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Redefining the Shape of Numbers and Three Forms of Calculation

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Abstract

This paper redefines the Shape of numbers, makes it more natural and concise, and the domain of definition is extended to ring. The inconvenient PCHG() and PH() are removed. The concept of subsets is also removed. The new definition can be used to calculate $\sum_{n=0}^{N-1}\prod_{i=1}^{M}\left(K_i+n\times D_i\right)$ $\sum_{n_{i,j}=0}^{j=N-1}\prod_{i=1}^{M}\left(K_i+n_{i,j}\times D_i\right)$, $n_{i,j}\le n_{i+1,j}$ or $n_{i,j}=n_{i+1,j}$; $K_i,D_i\in\text{ring}$. Three forms corresponding to three calculation methods are obtained. They can be used as a powerful tool for analysis. Some of the conclusions are: 1) Expressions and properties of two kinds of Stirling number, Lah number and Eulerian number; 2) Expression of power sum of natural numbers; 3) Vandermonde identity, Norlund identity; 4) New congruence and new proof of Wilson theorem; 5) $\sum_{n=1}^{P-1}n^{P-2}\equiv 0\ MOD\ P^2, P>3$; 6) $\sum_{C=0}^{C=M-1}(-1)^{M-1-C}\sum_{PM(PS)=M,PB(PS)=C}MIN(PS)=1$.

Subject Areas

Discrete Mathematics

Keywords

Shape of Numbers, Calculation Formula, Combinatorics, Congruence, Stirling Number, Eulerian Number

1. Introduction

Peng, J. has introduced Shape of numbers in [1] [2] [3] [4]: $(I_1,I_2,\cdots,I_M)$, $I_i\in Z$. There are $M-1$ intervals between adjacent numbers. $I_{i+1}-I_i\le 1$ means continuity; $I_{i+1}-I_i>1$ means discontinuity.

Shape of numbers: collect $(I_1,\cdots,I_M)$ with the same continuity and discontinuity at the same position into a catalog, call it a Shape.

A Shape has a min Item: $(K_1, K_2, \cdots)$. Use the symbol *PS* = [min Item] to represent it.

If $K_{i+1} - K_i = D > 1$, only $I_{i+1} - I_i \ge D$ is allowed.

If $K_{i+1} - K_i = D \le 1$, only $I_{i+1} - I_i = D$ is allowed.

The single $(I_1, \cdots, I_M)$ is an item. $I_1 \times \cdots \times I_M$ is the product. I_i is a factor.

Example 1.1:

$$PS = [2,3] \to (2,3),(3,4),(1000,1001) \in PS$$

$$PS = [-3,-1] \to (-3,-1),(-3,0),(-2,0),(-3,1),$$
$$(-2,1),(-1,1),(1000,2007) \in PS$$

$$PS = [1,4,4] \to (1,4,4),(1,5,5),(2,5,5),(1,6,6),$$
$$(2,6,6),(3,6,6) \in PS,(3,5,5) \notin PS$$

$$PS = [1,4,6,4] \to (1,4,7,5),(1,5,7,5),(2,5,7,5) \in PS,$$
$$(1,4,7,6),(3,5,7,5) \notin PS$$

PM(*PS*) = Count of factors.

PB(*PS*) = Count of discontinuities.

MIN(*PS*) = Min product: $MIN([1,2,3]) = 1 \times 2 \times 3$, $MIN([1,2,4]) = 1 \times 2 \times 4$

Basic Shape: $K_1 = 1$ and intervals = 1 or 2

BASE(*PS*) = *BS*: if (1) *PB*(*BS*) = *PB*(*PS*), (2) *PM*(*BS*) = *PM*(*PS*), (3) *BS* is a Basic Shape, (4) *BS* has discontinuity intervals at the same positions of *PS*.

PH(*PS*) = (Max Factor)-1-*PB*(*BS*)

IDX(*PS*) = *IDX* of *BS* = {Max factor of *BS*} + 1 = *PM*(*BS*) + *PB*(*BS*) + 1

$PS = [K_1, \cdots, K_M]$, $BS = [G_1, \cdots, G_M]$ then $K_{i+1} - K \le 1 \to G_{i+1} - G_i = 1$; $K_{i+1} - K > 1 \to G_{i+1} - G_i = 2$

Example 1.2:

$$PS = [1,2],[1001,1002] \to BASE(PS) = [1,2]$$

$$PS = [1,3],[-9,4],[1,K>2] \to BASE(PS) = [1,3]$$

$$PS = [1,3,4],[-8,4,5],[1,K>2,X=K+1] \to BASE(PS) = [1,3,4]$$

$$PS = [1,3,5],[0,4,9],[1,K>2,X>K+1] \to BASE(PS) = [1,3,5]$$

$$PS = [1001,1002] \to PH(PS) = 1002 - 1 - 0 = 1001,\ IDX(PS) = IDX([1,2]) = 3$$

$$PS = [0,4,9] \to PH(PS) = 9 - 1 - 2 = 6,\ IDX([0,4,9]) = IDX([1,3,5]) = 6$$

SET(*N*, *PS*) = *SET* of items $\in$ *PS* in [K_1, *N*-1], item's max factor ≤ *N*-1

[3] introduced the subset: fix some interval of discontinuities.

SET(*N*,*PS*,*PT*) = Subset of *SET*(*N*, *PS*), $BASE(PS) = [G_1, \cdots, G_M]$, $PT = [T_1, \cdots, T_M]$

$$= \begin{cases} T_{i+1} - T_i = 1 : G_{i+1} - G_i = 1, \text{ means } I_{i+1} - I_i = K_{i+1} - K_i \\ T_{i+1} - T_i = 1 : G_{i+1} - G_i = 2, \text{ means } I_{i+1} - I_i = K_{i+1} - K_i \\ T_{i+1} - T_i = 2 : G_{i+1} - G_i = 2, \text{ means } I_{i+1} - I_i >= K_{i+1} - K_i \end{cases} \quad (*)$$

PT only has the change at (*). When a change happens, make the interval fixed.

PCHG(*PS*, *PT*) = Count of change from *BASE*(*PS*) to *PT*

Example 1.3:

$$PCHG([1,3,5],[1,3,5]) = 0$$

$PCHG([1,3,5],[1,2,4]) = PCHG([1,4,7],[1,2,4]) = 1$, changed at T_1

$PCHG([1,3,5],[1,3,4]) = PCHG([1,4,7],[1,3,4]) = 1$, changed at T_2

$PCHG([1,3,5],[1,2,3]) = PCHG([1,8,10],[1,2,3]) = 2$, changed at T_1, T_2

|*SET*(*N*, *PS*, *PT*)| = Count of items in *SET*(*N*, *PS*, *PT*)

SUM(*N*, *PS*, *PT*) = Sum of all products in *SET*(*N*, *PS*, *PT*)

Example 1.4:

$$SUM(6,[1,2,4]) = 1\times2\times4 + 1\times2\times5 + 2\times3\times5$$

$$SUM(9,[1,4,7]) = SUM(9,[1,4,7],[1,3,5]) = 1\times4\times7 + 1\times4\times8 + 1\times5\times8 + 2\times5\times8$$

$$SUM(9,[1,4,7],[1,2,4]) = 1\times4\times7 + 1\times4\times8 + 2\times5\times8$$

[1] [2] [3] [4] came to the following conclusion:

(1.1) $$|SET(N,PS,PT)| = \binom{N - PH(PS) - PCHG(PS,PT) - 1}{PB(PT)+1}$$

The following uses count of $X \in K$ for count of $\{X_1, X_2, \cdots, X_M\} \in \{K_1, K_2, \cdots, K_M\}$

(1.2) Use the form $(T_1+K_1)(T_2+K_2)\cdots(T_M+K_M) = \sum X_1X_2\cdots X_M$, $X_i = T_i$ or K_i.

Don't swap the factors of $X_1X_2\cdots X_M$, then each $X_1X_2\cdots X_M$ corresponds to a expression = $A_q\binom{N-PH(PS)-PCHG(PS,PT)-1}{IDX(PT)-q}$, q = count of $X \in K$.

$$SUM(N,PS,PT) = \sum A_q \binom{N-PH(PS)-PCHG(PS,PT)-1}{IDX(PT)-q}$$

$$A_q = \prod_{i=1}^{M}(X_i + D_i),\quad D_i = \begin{cases} -m : X_i = T_i, m = \text{count of } \{X_1,\cdots,X_{i-1}\} \in K \\ +m : X_i = K_i, m = \text{count of } \{X_1,\cdots,X_{i-1}\} \in T \end{cases}$$

Example 1.5:

$PS = [K_1, K_2 \ge K_1 + 2, K_3 \ge K_2 + 2]$, $BS = BASE(PS) = [1,3,5]$,

$\to IDX(BS) = 6$,

$$\begin{aligned} \text{form} &= (1+K_1)(3+K_2)(5+K_3) \\ &= 1\times3\times5 + 1\times3\times K_3 + 1\times K_2\times5 + 1\times K_2\times K_3 + K_1\times3\times5 \\ &\quad + K_1\times3\times K_3 + K_1\times K_2\times5 + K_1\times K_2\times K_3 \end{aligned}$$

$$\begin{aligned} P &= N - PH(PS) - PCHG(PS,BS) - 1 \\ &= N - \{K_3 - 1 - 2\} - 0 - 1 = N - K_3 + 2 \end{aligned}$$

$$\to SUM(N,PS)$$
$$=1\times3\times5\binom{P}{6}+1\times3\times(K_3+2)\binom{P}{5}+1\times(K_2+1)\times(5-1)\binom{P}{5}$$
$$+1\times(K_2+1)\times(K_3+1)\binom{P}{4}+K_1\times(3-1)\times(5-1)\binom{P}{5}$$
$$+K_1\times(3-1)\times(K_3+1)\binom{P}{4}+K_1\times K_2\times(5-2)\binom{P}{4}$$
$$+K_1\times K_2\times K_3\binom{P}{3}$$

An item = $\{K_1+E_1,\cdots,K_M+E_M\}$, K is fixed, E is variable.

A product = $(K_1+E_1)\cdots(K_M+E_M)=\sum F_1F_2\cdots F_M$, $F_i=E_i$ or K_i

That is, a product can be broken down into 2^M parts.

Define $SUM\_K(N,PS,PT,PF=F_1\cdots F_M)$ = Sum of one part, PF indicates the part.

Rewrite 1.2), add {braces}:

$$SUM(N,PS,PT)=\sum\prod_{i=1}^{M}(X_i+D_i)\binom{A}{M_q},$$

$$X_i+D_i=\begin{cases}\{T_i-D_i\}:X_i=T_i,D_i=\text{count of }\{X_1,\cdots,X_{i-1}\}\in K\\ \{K_i\}+\{D_i\}:X_i=K_i,D_i=\text{count of }\{X_1,\cdots,X_{i-1}\}\in T\end{cases}$$

Expand $SUM()$ by {braces}:

(1.3) $SUM\_K(N,PS,PT,PF)$

$$=\sum\text{Expansion of }SUM(\ )\text{ with same }\{K_i\}\in PF=\sum\prod_{i=1}^{M}Y_i\binom{A}{M_q}$$

$$Y_i=\begin{cases}0:F_i=K_i,X_i=T_i\\ K_i:F_i=K_i,X_i=K_i\\ T_i-D_i:F_i=E_i,X_i=T_i,D_i=\text{count of }\{\cdots,X_{i-1}\}\in K\\ D_i:F_i=E_i,X_i=K_i,D_i=\text{count of }\{\cdots,X_{i-1}\}\in T\end{cases}$$

Example 1.6:

Use the example above

$$SUM\_K(\cdots E_1E_2E_3)=1\times3\times5\binom{P}{6}+\left[1\times3\times2+1\times1\times(5-1)\right]\binom{P}{5}+1^3\binom{P}{4}$$
$$SUM\_K(\cdots E_1E_2K_3)=1\times3\times K_3\binom{P}{5}+1\times1\times K_3\binom{P}{4}$$
$$SUM\_K(\cdots E_1K_2E_3)=1\times K_2\times(5-1)\binom{P}{5}+1\times K_2\times1\binom{P}{4}$$
$$SUM\_K(\cdots K_1E_2E_3)=K_1\times(3-1)\times(5-1)\binom{P}{5}+K_1\times(3-1)\times1\binom{P}{4}$$
$$SUM\_K(\cdots E_1K_2K_3)=1\times K_2\times K_3\binom{P}{4}$$
$$SUM\_K(\cdots K_1E_2K_3)=K_1\times(3-1)\times K_3\binom{P}{4}$$

$$SUM\_K(\cdots K_1K_2E_3) = K_1 \times K_2 \times (5-2)\binom{P}{4}$$

$$SUM\_K(\cdots K_1K_2K_3) = K_1 \times K_2 \times K_3\binom{P}{3}$$

SUM_K() can explain why *SUM*() has that strange form:

We can calculate every part of *SUM*() by some way without the form. There may be complex relationships between the parts, but their sum match a simple form.

If understand this: In 1.2), when T_i and D_i all increase L times

(1.4) $K_1 \times \cdots \times K_M + (L+K_1) \times \cdots \times (L+K_M) + \cdots + ((N-1)L+K_1) \times \cdots \times ((N-1)L+K_M)$

$PT = [1\times L, 2\times L, \cdots, M\times L]$, can use the form

$(T_1+K_1)\cdots(T_M+K_M) = \sum A_q\binom{N}{M+1-q}$, q = count of $X \in K$, 2^M items in total.

$$A_q = \prod_{i=1}^{M}(X_i+D_i),\quad D_i = \begin{cases} -mL : X_i = T_i, m = \text{count of } \{\cdots, X_{i-1}\} \in K \\ +mL : X_i = K_i, m = \text{count of } \{\cdots, X_{i-1}\} \in T \end{cases}$$

This paper starts from (1.4), tries to calculate

(1*) $K_1 \times \cdots \times K_M + (D_1+K_1) \times \cdots \times (D_M+K_M) + \cdots + ((N-1)D_1+K_1) \times \cdots \times ((N-1)D_M+K_M)$

In the process, the concept of Shape is greatly expanded.

The form of 1.2) is obtained by guess. If the correct form is found, the rest is mainly inductive proof. With the forms, we can analyze the expression and coefficient and get a lot of results.

2. Redefinition

Change domain from Z to Ring, $K_i, D_i \in \text{Ring}$. K_i no longer compares big or small.

M-series:

$\{K_1, D_1+K_1, 2D_1+K_1, 3D_1+K_1, \cdots, (N-1)D_1+K_1\} = \{K_1+n\times D_1\}$

$\{K_2, D_2+K_2, 2D_2+K_2, 3D_2+K_2, \cdots, (N-1)D_2+K_2\} = \{K_2+n\times D_2\}$

$\vdots$

$\{K_M, D_M+K_M, 2D_M+K_M, 3D_M+K_M, \cdots, (N-1)D_M+K_M\}$
$= \{K_M+n\times D_M\}$

An item = $(I_1, I_2, \cdots, I_M)$, I_1 come from serie1, I_2 come from serie2…

Use $PS = [K_1 : D_1, K_2 : D_2, \cdots, K_M : D_M]$ to represent the Shape.

$[K_1 : 1, K_2 : 1, \cdots, K_M : 1]$ is abbreviated as $[K_1, K_2, \cdots, K_M]$

If $K_i \in Z$, the new definition is similar to the old definition and allows $D_i \le 0$.

SET(*N*, *PS*, *PT*) = Set of some Items come from *M*-series.

$I_i = K_i + a \times D_i$, $I_{i+1} = K_{i+1} + b \times D_{i+1}$,

$$PT = [T_1 = 1, T_2, \cdots, T_M] = \begin{cases} T_{i+1} - T_i = 1 : \text{means } a = b \\ T_{i+1} - T_i = 2, \text{ means } a \le b \end{cases}$$

There is no longer the idea of subsets.

We have tried to extend the domain of *PT*, but when $M > 2$, no rules have been found yet.

Basic Shape: $K_1 = 1$ and intervals = 1 or 2, $K_i \in N$, $D_i = 1$

PT is always a Basic Shape.

$MIN(PS)$ = Min product of a Basic Shape = $\prod K_i$

$|SET(N, PS, PT)|$ = Count of items $\in$ $SET(N, PS, PT)$

$END(N, PS, PT)$ = Set of Items $\in$ $SUM(N, PS, PT)$ and $I_M = (N-1)D_M + K_M$.

$SUM(N, PS, PT)$ = Sum of products in $SET(N, PS, PT)$

$SUM(N, PS, [1, 2, \cdots, M])$ is abbreviated as $SUM(N, PS)$

$SUM(N, \cdots) = \text{Old Sum}(Max(K_i) + N, \cdots)$, *PH*() and *PCHG*() are no longer needed.

The old does not allow $SUM([2,3],[1,3])$, $SUM([3,2],[1,3])$. The new it is allowed.

$$SUM(N, PS) = (1^*) = \sum_{n=0}^{N-1} \prod_{i=1}^{M} (K_i + n \times D_i)$$

$$SUM(N, PS, [1, 3, 5, \cdots, 2M-1]) = \sum \prod_{i=1}^{M} (K_i + J_{i,j} \times D_i),$$

$J_{1,i} \le J_{2,i} \le \cdots \le J_{M,i}$, $J_{i,j} \in [0, N-1]$

$PM(PS) = M$; $PB(PT)$ = Count of discontinuities in *PT*

$$IDX(PT) = T_M + 1 = PB(PT) + PM(PT) + 1$$

2.1) $$|SET(N, PS, PT)| = \binom{N + PB(PT)}{PB(PT) + 1}$$

3. Form$_1$ of Calculation

Similar to [4], key points are:

Define $\nabla f(n) = f(n) - f(n-1)$: if $f(n) = \sum A_i \binom{N_i}{m_i}$, then $\nabla f(n) = \sum A_i \binom{N_i - 1}{m_i - 1}$

1) $$\sum_{n=0}^{N-1} n \binom{n+K}{M} = (M+1)\binom{N+K}{M+2} + (M-K)\binom{N+K}{M+1}$$

By definition:

2) $$\sum END(N, PS, PT) = \nabla SUM(N, PS, PT)$$

3) $$SUM(N, [PS, K_{M+1} : D_{M+1}], [PT, T_M = T_M + 1]) = \sum_{n=0}^{N-1} (K_{M+1} + n \times D_{M+1}) \times \sum END(n+1, PS, PT)$$

4) $$SUM(N, [PS, K_{M+1} : D_{M+1}], [PT, T_M = T_M + 2]) = \sum_{n=0}^{N-1} (K_{M+1} + n \times D_{M+1}) \times SUM(n+1, PS, PT)$$

$PS=\left[K_1:D_1,\cdots,K_M:D_M\right]$, $PT=\left[T_1,\cdots,T_M\right]$, $PS1=\left[PS,K_{M+1}:D_{M+1}\right]$

3.1) Use the Form$_1$ = $(T_1+K_1)\cdots(T_M+K_M)=\sum X_1\cdots X_M$,

$$SUM(N,PS,PT)=\sum A_q\binom{N+PB(PT)}{IDX(PT)-q},\ q=\text{count of }\ X\in K.$$

$$A_q=\prod_{i=1}^{M}B_i,\quad B_i=\begin{cases}(T_i-m)D_i;X_i=T_i,m=\text{count of }\{X_1,\cdots,X_{i-1}\}\in K\\ K_i+mD_i;X_i=K_i,m=\text{count of }\{X_1,\cdots,X_{i-1}\}\in T\end{cases}$$

[Proof]

Suppose $SUM(N,PS,PT)=\sum X_1\cdots X_M\binom{N+PB(PT)}{IDX(PT)-q}$

When $PT1=\left[PT,T_{M+1}=1+T_M\right]$

$$\begin{aligned}&SUM(N,PS1,PT1)\\&=\sum_{n=0}^{N-1}(K_{M+1}+n\times D_{M+1})\times\sum END(n+1,PS,PT)\\&=\sum_{n=0}^{N-1}(K_{M+1}+n\times D_{M+1})\times\nabla SUM(n+1,PS,PT)\\&=\sum_{n=0}^{N-1}(K_{M+1}+n\times D_{M+1})\times\sum X_1\cdots X_M\binom{n+PB(PT)}{IDX(PT)-q-1}\xrightarrow{(1)}\\&=\sum X_1\cdots X_M D_{M+1}(IDX(PT)-q)\binom{N+PB(PT)}{IDX(PT)-q+1}\\&\quad+\sum X_1\cdots X_M\left\{K_{M+1}+(IDX(PT)-q-1-PB(PT))D_{M+1}\right\}\binom{N+PB(PT)}{IDX(PT)-q}\\&\xrightarrow{PB(PT1)=PB(PT),IDX(PT1)=IDX(PT)+1,IDX(PT)=1+T_M=T_{M+1},IDX(PT)-PB(PT)-1=M}\\&=\sum X_1\cdots X_M D_{M+1}(T_{M+1}-q)\binom{N+PB(PT1)}{IDX(PT1)-q}\\&\quad+\sum X_1\cdots X_M\left\{K_{M+1}+(M-q)D_{M+1}\right\}\binom{N+PB(PT1)}{IDX(PT1)-(q+1)}\end{aligned}$$

The previous expression means $X_{M+1}=T_{M+1}$

M – q = Count of $\{X_1\cdots X_M\}\in T$. The following expression means $X_{M+1}=K_{M+1}$

→ Match the Form $(T_1+K_1)\cdots(T_M+K_M)\{T_{M+1}+K_{M+1}\}$.

When $PT1=\left[PT,T_{M+1}=2+T_M\right]$

$$\begin{aligned}&SUM(N,PS1,PT1)\\&=\sum_{n=0}^{N-1}(K_{M+1}+n\times D_{M+1})\times SUM(n+1,PS,PT)\\&=\sum_{n=0}^{N-1}(K_{M+1}+n\times D_{M+1})\times\sum X_1\cdots X_M\binom{n+1+PB(PT)}{IDX(PT)-q}\xrightarrow{(1)}\\&=\sum X_1\cdots X_M D_{M+1}(IDX(PT)-q+1)\binom{N+1+PB(PT)}{IDX(PT)-q+2}\\&\quad+\sum X_1\cdots X_M\left\{K_{M+1}+(IDX(PT)-q-1-PB(PT))D_{M+1}\right\}\\&\quad\times\binom{N+1+PB(PT)}{IDX(PT)-q+1}\xrightarrow{PB(PT1)=PB(PT)+1,IDX(PT1)=IDX(PT)+2}\end{aligned}$$

$$=\sum X_1\cdots X_M D_{M+1}\left(T_{M+1}-q\right)\binom{N+PB\left(PT1\right)}{IDX\left(PT1\right)-q}$$

$$+\sum X_1\cdots X_M\left\{K_{M+1}+\left(M-q\right)D_{M+1}\right\}\binom{N+PB\left(PT1\right)}{IDX\left(PT1\right)-\left(q+1\right)}$$

→ Match the Form $\left(T_1+K_1\right)\cdots\left(T_M+K_M\right)\left\{T_{M+1}+K_{M+1}\right\}$.

q.e.d.

The proof process can be extended to ring.

Example 3.1:

$PS=\left[7:13,-7.7:2.2,15:-23\right]$, $PT=\left[1,3,5\right]$, $\text{Form}_1=\left(1+7\right)\left(3-7.7\right)\left(5+15\right)$

$$SUM\left(N,PS,PT\right)$$
$$=-9867.0\binom{N+2}{6}+1084.6\binom{N+2}{5}+4044.7\binom{N+2}{4}-808.5\binom{N+2}{3}$$

$$-808.5=7\times\left(-7.7\right)\times 15$$

$$\begin{aligned}4044.7=&\left[\left(1-0\right)\times 13\right]\times\left(-7.7+2.2\times 1\right)\times\left(15-23\times 1\right)\\&+7\times\left[\left(3-1\right)\times 2.2\right]\times\left(15-23\times 1\right)\\&+7\times\left(-7.7\right)\times\left[\left(5-2\right)\times\left(-23\right)\right]\end{aligned}$$

$$\begin{aligned}1084.6=&\left[\left(1-0\right)\times 13\right]\times\left[\left(3-0\right)\times 2.2\right]\times\left(15-23\times 2\right)\\&+\left[\left(1-0\right)\times 13\right]\times\left(-7.7+2.2\times 1\right)\times\left[\left(5-1\right)\times\left(-23\right)\right]\\&+7\times\left[\left(3-1\right)\times 2.2\right]\times\left[\left(5-1\right)\times\left(-23\right)\right]\end{aligned}$$

$$-9867.0=\left[\left(1-0\right)\times 13\right]\times\left[\left(3-0\right)\times 2.2\right]\times\left[\left(5-0\right)\times\left(-23\right)\right]$$

$$SUM\left(2,PS,PT\right)$$
$$=7\times\left(-7.7\right)\times 15+7\times\left(-7.7-5.5\right)\times\left(-8\right)+20\times\left(-5.5\right)\times\left(-8\right)$$
$$=4044.7\binom{4}{4}-808.5\times\binom{4}{3}=810.7$$

$$SUM\left(3,PS,PT\right)$$
$$=SUM\left(2,PS,PT\right)+7\times\left(-7.7-5.5-3.3\right)\times\left(-31\right)$$
$$+20\times\left(-5.5-3.3\right)\times\left(-31\right)+33\times\left(-3.3\right)\times\left(-31\right)$$
$$=1084.6\binom{5}{5}+4044.7\binom{5}{4}-808.5\binom{5}{3}=13223.1$$

$$SUM\left(4,PS,PT\right)$$
$$=SUM\left(3,PS,PT\right)+7\times\left(-7.7-5.5-3.3-1.1\right)\times\left(-54\right)$$
$$+20\times\left(-5.5-3.3-1.1\right)\times\left(-54\right)+33\times\left(-3.3-1.1\right)\times\left(-54\right)$$
$$+46\times\left(-1.1\right)\times\left(-54\right)$$
$$=-9867.0\binom{6}{6}+1084.6\binom{6}{5}+4044.7\binom{6}{4}-808.5\binom{6}{3}=41141.1$$

Example 3.2:

$$PS=\left[\begin{pmatrix}7&3\\1&2\end{pmatrix}:\begin{pmatrix}1&3\\4&2\end{pmatrix},\begin{pmatrix}2&1\\1&2\end{pmatrix}:\begin{pmatrix}2&3\\1&2\end{pmatrix}\right],\quad PT=[1,3]$$

$$\text{Form}_1=\left(1+\begin{pmatrix}7&3\\1&2\end{pmatrix}\right)\left(3+\begin{pmatrix}2&1\\1&2\end{pmatrix}\right)$$

$$\begin{pmatrix}17&13\\4&5\end{pmatrix}=\begin{pmatrix}7&3\\1&2\end{pmatrix}\begin{pmatrix}2&1\\1&2\end{pmatrix}$$

$$\begin{pmatrix}44&70\\28&38\end{pmatrix}=1\times\left[\begin{pmatrix}2&1\\1&2\end{pmatrix}+\begin{pmatrix}2&3\\1&2\end{pmatrix}\right]\begin{pmatrix}1&3\\4&2\end{pmatrix}+\begin{pmatrix}7&3\\1&2\end{pmatrix}\times(3-1)\begin{pmatrix}2&3\\1&2\end{pmatrix}$$

$$\begin{pmatrix}15&27\\30&48\end{pmatrix}=1\times\begin{pmatrix}1&3\\4&2\end{pmatrix}\times3\times\begin{pmatrix}2&3\\1&2\end{pmatrix}$$

$$SUM(N,PS,PT)=\begin{pmatrix}15&27\\30&48\end{pmatrix}\binom{N+1}{4}+\begin{pmatrix}44&70\\28&38\end{pmatrix}\binom{N+1}{3}+\begin{pmatrix}17&13\\4&5\end{pmatrix}\binom{N+1}{2}$$

$$\begin{aligned}SUM(2,PS,PT)&=\begin{pmatrix}7&3\\1&2\end{pmatrix}\begin{pmatrix}2&1\\1&2\end{pmatrix}+\left[\begin{pmatrix}7&3\\1&2\end{pmatrix}+\begin{pmatrix}8&6\\5&4\end{pmatrix}\right]\begin{pmatrix}4&4\\2&4\end{pmatrix}\\&=\binom{2+1}{3}\begin{pmatrix}44&70\\28&38\end{pmatrix}+\binom{2+1}{2}\begin{pmatrix}17&13\\4&5\end{pmatrix}\\&=\begin{pmatrix}95&109\\40&53\end{pmatrix}\end{aligned}$$

$$\begin{aligned}SUM(3,PS,PT)&=SUM(2,PS,PT)+\left[\begin{pmatrix}7&3\\1&2\end{pmatrix}+\begin{pmatrix}8&6\\5&4\end{pmatrix}+\begin{pmatrix}9&9\\9&6\end{pmatrix}\right]\begin{pmatrix}6&7\\3&6\end{pmatrix}\\&=\binom{3+1}{4}\begin{pmatrix}15&27\\30&48\end{pmatrix}+\binom{3+1}{3}\begin{pmatrix}44&70\\28&38\end{pmatrix}+\binom{3+1}{2}\begin{pmatrix}17&13\\4&5\end{pmatrix}\\&=\begin{pmatrix}293&385\\166&230\end{pmatrix}\end{aligned}$$

A product = $(K_1+E_1)\cdots(K_M+E_M)=\sum F_1\cdots F_M$, $F_i=E_i$ or K_i

Here's another extension: Let $F_i=E_i$ or K_i or R_i, R_i means (K_i+E_i)

So a product can be broken down into $2^0,2^1,2^2,\cdots,2^M$ parts.

$SUM\_K(N,PS,PT,PF=F_1F_2\cdots F_M)$ = Sum of one part, PF indicates the part.

Rewrite 3.1), add {braces}:

$$SUM(N,PS,PT)=\sum\prod_{i=1}^{M}A_q\binom{A}{M_q},\quad A_q=\prod_{i=1}^{M}B_i$$

$$B_i=\begin{cases}\{(T_i-m)D_i\};X_i=T_i,m=\text{count of }\{\cdots,X_{i-1}\}\in K\\\{K_i\}+\{mD_i\};F_i\neq R_i,X_i=K_i,m=\text{count of }\{\cdots,X_{i-1}\}\in T\\\{K_i+mD_i\};F_i=R_i,X_i=K_i,m=\text{count of }\{\cdots,X_{i-1}\}\in T\end{cases}$$

Expand *SUM*() by {braces}:

3.2) $SUM\_K(N,PS,PT,PF)=\sum$ Expansion with same $\{K_i,R_i\}\in PF$

Use the similar method of [2] to prove.

4. Coefficient Analysis

Define $H_1(PS,PT,C)=A_{M-C}$ of 3.1), $C=$ Count of $X\in T$, $T\in$ Ring.

It's the coefficient in the expression. Here $T\in Ring$ is just for analysis.

$F_M^K=\sum M-$product with different factors $\in K$, the sum traverse all combinations.

$E_M^K=\sum M-$product with factors $\in K$, the sum traverse all combinations.

$F_M^{\{1,2,\cdots,N\}}$ is abbreviated as F_M^N, $E_M^{\{1,2,\cdots,N\}}$ is abbreviated as E_M^N;

By definition:

$$E_M^N=\sum_{N_1+\cdots+N_C=M}1^{N_1}2^{N_2}\cdots N^{N_C}$$

$$F_0^K=E_0^K=1;F_{M>|K|}^K=0$$

$$E_M^{N+1}=(N+1)E_{M-1}^{N+1}+E_M^N;F_M^{[K,K_{M+1}]}=K_{M+1}F_{M-1}^K+F_N^K$$

4.1) $H_1(K,T,0)=\prod K_i$, $H_1(K,T,M)=\prod T_i\prod D_i$

[4] has proved:

4.2) $H_1([D\times A:D,K_1:D_1,\cdots,K_M:D_M],[A,T_1,\cdots,T_M],C)$

$$=D\times A\times H_1([K_1:D_1,\cdots,K_M:D_M],[T_1,\cdots,T_M],C)$$
$$+D\times A\times H_1([K_1:D_1,\cdots,K_M:D_M],[T_1,\cdots,T_M],C-1)$$

this →

4.3) $SUM(N,[1,2,\cdots,n,K_1:D_1,\cdots,K_M:D_M],[1,2,\cdots,n,T_1,\cdots,T_M])$, $PT=[1,\cdots,n,T_1,\cdots,T_M]$ can use the form:

$$(T_1+K_1)\cdots(T_M+K_M)=n!\sum A_q\binom{N+PB(PT)+n}{IDX(PT)-q}$$

[2] has proved:

4.4) $H_1([D\times T_1:D,\cdots,D\times T_M:D],[T_1,\cdots,T_M],C)=D^M\binom{M}{C}\prod T_i$

$D=1$, this →

4.5) $SUM(N,PT,PT)=MIN(PT)\binom{N+PB(PT)+PM(PT)}{IDX(PT)}$

This is a generalization of $\sum_{n=0}^N\binom{n}{M}=\binom{N+1}{M+1}$

Eg: $SUM(2,[1,2,4])=1\times2\times4+1\times2\times5+2\times3\times5=1\times2\times4\binom{2+1+3}{5}$

4.6) $H_1(PS,PT,C)=\nabla^C SUM(C,PS,PT)$, $\nabla^{IDX(PT)}SUM(N>M,\cdots)=\prod T_i\prod D_i$

The last row value of the difference sequence is not arbitrary.

Comparison with 4.4), [4] has proved:

4.7) if $T_i+1=T_{i+1}$, then

$$\begin{cases} K_i+1=K_{i+1}, H_1\left(\left[D\times K_1:D,\cdots\right],PT,C\right)=D^M\begin{pmatrix}M\\C\end{pmatrix}T_1\cdots T_C K_{C+1}\cdots K_M \\ K_i-1=K_{i+1}, H_1\left(\left[D\times K_1:D,\cdots\right],PT,C\right)=D^M\begin{pmatrix}M\\C\end{pmatrix}T_1\cdots T_C K_1\cdots K_{M-C} \\ K_i+1=K_{i+1}, H_1\left(\left[D\times K_1:-D,\cdots\right],PT,C\right)=(-1)^C D^M\begin{pmatrix}M\\C\end{pmatrix}T_1\cdots T_C K_1\cdots K_{M-C} \\ K_i-1=K_{i+1}, H_1\left(\left[D\times K_1:-D,\cdots\right],PT,C\right)=(-1)^C D^M\begin{pmatrix}M\\C\end{pmatrix}T_1\cdots T_C K_{C+1}\cdots K_M \end{cases}$$

$$\begin{aligned}[x+y]_n &= \nabla SUM\left(y+1,\left[x-n+1,x-n+2,\cdots,x\right]\right) \\ &= \binom{y}{n}\binom{n}{n}n![x]_0+\binom{y}{n-1}\binom{n}{n-1}(n-1)![x]_1+\cdots+\binom{y}{0}\binom{n}{0}0![x]_n \\ &= \binom{n}{n}[y]_n[x]_0+\binom{n}{n-1}[y]_{n-1}[x]_1+\cdots+\binom{n}{0}[y]_0[x]_n\end{aligned}$$

→ Vandermonde identity [5]: $[x+y]_n=\sum_{k=0}^{n}\binom{n}{k}[x]_k[y]_{n-k}$

$$\begin{aligned}[x+y]^n &= \nabla SUM\left(-y+1,\left[x:-1,x+1:-1,\cdots,x+n-1:-1\right]\right) \\ &= (-1)^n\binom{-y}{n}\binom{n}{n}n![x]^0+(-1)^{n-1}\binom{-y}{n-1}\binom{n}{n-1}(n-1)![x]^1+\cdots \\ &= \binom{n}{n}[y]^n[x]^0+\binom{n}{n-1}[y]^{n-1}[x]^1+\cdots+\binom{n}{0}[y]^0[x]^n\end{aligned}$$

→ Norlund identity [5]: $[x+y]^n=\sum_{k=0}^{n}\binom{n}{k}[x]^k[y]^{n-k}$

$$\left(K_1+N\times D_1\right)\cdots\left(K_M+N\times D_M\right)$$

$$=\nabla SUM\left(N+1,\left[K_1:D_1..K_M:D_M\right]\right)=\sum_{q=0}^{M}A_q\binom{N}{q}$$

4.8) $\left(K_1+N\times D_1\right)\cdots\left(K_M+N\times D_M\right)$ can be decomposed to $\sum_{q=0}^{M}C_q\binom{N}{q}$ by 3.1)

$N=x$, $K_i=i-1$, $PS=\left[0,1,2,\cdots,M-1\right]$ →

$$[x]^M=\binom{x}{M}\binom{M}{M}M![M-1]_0$$
$$+\binom{x}{M-1}\binom{M}{M-1}(M-1)![M-1]_1\cdots\binom{x}{0}\binom{M}{0}0![M-1]_M$$

4.9) $[x]^M=\sum_{k=0}^{M}\binom{M}{k}[x]_k[M-1]_{M-k}$

$N=x$, $K_i=1-i$, $PS=\left[0:-1,-1:-1,\cdots,-M+1:-1\right]$ →

$$[-x]_M=(-1)^{M+0}\binom{X}{M}\binom{M}{M}M![M-1]_0$$
$$+(-1)^{(M-1)+1}\binom{X}{M-1}\binom{M}{M-1}(M-1)![M-1]_1$$

4.10) $\left[-x\right]_M = \sum_{k=0}^{M}(-1)^M \binom{M}{k}\left[x\right]_k \left[M-1\right]_{M-K}$

[4] has proved:

4.11) if $T_i + 1 = T_{i+1}$, then

$H_1\left(\left[D\times K_1 : D,\cdots,D\times K_M : D\right],\left[T_1,\cdots,T_M\right],C\right)$

$= D^C T_1 \cdots T_C \left[F_{M-C}^{D\times K} E_0^{D\times\{1,2,\cdots,C\}} + F_{M-C-1}^{D\times K} E_1^{D\times\{1,2,\cdots,C\}} + \cdots + F_0^{D\times K} E_{M-C}^{D\times\{1,2,\cdots,C\}}\right]$

This → K_i can exchange order in $SUM\left(N,\left[K_1,\cdots,K_M\right]\right)$.

In fact, K_i can exchange order in $SUM\left(N,\left[K_1 : D_1,\cdots,K_M : D_M\right]\right)$ by definition.

This →

4.12) if λ is a primitive unit root, $\lambda^M = 1$, then

$SUM\left(N,\left[\lambda^1,\lambda^2,\cdots,\lambda^M\right]\right)$

$= M!\binom{N}{M+1}E_0^M + (M-1)!\binom{N}{M}E_1^{M-1} + \cdots$

$+1!\binom{N}{2}E_{M-1}^1 + 0!\binom{N}{1}(-1)^{M+1}$

$\left(\lambda^1+1\right)\cdots\left(\lambda^M+1\right) = SUM\left(2,\left[\cdots\right]\right) - SUM\left(1,\left[\cdots\right]\right) = 1 + (-1)^{M+1}$

It's obvious when M is even; if M is odd then $\left(\lambda^1+1\right)\cdots\left(\lambda^{M-1}+1\right) = 1$

$\left(\lambda^1+2\right)\cdots\left(\lambda^M+2\right) = SUM\left(3,\left[\cdots\right]\right) - SUM\left(2,\left[\cdots\right]\right) = 2^M + (-1)^{M+1}$

It can be concluded from the definition:

4.13) 1) $H_1\left(\left[1,\cdots,M\right],\left[1,\cdots,2M-1\right],C\right)$

$= \sum_{PM(\ )=M,PB(\ )=C} MIN(PS) + \sum_{PM(\ )=M,PB(\ )=C-1} MIN(PS)$

2) $H_1\left(\left[2,\cdots,M\right],\left[3,\cdots,2M-1\right],C\right) = \sum_{PM(PS)=M,PB(PS)=C} MIN(PS)$

PS are Basic Shapes.

5. Special Functions

5.1) $SUM\left(N,\left[a:0\right]\right) = a\binom{N}{1}$

5.2) $SUM\left(N,\left[a:d\right]\right) = \sum_{n=0}^{N-1}(a+nd) = d\binom{N}{2} + a\binom{N}{1}$

5.3) $SUM\left(N,\left[1,2,\cdots,M\right]\right) = M!\binom{N+M}{M+1}$

5.4) $SUM\left(N,\left[1,2,\cdots,M\right],\left[1,3,\cdots,2M-1\right]\right) = \sum I_1 I_2 \cdots I_M$, $1 \le I_1 < I_2 < \cdots < I_M \le N+M-1$

5.5) $SUM\left(N,\left[1,1,\cdots,1\right],\left[1,2,\cdots,M\right]\right) = 1^M + 2^M + \cdots + N^M$

5.6) $SUM\left(N,\left[0:D_1,\cdots,0:D_M\right],PT\right) = SUM\_K\left(N,PS,PT,E_1\cdots E_M\right)$

5.7) $SUM\left(N,\left[1,1,\cdots,1\right],\left[1,3,\cdots,2M-1\right]\right) = E_M^N$

6. Stirling, Lah Number

$S_1(M, K)$, $S_2(M, K)$ is unsigned Stirling number. $L_{M,K}$ is Lah number.

[1] use 4.5) to calculate

$S_1(N, N-M) = F_M^{N-1} = \sum MIN(PS)\binom{N+PB(PT)+PM(PT)}{IDX(PT)}$, PS traverses all Basic Shapes, $PM(PS) = M$

This conforms to 4.13). In this paper, it can be written as:

6.1) $SUM(N,[1,2,\cdots,M],[1,3,\cdots,2M-1]) = S_1(N+M,N)$, this is 5.4)

$E_M^N \xrightarrow{def} \sum_{N_1+\cdots+N_C=M} 1^{N_1}2^{N_2}\cdots N^{N_C}$, It's a known property of $S_2(N+M,N)$

6.2) $SUM(N,[1,\cdots,1],[1,3,\cdots,2M-1]) = E_M^N = S_2(N+M,N)$

6.3) 1) $H_1([1,\cdots,1],[1,\cdots,M],K)$

$$= K!S_2(M+1,K+1) = K!\sum_{i=0}^{M-K}\binom{M}{i}S_2(M-i,K)$$

2) $H_1([1,\cdots,1],[2,\cdots,M],K) = (K+1)!S_2(M,K+1)$

[Proof]

Definition of $S_2(M, K)$ is

$$N^M = \sum_{K=0}^{M} S_2(M,K)[N]_k = \sum_{K=0}^{M} K!S_2(M,K)\binom{N}{K}$$

$$\begin{aligned}(N+1)^M &= \nabla SUM(N+1,[1,\cdots,1],[1,\cdots,M]) \\ &= \sum_{q=0}^{M} A_q\binom{N}{q} = \sum_{K=0}^{M} K!S_2(M,K)\binom{N+1}{K} \quad \rightarrow \\ &= \sum_{K=0}^{M} K!S_2(M,K)\binom{N}{K+1} + \sum_{K=0}^{M} K!S_2(M,K)\binom{N}{K}\end{aligned}$$

$H_1([1,\cdots,1],[1,\cdots,M],K) = K!S_2(M,K) + (K+1)!S_2(M,K+1)$ → the first equation

$\xrightarrow{4.11)} H_1(\cdots,K) = K!\left[F_{M-K}^{[1,\cdots,1]}E_0^K + \cdots + F_0^{[1,\cdots,1]}E_{M-K}^K\right] \xrightarrow{F_{M-K}^{[1,\cdots,1]}=\binom{M}{M-K}}$ the second equation

$$\begin{aligned}&\xrightarrow{4.2)} H_1([1,\cdots,1],[1,\cdots,M],K) \\ &= H_1([1,\cdots,1],[2,\cdots,M],K) + H_1([2,\cdots,M],K-1)\end{aligned} \rightarrow 2)$$

q.e.d.

This → $S_2(N+1,M) = \sum_{k=M-1}^{N}\binom{N}{k}S_2(K,M-1)$, which is recorded in [5]

Example 6.1:

Directly according to the definition of $H_1()$

$$H_1(\cdots,1) = (1\times2\times2\times\cdots\times2 + 1\times1\times2\times\cdots\times2 + \cdots + 1\times1\times\cdots\times1\times1)$$

$$\rightarrow S_2(M+1,2) = 2^M - 1$$

$$H_1(\cdots,M-1)=(M-1)!(1+2+\cdots+M)\to S_2(M+1,\ M)=\binom{M+1}{2}$$

6.4) $\binom{M}{K}\frac{M!}{K!}=S_1(M+1,K+1)S_2(K,K)+\cdots+S_1(M+1,M+1)S_2(M,K)$

[Proof]

$$H_1([1,\cdots,M],[1,\cdots,M],K)\xrightarrow{4.4)}=\binom{M}{K}M!$$

$$\xrightarrow{4.11)}=K!\left[F_{M-K}^{M}E_0^{K}+\cdots+F_0^{M}E_{M-K}^{K}\right]$$

$$=K!\left[S_1(M+1,K+1)S_2(K,K)+\cdots+S_1(M+1,M+1)S_2(M,K)\right]$$

q.e.d.

Definition of Lah number [5] is $[-X]_M=\sum_{k=0}^{M}L_{M,K}[x]_k\xrightarrow{4.10)}$

6.5) $L_{M,K}=(-1)^M\frac{M!}{K!}\binom{M-1}{K-1}$, this is recorded in [5].

7. Congruence Analysis

P is a prime number, we already know:

(7*) $S_1(P,P-K)=F_K^{P-1}\equiv 0\ MOD\ P, 0<K<P-1$

[1] has proved, it is easy to infer from 4.5):

7.1) $SUM(P-PB(PT)-PM(PT),PT,PT)$

$$=MIN(PT)\binom{P}{IDX(PT)}\equiv 0\ MOD\ P,\ IDX(PT)<P$$

E.g.: $1\times2+2\times3+3\times4\equiv1\times3+1\times4+2\times4\equiv0\ MOD\ 5$

This is the promotion of (7*).

[4] has proved, it is easy to infer from 3.1):

7.2) For arbitrary $K_i\in Z$

1) $M<P-1$, $SUM(P,[K_1:D,\cdots,K_M:D])\equiv 0\ MOD\ P$

2) $M=P-1$, $(D,P)=1$, $SUM(P,[K_1:D,\cdots,K_M:D])\equiv -1\ MOD\ P$

$SUM(P,[K_1,\cdots,K_M])=\prod K_i+\prod(1+K_i)+\cdots+\prod(P-1+K_i)$. Exclude products≡0 $MOD\ P$.

Example 7.1:

$1^Q+2^Q+\cdots+(P-1)^Q\equiv 0\ MOD\ P$, $Q<P-1$; $\equiv -1\ MOD\ P$, $Q=P-1$

$1^2\times2\times3+2^2\times3\times4+\cdots+(P-3)^2\times(P-2)\times(P-1)\equiv -1\ MOD\ P$, $P=5$; $\equiv 0\ MOD\ P$, $P>5$

$1^3\times2+2^3\times3+\cdots+(P-2)^3\times(P-1)\equiv -1\ MOD\ P$, $P=5$; $\equiv 0\ MOD\ P$, $P>5$

$1\times2^3+2\times3^3+\cdots+(P-2)\times(P-1)^3\equiv -1\ MOD\ P$, $P=5$; $\equiv 0\ MOD\ P$, $P>5$

$1^2\times2^2+2^2\times3^2+\cdots+(P-2)^2\times(P-1)^2\equiv -1 MOD\ P$, $P=5$; $\equiv 0 MOD\ P$, $P>5$

In 1), let $PS=[K_1=1,\cdots,K_M]$ is a Basic Shape and $K_M=P-1$

Rewrite $PS=[\![L_1L_2\cdots L_Q]\!]$, L_i = count of continuity. (L_i,L_{i+1}) means a discontinuity.

$[\![L_1 \cdots L_Q]\!]$ can been slided to $[L_Q, L_1, L_2, \cdots]$, $[L_{Q-1}, L_Q, L_1, L_2, \cdots]$ by $SUM(P, PS)\ MOD\ P$

[3] has proved:

7.3) $\sum MIN(PS) \equiv 0\ MOD\ P$

{*PS*} = All of the Basic Shapes $[1, \cdots, K_M = P-1] \neq [1, 2, \cdots, P-1]$ can slide to.

Example 7.2:

$$1\times(3\times4\times5)\times(7\times8)\times10+1\times3\times(5\times6\times7)\times(9\times10)$$
$$+(1\times2)\times4\times6\times(8\times9\times10)+(1\times2\times3)\times(5\times6)\times8\times10$$
$$=139260 \equiv 0\ MOD\ 11$$

This → [1] has proved:

7.4) $\sum MIN(PS) \equiv 0\ MOD\ P$, {*PS*} = All of the Basic Shapes with the same *PM*() and the same *PB*(), *PB*() > 0 and $K_M = P-1$

In Example 7.1, the *SUM*() is not symmetrical, it's part of some symmetrical express.

E.g.: $\left(1^2\times2^2+2^2\times3^2+3^3\times4^2\right)+\left(1^2\times3^2+2^2\times4^2+4^2\times1^2\right)$

$\equiv SUM(5,[1,1,2,2]) + SUM(5,[1,1,3,3])\ MOD\ 5$

Use *F*(*PS*) = The symmetrical express. Obviously: $F([K_1 \cdots K_M]) \equiv 0\ MOD\ P$, $M < P-1$;

7.5) $$F(\lambda(PS)) \equiv -\frac{\binom{P-1}{LEN(PS)}\frac{(\mu_1+\mu_2+\cdots)!}{\mu_1!\mu_2!\cdots}}{P-LEN(PS)}\ MOD\ P, M = P-1$$

[Proof]

Use *CNT*(*PS*) = Count of $PS \in F(PS)$ → $F(PS) \equiv -CNT(PS)\ MOD\ P$

Use *LEN*(*PS*) =Count of different $K_i \in PS$

Obviously:

$LEN(PS \in F(PS))$ is same, count of products ∈ *F*(*PS*) = *P*-*LEN*(*PS*)

CNT(*PS*) = [Count of products ∈ *F*(*PS*)]/(*P*-*LEN*(*PS*))

Use $\lambda(PS)$ to classify *PS*.

$$\lambda(PS) := 1^{\lambda_1}2^{\lambda_2}\cdots(P-1)^{\lambda_{p-1}}, PM(PS) = 1\times\lambda_1 + 2\times\lambda_2 + \cdots + (P-1)\times\lambda_{p-1}$$

$$\lambda(PS) := 4^1 : 1^4 + 2^4 + 3^4 + \cdots$$

$$\lambda(PS) := 1^4 : 1\times2\times3\times4 + \cdots$$

$$\lambda(PS) := 1^1 3^1 : 1^3\times2 + 2^3\times3 + \cdots; 1^3\times3 + 2^3\times4 + \cdots;$$
$$1\times2^3 + 2\times3^3 + \cdots; 1\times3^3 + 2\times4^3 + \cdots$$

$$\lambda(PS) := 2^2 : 1^2\times2^2 + 2^2\times3^2 + \cdots; 1^2\times3^2 + 2^2\times4^2 + \cdots; 1^2\times4^2 + 2^2\times5^2 + \cdots$$

$$\lambda(PS) := 1^2 2^1 : 1^2\times2\times3 + \cdots; 1\times2^2\times3 + \cdots; 1\times2\times3^2 + \cdots; 1^2\times3\times5 + \cdots$$

1) The combination number of *LEN*(*PS*) in $P-1=\binom{P-1}{LEN(PS)}$

2) Record $\lambda(PS)$ as $\{\mu_1,\mu_2,\cdots\}$

There are μ_1 numbers in $\lambda_1,\lambda_2,\cdots$ equal to $x>0$

There are μ_2 numbers in $\lambda_1,\lambda_2,\cdots$ equal to $y>0, y\neq x$

The combination number of $\mu_1,\mu_2,\cdots$ = $\dfrac{(\mu_1+\mu_2+\cdots)!}{\mu_1!\mu_2!\cdots}$

Count of products $\in$ *F*(*PS*) = $\binom{P-1}{LEN(PS)}\dfrac{(\mu_1+\mu_2+\cdots)!}{\mu_1!\mu_2!\cdots}$

q.e.d.

Example 7.3:

$$\left(1^3\times2+2^3\times3+3^3\times4\right)+\left(1^3\times3+2^3\times4+4^3\times1\right)+\left(1^3\times4+3^3\times1+4^3\times2\right)$$
$$+\left(1\times2^3+2\times3^3+3\times4^3\right)\equiv-\frac{4!}{2!2!}\times\frac{2!}{1!1!}\Big/(5-2)\equiv-4\ MOD\ P$$

$$\left(1^2\times2^2+2^2\times3^2+3^2\times4^2\right)+\left(1^2\times3^2+2^2\times4^2+4^2\times1^2\right)$$
$$\equiv-\frac{4!}{2!2!}\times\frac{2!}{2!}\Big/(5-2)\equiv-2\ MOD\ P$$

$$\left(1^2\times2\times3+2^2\times3\times4\right)+\left(1^2\times2\times4+3^2\times4\times1\right)+\left(1^2\times3\times4+4^2\times1\times2\right)$$
$$+\left(1\times2^2\times3+2\times3^2\times4\right)+\left(1\times2^2\times4+3\times4^2\times1\right)+\left(1\times2\times3^2+2\times3\times4^2\right)$$
$$\equiv-\frac{4!}{3!1!}\times\frac{3!}{2!1!}\Big/(5-3)\equiv-6\ MOD\ P$$

7.6) 1) $K!S_2(P-1,K)=K!E^K_{P-1-K}\equiv(-1)^{K+1}\ MOD\ P, 1\le K\le P-1$

2) $S_2(P,\ K)\equiv0\ MOD\ P, 1<K<P$

[Proof]

$$SUM(N,[1,2,\cdots,P-1])\xrightarrow{4.11)\ (7^*)}$$
$$\equiv(P-1)!\binom{N}{P}E_0^{P-1}+\cdots+1!\binom{N}{2}E^1_{P-2}+0!\binom{N}{1}(P-1)!\ MOD\ P$$

$$\nabla SUM(N,[1,\cdots,P-1])$$
$$\equiv(P-1)!\binom{N-1}{P-1}E_0^{P-1}+\cdots+1!\binom{N-1}{1}E^1_{P-2}+0!\binom{N-1}{0}(p-1)!\ MOD\ P$$

$$[P]_{P-1}=\nabla SUM(2,[1,\cdots,P-1])\equiv1+(P-1)!\equiv0\ MOD\ P$$
$$\rightarrow(P-1)!\equiv-1\ MOD\ P$$

This is a new proof of Wilson theorem.

$$1!E^1_{P-2}=1\equiv1\ MOD\ P$$

$$[P+1]_{P-1}=\nabla SUM(3,[1,\cdots,P-1])\equiv0\ MOD\ P\rightarrow2!E^2_{P-3}\equiv-1\ MOD\ P$$

$$[P+2]_{P-1}=\nabla SUM(4,[1,\cdots,P-1])\equiv0\ MOD\ P\rightarrow3!E^3_{P-4}\equiv1\ MOD\ P$$

…

→ 1)

$S_2(P,K)=S_2(P-1,K-1)+K\times S_2(P-1,K)$ →

$(K-1)!S_2(P,\ K)=(K-1)!S_2(P-1,K-1)+K!S_2(P-1,K)$
$\equiv(-1)^K+(-1)^{K+1}\equiv 0\ MOD\ P$ → 2)

q.e.d.

$$N^{P-1}\equiv 1\ MOD\ P=\sum_{K=0}^{P-1}K!S_2(P-1,K)\binom{N}{K}\xrightarrow{7.6),S_2(P-1,0)=0}$$

7.7) $-\binom{N}{P-1}+\binom{N}{P-2}+\cdots-\binom{N}{2}+\binom{N}{1}\equiv 1\ MOD\ P,(N,P)=1$

7.8) $\sum_{n=1}^{P-1}n^{P-2}\equiv 0\ MOD\ P^2,P>3$

[Proof]

$$P^{P-2}+\sum_{n=1}^{P-1}n^{P-2}=\sum_{n=1}^{P}n^{P-2}=SUM(P,[1,\cdots,1],[1,\cdots,P-2])$$

$$\xrightarrow{6.3)}=\sum_{k=0}^{P-2}k!S_2(P-1,k+1)\binom{P}{K+1}$$

$$=\sum_{k=1}^{\frac{P-1}{2}}\left[(k-1)!S_2(P-1,k)\binom{P}{k}+(P-k-1)!S_2(P-1,P-k)\binom{P}{P-k}\right]$$

$$\xrightarrow{7.6)}\equiv\sum_{k=1}^{\frac{P-1}{2}}\left[(-1)^{K+1}+(-1)^{P-K+1}\right]\binom{P}{K}$$

$$\equiv\sum_{k=1}^{\frac{P-1}{2}}P\times\lambda_k\binom{P}{K}MOD\ P\equiv 0\ MOD\ P^2$$

q.e.d.

8. Form₂ and Analysis

Rewrite (1) of section 3 as

$$\sum_{n=0}^{N-1}n\binom{n+K}{M}=(M+1)\binom{N+K+1}{M+2}-(1+K)\binom{N+K}{M+1}$$

$PS=[K_1:D_1,\cdots,K_M:D_M]$, $PT=[T_1,\cdots,T_M]$, use the same method of 3.1) to prove:

8.1) Use the $\text{Form}_2=(T_1+K_1)\cdots(T_M+K_M)=\sum X_1\cdots X_M$,

$SUM(N,PS,PT)=\sum A_q\binom{N+T_M-q}{IDX(PT)-q}$, $q=$ count of $X\in K$. $A_q=\prod_{i=1}^{M}B_i$

$$B_i=\begin{cases}(T_i-m)D_i;X_i=T_i,m=\text{count of }\{\cdots,X_{i-1}\}\in K\\K_i+(m-T_i)D_i;X_i=K_i,m=\text{count of }\{\cdots,X_{i-1}\}\in K\end{cases}$$

Define $H_2(PS,PT,C)=A_{M-q}$ of 8.1), $C=$ Count of $X\in T$

8.2) $H_2(PT,PT,q<PM(PT))=0$; $H_2(PT,PT,PM(PT))=\prod T_i$, This → 4.5)

8.3) $H_2([1,\cdots,M],[1,\cdots,2M-1],C)$
$=(-1)^{M-C-1}H_1([2,\cdots,M],[3,\cdots,2M-1],C-1)$

[Proof]

$$H_2([1,\cdots,1],[1,\cdots,2M-1],C)=\prod_{i=1}^{M}B_i,$$

$$B_i=\begin{cases}(T_i-m);X_i=T_i,m=\text{count of }\{\cdots,X_{i-1}\}\in K\\ 1+(m-2i+1);X_i=K_i,m=\text{count of }\{\cdots,X_{i-1}\}\in K\end{cases}\xrightarrow{m=i-1-q}$$

$$=\begin{cases}(T_i-m);X_i=T_i,m=\text{count of }\{\cdots,X_{i-1}\}\in K\\ -((i-1)+q);X_i=K_i,q=\text{count of }\{\cdots,X_{i-1}\}\in T\end{cases}$$

$$\to(-1)^{M-C}H_2(\cdots,C)+(-1)^{M-C-1}H_2(\cdots,C+1)$$

$$=H_1([1,\cdots,M],[1,\cdots,2M-1],C)\xrightarrow{4.2)}$$

$$=H_1([2,\cdots,M],[3,\cdots,2M-1],C)+H_1([2,\cdots,M],[3,\cdots,2M-1],C-1)$$

$$\to H_2([1,\cdots,M],[1,\cdots,2M-1],C)$$

$$=(-1)^{M-C-1}H_1([2,\cdots,M],[3,\cdots,2M-1],C-1)$$

q.e.d.

[6] obtains the unified expression of $S_1(N,N-M)$ and $S_2(N,N-M)$ by induction:

1) $S_1(N,N-M)=\sum_{k=1}^{M}A(M,k)\binom{N}{2M-K+1}$

2) $S_2(N,N-M)=\sum_{k=1}^{M}(-1)^{k-1}A(M,k)\binom{N+M-K}{2M-K+1}$

[Proof]

$$S_1(N,N-M)=SUM(N-M,[1,\cdots,M],[1,\cdots,2M-1]),$$

$\text{Form}_1=(T_2+K_2)\cdots(T_M+K_M)$, 4.3) →

$$=\sum_{q=0}^{M-1}H_1([2,\cdots,M],[3,\cdots,2M-1],q)\binom{N-M+(M-1)+1}{2M-(M-q)}\xrightarrow{k=M-q}$$

$$=\sum_{k=1}^{M}H_1([2,\cdots,M],[3,\cdots,2M-1],M-k)\binom{N}{2M-k}\ \to 1)$$

$$H_1([2,\cdots,M],[3,\cdots,2M-1],M-K)=A(M,K)$$

$S_2(N,N-M)=SUM(N-M,[1,\cdots,1],[1,\cdots,2M-1])$, use the Form_2

$$=\sum_{q=0}^{M}H_2([1,\cdots,1],[1,\cdots,2M-1],q)\binom{N-M+(2M-1)-(M-q)}{2M-(M-q)}$$

$$=\sum_{q=0}^{M}H_2([1,\cdots,1],[1,\cdots,2M-1],q)\binom{N-1+q}{M+q}\xrightarrow{H_2(\cdots,0)=0}$$

$$=\sum_{q=1}^{M}H_2([1,\cdots,1],[1,\cdots,2M-1],q)\binom{N-1+q}{M+q}\xrightarrow{K=M-q+1}$$

$$=\sum_{K=1}^{M}H_2([1,\cdots,1],[1,\cdots,2M-1],M-k+1)\binom{N+M-K}{2M-K+1}\xrightarrow{8.3)}$$

$$=\sum_{k=1}^{M}(-1)^{k-1}H_1([2,\cdots,M],[3,\cdots,2M-1],M-k)\binom{N+M-k}{2M-k+1}\ \to 2)$$

q.e.d.

8.4) 1) $\sum_{C=0}^{C=M-1}(-1)^{M-1-C}\sum_{PM(PS)=M,PB(PS)=C}MIN(PS)=1$

2) $\sum_{C=0}^{C=M-1}(-1)^{M-1-C}\sum_{PM(PS)=M,PB(PS)=C}(M+2+C)MIN(PS)=2^{M+1}-1$

PS are Basic Shapes

[Proof]

$S_2(M+1,1)=1$, this is a known property = $SUM(1,[1,\cdots,1],[1,\cdots,2M-1])$

$$\xrightarrow{\text{Use Form}_3}$$

$$=\sum A_q\binom{N+T_M-q}{IDX(PT)-q}=\sum A_q\binom{1+2M-1-q}{2M-q}=\sum A_q$$

$$\xrightarrow{A_0=0}\sum_{q=1}^{q=M}H_2(\cdots,q)\xrightarrow{8.3)}$$

$$=\sum_{C=0}^{C=M-1}(-1)^{M-1-C}H_1([2,\cdots,M],[3,\cdots,2M-1],C)\xrightarrow{4.13)}1)$$

$S_2(M+2,2)=2^{M+1}-1=SUM(2,[1,\cdots,1],[1,\cdots,2M-1])\rightarrow 2)$

q.e.d.

Example 8.1:

$M=1:1=1$

$M=2:1\times3-1\times2=1$

$M=3:1\times3\times5-(1\times3\times4+1\times2\times4)+1\times2\times3=1$

$$M=4:1\times3\times5\times7-(1\times3\times5+1\times3\times4+1\times2\times4)\times6$$
$$+(1\times2\times3+1\times3\times4+1\times2\times4)\times5-1\times2\times3\times4=1$$

$$M=5:1\times3\times5\times7\times9-(1\times3\times5\times7+1\times3\times4\times6+1\times2\times4\times6+1\times3\times5\times6)\times8$$
$$+(1\times2\times3\times5+1\times3\times4\times5+1\times2\times4\times5+1\times3\times5\times6+1\times2\times4\times6$$
$$+1\times3\times4\times6)\times7-(1\times2\times3\times4+1\times2\times3\times5+1\times2\times4\times5+1\times3\times4\times5)\times6$$
$$+1\times2\times3\times4\times5=1$$

It can be concluded from the definition:

8.5) $H_2([1,\cdots,1],[1,\cdots,M],C)=(-1)^{M-C}C!E_{M-C}^{C}=(-1)^{M-C}C!S_2(M,C)$

$N^M=\nabla SUM(N,[1,\cdots,1],[1,\cdots,M])$

$$\xrightarrow{\text{Form}_2}\nabla\sum A_q\binom{N+T_M-q}{IDX(PT)-q}=\sum A_c\binom{N+M-(M-C)-1}{M+1-(M-C)-1}$$

$$=\sum A_c\binom{N+C-1}{C}$$

8.6) $N^M=\sum_{K=0}^{M}K!S_2(M,K)\binom{N}{K}=\sum_{K=0}^{M}(-1)^{M-K}K!S_2(M,K)\binom{N+K-1}{K}$

It can be concluded from the definition:

8.7) $H_2([1+y,2+y,\cdots,M+y],[1,2,\cdots,M],C)=C![y]^{M-C}\binom{M}{C}$

$[x+y]^M=\nabla SUM(x,[1+y,2+y,\cdots,M+y])$

$$\xrightarrow{\text{Form}_2}\nabla\sum_{c=0}^{M}\binom{M}{c}C![y]^{M-c}\binom{x+M-(M-C)}{M+1-(M-C)}$$

$$=\nabla\sum_{c=0}^{M}\binom{M}{c}c![y]^{M-c}\binom{x+C}{C+1}=\sum_{c=0}^{M}\binom{M}{c}c![y]^{M-c}\binom{x+C-1}{C}$$

$$=\sum_{K=0}^{M}\binom{M}{K}[y]^{M-K}[x+k-1]_K=\sum_{K=0}^{M}\binom{M}{K}[y]^{M-K}[x]^K \rightarrow \text{Norlund identity}$$

9. Form3 and Eulerian Number

Rewrite (1) of section 3 as

$$\sum_{n=0}^{N-1} n\binom{n+K}{M}=(M-K)\binom{N+K+1}{M+2}+(1+K)\binom{N+K}{M+2}$$

$PS=[K_1:D_1,\cdots,K_M:D_M]$, $PT=[T_1,\cdots,T_M]$, use the same method of 3.1) to prove:

9.1) Use the $\text{Form}_3=(T_1+K_1)\cdots(T_M+K_M)=\sum X_1\cdots X_M$,

$SUM(N,PS,PT)=\sum A_q\binom{N+T_M-q}{IDX(PT)}$, $q=$ count of $X\in T$. $A_q=\prod_{i=1}^{M}B_i$,

$$B_i=\begin{cases}-K_i+[T_i-m]D_i; X_i=T_i, m=\text{count of }\{\cdots,X_{i-1}\}\in T\\ K_i+mD_i; X_i=K_i, m=\text{count of }\{\cdots,X_{i-1}\}\in T\end{cases}$$

Define $H_3(PS,PT,C)=A_q$ of 9.1), $C=$ Count of $X\in T$

$\left\langle{M \atop k}\right\rangle$ is Eulerian number. Worpitzky identity: $N^M=\sum_{k=0}^{M-1}\left\langle{M \atop k}\right\rangle\binom{N+k}{M}$

Already known 1) $\left\langle{M \atop M-q-1}\right\rangle=\left\langle{M \atop k}\right\rangle$,

2) $\left\langle{M \atop q}\right\rangle=(M-q)\left\langle{M-1 \atop q-1}\right\rangle+(q+1)\left\langle{M-1 \atop q}\right\rangle$

9.2) $H_3(PT,PT,q>0)=0$; $H_3(PT,PT,0)=\prod T_i$ This → 4.5)

9.3) $H_3([1,\cdots,1],[1,\cdots,M],q)=\left\langle{M \atop M-q-1}\right\rangle=\left\langle{M \atop q}\right\rangle$,

$H_3([0,\cdots,0],[1,\cdots,M],q)=\left\langle{M \atop q-1}\right\rangle$

[Proof]

Obviously: $H_3([K_1=1,\cdots,K_M],[T_1=1,\cdots,T_M],M)=0$

$$N^M=\nabla SUM(N,[1,\cdots,1])=\sum_{q=0}^{M}A_q\binom{N+M-q-1}{M}$$

$$=\sum_{q=0}^{M}A_q\binom{N+M-q-1}{M}=\sum_{k=0}^{M-1}\left\langle{M \atop k}\right\rangle\binom{N+k}{M}$$

q.e.d.

It's easy to deduce:

(*) $H_3(q)=H_2([0,\cdots,0],[1,\cdots,M],q)=\sum\prod B_i$,

$$B_i=\begin{cases}m+1: X_i=T_i, m=\text{count of }\{\cdots,X_{i-1}\}\in K\\ m: X_i=K_i, m=\text{count of }\{\cdots,X_{i-1}\}\in T\end{cases}$$

(*1) $H_3(q)=H_2(M-q-1)\rightarrow$ 1)

(*2) $H_3(q)=(M-q)H_3(q-1)+(q+1)H_3([0,\cdots,0],[1,\cdots,M-1],q)$ → 2)

(*3) $H_3(q)=\sum\prod B_i$, then $B_1+B_2+\cdots+B_M=q(M-q+1)$

$H_3([1,\cdots,1],[1,\cdots,M],1)$

$=1^{M-1}\times(M-1)+1^{M-2}\times(M-2)\times 2+1^{M-3}\times(M-3)\times 2^2+\cdots$

$=2^0\times M+2^1\times M+\cdots+2^{M-2}\times M-\left(2^0\times 1+2^1\times 2+\cdots+2^{M-2}\times(M-1)\right)$

$=M\left(2^{M-1}-1\right)-\sum_{i=0}^{M-2}2^i(i+1)=M\left(2^{M-1}-1\right)-\left(2^{M-1}-1\right)-\sum_{i=0}^{M-2}2^i i$

$=\left\langle\begin{matrix}M\\1\end{matrix}\right\rangle=2^M-(M+1)$, the final equation is a known property of $\left\langle\begin{matrix}M\\1\end{matrix}\right\rangle$→

9.4) $\sum_{i=0}^{M}2^i i=(M-1)2^{M+1}+2$

9.5) $\left\langle\begin{matrix}M\\q\end{matrix}\right\rangle=\sum_{t_1+\cdots+t_{q+1}=M-q-1}1^{t_1}2^{t_2}\cdots(q+1)^{t_{q+1}}(1+t_1)\cdots(1+t_1+\cdots+t_q),t_i\geq 0$

[Proof]

$H_3([0,\cdots,0],[1,\cdots,M],q+1)=\left\langle\begin{matrix}M\\q\end{matrix}\right\rangle=\sum\Pi(X\in T)\Pi(X\in K)$

If $X\in K$ is certain, then $X\in T$ is certain and $X_1\cdots X_M$ is certain.

When $X_1\cdots X_M>0$,then $X_1=T_1=1$, $K_i\in[1,q+1]$

Record $\Pi(X\in K)=1^{t_1}2^{t_2}\cdots(q+1)^{t_{q+1}}$, $t_1+t_2+\cdots+t_{q+1}=M-q-1$

Take out factors > 0, record as $\{P_1,P_2,\cdots\}$,(*)→

$\Pi(X\in T)=1^{P_1}(1+t_{p1})^{P_2-P_1}(1+t_{p1}+t_{p2})^{P_3-P_2}\cdots$, it can be rewritten as

$\Pi(X\in T)=(1+t_1)(1+t_1+t_2)(1+t_1+t_2+t_3)\cdots(1+t_1+\cdots+t_q)$

q.e.d.

This is the conclusion of [7], which is obtained by guess and proved by induction.

Example 9.1:

$\left\langle\begin{matrix}6\\2\end{matrix}\right\rangle=\sum_{t_1+t_2+t_3=3}1^{t_1}2^{t_2}3^{t_3}(1+t_1)(1+t_1+t_2)$

$=1^3 2^0 3^0(1+3)(1+3+0)+1^2 2^1 3^0(1+2)(1+2+1)+1^2 2^0 3^1(1+2)(1+2+0)$

$+1^1 2^1 3^1(1+1)(1+1+1)+1^1 2^2 3^0(1+1)(1+1+2)+1^1 2^0 3^2(1+1)(1+1+0)$

$+1^0 2^1 3^2(1+0)(1+0+1)+1^0 2^2 3^1(1+0)(1+0+2)$

$+1^0 2^3 3^0(1+0)(1+0+3)+1^0 2^0 3^3(1+0)(1+0+0)$

$=302$

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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