On the approximation of the integer least-squares success rate: which lower or upper bound to use?

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Abstract. The probability of correct integer estimation, the success rate, is an important measure when the goal is fast and high precision positioning with a Global Navigation Satellite System. Integer ambiguity estimation is the process of mapping the least-squares ambiguity estimates, referred to as the float ambiguities, to an integer value. It is namely known that the carrier phase ambiguities are integer-valued, and it is only after resolution of these parameters that the carrier phase observations start to behave as very precise pseudorange measurements.

The success rate equals the integral of the probability density function of the float ambiguities over the pull-in region centered at the true integer, which is the region in which all real values are mapped to this integer. The success rate can thus be computed without actual data and is very valuable as an a priori decision parameter whether successful ambiguity resolution is feasible or not.

The pull-in region is determined by the integer estimator that is used and therefore the success rate also depends on the choice of the integer estimator. It is known that the integer least-squares estimator results in the maximum success rate. Unfortunately, it is very complex to evaluate the integral when integer least-squares is applied. Therefore, approximations have to be used. In practice, for example, the success rate of integer bootstrapping is often used as a lower bound. But more approximations have been proposed which are known to be either a lower or upper bound of the actual integer least-squares success rate.

In this contribution an overview of the most important lower and upper bounds will be given. These bounds are compared theoretically as well as based on their performance. The performance is evaluated using simulations, since it is then possible to compute the 'actual' success rate. Simulations are carried out for the two-dimensional case, since its simplicity makes evaluation easy, but also for the higher-dimensional geometry-based case, since this gives an insight to the performance that can be expected in practice.

Keywords: ambiguity resolution, success rate, integer least-squares.

1 Introduction

Fast and high precision positioning with a Global Navigation Satellite System is only feasible when the very precise carrier phase observations can be used. Unfortunately, these observations are ambiguous by an unknown, integer number of cycles. These integer ambiguity parameters need to be resolved, before the carrier phase observations start to behave as very precise pseudorange measurements.

The procedure to solve the GNSS model is to first apply a standard least-squares adjustment so that a real-valued *float* solution is obtained. The next step is then to map the real-valued ambiguity estimates to integer values. Several integer estimators can be used for that purpose: integer rounding, bootstrapping (conditional rounding), or integer least-squares (ILS). The optimal choice is the latter, since this estimator maximizes the probability of correct integer estimation as was proven in Teunissen (1999). The last step is to correct the remaining real-valued parameters, such as the baseline parameters, by virtue of their correlation with the ambiguities, and then the so-called *fixed* solution is obtained. For that purpose, it is assumed that the integer-valued ambiguity estimates are deterministic. However, this is actually not the case and this assumption can only be made for a very high probability of correct integer estimation, i.e. success rate.

The success rate is thus a very important measure in order to decide whether or not an attempt should be made to fix

the ambiguities. The integer ambiguities can only be considered deterministic when the success rate is very close to one, and then evaluation by means of discernibility tests is possible.

The success rate equals the integral of the probability density function of the float ambiguities over the pull-in region, which is the region in which all real values are mapped to the same integer. The pull-in region is determined by the integer estimator that is used, and therefore the success rate also depends on the choice of the integer estimator. Unfortunately, it is very complex to evaluate the integral when integer least-squares is applied. Therefore, a number of approximations have been proposed. For example, the success rate of integer bootstrapping is often used, and is introduced as a lower bound in Teunissen (1999). In Teunissen (1998a) lower and upper bounds were obtained by bounding the region of integration. Another upper bound was given in Teunissen (2000) based on the Ambiguity Dilution of Precision. A final interesting lower bound is derived in Kondo (2003). These approximations of the ILS success rate are considered in this paper due to their improved performance. Note that in Thomsen (2000) some lower and upper bounds for the ILS success rate are evaluated. However, the evaluation is based only on two-dimensional examples, which showed that the bootstrapped lower bound, and the ADOP based upper bound performed very well.

This paper is organized as follows. The problem of integer estimation and the integer least-squares estimator are described in section 2. The lower and upper bounds for the success rate are presented in section 3. An evaluation of these bounds is made based on simulations in section 4.

2 Integer least-squares estimation

The general GNSS observation model can be written in the form:

$$y = Aa + Bb + e, \quad Q_y \tag{1}$$

where y is the random vector with m double difference code and phase observations, a the n-vector with unknown integer carrier phase ambiguities, i.e. $a \in \mathbb{Z}^n$, b is a p-vector with the unknown real-valued parameters, and e is the noise vector. The real-valued parameters are referred to as the baseline unknowns, although b may also contain for example atmospheric delays. The variance-covariance (vc-) matrix of the observation vector is given by Q_y.

Optimizing on the integer nature of the ambiguity parameters, (cf. Teunissen, 1999), involves solving a non-standard least-squares problem, referred to as *integer least-squares* (Teunissen, 1993). The solution of model (1) is then ob-

tained by the following minimization problem:

$$\min_{a,b}\|y - Aa - Bb\|_{Q_y}^2, \quad a \in \mathbb{Z}^n, b \in \mathbb{R}^n \tag{2}$$

where $\|\cdot\|_Q^2 = (\cdot)^T Q^{-1}(\cdot)$.

The following orthogonal decomposition can be used:

$$\|y - Aa - Bb\|_{Q_y}^2 =$$
$$\|\hat{e}\|_{Q_y}^2 + \|\hat{a} - a\|_{Q_{\hat{a}}}^2 + \|\hat{b}(a) - b\|_{Q_{\hat{b}|\hat{a}}}^2 \tag{3}$$

with the residual estimator $\hat{e} = y - A\hat{a} - B\hat{b}$, the conditional baseline estimator $\hat{b}(a) = \hat{b} - Q_{\hat{b}\hat{a}}Q_{\hat{a}}^{-1}(\hat{a} - a)$, and corresponding vc-matrix $Q_{\hat{b}|\hat{a}} = Q_{\hat{b}} - Q_{\hat{b}\hat{a}}Q_{\hat{a}}^{-1}Q_{\hat{a}\hat{b}}$.

It follows from eq.(3) that the solution of the minimization problem in eq.(2) is obtained using a three step procedure. The unconstrained least-squares solution is referred to as the float solution, with estimators $\hat{a}$ and $\hat{b}$, and residual vector $\hat{e}$. Taking into account the integer nature of the ambiguities means that the second term on the right-hand side of eq.(3) needs to be minimized and the last term is set to zero. This is the integer estimation step, providing the fixed ambiguities $\check{a}$:

$$\check{a} = \arg \min_{z \in \mathbb{Z}^n} \|\hat{a} - z\|_{Q_{\hat{a}}}^2 \tag{4}$$

Finally, solving for the last term in eq.(3) corresponds to fixing the baseline, $\check{b} = \hat{b} - Q_{\hat{b}\hat{a}}Q_{\hat{a}}^{-1}(\hat{a} - \check{a})$.

The integer estimation step involves a mapping from the n-dimensional space of reals to the n-dimensional space of integers. In the integer least-squares approach a subset $S_z \subset \mathbb{R}^n$ is assigned to each integer vector $z \in \mathbb{Z}^n$. This subset is called the pull-in region and is defined as the collection of all $x \in \mathbb{R}^n$ that are closer to z than to any other integer grid point in $\mathbb{R}^n$, where the distance is measured in the metric of $Q_{\hat{a}}$. The pull-in region that belongs to the integer a follows thus as:

$$S_a = \left\{ \hat{a} \in \mathbb{R}^n \mid \|\hat{a} - a\|_{Q_{\hat{a}}}^2 \leq \|\hat{a} - z\|_{Q_{\hat{a}}}^2, \forall z \in \mathbb{Z}^n \right\} \tag{5}$$

Since

$$\|\hat{a} - a\|_{Q_{\hat{a}}}^2 \leq \|\hat{a} - z\|_{Q_{\hat{a}}}^2$$
$$\iff (z - a)^T Q_{\hat{a}}^{-1}(\hat{a} - a) \leq \frac{1}{2}\|z - a\|_{Q_{\hat{a}}}^2, \forall z \in \mathbb{Z}^n$$

it follows that

$$S_a = \{\hat{a} \in \mathbb{R}^n \mid |w| \leq \frac{1}{2}\|c\|_{Q_{\hat{a}}}, \forall c \in \mathbb{Z}^n\} \tag{6}$$

with

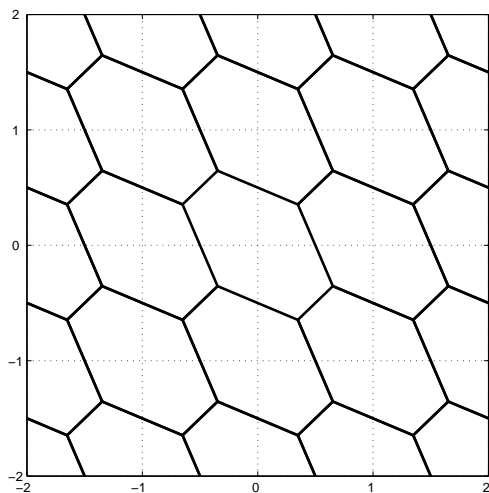$$w = \frac{c^T Q_{\hat{a}}^{-1}(\hat{a} - a)}{\|c\|_{Q_{\hat{a}}}} \tag{7}$$

Fig. 1 Example of the 2-D integer least-squares pull-in regions.

An example of the two-dimensional pull-in regions is shown in Fig. 1.

In order to arrive at the integer least-squares solution, an integer search is required. The ILS procedure is mechanized in the LAMBDA (Least-Squares AMBiguity Decorrelation Adjustment) method (see Teunissen, 1993, 1995; De Jonge and Tiberius, 1996). The search space is then defined as an n-dimensional ellipsoid centered at $\hat{a}$, its shape is governed by the vc-matrix $Q_{\hat{a}}$. Due to the high correlation between the individual ambiguities, the search space in the case of GNSS is extremely elongated, so that the search for the integer solution may take very long. Therefore a very important step is to first transform the search space to a more spherical shape by means of a decorrelation of the original float ambiguities. This decorrelation is attained by a transformation:

$$\hat{z} = Z^T \hat{a}, \quad Q_{\hat{z}} = Z^T Q_{\hat{a}} Z \tag{8}$$

This transformation needs to be admissible, which is said to be the case when both Z and its inverse have integer entries, so that the integer nature of the ambiguities is preserved. The determinant of Z is then equal to ± 1, so that the Z-transformation is volume-preserving with respect to the search space.

3 Lower and upper bounds of the ILS success rate

The probability of correct integer estimation in the case of integer least-squares equals:

$$P_s = P(\hat{a} = a) = \int_{S_a} f_{\hat{a}}(x) dx \tag{9}$$

with S_a given in (6), and $f_{\hat{a}}(x)$ the probability density function of the float ambiguities. In general it is assumed

that the float ambiguities are normally distributed. The pull-in region is unfortunately a very complex region, so that in practice approximations have to be used for the integer least-squares success rate. This section gives an overview of the most important lower and upper bounds that are available.

3.1 Lower bound based on bootstrapping

Integer bootstrapping means that the float ambiguities are conditionally rounded to the nearest integers. One starts with the most precise ambiguity, then corrects all other ambiguities by virtue of their correlation with this one, and continues with rounding the second ambiguity. This process is repeated until all n ambiguities are fixed. Integer bootstrapping is a very simple method of ambiguity resolution, and has a close to optimal performance after decorrelation of the ambiguities using the Z-transformation of the LAMBDA method. In Teunissen (1998b, 1999) it was shown that the integer least-squares estimator is optimal in the sense that the success rate is maximized, and it was proposed to use the success rate of integer bootstrapping therefore as a lower bound for the ILS success rate, since in Teunissen (1998c) it is shown that exact and easy computation of this bootstrapped success rate is possible:

$$P_s \geq P_{s,B} = \prod_{i=1}^{n} \left(2\Phi \left(\frac{1}{2\sigma_{i|I}} \right) - 1 \right) \tag{10}$$

where $\sigma_{i|I}$ the standard deviation of the ith ambiguity obtained through a conditioning on the previous $I = 1, \ldots, (i-1)$ ambiguities. And

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}v^2 dv\}$$

3.2 Lower and upper bounds based on bounding the integration region

In Teunissen (1998a) lower and upper bounds for the ILS success rate were obtained by bounding the integration region. Obviously, a lower bound is obtained if the integration region is chosen such that it is completely contained by the pull-in region, and an upper bound is obtained if the integration region is chosen such that it completely contains the pull-in region. The integration region can then be chosen such that the integral is easy-to-evaluate. In ibid the integration region for the lower bound is chosen as an ellipsoidal region $E_a \subset S_a$.

The upper bound can thus be obtained by defining a region $U_a \supset S_a$, with S_a as defined in (6). Note that the w in this expression can be geometrically interpreted as the orthogonal projection of $(\hat{a} - a)$ onto the direction vector c.

Hence, S_a is the intersection of *banded* subsets centered at a and having a width $\|c\|_{Q_{\hat{a}}}$. Any *finite* intersection of these banded subsets encloses S_a, and therefore the subset U_a could be chosen as

$$U_a = \{\hat{a} \in \mathbb{R}^n \mid |w_i| \le \frac{1}{2}\|c_i\|_{Q_{\hat{a}}}, \, i = 1, \ldots, p\} \supset S_a \tag{11}$$

with

$$w_i \sim N(0, 1)$$

The choice for p is still open, but a larger p will result in a sharper upper bound for the success rate. However, when $p > 1$ the w_i are correlated. This is handled by defining a p-vector v as:

$$v = (v_1, \ldots, v_p)^T \quad \text{with} \quad v_i = \frac{w_i}{\|c_i\|_{Q_{\hat{a}}}}$$

Then $U_a = \{\hat{a} \in \mathbb{R}^n \mid \overset{p}{\underset{i=1}{\cap}} |v_i| \le \frac{1}{2}\}$. The probability $P(\hat{a} \in U_a)$ equals therefore the probability that component-wise rounding of the vector v produces the zero vector. This means that $P(\hat{a} \in U_a)$ is bounded from above by the probability that conditional rounding, (cf. Teunissen, 1998c), produces the zero vector, i.e.:

$$P_s \le P(\hat{a} \in U_a) \le \prod_{i=1}^{p}\left[2\Phi(\frac{1}{2\sigma_{v_{i|I}}}) - 1\right] \tag{12}$$

with $\sigma_{v_{i|I}}$ the conditional standard deviation of v_i. The conditional standard deviations are equal to the diagonal entries of the matrix D from the LDL^T-decomposition of the vc-matrix of v. The elements of this vc-matrix are given as:

$$\sigma_{v_i v_j} = \frac{c_i^T Q_{\hat{a}}^{-1} c_j}{\|c_i\|^2_{Q_{\hat{a}}}\|c_j\|^2_{Q_{\hat{a}}}}$$

In order to avoid the conditional standard deviations becoming zero, the vc-matrix of v must be of full rank, and thus the vectors c_i, $i = 1, \ldots, p \le n$ need to be linearly independent.

The procedure for computation of this upper bound is as follows. LAMBDA is used to find the $q >> n$ closest integers $c_i \in \mathbb{Z}^n \setminus \{0\}$ for $\hat{a} = 0$. These q integer vectors are ordered by increasing distance to the zero vector, measured in the metric $Q_{\hat{a}}$. Start with $C = c_1$, so that $rank(C) = 1$. Then find the first candidate c_j for which $rank([c_1 \, c_j]) = 2$. Continue with $C = [c_1 \, c_j]$ and find the next candidate that results in an increase in rank. Continue this process until $rank(C) = n$.

In Kondo (2003) instead of the conditional variances, simply the variances of the v_i are used. Then the following is obtained:

$$\prod_{i=1}^{p}\left[2\Phi(\frac{1}{2\sigma_{v_i}}) - 1\right] = \prod_{i=1}^{p} P_{s,i} \tag{13}$$

with the $P_{s,i}$

$$P_{s,i} = \frac{2}{\sqrt{2\pi}\sigma_{v_i}} \int_{0}^{\frac{1}{2}} \exp\left\{-\frac{1}{2}\frac{x^2}{\sigma_{v_i}^2}\right\} dx \tag{14}$$

We know, (cf. Teunissen, 1998c), that

$$\prod_{i=1}^{p}\left[2\Phi(\frac{1}{2\sigma_{v_i}}) - 1\right] \le P(\hat{a} \in U_a) \tag{15}$$

This means that it is only guaranteed that Kondo's approximation of the success rate is a lower bound if $P(\hat{a} \in U_a)$ is equal to the success rate. This will be the case if p is chosen equal to half the number of facets that bound the ILS pull-in region. So, it is required to know this number, but in practice only the bounds are known:

$$n \le p \le 2^n - 1$$

If p is chosen to be smaller than half the number of bounding facets, it is not guaranteed that the approximation gives a lower bound. On the other hand, if p is chosen to be larger than required in order to guarantee that $P(\hat{a} \in U_a) = P(\hat{a} \in S_a)$, the lower bound is less strict since it is defined as a product of probabilities which are all smaller or equal to one. Note that p may become very large when many satellites are visible. For instance, with 6 visible satellites and two frequencies available, the number of unknown ambiguities for one epoch is $n = 10$, and $p \le 2^n - 1 = 1023$.

It is possible to find *all* adjacent integers, but it is computationally demanding. First, note that it is not always the case that the $2p$ adjacent integers are also the $2p$ closest integers. Therefore, a large set of integers c_i must be selected, in the same way as for the computation of the upper bound described above with $q >> 2(2^n - 1)$. For each integer in this set it must be checked if it is adjacent, which is the case if $\frac{1}{2}c_i$ lies on the boundary of both the pull-in regions S_0 and S_{c_i}. This is the case if:

$$\|\frac{1}{2}c_i - 0\|^2_{Q_{\hat{a}}} = \|\frac{1}{2}c_i - c_i\|^2_{Q_{\hat{a}}} = \min_{z\in\mathbb{Z}^n}\|\frac{1}{2}c_i - z\|^2_{Q_{\hat{a}}} \tag{16}$$

Note that if c_j is selected as adjacent integer, $-c_j$ must not be included in the set $C = [c_1 \ldots, c_p]$ adjacent integers that is used to compute the lower bound.

3.3 Upper bound based on ADOP

The Ambiguity Dilution of Precision (ADOP) is defined as a diagnostic that tries to capture the main characteristics of

Table 1 Two-dimensional example. Mean and maximum difference between success rate based on simulations and the lower and upper bounds. The success rate for which the maximum difference obtained is given in the last row.

	LB bootstr.	LB region	UB ADOP	UB region
mean difference	0.0045	0.0180	0.0012	0.0181
maximum difference	0.0104	0.1052	0.0029	0.0648
success rate	0.8046	0.3876	0.8331	0.5589

the ambiguity precision. It is given as:

$$ADOP = \sqrt{|Q_{\hat{a}}|^{\frac{1}{n}}} \qquad (17)$$

and has units of *cycles*. It is introduced in Teunissen (1997), and described and analyzed in Teunissen and Odijk (1997). The ADOP measure has some desirable properties. First, it is invariant for the class of admissible ambiguity transformation, e.g. ADOP is independent of the chosen reference satellite in the double difference approach, and ADOP will not change after the decorrelating Z-transformation of the ambiguities. When the ambiguities are completely decorrelated, the ADOP equals the geometric mean of the standard deviations of the ambiguities, hence it can be considered as a measure of the ambiguity precision.

In Teunissen (2000) it is proven that an upper bound for the ILS success rate based on the ADOP can be given as:

$$P_s \leq P\left(\chi^2(n,0) \leq \frac{c_n}{ADOP^2}\right) \qquad (18)$$

with

$$c_n = \frac{\left(\frac{n}{2}\Gamma(\frac{n}{2})\right)^{\frac{2}{n}}}{\pi}$$

This upper bound is identical to the one presented in Hassibi and Boyd (1998).

4   Evaluation of the bounds

In order to evaluate the lower and upper bounds as presented in section 3, simulations are used. In section 3.2 the lower bound based on bounding the integration region with an ellipsoidal region $E_a \subset S_a$ was briefly outlined. This bound is not included in the results presented here, since for all examples this lower bound performed badly.

The procedure is as follows. Since it is assumed that the float solution is normally distributed, the probabilities are independent of the mean, so one can use $N(0, Q)$ and draw samples from this distribution.

The first step is to use a random generator to generate n independent samples from the univariate standard normal distribution $N(0, 1)$, and then collect these in a vector s. This vector is transformed by means of $\hat{a} = Gs$, with G

equal to the Cholesky factor of $Q_{\hat{a}} = GG^T$. The result is a sample $\hat{a}$ from $N(0, Q_{\hat{a}})$, and this sample is used as input for integer least-squares estimation. If the output of this estimator equals the null vector, then it is correct, otherwise it is incorrect. This process can be repeated N number of times, and one can count how many times the null vector is obtained as a solution, say N_s times, and how often the outcome equals a nonzero integer vector, say N_f times. The approximations of the success rate and fail rate follow then as:

$$P_s = \frac{N_s}{N}, \quad P_f = \frac{N_f}{N}$$

In order to get good approximations, the number of samples N must be sufficiently large (see Teunissen, 1998a).

We will start here with the simple two-dimensional case. The dual-frequency geometry-free GPS model for a short baseline and for only one satellite-pair is used:

$$E\left\{\begin{pmatrix} p_1 \\ p_2 \\ \phi_1 \\ \phi_2 \end{pmatrix}\right\} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & \lambda_1 & 0 \\ 1 & 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \rho \\ a_1 \\ a_2 \end{pmatrix} \qquad (19)$$

where p_i and ϕ_i are the double difference (DD) code and phase observations on frequency L_i. Wavelengths are denoted as λ_i, the range as ρ, and the integer ambiguities as a_i. $E\{\cdot\}$ is the expectation operator. The variance-covariance matrix Q_y is chosen as a diagonal matrix, with undifferenced standard deviations of $\sigma_p = 15$ cm and $\sigma_\phi = 1.5$ mm for both frequencies. For the simulation 1,000,000 samples were used. The resulting lower and upper bounds are shown in table 2 (first row).

The same approach was followed by using:

$$Q_{\hat{a}} = \frac{1}{f} Q_{\hat{a},\text{ref}}, \quad 0 < f \leq 1$$

for different values of f, and $Q_{\hat{a},\text{ref}}$ the vc-matrix from the example described above. The results are shown in Fig. 2. The top panels show the two upper bounds and the success rates from the simulations. Obviously, the ADOP-based upper bound is very strict and is always much better than the upper bound based on bounding the integration region. The bottom panels show the lower bounds. It follows that for lower success rates (< 0.93) the bootstrapped success rate is the best lower bound. For higher success rates (right

Journal of Global Positioning Systems

Table 2 Approximated success rates using simulation, the lower bounds based on bootstrapping (LB bootstr.) and bounding the integration region (LB region), and the upper bounds based on ADOP (UB ADOP) and bounding the integration region (UB region). Number of satellites (no. SV) and the ionospheric standard deviation (σ_I) are given in the first columns.

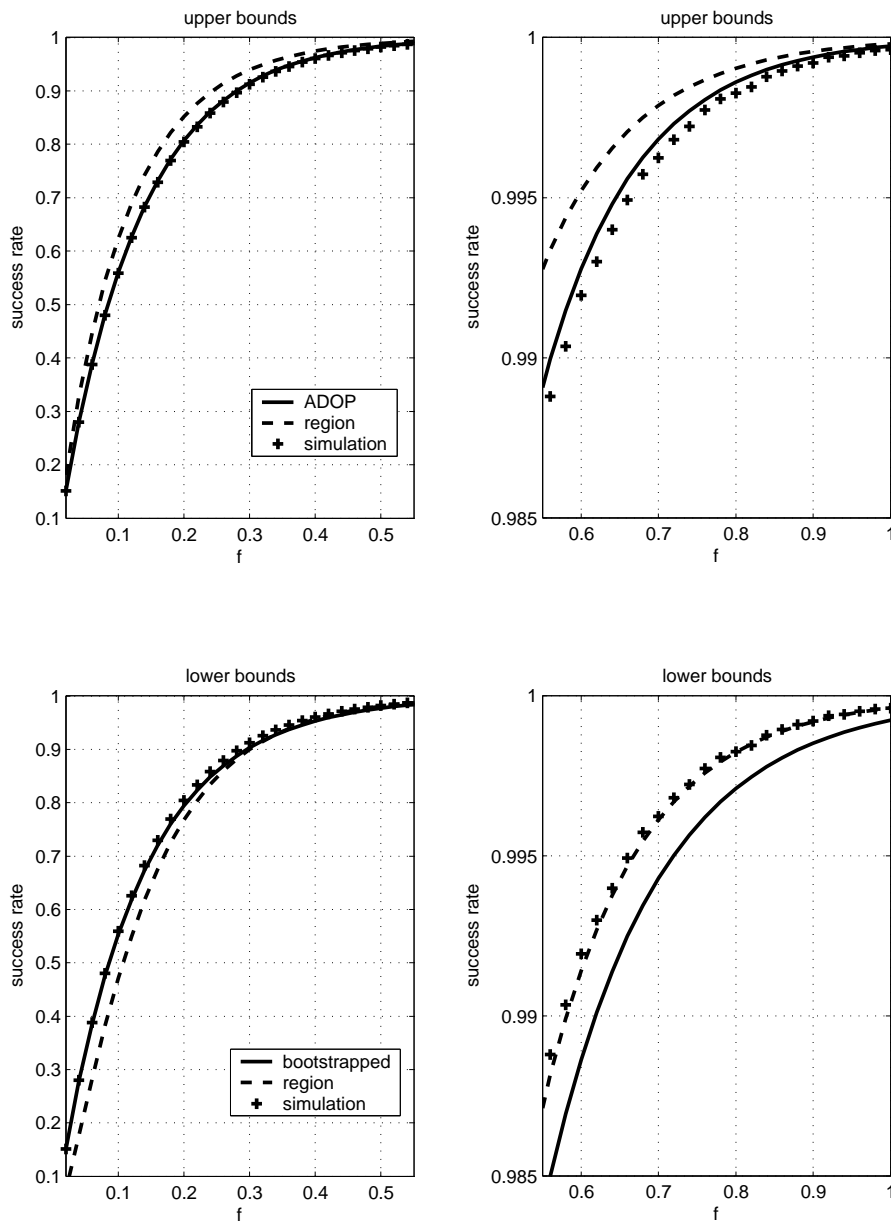| no. SV	σ_I[cm]	simulation	LB bootstr.	LB region	UB ADOP	UB region
2	0	0.9996	0.9992	0.9996	0.9997	0.9998
4	0	0.8175	0.7494	0.6976	0.8480	0.9420
4	1	0.4420	0.4097	0.1177	0.4749	0.6256
5	0	0.9989	0.9979	0.9989	1.0000	0.9990
5	1	0.8744	0.8337	0.8109	0.9470	0.9388
6	1	0.9886	0.9759	0.9881	0.9994	0.9922
6	3	0.4763	0.4416	0.1256	0.6808	0.6608

Fig. 2 Upper and lower bounds for the success rate in the 2-D case as function of f with vc-matrix $\frac{1}{f}Q_{\hat{a},\mathrm{ref}}$. Left: whole range of success rates; Right: only for high success rates.

Table 3 Overview of lower and upper bounds for the ILS success rate considered in this contribution.

bound	based on	references
$P_s \geq \prod_{i=1}^{n} \left(2\Phi\left(\frac{1}{2\sigma_{i\|I}}\right) - 1\right)$	bootstrapping	Teunissen (1998b, 1998c, 1999)
$P_s \geq \prod_{i=1}^{p} \left[2\Phi(\frac{1}{2\sigma_{v_i}}) - 1\right]$	bounding integration region	Kondo (2003)
$P_s \leq P\left(\chi^2(n,0) \leq \frac{c_n}{ADOP^2}\right)$	ADOP	Hassibi and Boyd (1998); Teunissen (2000)
$P_s \leq \prod_{i=1}^{p} \left[2\Phi(\frac{1}{2\sigma_{v_i\|I}}) - 1\right]$	bounding integration region	Teunissen (1998a)

panel), the lower bound proposed by Kondo works very well and is better than the bootstrapped lower bound. Note that the range of success rates in the right panel is very small.

Table 1 shows the maximum and mean differences of the lower and upper bounds with the success rate from simulation. From these differences it follows that the bootstrapped lower bound and the ADOP-based upper bound are best.

Because of its simplicity the geometry-free model is very suitable for a first evaluation, though it is of course more useful to know how well the approximations work in practice. Therefore, simulations were carried for several geometry-based models. The GPS constellation was based on the Yuma almanac for GPS week 184 and a cut-off elevation of 15°. Undifferenced standard deviations of $\sigma_p = 30$cm and $\sigma_\phi = 3$mm were used for both frequencies. The GPS model was set up for a single epoch for three different times, for which 4, 5 and 6 satellites were visible respectively. A short to medium baseline length was chosen by varying the ionospheric standard deviation σ_I. For the simulation 500,000 samples were used. The resulting lower and upper bounds are shown in table 2.

The results show that Kondo's lower bound works very well for a high success rate, but in general the bootstrapped lower bound is much better. It is difficult to say which upper bound is best. For the examples with only four visible satellites the ADOP-based upper bound is better than the one obtained by bounding the integration region, but in the examples with more satellites the latter is somewhat better. All bounds are best in the case of high precisions, i.e. high success rates.

5 Concluding remarks

In this contribution two lower bounds and two upper bounds for the integer least-squares success rate were presented. An overview of the bounds is given in table 3.

The performance of the different bounds was evaluated by comparing their outcomes for several geometry-free and geometry-based examples with the success rate that is obtained by using simulation.

In general, the bootstrapped lower bound gives the best results. When the success rate is high, the lower bound proposed by Kondo (2003) based on bounding the integration region may work better.

It can be concluded that Kondo's lower bound seems to be useful only in a few cases. Firstly, to obtain a strict lower bound the precision should be high, so that the success rate is high. Even then, it depends on the minimum required success rate whether it is really necessary to use the approximation: if the bootstrapped success rate is somewhat lower than this minimum required success rate, Kondo's approximation can be used to see if it is larger. The minimum required success rate could be chosen such that the fixed ambiguities can be considered deterministic. In this case, the discernibility tests as used in practice, such as the ratio test, can be used.

An advantage of the bootstrapped success rate is that it is very easy to compute, since the conditional variances are already available when using the LAMBDA method. The computation of Kondo's lower bound may be slightly more complex, since for high-dimensional problems the number of facets that bound the pull-in region can be very large, and this number needs to be known in order to guarantee that a (strict) lower bound is obtained.

With respect to the upper bounds, one can have a little more confidence in the ADOP-based bound, since its overall performance, based on all examples, is slightly better. However, in the geometry-based case, the upper bound based on bounding the integration region often performs somewhat better. An advantage of the ADOP-based upper bound is that it is easy to compute, whereas for the upper bound based on bounding the integration region one has the problem of determining the n closest independent integers to the zero vector.

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