi.org/10.1515/acv-2016-0049

[22] Huang, L., Wang, L. and Feng, S. (2021) Ground State Solutions for Fractional Schrödinger-Choquard-Kirchhoff Type Equations with Critical Growth. *Complex Variables and Elliptic Equations*, **67**, 1624-1638.

https://doi.org/10.1080/17476933.2021.1890051

[23] Li, Q., Teng, K. and Wu, X. (2018) Ground States for Fractional Schrödinger Equations with Critical Growth. *Journal of Mathematical Physics*, **59**, Article ID: 033504. https://doi.org/10.1063/1.5008662

[24] Di Nezza, E., Palatucci, G. and Valdinoci, E. (2012) Hitchhiker's Guide to the Fractional Sobolev Spaces. *Bulletin des Sciences Mathématiques*, **136**, 521-573. https://doi.org/10.1016/j.bulsci.2011.12.004

[25] Lieb, E.H. and Loss, M. (2001) Analysis, Second Edition. American Mathematical Society, Providence. https://doi.org/10.1090/gsm/014

[26] Szulkin, A. and Weth, T. (2010) The Method of Nehari Manifold. In: Gao, D.Y. and Motreanu, D., Eds., *Handbook of Nonconvex Analysis and Applications*, International Press, Somerville, 597-632.

[27] Willem, M. (1996) Minimax Theorems. Progress in Nonlinear Differential Equations and their Applications Vol. 24, Birkhäuser, Boston. https://doi.org/10.1007/978-1-4612-4146-1

[28] Zhang, H., Xu, J. and Zhang, F. (2015) Existence and Multiplicity of Solutions for Superlinear Fractional Schrdinger Equations in $\mathbb{R}^N$. *Journal of Mathematical Physics*, **56**, Article ID: 091502. https://doi.org/10.1063/1.4929660

[29] Tao, F. and Wu, X. (2017) Existence and Multiplicity of Positive Solutions for Fractional Schrödinger Equations with Critical Growth. *Nonlinear Analysis*: *Real World Applications,* **35**, 158-174. https://doi.org/10.1016/j.nonrwa.2016.10.007