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Limits to Public Debt in a Diamond-Type OLG Model of Involuntary Unemployment under Inflexible Aggregate Investment

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Abstract

Governments in advanced countries are currently striving to combat the disastrous economic effects of the shortage of energy supply by providing generous public subsidies to households and firms. As a result, the deficits of federal governments are further increasing after the explosion of public debts due to SARS-CoV-2 related government expenditures and collapsing tax revenues. Unemployment rates in many advanced countries while receding-due to extremely expansionary fiscal and monetary stances remain significant. Thus, the question arises as to whether, in the face of involuntary unemployment, limits to public debt can and/or ought to be respected, or simply disregarded. It is the aim of this research to answer this question within the scope of a Diamond-type overlapping generations (OLG) model of involuntary unemployment under inflexible aggregated investment. It is found that limits to public debt to output ratios exist; and their numerical values are calculated. Moreover, a debt threshold pops up whereby larger public debt diminishes output growth. In fact, the numerical value of the debt threshold is found to be close to World Bank estimates.

Keywords

Limits to Public Debt, Diamond-Type OLG Model, Involuntary Unemployment, Inflexible Aggregate Investment

1. Introduction

The concept of involuntary unemployment figures prominently in Keynes (1936)'s famous "General Theory of Employment, Interest, and Money". As is well-known, involuntary unemployment is traced back to a lack of aggregate demand (aggregate demand failures). However, even after several decades lasting discussion among macro-economists, there is still no consensus among mainstream economists concerning the reasons for aggregate demand remaining below full-employment output in a perfectly functioning market economy. Up to now, mainstream macro-economists adhere to the New Keynesian approach of micro-founded, dynamic general equilibrium models. Here, prices and wages are thought to adapt sluggishly to market imbalances due to imperfect competition and other forms of market failure (see Taylor (1979, 1980); Mankiw (1985), Akerlof & Yellen (1985); Blanchard & Kiyotaki (1987); Ball & Romer (1990); for a survey see Dixon (2000)).

A minority of macro-oriented general equilibrium modelers employ dynamic, intertemporal general equilibrium with perfect competition in factor and output markets to feature involuntary unemployment. Implicitly based on Morishima (1977)'s seminal insight that involuntary unemployment under perfect competition in output and factor markets may be traced back to the existence of an aggregate investment function independent of aggregate savings, Magnani (2015) modeled involuntary unemployment in a Solow (1956)-type neoclassical growth model. Following the lead of Magnani (2015), Farmer & Kuplen (2018) modeled growth and involuntary unemployment in a Diamond-type OLG economy with production, physical and human capital accumulation in which steady-state growth becomes endogenous. Human capital accumulation is modeled à la Glomm & Ravikumar (1992) and Lin (2000).

Use is made of this intertemporal general equilibrium model to address current macro-economic policy challenges in advanced countries. As is well-known, their governments are currently striving to combat the disastrous economic effects of the shortage of energy supply by providing generous public subsidies to households and firms. As a result, the deficits of federal governments are further increasing after the explosion of public debts due to SARS-CoV-2 related government expenditures and collapsing tax reveneues. Unemployment rates in many advanced countries while reciding due to extremely expansionary fiscal and monetary stances remain significant. Thus, the question arises as to whether, in the face of involuntary unemployment, limits to public debt can and/or ought to be respected, or simply disregarded.

In order to be able to answer this question analytically and by providing numerical values to debt limits we use a log-linear utility and Cobb-Douglas production function version of Diamond's OLG model with internal public debt extended by human capital accumulation and modified when aggregate investment demand is governed by a savings-independent investment function.[1] It is found that even in face of involuntary unemployment finite limits to the debt to output ratio exists. Moreover, considering crowding out effects of private and public savings on aggregate investment in line with Magnani (2015) thresholds to the public debt to output ratio are detected beyond which output growth de-

[1]To some extent our basic model can be considered as complementary to Tanaka (2020)'s three-period OLG model of involuntary unemployment without real capital and investment.

clines with a larger debt to output ratio. In fact, a numerical threshold pops up which accords rather well with empirical estimates made by the World Bank. While these model-based results clearly represent progress in intertemporal general equilibrium modeling of involuntary unemployment, establishing or identifying the micro-foundations of the aggregate investment function remains challenging.

The structure of the paper is as follows. The next section presents the model setup. This is followed by derivation of the intertemporal equilibrium dynamics and demonstration of the existence and dynamic stability of steady states. We then investigate the existence of limits to the public debt to GDP ratio and of thresholds to this ratio beyond which higher debt decreases GDP growth. Some conclusions are drawn in the final section of the paper.

2. The Set-Up of the Basic Model

As in de la Croix & Michel (2002), we consider an economy of infinite horizon which is composed of infinitely lived firms, finitely lived households, and an infinitely lived government. In each period $t = 0,1,2,\cdots$ a new generation, called generation *t*, enters the economy. A continuum of $L_t > 0$ units of identical agents comprise generation *t*.

As mentioned above, to be able to address the question of how fiscal policy impacts long-run growth we extend Diamond (1965)'s basic OLG model by introducing human capital accumulation. In order to point out the growth enhancing effects of human capital accumulation most clearly, it is assumed here that there is no population growth g^L, *i.e.*, $g^L = 0 \Leftrightarrow G^L \equiv 1 + g^L = 1$, and no exogenous growth in labor efficiency denoted as g^a, *i.e.*, $g^a = 0 \Leftrightarrow G^a \equiv 1 + g^a = 1$. As a result of the first assumption, the number of households, L_t, remains constant over time: $L_t = L_{t-1} = L$.

Each household consists of one agent and the agent acts intergenerationally egoistic: The old agent does not take care of the young agent and the young agent does not take care of the old agent. They live two periods long, namely youth (adult) and old age. In youth age, each household starts with human capital h_t, accumulated by the household in period $t-1$. Individual human capital is inelastically supplied to firms which remunerate the real wage rate w_t in exchange for the labor supply. The former denotes the units of the produced good per efficiency unit of labor.

In contradistinction to the original Diamond (1965) OLG model, not the total labor supply is employed but only $(1-u_t)L_t$, where $0 \le u_t < 1$ denotes the unemployment rate. The number of unemployed people is thus $u_t L_t$.

The government collects taxes on wages, quoted as a fixed proportion of wage income, $\tau_t w_t h_t, 0 < \tau_t < 1$. The unemployed do not pay any taxes. Young, employed agents, denoted by superscript *E*, split the net wage income $(1-\tau_t) w_t h_t$ each period between current consumption $c_t^{1,E}$ and savings s_t^E. Savings of the employed are invested in real capital in period *t* per employed capita, $I_t^D / L_t (1-u_t)$, which is demanded by employed households in youth, and in real government bonds per employed capita, $B^D_{t+1}/L_t(1-u_t)$, which is also demanded by employed households in youth. For simplicity, we assume a depreciation rate of one with respect to real capital.

In old age, the employed household supplies inelastically $K^S_{t+1}/L_t(1-u_t)=I^D_t/L_t(1-u_t)$ to firms, and $B^S_{t+1}/L_t(1-u_t)=B^D_{t+1}/L_t(1-u_t)$ to young households in period $t+1$. Thus, the per capita savings of employed people are invested as follows: $s^E_t=K^S_{t+1}/L_t(1-u_t)+B^S_{t+1}/L_t(1-u_t)$. Similarly, the per capita savings of the unemployed household are invested as follows: $s^U_t=K^S_{t+1}/L_tu_t+B^S_{t+1}/L_tu_t$. In old age, both employed and unemployed households consume their gross return on assets: $c^{2,E}_{t+1}=q_{t+1}K^S_{t+1}/L_t(1-u_t)+(1+i_{t+1})B^S_{t+1}/L_t(1-u_t)$, respectively $c^{2,U}_{t+1}=q_{t+1}K^S_{t+1}/L_tu_t+(1+i_{t+1})B^S_{t+1}/L_tu_t$, where $c^{2,E}_{t+1}$, and $c^{2,U}_{t+1}$, represent consumption of the employed, respectively unemployed, in old age. q_{t+1} denotes the gross rental rate on real capital, and i_{t+1} denotes the real interest rate on government bonds in period $t+1$. To remain as simple as possible, we assume that rental and interest income are not taxed.

A log-linear intertemporal utility function slightly generalized in comparison to Diamond (1965: p. 1134)'s leading example represents the intertemporal preferences of all two-period lived households. As usual, this simple specification aims at closed-form solutions for the intertemporal equilibrium dynamics (see e.g., de la Croix & Michel (2002: pp. 181-184)).

The typical younger, employed household maximizes the following intertemporal utility function subject to the budget constraints of the active period (i) and the retirement period (ii):

$$\text{Max} \to \varepsilon \ln c^{1,E}_t + \beta \ln c^{2,E}_{t+1}$$

subject to:

i) $c^{1,E}_t + I^{D,E}_t/L_t(1-u_t)+B^{D,E}_{t+1}/L_t(1-u_t)=w_th_t(1-\tau_t)$,

ii) $c^{2,E}_{t+1}=q_{t+1}K^{S,E}_{t+1}/L_t(1-u_t)+(1+i_{t+1})B^{S,E}_{t+1}/L_t(1-u_t)$,
$K^{S,E}_{t+1}=I^{D,E}_t$, $B^{S,E}_{t+1}=B^{D,E}_{t+1}$.

Here, $0<\varepsilon\le 1$ depicts the utility elasticity of employed household's consumption in youth, while $0<\beta<1$ denotes the subjective future utility discount factor. For the log-linear utility function above, a unique, interior solution of the optimization problem exists. Hence, one may solve the old-age budget constraint for $B^{S,E}_{t+1}/L_t(1-u_t)$ and insert the result into the young-age, employed budget constraint of (i), and thus obtain:

$$c^{1,E}_t + c^{2,E}_{t+1}/(1+i_{t+1}) + \left[1-q_{t+1}/(1+i_{t+1})\right]K^{S,E}_{t+1}/L_t(1-u_t)=w_th_t(1-\tau_t). \quad (1)$$

Obviously, a strictly positive and finite solution to maximizing the intertemporal utility function subject to Constraint (1) requires that the following no-arbitrage condition holds:

$$q_{t+1}=1+i_{t+1}. \quad (2)$$

The no-arbitrage Condition (2) implies that $K^{S,E}_{t+1}/L_t(1-u_t)$ is optimally indeterminate, and the first-order conditions for a maximum solution read as follows:

$$c_t^{1,E} + c_{t+1}^{2,E} / \left(1 + i_{t+1}\right) = \left(1 - \tau_t\right) w_t h_t , \tag{3}$$

$$\left(\beta / \varepsilon\right) c_t^{1,E} = c_{t+1}^{2,E} / \left(1 + i_{t+1}\right). \tag{4}$$

Solving Equations (3) and (4) for $c_t^{1,E}$ and $c_{t+1}^{2,E}$ yields the following optimal consumption for employed people in youth and old age:

$$c_t^{1,E} = \left[\varepsilon / \left(\varepsilon + \beta\right)\right]\left(1 - \tau_t\right) w_t h_t , \tag{5}$$

$$c_{t+1}^{2,E} = \left[\beta / \left(\varepsilon + \beta\right)\right]\left(1 + i_{t+1}\right)\left(1 - \tau_t\right) w_t h_t . \tag{6}$$

Since $s_t^E = K_{t+1}^{S,E} / L_t\left(1 - u_t\right) + B_{t+1}^{S,E} / L_t\left(1 - u_t\right)$, we find for the utility maximizing savings:

$$s_t^E = \left[\beta / \left(\varepsilon + \beta\right)\right]\left(1 - \tau_t\right) w_t h_t . \tag{7}$$

The typical younger, unemployed household maximizes the following intertemporal utility function subject to the budget constraints of the active period (i) and the retirement period (ii):

$$\text{Max} \rightarrow \varepsilon \ln c_t^{1,U} + \beta \ln c_{t+1}^{2,U}$$

subject to:

i) $c_t^{1,U} + I_t^{D,U} / L_t u_t + B_{t+1}^{D,U} / L_t u_t = \varsigma_t$,

ii) $c_{t+1}^{2,U} = q_{t+1} K_{t+1}^{S,U} / L_t u_t + \left(1 + i_{t+1}\right) B_{t+1}^{S,U} / L_t u_t$, $K_{t+1}^{S,U} = I_t^{D,U}$, $B_{t+1}^{S,U} = B_{t+1}^{D,U}$.

Again, $0 < \varepsilon \leq 1$ denotes the utility elasticity of consumption in unemployed youth, while $0 < \beta < 1$ depicts the subjective future utility discount factor and ς_t denotes the unemployment benefit per capita unemployed.

As above, the log-linear intertemporal utility function ensures the existence of a unique, interior solution for the above optimization problem. Hence, one may again solve the old-age budget constraint for $B_{t+1}^{S,U} / L_t u_t$ and insert the result into the young-age, unemployed budget constraint of (i), and thus obtain:

$$c_t^{1,U} + c_{t+1}^{2,U} / \left(1 + i_{t+1}\right) + \left[1 - q_{t+1} / \left(1 + i_{t+1}\right)\right] K_{t+1}^{S,U} / L_t u_t = \varsigma_t . \tag{8}$$

The no-arbitrage Condition (2) implies that $K_{t+1}^{S,U} / L_t u_t$ is optimally indeterminate, and the first-order conditions for a maximum solution read as follows:

$$c_t^{1,U} + c_{t+1}^{2,U} / \left(1 + i_{t+1}\right) = \varsigma_t , \tag{9}$$

$$\left(\beta / \varepsilon\right) c_t^{1,U} = c_{t+1}^{2,U} / \left(1 + i_{t+1}\right). \tag{10}$$

Solving Equations (9) and (10) for $c_t^{1,U}$ and $c_{t+1}^{2,U}$ yields the following optimal consumption in youth and old age:

$$c_t^{1,U} = \left[\varepsilon / \left(\varepsilon + \beta\right)\right] \varsigma_t , \tag{11}$$

$$c_{t+1}^{2,U} = \left[\beta / \left(\varepsilon + \beta\right)\right]\left(1 + i_{t+1}\right) \varsigma_t . \tag{12}$$

Since $s_t^U = K_{t+1}^{S,U} / L_t u_t + B_{t+1}^{S,U} / L_t u_t$, we find for the utility maximizing savings:

$$s_t^U = \left[\beta / \left(\varepsilon + \beta\right)\right] \varsigma_t . \tag{13}$$

All firms are endowed with an identical (linear-homogeneous) Cobb-Douglas production function which reads as follows:

$$Y_t = M\left(h_t N_t\right)^{1-\alpha}\left(K_t\right)^{\alpha},\ 0<\alpha<1, M>0. \tag{14}$$

Here, Y_t denotes aggregate output or gross domestic product (GDP), $M>0$ stands for total factor productivity, N_t represents the number of employed laborers, while K_t denotes the input of capital services, all in period *t*, and $1-\alpha$ (α) depicts the production elasticity (= production share) of labor (capital) services.

Maximization of $Y_t - w_t h_t N_t - q_t K_t$ subject to Cobb-Douglas production (14) implies the following first-order conditions:

$$(1-\alpha)M\left[K_t/\left(h_t N_t\right)\right]^{\alpha} = w_t, \tag{15}$$

$$\alpha M\left[K_t/\left(h_t N_t\right)\right]^{(\alpha-1)} = q_t. \tag{16}$$

However, since the number of employed workers is $N_t = L\left(1-u_t\right)$, we can rewrite the profit maximization Conditions (15) and (16) as follows:

$$(1-\alpha)M\left[K_t/\left(h_t L\left(1-u_t\right)\right)\right]^{\alpha} = w_t, \tag{17}$$

$$\alpha M\left[K_t/\left(h_t L\left(1-u_t\right)\right)\right]^{(\alpha-1)} = q_t. \tag{18}$$

Finally, the GDP function can be rewritten as follows:

$$Y_t = M\left(h_t L\left(1-u_t\right)\right)^{1-\alpha}\left(K_t\right)^{\alpha}. \tag{19}$$

As in Diamond (1965), the government does not optimize, but is subject to the following constraint period by period:

$$B_{t+1} = \left(1+i_t\right)B_t + \Delta_t + L_t u_t \varsigma_t + \Gamma_t - \tau_t\left(1-u_t\right)w_t h_t L, \tag{20}$$

where B_t denotes the aggregate stock of real public debt at the beginning of period *t*, Γ_t denotes human capital investment (HCI) expenditures, and Δ_t denotes all non-HCI expenditures of the government exclusive of government's unemployment benefits $L_t u_t \varsigma_t$ per period.

In line with Glomm & Ravikumar (1992) human capital in period *t* is determined by human capital of the generation entering the economy in period $t-1$, and by government's HCI spending in period $t-1$, Γ_{t-1}:

$$h_t = H_0\left(h_{t-1}\right)^{\mu}\left(\Gamma_{t-1}/L\right)^{1-\mu},\ H_0 = \underline{H} > 0,\ 0<\mu<1, \tag{21}$$

whereby $\underline{H}$ indicates a level parameter, μ depicts the production elasticity of human capital, and $1-\mu$ denotes the production elasticity of public HCI spending. The macroeconomic version of Equation (21) is obtained by multiplying it on both sides by *L*:

$$Lh_t \equiv H_t = H_0\left(Lh_{t-1}\right)^{1-\mu}\left(\Gamma_{t-1}\right)^{\mu} \equiv H_0\left(H_{t-1}\right)^{1-\mu}\left(\Gamma_{t-1}\right)^{\mu}. \tag{22}$$

The economy grows, even in the absence of population growth and exogenous progress in labor efficiency. Using the GDP growth factor $G_t^Y \equiv Y_{t+1}/Y_t$ as well as Equations (19) and (22), the GDP growth factor can be written as follows:

$$G_{t+1}^{Y}=\frac{H_{t+1}}{H_t}\frac{\left(1-u_{t+1}\right)^{1-\alpha}}{\left(1-u_t\right)^{1-\alpha}}\frac{\left(k_{t+1}\right)^{\alpha}}{\left(k_t\right)^{\alpha}},\ k_t\equiv\frac{K_t}{H_t}. \tag{23}$$

As Magnani (2015: pp. 13-14) rightly states, aggregate investment in Solow (1956)'s neoclassical growth model is not micro-, but macro-founded since it is determined by aggregate savings. The same holds true in Diamond (1965)'s OLG model of neoclassical growth where perfectly flexible aggregate investment is also determined by aggregate savings of households. Deviating from those neoclassical growth models, Morishima (1977) and more recently Magnani (2015: p. 14) claims that "investments are determined by an independent investment function." This function is specified in discrete time as follows:

$$I_t^D=\phi H_t\left(1+i_t\right)^{-\theta},\ \phi>0,\theta\geq 0. \tag{24}$$

The positive parameter ϕ reflects "Keynesian investors' animal spirits" (Magnani, 2015: p. 14) while θ denotes the interest-factor elasticity of aggregate investment demand I_t^D.

In addition to the restrictions imposed by household and firm optimizations and the government budget constraint, markets for labor, capital services and assets, ought to clear in all periods (the market for the output of production is cleared by means of Walras' Law[2]).

$$L_t\left(1-u_t\right)=N_t,\forall t. \tag{25}$$

$$K_t^{S,E}+K_t^{S,U}=K_t^S=K_t,\ \forall t. \tag{26}$$

$$B_t^{D.E}+B_t^{D,U}=B_t^{S,E}+B_t^{S,U}=B_t,\ \forall t. \tag{27}$$

3. Intertemporal Equilibrium, Existence and Stability of Steady States in Basic Model

Intertemporal equilibrium. As a first step, the unemployment rate in period t (= intertemporal equilibrium unemployment rate) is derived. To this end, we use the output market clearing identity:

$$\begin{aligned}&P_tL_t\left(1-u_t\right)c_t^{1,E}+P_tL_{t-1}\left(1-u_{t-1}\right)c_t^{2,E}+P_tL_tu_tc_t^{1,U}+P_tL_{t-1}u_{t-1}c_t^{2,U}\\&+P_tI_t^{D,E}+P_tI_t^{D,U}+P_t\Gamma_t+P_t\Delta_t=P_tY_t.\end{aligned} \tag{28}$$

Starting with identity (28), we insert Equations (19), Equation (5) and constraint (ii) from employed household's optimization problem for period t as well as Equation (11) and constraint (ii) from unemployed household's optimization problem for period t. In addition, we also insert Equation (24), with $I_t^D=I_t^{D,E}+I_t^{D,U}$ and add the market clearing Conditions (26) and (27). In this way, the following equation for $P_t=1,\forall t$ [3] is obtained:

$$\begin{aligned}&M\left(h_tL\left(1-u_t\right)\right)^{1-\alpha}\left(K_t\right)^{\alpha}\\&=\varepsilon/\left(\varepsilon+\beta\right)\left(1-\tau_t\right)w_th_tL_t\left(1-u_t\right)+\varepsilon/\left(\varepsilon+\beta\right)\varsigma_tL_tu_t\\&\quad+q_tK_t+\left(1+i_t\right)B_t+\phi H_t\left(1+i_t\right)^{-\theta}+\Gamma_t+\Delta_t.\end{aligned} \tag{29}$$

[2]The proof of Walras' law can be found in Farmer & Kuplen (2018: pp. 10-11).
[3]Due to Walras' Law.

On dividing Equation (29) on both sides by Y_t, this equation turns into the following equation:

$$\begin{aligned}1 &= \varepsilon/(\varepsilon+\beta)(1-\tau_t)w_t h_t L_t(1-u_t)/Y_t + \varepsilon/(\varepsilon+\beta)\varsigma_t L_t u_t/Y_t \\ &\quad + q_t K_t/Y_t + (1+i_t)B_t/Y + \phi H_t(1+i_t)^{-\theta}/Y + \Gamma_t/Y + \Delta_t/Y.\end{aligned} \tag{30}$$

Rewriting profit maximization Condition (15) as

$$w_t h_t L(1-u_t) = (1-\alpha)(K_t)^{\alpha}(h_t L(1-u_t))^{1-\alpha} = (1-\alpha)Y_t, \tag{31}$$

and using the definitions $v_t \equiv K_t/Y_t$, $b_t \equiv B_t/Y_t$, $\delta_t \equiv \Delta_t/Y_t$, $\gamma_t \equiv \Gamma_t/Y_t$ and $\xi_t \equiv (\varsigma_t L_t u_t)/Y_t$, Equation (30) can be rewritten as follows:

$$\begin{aligned}1 &= \varepsilon/(\varepsilon+\beta)(1-\tau_t)(1-\alpha) + \varepsilon/(\varepsilon+\beta)\xi_t + q_t v_t \\ &\quad + (1+i_t)b_t + \phi(1/k_t)(1+i_t)^{-\theta} v_t + \gamma_t + \delta_t.\end{aligned} \tag{32}$$

The capital-output ratio v_t is related to the real-capital to human-capital ratio k_t as follows:

$$v_t = \frac{K_t}{M(H_t)^{1-\alpha}(1-u_t)^{1-\alpha}(K_t)^{\alpha}} = \frac{(K_t)^{1-\alpha}}{M(H_t)^{1-\alpha}(1-u_t)^{1-\alpha}} = \frac{(k_t)^{1-\alpha}}{M(1-u_t)^{1-\alpha}}, \tag{33}$$

which implies:

$$q_t = \alpha/v_t. \tag{34}$$

By use of the no-arbitrage Condition (2) as well as of Equations (33) and (34) Equation (32) turns out to be:

$$\begin{aligned}1 &= \varepsilon/(\varepsilon+\beta)(1-\tau_t)(1-\alpha) + \varepsilon/(\varepsilon+\beta)\xi_t + \alpha[1+b_t/v_t] \\ &\quad + \phi(\alpha/v_t)^{-\theta} M^{-1/(1-\alpha)}(1-u_t)^{-1} v_t^{-\alpha/(1-\alpha)} + \gamma_t + \delta_t.\end{aligned} \tag{35}$$

Next, it is apt to specify how the government determines its intertemporal policy profile. To this end, we assume that government consumption expenditures per GDP, δ_t, government human capital investment expenditures per GDP, γ_t, and unemployment benefits per GDP, ξ_t, are time-stationary, *i.e.*, $\delta_t = \delta_{t+1} = \delta, \gamma_t = \gamma_{t+1} = \gamma, \forall t$ and $\xi_t = \xi_{t+1} = \xi, \forall t$. As in Diamond (1965: p. 1137) we furthermore assume that the government runs a "constant-stock" fiscal policy: $b_{t+1} = b_t = b$. The budget constraint of the government written in per GDP terms reads then as follows:

$$\begin{aligned}\frac{B_{t+1}}{Y_{t+1}}\frac{Y_{t+1}}{Y_t} &\equiv bG_t^Y = (1+i_t)B_t/Y_t + \Gamma_t/Y_t + \frac{\Delta_t}{Y_t} + \frac{L_t u_t \varsigma_t}{Y_t} - \frac{\tau_t(1-u_t)w_t h_t L}{Y_t} \\ &\equiv \alpha b/v_t + \gamma + \delta + \xi - \tau_t(1-\alpha).\end{aligned} \tag{36}$$

Equation (36) implies that the wage-tax rate ought to become endogenous and is determined by the following equation:

$$\tau_t = \left[(\alpha/v_t - G_t^Y)b + \gamma + \delta + \xi\right]/(1-\alpha). \tag{37}$$

Inserting τ_t from Equation (37) into Equation (32), leads to the following result:

$$1=\varepsilon/(\varepsilon+\beta)\left[1-\alpha(1+b/v_t)-\gamma-\delta-\xi+bG_t^Y\right]+\varepsilon/(\varepsilon+\beta)\xi$$
$$+\alpha\left[1+b_t/v_t\right]+\gamma+\delta+\phi(\alpha/v_t)^{-\theta}M^{-1/(1-\alpha)}(1-u_t)^{-1}v_t^{-\alpha/(1-\alpha)}. \tag{38}$$

Collecting terms and simplifying the resulting expression yields the following equation for $(1-u_t)$:

$$1-u_t=\left\{(\varepsilon+\beta)\phi\alpha^{-\theta}M^{-1/(1-\alpha)}v_t^{[\theta(1-\alpha)-\alpha]/(1-\alpha)}\right\}$$
$$\Big/\left\{\beta\left[1-\gamma-\delta-\alpha(1+b/v_t)\right]-\varepsilon bG_t^Y\right\}. \tag{39}$$

In terms of the transformed variables, the growth factor of human capital reads as follows:

$$\frac{H_{t+1}}{H_t}=H_0(H_t)^{\mu-1}\gamma^{1-\mu}(Y_t)^{1-\mu}$$
$$=H_0(H_t)^{\mu-1}\gamma^{1-\mu}M^{1-\mu}(K_t)^{\alpha(1-\mu)}(H_t)^{(1-\alpha)(1-\mu)}(1-u_t)^{(1-\alpha)(1-\mu)} \tag{40}$$
$$=H_0M^{(1-\mu)/(1-\alpha)}\gamma^{1-\mu}(v_t)^{\alpha(1-\mu)/(1-\alpha)}(1-u_t)^{1-\mu}.$$

The GDP growth factor in terms of the capital-output ratio can be rewritten as follows:

$$G_t^Y=\frac{H_{t+1}}{H_t}\left(\frac{v_{t+1}}{v_t}\right)^{\alpha/(1-\alpha)}\frac{1-u_{t+1}}{1-u_t}$$
$$=H_0M^{(1-\mu)/(1-\alpha)}\gamma^{1-\mu}(v_{t+1})^{\alpha/(1-\alpha)}(v_t)^{-\alpha\mu/(1-\alpha)}(1-u_{t+1})(1-u_t)^{-\mu}. \tag{41}$$

By using the intertemporal equilibrium condition $K_{t+1}^S=I_t^D$, one obtains the following equation for the dynamics of the capital-output ratio:

$$v_{t+1}G_t^Y=\phi\frac{H_t}{K_t}\frac{K_t}{Y_t}q_t^{-\theta}=\phi\frac{1}{k_t}v_tq_t^{-\theta}=\alpha^{-\theta}\phi M^{\frac{-1}{1-\alpha}}(v_t)^{\frac{\theta(1-\alpha)-\alpha}{1-\alpha}}(1-u_t)^{-1}. \tag{42}$$

The final steps needed to arrive at the equation of motion for the capital-output ratio entail, first, inserting the GDP growth factor Equation (41) into Equation (42). This procedure yields:

$$\frac{v_{t+1}^{1/(1-\alpha)}}{1-u_{t+1}}=\left[\phi/(\alpha^{\theta}H_0)\right]\gamma^{\mu-1}M^{(\mu-2)/(1-\alpha)}v_t^{[\theta(1-\alpha)-\alpha(1-\mu)]/(1-\alpha)}(1-u_t)^{\mu-1}. \tag{43}$$

Next, after inserting the growth factor Equation (41) into Equation (39) and rearranging, we arrive at the following intermediate result:

$$v_{t+1}^{\alpha/(1-\alpha)}(1-u_{t+1})$$
$$=(\varepsilon bH_0)^{-1}\gamma^{\mu-1}M^{(1-\mu)/(\alpha-1)}(1-u_t)^{\mu-1}v_t^{(\alpha\mu)/(1-\alpha)} \tag{44}$$
$$\times\left\{\beta(1-u_t)\left[1-\alpha(1+b/v_t)-\gamma-\delta\right]-(\beta+\varepsilon)\phi M^{-1/(1-\alpha)}v_t^{[\theta(1-\alpha)-\alpha]/(1-\alpha)}\right\}.$$

Solving Equation (44) for $1-u_{t+1}$ and inserting the result into Equation (43) then yields, after re-arranging, the first equation of motion:

$$v_{t+1}=\frac{\varepsilon b}{\beta\phi^{-1}\alpha^{\theta}M^{1/(1-\alpha)}\left[1-\alpha(1+b/v_t)-\gamma-\delta\right](1-u_t)v_t^{[\alpha-\theta(1-\alpha)]/(1-\alpha)}-\beta-\varepsilon}. \tag{45}$$

Reinserting the dynamic Equation (45) into Equation (43) and solving for $1-u_{t+1}$ generate the second equation of motion:

$$u_{t+1}=1-\phi\left(\alpha^{\theta}H_0\right)^{-1}\gamma^{\mu-1}(\varepsilon b)^{-1/(1-\alpha)}M^{(2-\mu)/(\alpha-1)}v_t^{[\theta(1-\alpha)-\alpha(1-\mu)]/(1-\alpha)}(1-u_t)^{\mu-1}\times\left\{\alpha^{\theta}\beta\phi^{-1}M^{1/(1-\alpha)}(1-u_t)\left[1-\alpha(1+b/v_t)-\gamma-\delta\right]v_t^{[\alpha-\theta(1-\alpha)]/(1-\alpha)}-(\beta+\varepsilon)\right\}^{1/(1-\alpha)}. \quad (46)$$

Existence of steady states. The steady states of the equilibrium dynamics depicted by the difference Equations (45) and (46) are defined as $\lim_{t\to\infty}v_t=v$ and $\lim_{t\to\infty}u_t=u$. Explicit steady-state solutions are not possible. Thus, we are in need to resort to an intermediate value theorem to prove the existence of at least one feasible steady-state solution $v_{\min}<v<\infty$ and $0<u<1$.

To this end, for given structural and policy parameters (except *b*), the maximal sustainable debt to GDP parameter is defined as $b^{\max}$, and the minimal capital-output ratio as $v_{\min}$ which ensure full employment. On inserting G^Y from the steady-state version of Equation (41) into the steady-state version of Equation (42) with $u=0$, $v_{\min}$ can then explicitly be determined as follows:

$$v_{\min}=\left[\alpha^{-\theta}\gamma^{\mu-1}\phi\left(H_0\right)^{-1}M^{(\mu-2)/(1-\alpha)}\right]^{(1-\alpha)/[1+\alpha(1-\mu)-\theta(1-\alpha)]}. \quad (47)$$

Using the steady-state version of Equation (39) with $u=0$, $b=b^{\max}$ can be calculated as follows:

$$b^{\max}=\left[\beta(1-\alpha-\gamma-\delta)-\alpha^{-\theta}(\beta+\varepsilon)\phi M^{-1/(1-\alpha)}v_{\min}^{[\theta(1-\alpha)-\alpha]/(1-\alpha)}\right]\Big/\left[\alpha\beta\left(v_{\min}\right)^{-1}+\varepsilon H_0M^{(1-\mu)/(1-\alpha)}\gamma^{1-\mu}\left(v_{\min}\right)^{\alpha(1-\mu)/(1-\alpha)}\right], \quad (48)$$

whereby, to ensure a strictly larger than zero $b^{\max}$, it is assumed that:
$\beta(1-\alpha-\gamma-\delta)>\alpha^{-\theta}(\beta+\varepsilon)\phi M^{-1/(1-\alpha)}v_{\min}^{[\theta(1-\alpha)-\alpha]/(1-\alpha)}$.

For the proof of the existence of at least one $0<u<1$ and $v_{\min}<v<\infty$ the steady-state versions of Equations (45) and (46) are used. This results in:

$$v=\frac{\varepsilon b}{\alpha^{\theta}\beta\phi^{-1}M^{1/(1-\alpha)}\left[1-\alpha(1+b/v)-\gamma-\delta\right](1-u)v^{[\alpha-\theta(1-\alpha)]/(1-\alpha)}-\beta-\varepsilon}. \quad (49)$$

$$1-u=\phi\left(\alpha^{\theta}H_0\right)^{-1}\gamma^{\mu-1}(\varepsilon b)^{-1/(1-\alpha)}M^{(2-\mu)/(\alpha-1)}v^{[\theta(1-\alpha)-\alpha(1-\mu)]/(1-\alpha)}(1-u)^{\mu-1}\times\left\{\alpha^{\theta}\beta\phi^{-1}M^{1/(1-\alpha)}(1-u)\left[1-\alpha(1+b/v)-\gamma-\delta\right]v^{[\alpha-\theta(1-\alpha)]/(1-\alpha)}-(\beta+\varepsilon)\right\}^{1/(1-\alpha)}. \quad (50)$$

By substituting $\alpha^{\theta}\beta\phi^{-1}M^{1/(1-\alpha)}(1-u)\left[1-\alpha(1+b/v_t)-\gamma-\delta\right]v^{\alpha/(1-\alpha)}-\beta-\varepsilon$ in (50) for $\varepsilon b/v$, Equation (50) can be reduced to the following simpler equation:

$$1-u=\alpha^{-\theta}\gamma^{\mu-1}\phi H_0^{-1}M^{(\mu-2)/(1-\alpha)}(1-u)^{\mu-1}v^{[\theta(1-\alpha)-\alpha(1-\mu)-1]/(1-\alpha)} \quad (51)$$

Using the short cut $1-u\equiv w$, the two Equations (49) and (50) can be explicitly solved for w as follows:

$$w_1 = \frac{\alpha^{-\theta}\phi M^{-1/(1-\alpha)} v^{[\theta(1-\alpha)-\alpha]/(1-\alpha)}\left[\beta+\varepsilon(1+b/v)\right]}{\beta\left[1-\alpha(1+b/v)-\gamma-\delta\right]}, \tag{52}$$

$$w_2 = \left[\alpha^{-\theta}\gamma^{\mu-1}\phi H_0^{-1} M^{(\mu-2)/(1-\alpha)} v^{[\theta(1-\alpha)-\alpha(\mu-1)-1]/(1-\alpha)}\right]^{1/(2-\mu)}. \tag{53}$$

Hereby, w_1 represents the solution of Equation (49) for *w*, while w_2 exhibits the solution of Equation (50) for *w*. A steady-state solution exists if $w_1(v) = w_2(v)$ for at least one $v_{\min} < v < \infty$.

Proposition 1. Suppose there exist $\eta > 0$ and $v^{\max} > 0$ such that $w_1(v^{\max}) = w_2(v^{\max}) + \eta$. Then, the solution of $w_1(v) = w_2(v)$ for at least one $v_{\min} < v < v^{\max}$ exists and represents a steady state of the equilibrium Dynamics (45) and (46) with $0 < w < 1, (0 < u < 1)$.

Proof. See Farmer & Kuplen (2018: pp. 19-20).

Dynamic stability of steady states. The next step is to investigate the local dynamic stability of the unique steady-state solution. To this end, the intertemporal equilibrium Equations (39), (41), and (42) are totally differentiated with respect to $v_{t+1}, w_{t+1}, G_t^Y, v_t, w_t$. Then, the Jacobian matrix $J(v,w)$ of all partial differentials with respect to v_t and w_t is formed as follows:

$$J(v,w) \equiv \begin{bmatrix} \frac{\partial v_{t+1}}{\partial v_t}(v,w) & \frac{\partial v_{t+1}}{\partial w_t}(v,w) \\ \frac{\partial w_{t+1}}{\partial v_t}(v,w) & \frac{\partial w_{t+1}}{\partial w_t}(v,w) \end{bmatrix}, \tag{54}$$

with

$$\frac{\partial v_{t+1}}{\partial v_t} \equiv j_{11} = -\frac{\varepsilon b G^Y\left[\alpha-\theta(1-\alpha)\right]+(1-\alpha)\beta b q+\left[\alpha-\theta(1-\alpha)\right](\beta+\varepsilon)G^Y v}{(1-\alpha)\varepsilon b G^Y},$$

$$\frac{\partial v_{t+1}}{\partial w_t} \equiv j_{12} = -\frac{v\left[\varepsilon b+(\beta+\varepsilon)v\right]}{\varepsilon b w} < 0,$$

$$\frac{\partial w_{t+1}}{\partial v_t} \equiv j_{21} = \frac{w\left\{\left[\alpha-\theta(1-\alpha)\right](\beta+\varepsilon)G^Y v+(1-\alpha)\beta b q+\alpha\varepsilon b G^Y\left[\alpha-\theta(1-\alpha)+\mu(1-\alpha)\right]\right\}}{(1-\alpha)^2\varepsilon b G^Y v},$$

$$\frac{\partial w_{t+1}}{\partial w_t} \equiv j_{22} = \frac{\varepsilon b\left[\alpha+\mu(1-\alpha)\right]+(\beta+\varepsilon)v}{\varepsilon b(1-\alpha)} > 0.$$

The sign of j_{12} is unambiguously smaller than zero, while the sign of j_{22} is always larger than zero. The signs of j_{11} and j_{21} depend on whether $\theta > \alpha/(1-a)$ or $\theta \le \alpha/(1-a)$. In the former case the sign of j_{11} is smaller than zero, whereby the sign of j_{21} is larger than zero. In the latter case, the signs of these entries of the Jacobian (54) are in general ambiguous. To evaluate the dynamic stability of the equilibrium dynamics in neighborhood of the steady-state solution, the eigenvalues of the Jacobian matrix (54) are needed. To this end, the trace $\mathrm{Tr}J(v,w)$, the determinant $\mathrm{Det}J(v,w)$ and $1-\mathrm{Tr}J(v,w)+\mathrm{Det}J(v,w)$ need to be calculated.

$$\mathrm{Tr}J(v,w)=\theta+\mu-\frac{q\beta}{\varepsilon G^{Y}}+\frac{(\beta+\varepsilon)(1+\theta)v}{\varepsilon b}, \tag{55}$$

$$\mathrm{Det}J(v,w)=\frac{\beta(1-\mu)q}{\varepsilon G^{Y}}+\theta\mu\left[1+\frac{(\beta+\varepsilon)v}{\varepsilon b}\right]>0, \tag{56}$$

$$\begin{aligned}&1-\mathrm{Tr}J(v,w)+\mathrm{Det}J(v,w)\\&=(1-\theta)(1-\mu)+\frac{q\beta(2-\mu)}{\varepsilon G^{Y}}-\frac{(\beta+\varepsilon)[1+\theta(1-\mu)]v}{\varepsilon b}.\end{aligned} \tag{57}$$

The sign of the trace turns out to be, in general, indeterminate, while the determinant of the Jacobian (54) is larger than zero. Moreover, the sign of $1-\mathrm{Tr}J(v,u)+\mathrm{Det}J(v,u)$ is in general ambiguous. However, for a broad set of feasible parameters, all of which are in accordance with the assumptions used thus far, the trace is larger than zero (larger than 2) and $1-\mathrm{Tr}J(v,u)+\mathrm{Det}J(v,u)<0$.

Proposition 2. Suppose the assumptions of Proposition 1 hold. Then, the calculation of the eigenvalues λ_1 and λ_2 of Jacobian (54) at the steady-state solution $0<w<1, (0<u<1)$ and $v_{\min}<v<v^{\max}$ shows that for a broad set of feasible parameter combinations with $b<b^{\max}$, $\lambda_1>1$ and $0<\lambda_2<1$.

In other words: the steady-state solution in the present endogenous growth model with involuntary unemployment represents a non-oscillating, monotone saddle point with v_t as a slowly moving variable and $w_t(u_t)$ as jump variable. With $v_0=\underline{v}>0$ historically given, $w_0(u_0)$ jumps onto the saddle-path along which both variables converge monotonically towards the steady-state solution.

4. Limits to Public Debt Even in the Face of Involuntary Unemployment

As the responses of ruling governments during the Corona pandemic and now in the face of the energy shortage show: Extremely generous public expenditure respective subsidy programs have been initiated to support economically struggling households and firms, even in the face of collapsing public revenues as a result of rising unemployment, failing business earnings and tax deferrals. Consequently, the deficits of federal and state governments are further exploding. Although unemployment rates decreased during 2021 and the first halve of 2022, the question remains as to whether, in the face of remaining high involuntary unemployment, limits to public debt can and/or ought to be respected or disregarded.

Full-employment, neo-classical growth models characteristically feature limits to public debt. To the best of this author's knowledge of that model type, Rankin & Roffia (2003) were the first to detect limits to government debt in a Diamond-type OLG economy with internal public (= government) debt. Rankin & Roffia (2003: p. 218) claim that "even with a constant stock of government debt, fiscal policy may be unsustainable because a steady state of the economy with non-degenerate values of the variables *may not exist*" (Italics in original). The private-sector capital-labor ratio (= aggregate capital intensity) associated with this unsustainable government debt level represents an "interior maximum", in contrast to a position of "degeneracy", in which, owing to the excessively high level of government debt, capital intensity approaches zero. Rankin & Roffia (2003) found in their version of Diamond (1965)'s OLG model with a log-linear intertemporal utility function and a Cobb-Douglas production function that "maximum sustainable debt is generally reached at an interior maximum rather than at a degeneracy (Rankin & Roffia, 2003: p. 220)." At this interior maximum the intertemporal equilibrium dynamics of the perfectly competitive market economy experience a saddle-note bifurcation (Azariadis, 1993: p. 152), eliciting a process of explosive capital decumulation and ending in a sudden implosion of the world economy.

Although invaluable, Rankin & Roffia (2003)'s contribution is, however, restricted to an economy in which the labor supply is always fully employed. This precludes the investigation of limits to government debt in a world economy with involuntary unemployment. Thus, it seems to be expedient to use our model with inflexible aggregate investment and involuntary unemployment presented above in order to inquire theoretically whether limits to public debt exist even in the face of significant involuntary unemployment.

In the more recent empirical literature concerning how public debt affects GDP growth (for an overview see (Hayati et al., 2019)), absolute limits to the public debt to GDP ratio do not feature prominently. However, and in line with the somewhat disputed study by Reinhart et al. (2015), the existence of a threshold to the debt to GDP ratio, beyond which GDP growth declines, is now attracting considerable attention. However, as will be shown below, the model presented above does not facilitate the existence of a threshold beyond which the GDP growth rate decreases. It is again Magnani (2015: pp. 18-20) who indicates why this may be the case by pointing out that, empirically, private and public consumption and private and public savings crowd out aggregate investment. Thus, by introducing a crowding-out effect for private and public savings on aggregate investment, it will be investigated below, given appropriate values for the crowding-out parameter and the interest-factor elasticity of aggregate investment, that a debt threshold can exist beyond which GDP growth declines with rising public debt. It is also revealed that, in a numerically specified version of the model, the numerical values found for the threshold and the growth detracting effect of larger public debt are in fact quite close to the empirical estimations provided by the World Bank. To this end, several structural parameters of the model are calibrated with respect to time-average figures for the global GDP growth rate, the global real interest rate, the global saving and investment ratios, all for the period 1995 to pre-pandemic 2020. Together with time averages for the same period of the policy parameters, the absolute limits for the public debt to GDP ratio are then numerically derived. Then, in line with Magnani (2015), the crowding out of private and public savings on aggregate investment is introduced, and questions concerning the existence and numerical value of a debt ratio threshold, beyond which GDP growth declines, are explored.

As a first step we proceed by investigating absolute limits to government debt under involuntary unemployment. It is immediately apparent that the natural candidate for a limit to government debt is $b^{\max}$. To derive a numerical value for $b^{\max}$, one needs to resort to a numerical specification of all structural parameters and of the policy parameters γ and δ in line with 25 years of relevant time-average data. Such data cover the empirical development of the growth rate of global real GDP, the long-term real interest rate, the global average of national saving rates and the global average of investment to GDP ratios for the period 1995-2020. Taking the data from the World Economic Outlook reports IMF (2008), IMF (2014), IMF (2020), the time-average of the global real GDP growth rate is found to amount to about 3.4% yearly, which is equal to a growth factor of 2.3 over a period of 25 years (= basic model period), the time-average of the (long-term) real interest rate is 2% yearly, which is equal to a growth factor of 1.64 for the basic model period, the time-average of national saving rates (= ratio of GDP minus aggregate private and government consumption expenditures to GDP) is 0.22, and that of investment ratios is also 0.22 IMF (2008), IMF (2014), IMF (2020). In line with the empirical time average for global public debt to GDP ratios between 1995 and 2020, $b = 0.03$ which is equivalent to an annual public debt ratio of 75%. For the human capital investment ratio γ, a value of 0.05 is assumed, and the non-HCI public expenditure ratio is set at $\delta = 0.2$. The time-average of the median of national unemployment rates over the past 25 years was found to be 6% (ILO, 2020). Thus, $u = 0.06$. Finally, the level parameter *M*, is set at 5, in accordance with Auerbach & Kotlikoff (1998), and $\mu = 0.5$, as in Glomm & Ravikumar (1992).

To perform the calibration exercise, first the government budget constraint in steady state (37) is needed. This is reproduced here for the convenience of the reader:

$$\tau = \left[b\left(q - G^{Y} \right) + \gamma + \delta \right] / (1 - \alpha). \tag{58}$$

Equilibrium Equation (35) is substituted for the following equilibrium equation in steady state:

$$\begin{aligned} 1 = {} & \frac{\varepsilon}{\beta + \varepsilon}(1 - \alpha)(1 - \tau) + \alpha\left(1 + b/v\right) + \gamma + \delta \\ & + \alpha^{-\theta} \phi M^{-1/(1-\alpha)} v^{\left[\theta(1-\alpha) - \alpha\right]/(1-\alpha)} (1 - u)^{-1}. \end{aligned} \tag{59}$$

To be able to calibrate the remaining parameters with respect to the empirical data for the past 25 years, two additional equations are needed. The first one represents the steady-state version of the growth-factor equation which reads as follows:

$$G^{Y} = H_0 M^{(1-\mu)/(1-\alpha)} \gamma^{1-\mu} v^{\left[\alpha(1-\mu)\right]/(1-\alpha)} (1 - u)^{1-\mu}. \tag{60}$$

Inserting G^{Y} from Equation (60) into the steady-state version of dynamic Equation (42) yields the required second equation as follows:

$$H_0 M^{(1-\mu)/(1-\alpha)} \gamma^{1-\mu} v^{[\alpha(1-\mu)]/(1-\alpha)} (1-u)^{1-\mu} = \alpha^{-\theta} \phi M^{\frac{-1}{1-\alpha}} v^{\frac{\theta(1-\alpha)-1}{1-\alpha}} (1-u_t)^{-1}. \quad (61)$$

Equations (58)-(61) together with $\alpha/v = 1.64$, $G^Y = 2.3$, $0.22 = \phi\alpha^{-\theta} \times M^{-1/(1-\alpha)} v^{[\theta(1-\alpha)-\alpha]/(1-\alpha)} (1-u)^{-1}$, and $0.22 = \beta/(\beta+\varepsilon) \times [1-\alpha(1+b/v) - \gamma - \delta] - \beta/(\beta+\varepsilon) b G^Y$, thus form six equations which may be used to calibrate the parameters α, β, ϕ, H_0 and the endogenous variables v and G^Y to the data. The result of this calculation exercise, together with $\varepsilon = 0.5$, reads as follows: $\alpha = 0.15687$, $\beta = 0.44608$, $\phi = 0.9014$, $H_0 = 5.08152$, $v = 0.09565$, $G^Y = 2.3$. In attempting to determine the value of the remaining parameter, the interest-factor elasticity of aggregate investment, three different scenarios are considered: 1) The "Vulgar-Keynesian" scenario under $\theta = 0$, 2) the moderately interest-elastic Keynesian scenario $\theta < \alpha/(1-\alpha)$, and 3) the highly interest-elastic, "Neo-Classical" scenario $\theta = 1$. The latter is based on the careful empirical study of Guiso et al. (2002), who, using Italian firm-level data, found that the elasticity of capital demand with respect to the user cost of capital equals −1.

Before the values for maximum sustainable debt ratios are presented, one needs to note that different values of the interest factor-elasticity of investment demand do not change the parameters calibrated, except the animal spirits parameter ϕ. It turns out from the calibration exercises that, as a rule, a higher ϕ is associated with a higher θ: For the Vulgar-Keynesian scenario, ϕ is as reported above; in the moderately interest-elastic scenario, where $\theta = 0.186$, we find that $\phi = 0.98828$, and finally in the Neo-Classical scenario, $\phi = 1.47831$. Proposition 3 reports the values of maximum sustainable debt ratios and how the GDP growth factor responds to a higher sustainable government debt ratio below the maximum level.

Proposition 3. Suppose that the values of all structural and policy parameters except θ and ϕ are as reported above. Then, in the Vulgar-Keynesian scenario $b^{\max} = 0.03379$ (≈84.33% yearly), for the moderately interest-elastic scenario $b^{\max} = 0.03494$ (≈87.33% yearly), and in the Neo-Classical scenario $b^{\max} = 0.0531$ (≈132.5% yearly). Moreover, in all scenarios, both the capital-output ratio and the unemployment rate decrease with rising government debt ratios up to the maximum sustainable debt ratio. Finally, the GDP growth factor increases with rising public debt ratios up to the maxima, except in the Neo-Classical scenario, where the GDP growth factor does not respond at all to rising debt ratios.

Proposition 3 implies that in the OLG model of the global economy presented thus far, there does not exist a threshold for the public-debt ratio beyond which the GDP growth factor decreases with rising debt ratios. Thus, it seems to be apt to look for a modification of the aggregate investment function in order to be able to investigate the research question elaborated in the introduction to this section. Fortunately, Magnani (2015: p. 17) also provides the requested modification of the aggregate investment function by noticing that in his base model (= the Vulgar-Keynesian model presented above) "an increase in the saving rate has a negative effect on employment and on real GDP, both in short and long run" a result "which is not consistent with the empirical evidence." To accord with the data, Magnani (2015: pp. 18-20) proposes a re-specification of the aggregate investment function which takes the crowding-out effects of private and public consumption, as well as those of private and public savings, on aggregate investment into account. In the present analysis, the focus is placed on the crowding-out effects of saving, and in compliance with the suggestion made by Magnani (2015: pp. 18-20), the aggregate investment ratio is rewritten as follows:

$$\begin{aligned} I_t^D/Y_t &= \alpha^{-\theta}\phi M^{-1/(1-\alpha)}\left(v_t\right)^{\left[\theta(1-\alpha)-\alpha\right]/(1-\alpha)}\left(1-u_t\right)^{-1} \\ &\quad + \nu\left(S_t/Y_t - S/Y + S_t^P/Y - S^P/Y\right), \text{ with} \\ S_t/Y_t &= \left[\beta/(\beta+\varepsilon)(1-\alpha)(1-\tau_t)+\beta/(\beta+\varepsilon)\xi\right]\left[\frac{1-u}{1-u_t}\right]^{1-\alpha}, \\ S/Y &= \beta/(\beta+\varepsilon)(1-\alpha)(1-\tau)+\beta/(\beta+\varepsilon)\xi, \\ S_t^P/Y_t &= b^{new}\left(1-G_t^Y\right),\ S/Y = b\left(1-G^Y\right). \end{aligned} \tag{62}$$

Here, the endogenous variables without a time subscript denote the steady-state values associated with the initial public-debt ratio *b*, while the endogenous variables with a time subscript feature the values associated with $b < b^{new} < b^{\max}$. Clearly, S_t represents aggregate private savings, while S_t^P is public savings and equals the difference between government's revenues and expenditures. Finally, $0 \le \nu \le 1$ features the crowding-out parameter with $\nu = 0$ signifying no crowding out at all, and $\nu = 1$ signifying complete crowding out, as in Diamond (1965)'s Neo-Classical growth model. Magnani (2015: pp. 29-30)'s econometric analysis using 1955-2012 data from six OECD countries results in a value of $0.6 \le \nu \le 0.8$.

Using the modified aggregate investment function (62) in equations (39) and (41), and sticking to $\nu = 0.8$ it first turns out that for $0 \le \theta < \alpha/(1-\alpha)$ and the associated ϕ values, the maximum sustainable debt values increase, e.g., from $b^{\max} = 84.33\%$ to $b^{\max} = 110.5\%$ in the case of the Vulgar Keynesian scenario, and from $b^{\max} = 87.33\%$ to $b^{\max} = 126.38\%$ for $\theta = 0.186$, although, even with a rather large crowding-out parameter, a higher public debt ratio below the maximum level always raises the GDP growth factor. In other words: Under conditions of completely inelastic or moderately interest-elastic aggregate investment even rather strong crowding out of private and public savings with respect to investment does not seriously dampen the growth-generating effect of a larger public-debt ratio below the now higher debt limits. That the debt limits are higher makes sense, since with strong crowding out the unemployment-reducing effect of larger public debt is lower than with less or no crowding out.

This leads us to the scenarios with $\alpha/(1-\alpha) < \theta \le 1$, in particular, to the Neo-Classical scenario with $\theta = 1$ and $\phi = 1.47831$. First, it is interesting to note that again the $b^{\max}$ values are in general larger than in the case of no investment crowding-out. Secondly, and more interesting, for $0.3 \le \theta \le 1$ and the associated ϕ values a threshold for the debt ratio appears beyond which more public debt reduces GDP growth. Taking, e.g. $\theta = 0.3$ and, $\phi = 1.04562$ we obtain a $b_{threshold} = 0.0321(\approx 80.25\%)$, beyond which $\mathrm{d}G^Y/\mathrm{d}b < 0$. Moreover, it is interesting to ascertain the extent to which strong GDP growth is diminished when the debt ratio is raised by one percentage point. Experimenting with differing values of $0.3 < \theta \leq 1$ reveals that the growth detracting effect of larger public debt is strongest when $\theta = 1$. It is worth summarizing the results thus far in the following proposition 4.

Proposition 4. Suppose that the structural and policy parameters, except θ and ϕ, exhibit the values as in Proposition 3. Assume $\nu = 0.8$. Then, for all $\alpha/(1-\alpha) < \theta \leq 1$, and the associated ϕ values, thresholds for the global public debt ratio exist beyond which larger debt reduces GDP growth. Moreover, the growth-detracting effect is larger, the higher the value for θ. For $\theta = 1$, $\phi = 1.47831$, the debt ratio threshold $b_{threshold} = 0.03(\approx 75\%)$ and the growth detracting effect equals 0.013 percentage points where the debt ratio is raised from 75% to 76%.

It is comforting to see that these numerical values for larger debt, both for the debt threshold and the growth detracting effect, come close to the values estimated in World Bank studies.

5. Conclusion

This paper aimed at investigating the existence of limits to the public debt to output ratio, their numerical values as well as debt threshold numbers beyond which GDP growth diminishes with rising public debt in a Diamond (1965)-type OLG growth model with internal public debt and human capital accumulation. In our intertemporal, macro-equilibrium model, perfect competition prevails in all markets. Our macro-oriented general equilibrium model thus deviates from new-Keynesian macro-models in which involuntary unemployment is traced back to inflexible wages, output prices and interest rates. In line with Morishima (1977) and Magnani (2015), involuntary unemployment now occurs since aggregate investment is not perfectly flexible (as in Solow (1956) and Diamond (1965)'s neo-classical growth models). Aggregate investment expenditures are inflexible due to investors' animal spirits à la Keynes (1936), and the aggregate investment function is independent of aggregate savings. For a perfectly flexible and perfectly competitive market system, this additional function makes the system of general equilibrium equations over-determinate, and results in an inconsistency which can only be resolved if one of the market-clearing conditions is dropped. The obvious candidate here is the labor market-clearing condition. Once this is done, the system of general-equilibrium equations becomes determinate again. Moreover, the unemployment rate becomes endogenous through the identity that the unemployment rate equals one minus the number of employed people relative to the number of employment-seeking people. It is thus aggregate demand (including investment demand) which governs aggregate production, and not aggregate supply.

Following Magnani (2015)'s lead, such atemporal macro-economic reasoning has been integrated in the present paper into Diamond (1965)'s OLG model with internal public debt and extended to cover human capital accumulation in line with Farmer & Kuplen (2018). Human capital accumulation makes GDP growth endogenous in steady state such that limits to public debt to GDP ratios in the face of involuntary unemployment can be explored.

Having proven in Propositions 1 and 2 the existence and dynamic stability of the unique steady-state solution, numerical values to the debt limits are attributed. To this end, the structural parameters of the OLG model are calibrated such that the model reproduces the time-average values for the growth rate of global GDP, the global real interest rate, the global saving and investment ratios, all for the period 1995 to pre-Corona 2020, together with the time-averages of policy parameters for the same period. This procedure provides maximum sustainable debt to GDP ratios of about 100% yearly. This debt limit is significantly lower than the corresponding limit in a model where aggregate savings determine perfectly flexible aggregate investment. Furthermore, in our model with involuntary unemployment, as opposed to models with full employment, the economy does not need to implode when the debt ratio approaches its limit.

As Proposition 3 makes clear, the magnitude of the interest factor elasticity of aggregate investment demand and the associated animal spirits parameter are decisive in determining the maximum-sustainable debt ratios. Under interest-factor inelastic or moderately elastic investment demand, the debt limits remain below 100%. In contrast, under unit interest-factor elasticity (the Neo-Classical case) the debt limit is significantly higher than 100%. In addition, as the economy approaches the debt limits, we find that, apart from the Neo-Classical case, higher debt to GDP ratios raise the GDP growth factor.

Using the basic model, it is not possible to ascertain whether there is a debt-ratio threshold beyond which growth decreases. In order to investigate this question, the model was therefore extended to incorporate the impact of private and public savings on aggregate investment, in line with Magnani (2015). Under a rather high crowding-out parameter, and unit interest-factor elasticity of investment demand, the model produces a debt threshold. The value of such a threshold, as well as that of the associated growth-detracting effect, are found to be only slightly lower than corresponding World Bank estimates (Proposition 4).

While the results of propositions 3 and 4 clearly confirm that (1) there exist limits to public debt even in the face of involuntary unemployment and (2) that there is a debt to GDP ratio beyond which higher debt diminishes GDP growth the main challenge for future research remains as to provide micro-foundations for inflexible aggregate investment.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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