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A New Version of the Lambda-CDM Cosmological Model, with Extensions and New Calculations

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Abstract

This article gives a state-of-the-art description of the cosmological Lambda-CDM model and in addition, presents extensions of the model with new calculations of background and CMB functions. Chapters 1-4 describe the background part of the model, *i.e.* the evolution of scale factor and density according to the Friedmann equations, and its extension, which results in a correction of the Hubble parameter, in agreement with new measurements (Cepheids-SNIa and Red-Giants). Based on this improved background calculation presented in chapters 5-9 the perturbation part of the model, *i.e.* the evolution of perturbation and structure according to the perturbed Einstein equations and continuity-Euler equations, and the power spectrum of the cosmic microwave background (CMB) is calculated with a new own code.

Keywords

Lambda-DCM, Friedmann Equations, CMB, Metric Perturbation, Hubble Parameter

1. Introduction

The Lambda-CDM model is widely accepted as the valid description of universe on large scales and its evolution history. It is based on General Relativity and consists of two parts:

- Background part with the ansatz Robertson-Walker (RW) metric, based on Friedmann equations and equations-of-state for the different component particles. It describes the evolution of scale factor and density without perturbations, *i.e.* without local structure (like galaxies and galaxy groups);

- Perturbation part with the ansatz perturbed RW-metric and locally perturbed density, velocity, and pressure of the component particles. It describes the time-evolution and (quasi-random perturbed spatial distribution) of density, velocity, and pressure, *i.e.* the actual structure of the universe on inter-galactic scale.

The parameters of the perturbed model are fitted in chap. 10 with the CMB spatial spectrum measured by Planck.

We present here in chap. 2-5 the background part with Friedmann equations and equations-of-state for the components with two notable extensions: explicit temperature dependence and classical gas as baryon eos. From this follows a new solution and own calculation in chap. 5, which offers an explanation for the apparent experimental discrepancy concerning the Hubble parameter.

Based on the improved background calculation, we present the perturbation part in chap. 6-10, with the derivation of the CMB spectrum, and new calculation of it.

2. Friedmann Equations

In this chapter, we present in concise form the basic equations (Friedmann equations) and equations of state (eos) for density and pressure with their different components radiation γ, neutrinos ν, electrons e, protons p, neutrons n (respectively baryons b), cold-dark-matter cdm d. The presentation relies basically on the four monographies [1] [2] [3] [4], with two notable extensions.

-Temperature

The eos depend explicitly on temperature T, resp. thermal energy $E_{th} = k_B T$, and **thermal energy is introduced as a function of time** $E_{th}(t)$, as all other variables, and has to be calculated.

-Baryon eos

The baryons are **modeled as classical gas**, and not as dust with zero pressure. We shall see in the background calculation in chap. 5, that this **model increases the value of the Hubble parameter**, which basically solves the Hubble-discrepancy problem.

2.1. Friedmann Equations and Metric

The metric which fulfills the conditions of space homogeneity and isotropy is the Robertson-Walker (RW) metric [1] [2] [3] [4]:

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + a^2(t)\left(\frac{\mathrm{d}r^2}{1-kr^2/R_H^2} + r^2\mathrm{d}\Omega^2\right) \tag{1}$$

with Hubble radius $R_H = \frac{c}{H_0} = 1.37\times10^{26}\,\mathrm{m}$ (Planck value), and scale factor $a(t)$.

The Einstein equations [1] [5] [6] [7] [8] for this metric are the two original Friedmann equations a and b (with $\dot{a} = \frac{\mathrm{d}a}{\mathrm{d}t}$) and two derived equations c (acceleration eq.) and d (density equation):

$$\left(\frac{\dot{a}}{ac}\right)^2+\frac{k}{a^2}-\frac{\Lambda}{3}=\frac{\kappa}{3}\rho c^2, \tag{2a}$$

$$\frac{2\ddot{a}}{ac^2}+\left(\frac{\dot{a}}{ac}\right)^2+\frac{k}{a^2}-\Lambda=-\kappa P, \tag{2b}$$

$$\frac{\ddot{a}}{ac^2}-\frac{1}{3}\Lambda=-\frac{\kappa}{2}\left(P+\frac{\rho c^2}{3}\right) \text{ derived from a, b} \tag{2c}$$

$$\frac{\dot{\rho}a}{3}+\dot{a}\left(\frac{P}{c^2}+\rho\right)=0 \text{ derived: density equation} \tag{2d}$$

with dimensionless variables using Planck-values: Hubble constant $H_0=67.74\,\text{km}\cdot\text{s}^{-1}\cdot\text{Mpc}^{-1}$, normalized Hubble constant $h=0.6774$,

Einstein constant $\kappa=\frac{8\pi G}{c^4}$, $\kappa c^2\rho_{crit,0}=\frac{\rho_{crH}}{R_H^2}$, relative pressure $P_r=\frac{P}{c^2\rho_{crit,0}}=\frac{P}{\rho_{Ecrit,0}}=P\kappa R_H^2$, relative cosmological constant $\Lambda_1=\Lambda R_H^2$, relative density $\Omega=\frac{\rho}{\rho_{crit,0}}$ with critical density today

$$\rho_{Ecrit,0}=c^2\rho_{crit,0}=\frac{3}{\kappa R_H^2},$$

$$\rho_{crit,0}=\frac{3}{\kappa R_H^2c^2}=\frac{3H_0^2}{8\pi G}=0.862\times10^{-26}\,\text{kg}\cdot\text{m}^{-3}=\frac{5.0m_p}{R_H^3}\left(1.37\times10^{26}\right)^3$$
$$=13.0\times10^{78}\frac{m_p}{R_H^3}=5.0\,\text{nucleon}/\text{m}^3$$

$$\rho_{crH}=\kappa c^2R_H^2\rho_{crit,0}=3$$

$$\rho_{Ecrit,0}=5.0\times0.963\frac{\text{GeV}}{\text{m}^3}=4.81\frac{\text{GeV}}{\text{m}^3},$$

Hubble radius $R_H=\frac{c}{H_0}=1.37\times10^{26}\,\text{m}$

The Friedmann equations can be reformulated dimensionless with $x_0=tc$, $a'=\frac{\mathrm{d}a}{\mathrm{d}x_0}$, $\rho_{crH}=3$

$$\left(\frac{a'}{a}\right)^2+\frac{k}{a^2}-\frac{\Lambda_1}{3R_H^2}-\frac{1}{3}\frac{\rho_{crH}}{R_H^2}\Omega=0,\ \textit{i.e.}\ \left(\frac{a'}{a}\right)^2+\frac{k}{a^2}-\frac{\Lambda_1}{3R_H^2}-\frac{\Omega}{R_H^2}=0$$

$$\frac{2a''}{a}+\left(\frac{a'}{a}\right)^2+\frac{k}{a^2}-\frac{\Lambda_1}{R_H^2}+\frac{P_r}{R_H^2}=0$$

$$\frac{\rho_r'a}{3}+a'\left(P_r+\rho_r\right)=0$$

rescaled with $\frac{a}{R_H}\to a$

$$\left(a'\right)^2+k-\frac{\Lambda_1}{3}a^2-\rho_ra^2=0 \quad \text{sF1} \tag{3a}$$

$$a''a - \frac{1}{3}\Lambda_1 a^2 = -\frac{3a^2}{2}\left(P_r + \frac{\rho_r}{3}\right) \quad \text{sF2} \tag{3b}$$

$$a''a + 2(a')^2 + 2k - \Lambda_1 a^2 + \frac{3}{2}(P_r - \rho_r)a^2 = 0 \quad \text{sF3} \tag{3c}$$

$$\frac{\rho_r' a}{3} + a'(P_r + \rho_r) = 0 \quad \text{sF4} \tag{3d}$$

density eq
with

$$\Omega_{mr} = \frac{\rho_{mat} + \rho_{rad}}{\rho_{Ecrit,0}}, \quad \Omega_{mr} = \Omega_{mr,0}\left(\frac{H}{H_0}\right)^2, \quad \Omega_{Ecrit} = \frac{3}{\kappa}\left(\frac{H}{c}\right)^2,$$

$$\Omega_\Lambda = \frac{\Lambda}{3}\left(\frac{c}{H}\right)^2, \quad \Omega_k = -k\left(\frac{c}{H}\right)^2 R_0^2.$$

Conformal Friedmann equations

In conformal time η, $\mathrm{d}\eta = \frac{\mathrm{d}t}{a}$, with comoving distance in η:

$\chi(\eta) = c\int_{t_1}^{t_0} \frac{\mathrm{d}t}{a(t)} = c\int_{\eta_1}^{\eta} \mathrm{d}\eta$, or with redshift $z = \frac{1}{a} - 1$: $\chi(z) = c\int_0^z \frac{\mathrm{d}z}{H(z)}$, follow the Friedmann conformal dimensionless equations [2] [3] [4] after rescaling $\frac{a}{R_H} \to a$, $c = 1$, conformal Friedmann equations:

$$(a')^2 + \frac{kc^2a^2}{R_H^2} = \frac{\Lambda c^2 a^4}{3} + \frac{8\pi G}{3}\rho a^4$$

$$a'' + \frac{kc^2 a}{R_H^2} = \frac{4\pi G}{3c^2}\left(\rho c^2 - 3P\right)a^3 + \frac{\Lambda c^2 a^3}{3}$$

and rescaled conformal:

$$(a')^2 + ka^2 = \frac{\Lambda_1 a^4}{3} + \frac{\rho_{crH}}{3}\rho a^4 \quad \text{scF1} \quad \frac{(a')^2}{a^2} = -k + \frac{\Lambda_1 a^2}{3} + \frac{\rho_{crH}}{3}\rho a^2 \tag{4a}$$

$$a'' + ka = \frac{\rho_{crH}}{6}(\rho - 3P)a^3 + \frac{\Lambda_1 a^3}{3} \quad \text{scF2} \tag{4b}$$

Friedmann radial equation

It is convenient to reformulate the first Friedmann equation in the form of velocity-potential equation, which we call here Friedmann radial equation [1] [2] [3] [4] [9].

We get the Friedmann radial equation

$$(\dot{a})^2 - \frac{K_s}{a^2} - \frac{K_m}{a} - \frac{\Lambda}{3}a^2 + k = 0 \tag{5}$$

it follows the potential form $\frac{\dot{a}^2}{c^2} + V(a) = -k$ with $c = 1$

$$V(a) = -\frac{K_s}{a^2} - \frac{K_m}{a} - \frac{\Lambda}{3}a^2$$

with Planck data we have

$$K_m = 0.423\times10^{26}\ \text{m}\ ,\quad K_s = 1.01\times10^{48}\ \text{m}^2\ ,\quad \Lambda = 1.1\times10^{-52}\ \text{m}^{-2}$$

dimensionless

$$K_{m1} = K_m / R_H = \Omega_{m,0} = 0.309$$

$$K_{s1} = K_s / R_H^2 = \Omega_{rad,0} = \Omega_{\gamma,0} + \Omega_{\nu,0} = 0.54\times10^{-4} + 0.0012 = 0.00125$$

$$\Lambda_1 = \Lambda R_H^2 = 1.1\times1.37^2 = 2.06$$

from this we get the dimensionless Friedmann radial equation

$$(\dot{a})^2 - \frac{K_{s1}}{a^2} - \frac{K_{m1}}{a} - \frac{\Lambda_1}{3}a^2 + k = 0 \qquad (5a)$$

2.2. Relative Density and Pressure (Relative to $c^2\rho_{crit,0}$)

In the following, we present the eos for the components radiation γ, neutrinos ν, electrons e, protons p, neutrons n, cdm d [2] [3] [4] [10] [11].

Relative density & pressure baryons b, CDM c, matter density $\rho_{m,r}$ dependent (Eth independent variable)

With thermal energy $E_{th} = k_B T$ matter density $\rho_{m,r} = \frac{K_{m1}}{a^3}$, b = baryon, c = cdm (cold dark matter)

$$\rho_{m,r}(a) = \rho_b + \rho_c\ ,\quad \rho_b(\rho_{m,r}) = \rho_{m,r}\frac{\Omega_{b,0}}{\Omega_{b,0}+\Omega_{c,0}}\ ,\quad \rho_c(\rho_{m,r}) = \rho_{m,r}\frac{\Omega_{c,0}}{\Omega_{b,0}+\Omega_{c,0}}\ ,$$

we have for the pressure before (1) and after (2) nucleosynthesis

$$P_{b,2}(\rho_b, E_{th}) = \rho_b\frac{E_{th}}{m_p c^2}\ ,\quad E_{th} > E_{c,ns}\ \ \text{ideal gas},\quad E_{mp} = m_p c^2 = 0.938\ \text{GeV}\ ,$$

using today's He-H-ratio $Y_{H,He} = \frac{\rho_{He}}{\rho_H} = \frac{4n_{He}}{n_H} = 0.25$, $\frac{\rho_{He}}{\rho_H} = \frac{4n_{He}}{n_H} = 0.25$

$$P_{b,1} = \frac{1+Y_{H,He}/4}{1+Y_{H,He}}\rho_b\frac{E_{th}}{m_p c^2} = 0.85\rho_b\frac{E_{th}}{m_p c^2}\ ,\quad E_{th} < E_{c,ns}\ ,\quad E_{c,ns} = 100\ \text{keV}\ ,$$

with the soft-1-0-step function for state-transition at ns = nucleosynthesis with transition energy $E_{c,ns} = 100\ \text{keV}$ (see chap. 9) we get the pressure

$$P_b(\rho_b, E_{th}) = P_{b,2}(\rho_b, E_{th}) + \left(P_{b,1}(\rho_b, E_{th}) - P_{b,2}(\rho_b, E_{th})\right)\Theta_{1-0}\left(E_{th}, E_{c,ns}, \delta_0 E_{c,ns}\right),$$

$$\delta_0 = 0.1,$$

$$P_c(\rho_c, E_{th}) = 0\ .$$

Relative density & pressure neutrinos

We have for neutrino density and pressure before (1) and after (2) neutrino decoupling [12] with threshold energy $E_{c,\nu} = 1\ \text{MeV}$:

$$\rho_{\nu,1}(\rho_b, E_{th}) = \frac{\Omega_{\nu b} n_b \frac{E_{th}}{c^2}}{\rho_{crit,0}} = \Omega_{\nu b}\rho_b\frac{E_{th}}{m_p c^2}\ ,\quad n_\nu = \Omega_{\nu b} n_b$$

$$\rho_{\nu,2}(\rho_b, E_{th}) = \Omega_{\nu b}\rho_b \frac{E_{th}}{m_p c^2}, \quad E_{th} > E_{c,\nu}, \text{ in thermal equilibrium,}$$

$$\rho_{\nu,1}(\rho_b, E_{th}) = \Omega_{\nu b}\rho_b \frac{E_{c.\nu}}{m_p c^2}\left(\frac{E_{c.\nu}}{E_{th}}\right)^{-3}, \quad E_{th} < E_{c,\nu} \quad \text{decrease with } \sim \tilde{a}^{-3}$$

$P_\nu(\rho_\nu) = \frac{1}{3}\rho_\nu$, parameters today $\Omega_{\nu,0} \approx 10^{-9}$, $T_{\nu,0} = 1.95\text{ K}$,

$E_{th,\nu 0} = k_B T_{\nu,0} = \frac{1.95\text{ K}}{300\text{ K}} \times 0.026\text{ eV} = 1.69 \times 10^{-4}\text{ eV}$, it follows

$$\Omega_{\nu,b} = \frac{n_{\nu,0}}{n_{b,0}} = \frac{\Omega_{\nu,0}}{\Omega_{b,0}}\frac{m_p c^2}{k_B T_{\nu,0}} = \frac{10^{-9}}{0.049}\frac{0.938\text{ GeV}}{1.69\times10^{-4}\text{ eV}} = 1.13\times10^5.$$

Relative density & pressure photons

The Stefan-Boltzmann law gives

$$\rho(T) = aT^4, \quad a = 7.56\times10^{-16}\frac{\text{J}}{\text{m}^3\cdot\text{K}^4} = 4.717\frac{\text{MeV}}{\text{m}^3\cdot\text{K}^4}, \quad a = 51.9\frac{4\pi k_B^4}{c^3 h^3} \tag{6}$$

$$\rho(E_{th}) = a_{SB}E_{th}^4$$

$$a_{SB} = \frac{207.6\pi}{h^3c^3} = \frac{a}{k_B^4} = \frac{7.56\times10^{-16}}{\left(1.38\times10^{-23}\right)^4}\frac{1}{\text{J}^3\cdot\text{m}^3}$$

$$= \frac{2.08\times10^{76}}{\left(6.24\times10^{18}\right)^3}\frac{1}{\text{eV}^3\cdot\text{m}^3} = 0.856\times10^{20}\frac{1}{\text{eV}^3\cdot\text{m}^3}$$

$$a_{SB} = 0.856\times10^{20}\frac{1}{\text{eV}^3\cdot\text{m}^3} = \frac{1}{\text{eV}^4}0.856\times10^{11}\frac{\text{GeV}}{\text{m}^3} = \frac{1}{\text{eV}^4}0.178\times10^{11}\rho_{Ecrit,0}$$

$$a_{SB0} = \frac{a_{SB}}{\rho_{Ecrit,0}} = \frac{1}{\text{eV}^4}0.178\times10^{11}.$$

Before photon decoupling the photon energy density is

$$\rho_\gamma(E_{th}) = a_{SB0}E_{th}^4, \quad P_\gamma(\rho_\gamma) = \frac{1}{3}\rho_\gamma$$

after photon decoupling at $E_{th} = E_{c,dc}$, $E_{c,dc} = 0.25\text{ eV}$, Planck $z_{dc} = 1090$, it becomes

$$\rho_\gamma(a, E_{th}) = a_{SB}\left(E_{c,dc}\frac{a(t_{c,dc})}{a}\right)^4, \quad E_{th} < E_{c,dc}, \quad a(t_{c,dc}) = \frac{1}{z_{dc}+1} = \frac{1}{1091}$$

at e-pair production and above photons lose energy and keep a mean energy

$$E \ge m_e c^2, \quad E_{th} \approx 2m_e c^2$$

at p-pair production and above photons lose energy and keep a mean energy

$$E \ge m_p c^2, \quad E_{th} \approx 2m_p c^2.$$

Temperature jumps at phase transitions

At recombination $E_{th} = E_{c,re}$, $E_{c,re} = 0.29\text{ eV}$ temperature goes up due to free electrons forming atoms with baryons,

before recombination:

$$n = n_b + n_e = 2n_b, \quad n_b = n_e, \quad E_{th} = E_{c,re} \frac{a(t_{c,re})}{a(t)}, \quad a(t_{c,re}) = \frac{1}{z_{re}+1} = \frac{1}{1271},$$

$$z_{re} = 1270, \quad t_{c,re} = 1.16 \times 10^{13}$$

after recombination: Saha equation:

$$X_e(E_{th}) = \frac{n_e}{n_e + n_H} = \frac{n_e}{n_b} = \frac{-1+\sqrt{1+4f(E_{th})}}{2f(E_{th})} \tag{7}$$

$$n = n_b + n_e = n_b\left(1 + X_e(E_{th})\right), \quad E_{H,re} = 13.6\ \text{eV}$$

$$f(E_{th}) = 4\zeta(3)\sqrt{\frac{2}{\pi}}\eta\left(\frac{E_{th}}{m_e c^2}\right)^{3/2}\exp\left(\frac{E_{H,re}}{E_{th}}\right) = 2.26\times10^{-9}\left(\frac{E_{th}}{m_e c^2}\right)^{3/2}\exp\left(\frac{E_{H,re}}{E_{th}}\right).$$

The equation for E_{th} after recombination with $E_H = E_{H,re}$, $E_m = m_e c^2$ is:

$$\frac{\mathrm{d}E_{th}}{\mathrm{d}a} = -\frac{E_{th0}}{a^2} - E_{H,re}\frac{\mathrm{d}X_e}{\mathrm{d}f}\frac{\mathrm{d}f}{\mathrm{d}a}\frac{\mathrm{d}E_{th}}{\mathrm{d}a}, \quad \frac{\mathrm{d}E_{th}}{\mathrm{d}a}\left(1 + E_{H,re}\frac{\mathrm{d}X_e}{\mathrm{d}f}\frac{\mathrm{d}f}{\mathrm{d}a}\right) = -\frac{E_{th0}}{a^2}$$

with solution $E_{th,a}(a)$ [13] shown in **Figure 1**.

$$E_{th,a}(1) = E_{th,0} = 0.000663\ \text{eV}, \quad E_{th,a}\left(a_{re} = 1/(z_{re}+1)\right) = 0.2842\ \text{eV} \approx E_{c,re}.$$

At nucleo-synthesis $E_{th} = E_{c,ns}$, $E_{c,ns} = 100\ \text{keV}$ temperature goes up due to helium synthesis with energy released $E_{He,ns} = 12\ \text{MeV}$, thermal energy behavior is analogously for $E_{c,re} < E_{th} < E_{He,ns}$, $z_{re} = 4\times10^8$

$$E_{th} \approx E_{c,ns}\frac{a(t_{c,ns})}{a(t)}\left(1 + 0.021\left(\left(\frac{E_{c,ns}a(t_{c,ns})}{m_p c^2 a(t)}\right)^{-3/4}\exp\left(-\frac{E_{He,ns}a(t)}{2E_{c,ns}a(t_{c,ns})}\right) - \left(\frac{E_{c,ns}a(t_{c,ns})}{m_p c^2 a(t_{He,ns})}\right)^{-3/4}\exp\left(-\frac{E_{He,ns}a(t_{He,ns})}{2E_{c,ns}a(t_{c,ns})}\right)\right)\right)$$

where the baryon temperature depends on the photon temperature

$$T_b' = -2\frac{a'}{a}T_b + \frac{8}{3}\frac{m_b}{m_e}\frac{\bar{\rho}_\gamma}{\bar{\rho}_b}an_e\sigma_T\left(T_\gamma - T_b\right) \quad \text{with} \quad a' = \frac{\mathrm{d}a}{\mathrm{d}\eta} \quad [14].$$

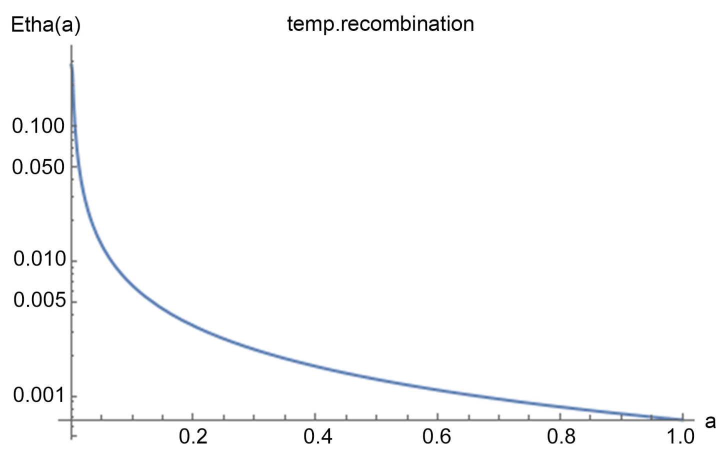


Figure 1. Temperature after recombination $E_{th,a}(a)$ in eV.

Density electrons

The density of electrons is described by the Peebles equation with the parameters

$C_r(T) \equiv \frac{\Lambda_{2\gamma}+\Lambda_\alpha}{\Lambda_{2\gamma}+\Lambda_\alpha+\beta_\alpha}$, $\Lambda_{2\gamma} = 8.227\ \mathrm{s}^{-1}$,

$\Lambda_\alpha = \frac{27}{128\zeta(3)} \frac{H(T)}{(1-X_e)(n_b/n_\gamma)(k_B T/E_I)^3}$, $\beta_\alpha = \beta(T)\exp\left(\frac{3E_I}{4k_B T}\right)$,

$E_I = 13.6\ \mathrm{eV}$ = hydrogen ionization energy, 1s ionization rate, $n_{1s} \approx (1-X_e) n_b$,

$n_b = \eta n_\gamma$, $\lambda_\alpha = \frac{8\pi\hbar c}{3E_I}$ Lyman wavelength,

$$\beta(T) = \langle \sigma v \rangle \left(\frac{m_e c^2 k_B T}{2\pi\hbar^2 c^2}\right)^{3/2} \exp\left(-\frac{E_I}{k_B T}\right)$$

$$\alpha(T) \approx 9.8 \frac{\alpha^2}{(m_e c^2)^2} \left(\frac{E_I}{k_B T}\right)^{1/2} \log\left(\left(\frac{E_I}{k_B T}\right)\right)$$

we get the Peebles equation ([4] 3.153) for the hydrogen ionization percentage

$$\frac{\mathrm{d}X_e}{\mathrm{d}z} = -\frac{C_r(T)}{H(z)(1+z)}\left(\left(\frac{m_e c^2 k_B T}{2\pi}\right)^{1/2} (1-X_e)\exp\left(-\frac{E_I}{k_B T}\right) - \alpha(T)\frac{n_b}{n_\gamma}\frac{2\zeta(3)}{\pi^2}(k_B T)^3 X_e^2\right) \quad (8)$$

where

$$H(z) = \sqrt{\Omega_m} H_0 (1+z)^{3/2}\left(1+\frac{1+z}{1+z_{eq}}\right), \quad H_0 \approx 1.5\times 10^{-33}\ \mathrm{eV}$$

$$T = (1+z)\, 0.235\ \mathrm{eV}.$$

We get for the electron density before (1) and after (2) recombination

$$\rho_{e,1}(\rho_b, E_{th}) = \rho_b \frac{E_{th}}{m_p c^2}, \quad E < E_{c,ep}, \quad E_{c,ep} = m_e c^2 = 511\ \mathrm{keV}$$

$$n_{e+} \approx \frac{n_b^2}{n_\gamma} 0.17\alpha \left(\frac{E_{th}}{m_e c^2}\right)^2 = \frac{n_b^2}{n_\gamma}\left(\frac{E_{th}}{m_e c^2}\right)^2 1.2\times 10^{-3}, \quad n_b = \Omega_{b,0}\frac{\rho_{crit,0}}{m_p}$$

$$\frac{n_b}{n_\gamma} = \frac{n_{b,0}}{n_{\gamma,0}}\frac{a_0^3 E_{th,0}^3}{a^3 E_{th}^3} = \frac{n_{b,0}}{n_{\gamma,0}} = \frac{0.242\ \mathrm{m}^{-3}}{0.41\times 10^{-3}\ \mathrm{m}^{-3}} = 590 \quad \text{scale-independent}$$

follows $\frac{n_{e+}}{n_b} \approx \frac{n_b}{n_\gamma} 0.17\alpha\left(\frac{E_{th}}{m_e c^2}\right)^2 = \left(\frac{E_{th}}{m_e c^2}\right)^2 0.708$,

$$\rho_{e,2}(\rho_b, E_{th}) = \rho_b\left(1+\frac{2n_{e+}}{n_b}\right)\frac{E_{th}+m_e c^2}{m_p c^2}, \quad E > E_{c,ep}$$

due to Saha equation

$$\rho_{e,0}(\rho_b, E_{th}) \approx \rho_{e,1}(\rho_b(t_{c,re}), E_{c,re}) \exp\left(E_{H,re}\left(\frac{1}{E_{c,re}} - \frac{1}{E_{th}}\right)\right)$$
$$= \rho_b(a(t_{c,re})) \frac{m_e}{m_p} \exp\left(E_{H,re}\left(\frac{1}{E_{c,re}} - \frac{1}{E_{th}}\right)\right)$$

alternatively

$$n_e = n_b X_e(E_{th}), \quad \rho_e = \frac{m_e c^2}{m_p c^2} \rho_b X_e(E_{th})$$

$$E < E_{c,re}, \quad E_{c,re} = 0.29\ \text{eV}, \quad \rho_b(t_{c,re}, E_{c,re}) = \Omega_{b,0} z_{re},$$

$$\Omega_{b,0} = 0.0486, \quad z_{re} = 1270, \quad a(t_{c,re}) = \frac{1}{z_{re}+1} = \frac{1}{1271}.$$

Fermi pressure electrons

The pressure of electrons is the Fermi pressure P_{Fe} of a (spin_1/2) fermion gas

$$P_e(\rho_e, E_{th}) = P_{Fe}(\rho_e, E_{th})$$

with low- and high-density limits $P_1 = \frac{1}{5} n p_F c$, $P_2 = \frac{2}{5} n E_F$.

Fermi energy $E_F = \sqrt{(p_F c)^2 + (m_e c^2)^2}$, $p_F c = \hbar c (3\pi^2 n)^{1/3}$

$$P_{Fe}(\rho, E) = P_2(\rho) + (P_1(\rho) - P_2(\rho)) \Theta_{1-0}(E, m_e c^2, \delta_0 m_e c^2) \tag{9}$$

$$\rho_{cr} = \rho_{crit,0} c^2 = 0.77 \times 10^{-10}\ \text{J} \cdot \text{m}^{-3} = 0.484 \times 10^3\ \text{MeV} \cdot \text{m}^{-3}$$

$$n_{p,0} = \frac{\rho_{crit,0} c^2 \Omega_{b,0}}{m_p c^2} = \frac{0.484 \times 10^3\ \text{MeV} \cdot \text{m}^{-3} \times 0.047}{0.938\ \text{GeV}} = 0.0242\ \text{m}^{-3}$$

$$\hbar c = 1.96 \times 10^{-16}\ \text{GeV} \cdot \text{m} = 1.96 \times 10^{-5}\ \text{eV} \cdot \text{m}$$

$$n_e = \frac{\rho_{crit,0} c^2 \rho_e}{m_e c^2} = n_{p,0} \frac{m_p}{\Omega_{b,0} m_e} \rho_e = 0.0242\ \text{m}^{-3} \rho_e 39.0 \times 10^3 = 943.8 \rho_e$$

$$\frac{n_e}{n_{p,0}} = \frac{m_p}{\Omega_{b,0} m_e} \rho_e = 339055.6 \rho_e.$$

For electrons we get the expressions

$$P_1 = \frac{1}{5} \frac{n p_F c}{\rho_{crit,0}} = \frac{1}{5}\left(\frac{n_e \Omega_{b,0}}{n_{p,0}}\right) \frac{p_F c}{m_p c^2} = \frac{1}{5}\left(\frac{m_p}{m_e} \rho_e\right) \frac{p_F c}{m_p c^2} = \frac{1}{5}\left(\frac{m_p}{m_e} \rho_e\right) \frac{201.78 (\rho_e)^{1/3}}{m_p c^2}$$

$$P_2 = \frac{2}{5} \frac{n E_F}{\rho_{crit,0}} = \frac{1}{5}\left(\frac{n_e \Omega_{b,0}}{n_{p,0}}\right) \frac{E_F}{m_p c^2} = \frac{1}{5}\left(\frac{m_p}{m_e} \rho_e\right) \frac{\sqrt{(p_F c)^2 + (m_e c^2)^2}}{m_p c^2}$$

$$p_F c = \hbar c (3\pi^2 n_{e,0})^{1/3} (\rho_e)^{1/3} 33.91$$
$$= 1.96 \times 10^{-5}\ \text{eV} \cdot \text{m} (3\pi^2 0.947 \times 10^3\ \text{m}^{-3})^{1/3} 33.91 (\rho_e)^{1/3}$$
$$= 201.78 (\rho_e)^{1/3}\ \text{eV}$$

$$p_F c = 201.78 (\rho_e)^{1/3}\ \text{eV}.$$

State transitions radiation γ, neutrinos ν, electrons e, protons p, neutrons n, cdm d.

Generally, the density state transition from ρ_1 to ρ_2 at transition temperature T_c (transition thermal energy $E_c = k_B T_c$) has the form

$$\rho(E) = \rho_2 + (\rho_1 - \rho_2)\Theta_{1-0}(E, E_c, \delta E_c),$$

with soft-0-1-step function $\Theta_{0-1}(E, E_c, \delta E_c) = \dfrac{1}{1+\exp\left(\dfrac{E_c - E}{\delta E_c}\right)}$,

with soft-1-0-step function $\Theta_{1-0}(E, E_c, \delta E_c) = \dfrac{1+\exp\left(-\dfrac{E_c}{\delta E_c}\right)}{1+\exp\left(\dfrac{E - E_c}{\delta E_c}\right)}$,

where δE_c is the standard deviation of E_c.

We can set approximately $\dfrac{\delta E_c}{E_c} = \dfrac{\delta T_c}{T_c} \approx \dfrac{\delta T_0}{T_0}$, where (measured in CMB) $\dfrac{\delta T_0}{T_0} = \dfrac{\Delta T_{\gamma,0}}{T_{\gamma,0}} \approx \dfrac{30\ \mu\text{K}}{2.72\ \text{K}} = 1.1\times 10^{-5}$.

2.3. Transition Thermal Energies and Eos

-neutrino decoupling $E_{c,\nu} = 1\ \text{MeV}$, $t_{c,\nu} = 1\ \text{s}$, $\rho_{1c,\nu} = \rho_{1,\nu}(t_{c,\nu})$,

$$\rho_{1,\nu}(E_{th}) = E_{th},\quad \rho_{2,\nu}(E_{th}, a) = \rho_{1c,\nu}\left(\frac{a}{a(t_{c,\nu})}\right)^4;$$

-e-p-annihilation

$E_{c,ep} = 0.5\ \text{MeV}$, $t_{c,ep} = 6\ \text{s}$, $n_\gamma = a_{SB}E_{th}^4$ for all t $a = 7.56\times 10^{-16}\ \text{J/m}^2\cdot\text{K}^4$, $a = 51.9\dfrac{4\pi k_B^4}{c^3h^3}$

$\rho_{1,e} = \left(n_b + n_{e+}(t_{c,ep})\right)m_e$, $\rho_{2,e} = n_b m_e$ with

$$n_{e+} \approx \frac{n_b^2}{n_\gamma}0.17\alpha\left(\frac{E_{th}}{m_e c^2}\right)^2 = \frac{n_b^2}{n_\gamma}\left(\frac{E_{th}}{m_e c^2}\right)^2 1.2\times 10^{-3};$$

-photon recombination

$$E_{c,re} = 0.29\ \text{eV},\quad t_{c,re} = 290\ \text{ky},\quad \rho_{2c,re} = \rho_{1c,re} + n_b(t_{c,re})E_{c,re}$$

$$\rho_{1,e} = n_b m_e,\quad \rho_{2,e} = \frac{1}{2}\rho_{1,e}\exp\left(\frac{E_{th} - E_{c,re}}{E_{th}}\right);$$

-photon decoupling

$$E_{c,\gamma} = 0.25\ \text{eV},\quad t_{c,\gamma} = 370\ \text{ky},\quad \rho_{1c,\gamma} = \rho_{1,\gamma}(t_{c,\gamma}),$$

$$\rho_{1,\gamma}(E_{th}) = E_{th},\quad \rho_{2,\gamma}(E_{th}, a) = \rho_{1c,\gamma}\left(\frac{a}{a(t_{c,\gamma})}\right)^4;$$

-nucleo-synthesis helium

$E_{c,ns} = 100\text{ keV}$, $t_{c,ns} = 3\text{ min}$, $4\text{p}^+ + 2\text{e}^- \to \text{He}^{2+}$, ratio $\frac{\rho_{He}}{\rho_p} = 0.25$, eos transition $1 \to 2$ with ideal gas $P_1 = n_b E_{th} = \rho_b \frac{E_{th}}{m_p}$, $t < t_{c,ns}$, with ideal gas $P_2 = n_{b,1}\left(0.75 + 0.25/4\right) E_{th} = n_{b,1} 0.81 E_{th} = 0.81 \rho_b \frac{E_{th}}{m_p}$, $t < t_{c,ns}$.

3. Parameters

The simple ΛCDM model is based on seven parameters: physical baryon density parameter $\Omega_b h^2$; physical matter density parameter $\Omega_m h^2$; the age of the universe t_0; scalar spectral index n_s; curvature fluctuation amplitude A_s; and reionization optical depth τ, dark energy density Ω_Λ.

The parameters of the ΛCDM are given in the following table (Table 1).

11 independent parameters: $\Omega_b h^2$, $\Omega_c h^2$, t_0, n_s, Δ_R^2, τ, Ω_ρ, w, Σm_ν, $N_{eff}(\nu)$, A_s;

7 fixed parameters r, $\mathrm{d}n_s/\mathrm{d}\ln k$, H_0, Ω_b, Ω_c, Ω_m, Ω_Λ;

5 calculated parameters ρ_{crit}, σ_8, z_{dec}, t_{dec}, z_{re};

13 total parameters Ω_b, Ω_c, t_0, n_s, A_s, τ, Ω_Λ, w, Σm_ν, $N_{eff}(\nu)$, r, $\mathrm{d}n_s/\mathrm{d}k$, H_0;

derived parameters ρ_{crit}, σ_8, z_{dec}, t_{dec}, z_{re}, $\omega_b = \Omega_b h^2$, $\omega_m = \Omega_m h^2$.

Table 1. Planck Collaboration Cosmological parameters [15].

	Description	Symbol	Value
Independent parameters 11	Physical baryon density parameter	$\Omega_b h^2$	0.02230 ± 0.00014
	Physical dark matter density parameter	$\Omega_c h^2$	0.1188 ± 0.0010
	Age of the universe	t_0	13.799 ± 0.021 × 10^9 years
	Scalar spectral index	n_s	0.9667 ± 0.0040
	Curvature fluctuation amplitude, $k_0 = 0.002\ \text{Mpc}^{-1}$	Δ_R^2	2.441 + 0.088 − 0.092 × 10^{-9}
	Reionization optical depth	τ	0.066 ± 0.012
Fixed parameters 7	Total density parameter	Ω_{tot}	1
	Equation of state of dark energy	w	−1
	Sum of three neutrino masses	Σm_ν	0.06 eV/c^2
	Effective number of relativistic degrees of freedom	N_{eff}	3.046
	Scalar amplitude	A_s	(2.215 ± 0.13)
	Tensor/scalar ratio	r	0
	Running of spectral index	$\mathrm{d}n_s/\mathrm{d}\ln k$	0
Calculated values 5	Hubble constant	H_0	67.74 ± 0.46 km·s^{-1}·Mpc^{-1}
	Baryon density parameter	Ω_b	0.0486 ± 0.0010
	Dark matter density parameter	Ω_c	0.2589 ± 0.0057
	Matter density parameter	Ω_m	0.3089 ± 0.0062
	Dark energy density parameter	Ω_Λ	0.6911 ± 0.0062
	Critical density	ρ_{crit}	(8.62 ± 0.12) × 10^{-27} kg/m^3
	Fluctuation amplitude at 8 h^{-1} Mpc	σ_8	0.8159 ± 0.0086
	Redshift at decoupling	z_*	1089.90 ± 0.23
	Age at decoupling	t_*	377,700 ± 3200 y
	Redshift of reionization (with uniform prior)	z_{re}	8.5 + 1.0 − 1.1

The additional parameters of the extended ΛCDM are given in the second table (Table 2).

Some specifications

The amplitude A_s, is determined by the CMB power spectrum

$$\Delta_R^2\left(k^2\right) = A_s\left(\frac{k}{k_0}\right)^{n_s-1}, \quad k_0 \approx 0.05\ \text{Mpc}^{-1}.$$

The relative current Hubble parameter is $h = \frac{H_0}{100}$.

The fluctuation amplitude is defined by $\sigma_8 = \sigma\left(\rho_{mat}, R\right)_{R=8h^{-1}\,\text{Mpc}}$, where $\sigma\left(\rho_{mat}, R\right) = \text{stdev}\left(\rho_{mat}\right)$ smoothed by distance R ([2]).

Key cosmological events

Key cosmological events calculated from the ΛCDM model with temperature, energy scale and cosmic time are given below [4] [16] in **Table 3**.

Table 2. Extended model parameters [15].

Description	Symbol	Value
Total density parameter	Ω_{tot}	1.0023 + 0.0056 − 0.0054
Equation of state of dark energy	w	−0.980 ± 0.053
Tensor-to-scalar ratio	r	<0.11, k_0 = 0.002 Mpc^{-1} (2σ)
Running of the spectral index	$\mathrm{d}n_s/\mathrm{d}\ln k$	−0.022 ± 0.020, k_0 = 0.002 Mpc^{-1}
Physical neutrino density parameter	$\Omega_\nu h^2$	<0.0062
Sum of three neutrino masses	Σm_ν	<0.58 eV/c^2 (2σ)

Table 3. Key cosmological events ([4], chap. 2).

Event	Temperature	Energy	Time
Inflation ends	10^{29} K	10^{16} GeV	10^{-35} s
CDM decouples, GUT scale	10^{29} K	10^{15} GeV	10^{-36} s
Baryons form	10^{16} K	1 TeV?	10^{-12} s
El-weak force	10^{15} K	100 GeV	10^{-11} s
Hadrons form	10^{12} K	150 MeV	10^{-5} s
Neutrinos decouple	10^{10} K	1 MeV	1 s
Nuclei form	10^{9} K	100 keV	200s
Atoms form	3460 K	0.29 eV	290 ky
Photons decouple	2970 K	0.25 eV	370 ky
First stars	50 K	4 meV	100 My
First galaxies	12 K	1 meV	400 My
Dark energy domination	3.8 K	0.33 meV	9 Gy
Now	2.7 K	0.24 meV	13.8 Gy

4. Inflation

The "naive" so called Hot-Big-Bang model has several aspects, which are in disagreement with cosmological observations.

Hot Big-bang problems

- the observed homogeneity of the present universe (distances > 200 Mly) should arise from arbitrary initial conditions: **horizon problem**;

- the observed curvature is small: **flatness problem**;

- the observed correlation regions in the CMB have supraluminal distance: **superhorizon correlations**.

Cosmological inflation

In the approximation that the expansion is exactly exponential, the horizon is static, *i.e.* $H = \frac{\dot{a}}{a} \approx const$, and we have an inflating universe [17]. This inflating universe can be described by the de-Sitter metric [1] [2] [3] [5]

$$\mathrm{d}s^2 = -\left(1-\Lambda r^2\right)c^2\mathrm{d}t^2 + \frac{1}{1-\Lambda r^2}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \tag{10a}$$

For the case of exponential expansion, the equation of state is $P = -\rho$, with world radius

$$R(t) = R_0 \exp\left(ct\sqrt{\frac{\Lambda}{3}}\right) \tag{10b}$$

The expansion generates an almost-flat and large-scale-homogeneous universe, as it is observed today.

Furthermore, horizon $R_H = \dot{a}^{-1} = (Ha)^{-1}$ reaches a minimum at the end of inflation, and then rises again, this explains superluminal correlations in the present universe.

Inflation in Ashtekar-Kodama quantum gravity [18]

Inflation takes place between $r_i = l_p = 1.61\times10^{-35}\ \mathrm{m}$ and $R_{\inf} = r_{gr} = 3.1\times10^{-5}\ \mathrm{m}$ with expansion factor $f_{\inf} = \exp\left(r_{\inf}\sqrt{\frac{\Lambda}{3}}\right) = 1.9\times10^{30}$, $r_{\inf} = 2\times10^{-26}\ \mathrm{m}$,

$E_{\inf} = \frac{\hbar c}{r_{\inf}} = \frac{1.96\times10^{-16}\ \mathrm{GeV}}{2\times10^{-26}\ \mathrm{m}} = 0.98\times10^{10}\ \mathrm{GeV}$, $t_{\inf} = \frac{r_{\inf}}{c} = 0.66\times10^{-34}\ \mathrm{s}$,

$R_{\inf} = 10^{-2}\ \mathrm{m}$.

Inflation with standard assumptions ([4], chap. 4)

$r_i = 3\times10^{-28}\ \mathrm{m}$, $t_{\inf} = 10^{-36}\ \mathrm{s}$, $f_{\inf} = 10^{30}$, $a_{\inf} = 10^{-28}$, $R_{\inf} = 3\times10^{2}\ \mathrm{m}$,

$$f_{\inf} = \exp\left(r_{\inf}\sqrt{\frac{\Lambda}{3}}\right),\quad \Lambda = 3\left(\frac{\log\left(f_{\inf}\right)}{r_{\inf}}\right)^2 = 1.4\times10^{60}\ \mathrm{m}^{-2},$$

$$H = \sqrt{\frac{\Lambda}{3}} = \frac{\log\left(f_{\inf}\right)}{r_{\inf}} = 6.9\times10^{29}\ \mathrm{m}^{-1}.$$

Assessment of the inflation factor ([3], chap. 4),

f = end inflation, *i* = start inflation, *eq* = matter-radiation-equality, *0* = today, *ER* = *f* = expansion rate

$$\frac{a(t_f)}{a(t_{in})} = \exp N\,,\quad N \gg \log\left(\frac{T_f}{T_{eq}}\right) + \frac{1}{2}\log\left(\frac{T_{eq}}{T_0}\right),$$

$$T_f \simeq 10^{16}\,\text{GeV}\,,\quad T_{eq} \simeq 1\,\text{eV}\,,\quad T_0 \simeq 10^{-4}\,\text{eV}$$

$$N \ge 60\,,\quad \Delta t \ge \frac{60}{H(t_f)} \simeq 60\sqrt{\frac{3}{8\pi G\rho_{ER}}}\left(\frac{T_0}{T_f}\right)^2 \approx 10^{-37}\,\text{s}\,.$$

Inflaton model $\phi(t,x)$ with GR-action

The action is ([3], chap. 4)

$$S = \int \mathrm{d}^4x\sqrt{-g}\left(L_{EH} + L_\phi\right)$$

with the Einstein-Hilbert action of GR

$$S_{EH} = \int\left(\frac{R-2\Lambda}{2\kappa}\right)\sqrt{-g}\,\mathrm{d}^4x$$

$$L_{EH} = \frac{R-2\Lambda}{2\kappa}$$

and the inflaton action

$$S_\phi = \int \mathrm{d}^4x\sqrt{-g}\left(\frac{\hbar c}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right)$$

$$L_\phi = \frac{\hbar c}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)$$

with energy-momentum $T_{\mu\nu} = \hbar c\partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{\hbar c}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right)$

$$T_0^0 = \hbar c\frac{\dot\phi^2}{2} + V(\phi)\,,\quad T_i^j = -\delta_i^j\left(\hbar c\frac{\dot\phi^2}{2} - V(\phi)\right).$$

For RW-metric the action is $S = \int \mathrm{d}^4x\sqrt{-g}\left(\hbar c\left(-\frac{\dot\phi^2}{2} + \frac{1}{2a^2}(\nabla\phi)^2\right) - V(\phi)\right)$

with eom = Klein-Gordon equation $\ddot\phi + 3H\dot\phi + \frac{1}{\hbar c}\frac{\mathrm{d}V(\phi)}{\mathrm{d}\phi} = 0$

which represents an oscillator with Hubble-friction $3H\dot\phi$

and energy density $\rho_\phi = \hbar c\frac{\dot\phi^2}{2} + V(\phi)$,

and pressure $P_\phi = \hbar c\frac{\dot\phi^2}{2} - V(\phi)$ (4.50).

If $E_{kin} \equiv \frac{1}{2}\dot\phi^2 \ll E_{pot} \equiv V(\phi)$, $E_{kin} = \hbar c\frac{\dot\phi^2}{2} \ll E_{pot} = V(\phi)$, we have $P_\phi \approx -\rho_\phi$ *i.e.* equation-of-state of dark energy Ω_Λ generating temporary inflation.

We get the **Friedmann equations** (radiation-matter density ρ_{rm} added)

$$H^2 = \frac{\kappa}{3}\rho_E = \frac{\kappa}{3}\left(\hbar c\frac{\dot\phi^2}{2} + V(\phi) + \rho_{rm}\right) \tag{11a}$$

$$\dot H = -\frac{\kappa}{2}\left(\rho_\phi + P_\phi - \rho_{rm} - P_{rm}\right) = -\frac{\kappa}{2}\left(\hbar c\dot\phi^2 - \frac{4}{3}\rho_{rm}\right) \tag{11b}$$

and the **Klein-Gordon equation**

$$\ddot{\phi} + 3H\dot{\phi} + \frac{1}{\hbar c}\frac{\mathrm{d}V(\phi)}{\mathrm{d}\phi} = 0 \tag{11c}$$

We get dimensionless 2 equations in Planck-units $l_{Pl} = 1.62\times10^{-35}\ \text{m}$,

$$\rho_{rm} = \frac{3}{8\pi}H^2 - \frac{\dot{\phi}^2}{2} - V(\phi)$$

Friedmann $\dot{H} = -4\pi\left(\dot{\phi}^2 - \frac{4}{3}\left(\frac{3}{8\pi}H^2 - \frac{\dot{\phi}^2}{2} - V(\phi)\right)\right) = -4\pi\left(\frac{3\dot{\phi}^2}{2} - \frac{H^2}{2\pi} + \frac{4}{3}V(\phi)\right)$.

Klein-Gordon $\ddot{\phi} + 3H\dot{\phi} + \frac{\mathrm{d}V(\phi)}{\mathrm{d}\phi} = 0$.

Slow-roll approximation

If $E_{kin} \equiv \frac{1}{2}\dot{\phi}^2 \ll E_{pot} \equiv V(\phi)$ or $\varepsilon_H \ll 1$, $\varepsilon_H \equiv -\frac{\dot{H}}{H^2}$ (slow-roll parameter 1), and almost constant velocity, $\eta_H = -\frac{\ddot{\phi}}{H\dot{\phi}} \ll 1$ (slow-roll parameter 2), we have persisting slow-roll condition $\varepsilon_H \ll 1$, $\eta_H \ll 1$ (**slow-roll approximation**), which yields approximate fundamental equations with approximations $3H\dot{\phi} \approx -V'$ and $3H^2 \approx 8\pi GV$ and $\varepsilon_H = -\frac{\dot{H}}{H^2} = -\frac{V'}{2V}\frac{\dot{\phi}}{H} = \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2$ and $\eta_H = -\frac{\ddot{\phi}}{H\dot{\phi}} = \frac{V''}{3H^2} = \frac{1}{8\pi G}\left(\frac{V''}{V}\right)$ and for the scale factor

$$a(t) = a(t_{in})\exp\left(\int_{t_{in}}^{t} H(t)\mathrm{d}t\right) = a(t_{in})\exp\left(-8\pi G\int_{t_{in}}^{t}\frac{V}{V'}\mathrm{d}\phi\right).$$

Square potential

We use the square potential $V(\phi) = c_1 + c_2(\phi-\phi_0)^2$, $c_1 = 1.16\times10^{-124}$, slow-roll condition: $c_1 \ll c_2$ with the minimum value $V(\phi_0) = c_1 = \frac{\Lambda}{\kappa} = 1.16\times10^{-124}$ and $r_{\inf} = 2\times10^{-26}\ \text{m}$, we get the following relations:

$$a(t) = a(t_{in})\exp\left(\int_{t_{in}}^{t} H(t)\mathrm{d}t\right) = a(t_{in})\exp\left(-8\pi\int_{t_{in}}^{t}\frac{V}{V'}\mathrm{d}\phi\right)$$

$$a(t) = a(t_{in})\exp\left(4\pi\int_{0}^{\phi_0}(\phi-\phi_0)\mathrm{d}\phi\right) = a(t_{in})\exp\left(2\pi\phi_0^2\right)$$

$$\phi_0 = \sqrt{\frac{1}{2\pi}\log\left(\frac{a(t)}{a(t_{in})}\right)} = \sqrt{\frac{1}{2\pi}\log(f_{\inf})} = 3.31$$

$$\varepsilon_H = \frac{1}{16\pi}\left(\frac{V'}{V}\right)^2 = \frac{1}{16\pi}\left(\frac{2}{\frac{c_1}{c_2(\phi-\phi_0)} + (\phi-\phi_0)}\right)^2 \approx \frac{1}{4\pi}\frac{1}{(\phi-\phi_0)^2}$$

$$\eta_H = \frac{1}{8\pi}\left(\frac{V''}{V}\right) = \frac{1}{8\pi}\left(\frac{2c_2}{c_1 + c_2(\phi-\phi_0)^2}\right) \approx \frac{1}{4\pi}\frac{1}{(\phi-\phi_0)^2}$$

$$\rho_{rm} = \frac{3}{8\pi}H^2 - \frac{\dot{\phi}^2}{2} - V(\phi)$$

for $t \to \infty$, $\dot{\phi} = \delta c_1 \ll 1$, $H = H_0$, $\phi \to \phi_0$, $\rho_{rm} = \left(\frac{3}{8\pi}H_0^2 - c_1\right) = 0$,

so condition for convergence is: $c_1 = \frac{3}{8\pi}H_0^2$.

The fundamental equations become

Friedmann $\dot{H} = -4\pi\left(\dot{\phi}^2 - \frac{4}{3}\left|\frac{3}{8\pi}H^2 - \frac{\dot{\phi}^2}{2} - V(\phi)\right|\right)$;

Klein-Gordon $\ddot{\phi} + 3H\dot{\phi} + \frac{\mathrm{d}V(\phi)}{\mathrm{d}\phi} = 0$;

slow-roll $\dot{H} \approx -6\pi\dot{\phi}^2$;

3 boundary conditions for $t = l_{Pl} = 1$: $H(1) = H_1$, $\phi(1) = \phi_1$, $\dot{\phi}(1) = \phi_{d1}$;

with 3 potential parameters c_1, c_2, ϕ.

Example: $\delta c_1 = 0.05$, $H_0 = 5$, $\phi_0 = 2.3$, $c_1 = 3$, $c_2 = 1$ [13].

Below in **Figure 2** and **Figure 3** are inflaton amplitude and Hubble parameter.

5. Background Calculations

There are basically two possible ways for background calculation:

-numerical solution of two Friedmann equations in two variables, calculating backward from boundary conditions at present time x_0;

-analytical solution, where the second equation is solved analytically, and inserted into the first, which gives an integral, which is calculated numerically.

The numerical solution encounters the problem of limited convergence: it stops at some time x_c.

The analytical solution avoids the convergence problem, and **this solution scheme is used in the calculation** of results presented below.

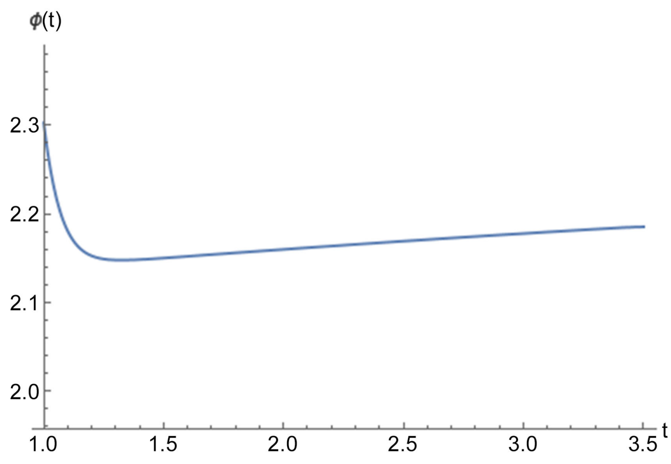


Figure 2. Inflaton amplitude $\phi(t)$.

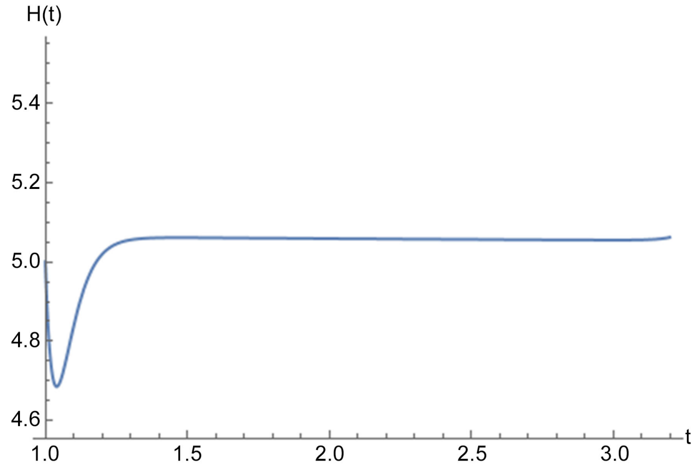


Figure 3. Hubble parameter $H(t)$.

5.1. Numerical Solution

We solve for dimensionless function variables a, ρ_r, in dimensionless relative time variable $x = \frac{tc}{R_H}$, limits $0 \le x \le x_{00} = 0.96$, where the upper limit is the relative cosmic time today $x_{00} = \frac{ct_0}{R_H} = \frac{R_0}{R_H} = 0.96$, from Planck data $t_0 = 13.9 \times 10^9$ y, with boundary conditions: $\rho_r(x_0) = \Omega_{m,0} + \Omega_{rad,0}$, $a(x_0) = 1$, $a'(x_0) = 1$ (because $H(x_0) = R_H$) from $a'(x_0) = 1$ follows $k_0 = -0.0042$ which is compatible with Planck data

$$(a')^2 + k_0 - \frac{\Lambda_1}{3} a^2 - \rho_r a^2 = 0 \quad \text{sF1} \tag{3a}$$

$$a''a - \frac{1}{3}\Lambda_1 a^2 = -\frac{a^2 \rho_{crH}}{2}\left(P_r + \frac{\rho_r}{3}\right) \quad \text{sF2} \tag{3b}$$

$$a''a + 2(a')^2 + 2k - \Lambda_1 a^2 + \frac{\rho_{crH}}{2}(P_r - \rho_r)a^2 = 0 \quad \text{sF3} \tag{3c}$$

$$\frac{\rho_r' a}{3} + a'(P_r + \rho_r) = 0 \quad \text{sF4} \tag{3d}$$

The two independent (3c and 3d is derived) Equations (3a, 3d) are non-linear second-order differential equations quadratic in the variables a, ρ_r.

Alternatively, one can solve for function variables a, $E_{th} = k_B T$, the latter with thermal energy $E_{th} = k_B T$, photon density $\rho_\gamma = a_{SB0} E_{th}^4$, $P_\gamma(\rho_\gamma) = \frac{1}{3}\rho_\gamma$, matter density $\rho_{mat} = \rho_b + \rho_c = \frac{K_m a}{K_s + K_m a}\rho_r$, baryon density $\rho_b = \rho_{mat}\frac{\Omega_{b,0}}{\Omega_{b,0} + \Omega_{c,0}}$, cold-dark-matter (cdm) density $\rho_c = \rho_{mat}\frac{\Omega_{c,0}}{\Omega_{b,0} + \Omega_{c,0}}$

$$P_b(\rho_b, E_{th}) = \rho_b \frac{E_{th}}{m_p c^2}.$$

The additional equation for pressure is the equation-of-state (eos) for the pressure P_r: $P_r = P(a, \rho_r)$.

Solution 1

One solves numerically [9] [13] [19] (3ac) with boundary conditions $a(x_0)=1$, $a'(x_0)=1$ as algebraic-differential equations for function variables *a*, $E_{th} = k_B T$. The solution exists until $x_{1c} = 0.14$, where numerical integration stops converging.

Solution 2

One solves numerically [9] [13] [19] (3ad) with boundary conditions $a(x_0)=1$, $a'(x_0)=1$ as differential equations for function variables a, ρ_r. The solution exists until $x_{1c} = 0.0196$, where numerical integration stops converging.

Plot *a*(*x*) is shown below [13] in **Figure 4**.

The solution limit $x_{1c} = 0.0196$ indicates the transition from matter-dominated to the radiation-dominated regime, which happens approximately at photon decoupling time $t_{re} = 370\text{ ky}$, $x_{re} = 0.000026$. For $x \le x_{1c}$ solution is continued by pure radiation density ([13]).

Solution 3

One solves numerically [13] (3a) with boundary conditions $a(x_0)=1$, $a'(x_0)=1$ as differential equation for function variable *a*, with ansatz for $\rho_r = \frac{K_s}{a^4} + \frac{K_m}{a^3}$. This is the usual solution method for background functions, used in CAMB [20] and in CMBquick ([21] [22]).

The solution exists until $x_{1c} = 0.0055$, where numerical integration stops converging, and the solution becomes complex (*i.e.* $\text{Im}(a) \ne 0$).

Plot *a*(*x*) is shown below [13] in **Figure 5**.

The solution limit $x_{1c} = 0.0055$ indicates the transition from matter-dominated to the radiation-dominated regime, which happens approximately at photon decoupling time $t_{re} = 370\text{ ky}$, $x_{re} = 0.000026$. For $x \le x_{1c}$ solution is continued by pure radiation density ([13] [20] [22]).

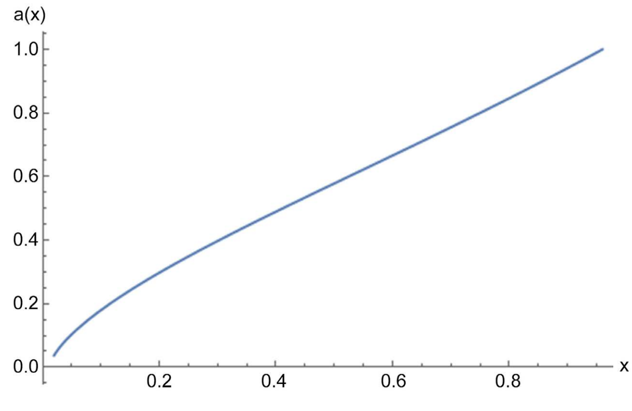


Figure 4. The scale factor *a(x)* in dependence of relative time $x = \frac{tc}{R_H}$, numerical solution 2.

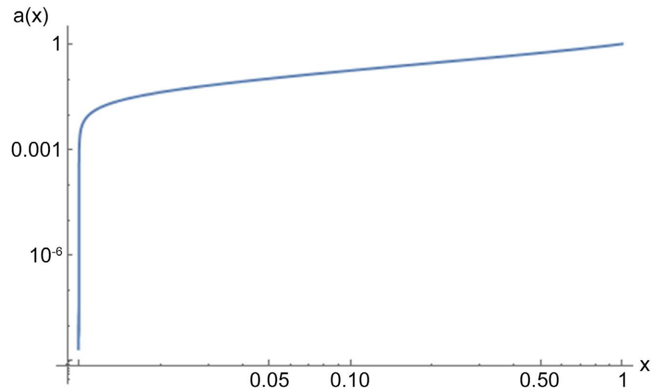


Figure 5. The scale factor *a(x)* in dependence of relative time $x = \frac{tc}{R_H}$, numerical solution 3.

5.2. Analytic Solution

The analytic solution scheme transforms the two basic equations into a parameterized integral $x(a)$, which is the inverted scale factor $a(x)$.

In order to calculate the thermal energy, we apply an iteration, we calculate the temperature $E_{th}(a)$ from $\rho_{rad} \equiv \rho_\gamma + \rho_\nu = \frac{K_s}{K_s + K_m a}\rho_r$, using the solution $a(x)$ in the next iteration: $E_{th}^{(n+1)} = E_{th}^{(n)}\left(a^{(n)}(x)\right)$, as shown in the schematic in chap. 11.

The zero iteration is the "naive" thermal energy $E_{th}^{(0)} = E_{th,0}/a$.

The variables are scale factor and density a, ρ_r.

The boundary conditions are $\rho_r(x_0) = \Omega_{m,0} + \Omega_{rad,0}$, $a(x_0) = 1$, $a'(x_0) = 1$, from $a'(x_0) = 1$ follows $k = -0.0042$ which is compatible with Planck data

$$(a')^2 + k_0 - \frac{\Lambda_1}{3}a^2 - \rho_r a^2 = 0 \quad \text{sF1} \tag{3a}$$

$$\frac{\rho_r{}'a}{3} + a'(P_r + \rho_r) = 0 \quad \text{sF4} \tag{3d}$$

The two Equations (3ad) are non-linear first-order differential equations quadratic in the variables a, ρ_r.

The third equation is the equation-of-state (eos) for the pressure P_r: $P_r = P(a, \rho_r)$.

The density and pressure have the form: relative energy density $\rho_r = \rho_b + \rho_\gamma + \rho_c + \rho_e + \rho_\nu$ for baryons, photons, dark matter, free electrons, neutrinos, relative pressure $P_r = P_b + P_\gamma + P_c + P_e + P_\nu$, where radiation pressure $P_{rad} = P_\gamma + P_\nu = \frac{\rho_\gamma + \rho_\nu}{3}$, and matter pressure (neglecting electrons) is the baryon ideal gas pressure $P_{mat} = P_b = \rho_b \frac{k_B T}{m_b c^2}$, for under-nuclear temperature $k_B T \ll m_b c^2 = 0.94\ \text{GeV}$ the baryon matter is dust-like, *i.e.* pressure is almost zero.

The densities have the form

$$\rho_r = \rho_{mat} + \rho_{rad}$$

$$\rho_{mat} = \rho_b + \rho_c = \frac{K_m a}{K_s + K_m a}\rho_r, \quad \rho_{rad} = \rho_\gamma + \rho_\nu = \frac{K_s}{K_s + K_m a}\rho_r$$

$$\rho_c = \rho_{mat}\frac{\Omega_{c,0}}{\Omega_{b,0} + \Omega_{c,0}}, \quad \rho_b = \rho_{mat}\frac{\Omega_{b,0}}{\Omega_{b,0} + \Omega_{c,0}}$$

$$\rho_\gamma = a_{SB0}E_{th}^4, \quad \rho_\nu = \frac{\Omega_{\nu,0}}{a^3}$$

We calculate the temperature $E_{th}(a)$ from $\rho_{rad} \equiv \rho_\gamma + \rho_\nu = \frac{K_s}{K_s + K_m a}\rho_r$ (12a)

$$i.e.\quad E_{th}(a) = \frac{1}{a_{SB0}^{1/4}}\left(\frac{K_s}{K_s + K_m a}\rho_r(a) - \frac{\Omega_{\nu,0}}{a^3}\right)^{1/4} \tag{12a1}$$

and all the pressure becomes a function of *a*,

$$P_r(a,\rho_r) = P_{rad} + P_{mat} = \left(\frac{K_s}{K_s + K_m a} + \frac{K_m a}{K_s + K_m a}\frac{\Omega_{b,0}}{\Omega_{b,0} + \Omega_{c,0}}\frac{E_{th}(a)}{m_b c^2}\right)\rho_r \tag{12b}$$

$$i.e.\quad \frac{P_r}{\rho_r} = P_\rho(a) = \left(\frac{K_s}{K_s + K_m a} + \frac{K_m a}{K_s + K_m a}\frac{\Omega_{b,0}}{\Omega_{b,0} + \Omega_{c,0}}\frac{E_{th}(a)}{m_b c^2}\right)$$

then we can integrate (3d) in a:

$$\log(\rho_r(a)) = \frac{\rho_r{}'a}{3} + a'(P_r + \rho_r) = -\int_0^a \mathrm{d}a\left(\frac{3 + P_\rho(a)}{a}\right) + c_1 \tag{12c}$$

and then can integrate (3a) in a:

$$x(a) = \int_0^a \mathrm{d}a\, a\sqrt{\frac{\Lambda_1}{3} + \rho_r(a) - \frac{k_0}{a^2}} + c_2, \tag{12d}$$

where c_1 and c_2 are set to fulfill the boundary conditions

$$\rho_r(x_0) = \Omega_{m,0} + \Omega_{rad,0}, \quad a(x_0) = 1, \quad \Omega = \frac{\rho}{\rho_{crit,0}}$$

5.3. Background Results

Results for density and relative time in dependence of scale factor $\rho_r(a)$, $x(a)$, are shown below [13].

Relative density in $\rho_{crit,0}$ units is shown over scale factor *a*, in double-logarithmic plot **Figure 6**.

There is a **critical point** $a_T \approx 0.5\times10^{-4}$, where the density changes its behavior, it coincides roughly with the critical point in temperature. The corresponding time is $x_T \approx 10^{-8}$, thermal energy $E_{th} \approx 1\,\text{eV}$.

The analytic solution yields directly the inverse scale factor function $x(a)$, it shown in **Figure 7**.

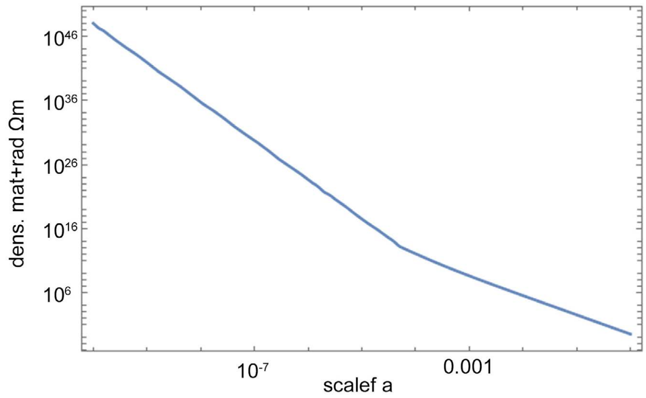


Figure 6. The density $\rho_r(a)$ in dependence of scale factor *a*, analytic solution.

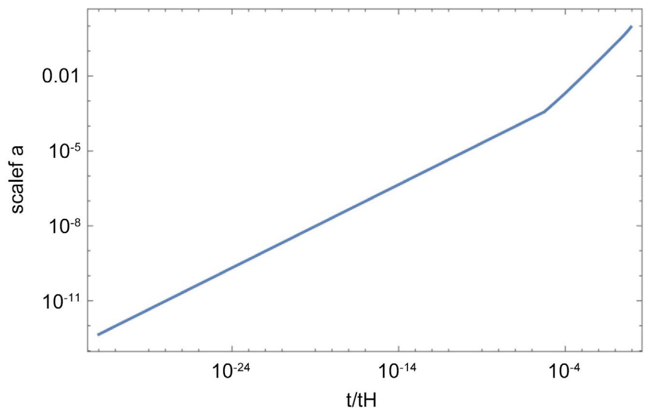


Figure 7. Relative time $x = \frac{tc}{R_H}$ and scale factor *a*, analytic solution.

There is a **critical point at photon decoupling**, $a_{dec} = 0.9 \times 10^{-3}$, $x_{dec} = 0.3 \times 10^{-4} \triangleq 370\ \text{ky}$, redshift $z_{dec} = 1090$, thermal energy $E_{th} = 0.25\ \text{eV}$.

The scale factor changes its power-law dependence on time:

$$a(x) \cong \begin{cases} x, & x > x_{dec} \\ x^{1/2}, & x < x_{dec} \end{cases}$$

It is useful to compare the result for $x(a)$ from the analytical solution and the standard CAMB solution ([13] [20]) **Figure 8**. The two curves separate roughly at $a_{dec} = 0.9 \times 10^{-3}$, the CAMB curve continues approximately linearly, whereas in the analytical solution time decreases quadratically, $x(a) \cong a^2$.

The plots of density $\rho_r(a)$ (blue) and radiation density $\rho_{rad}(a)$ are shown in comparison below ([13]) in **Figure 9**. As expected, we have radiation dominance roughly for $a < a_{dec}$, and matter dominance for $a > a_{dec}$.

The Hubble parameter is approximately linear in *x*, as it should be. However, there is a small deviation at critical point $x_{cH} \approx 10^{-8}$, scale factor $a_{cH} \approx 0.5 \times 10^{-4}$, redshift $z_{cH} \approx 1/a \approx 20000$.

This is apparently responsible for the small correction of the present Hubble constant H_0, compared to CAMB solution.

The plot of the Hubble parameter is shown in **Figure 10**.

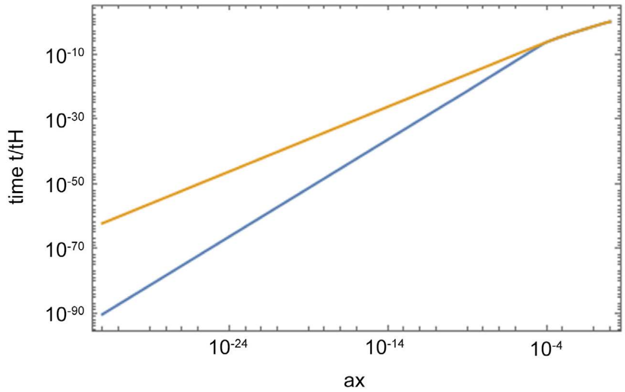


Figure 8. Relative time $x = \frac{tc}{R_H}$ in dependence of scale factor *a*, analytic solution (blue), CAMB-solution (orange).

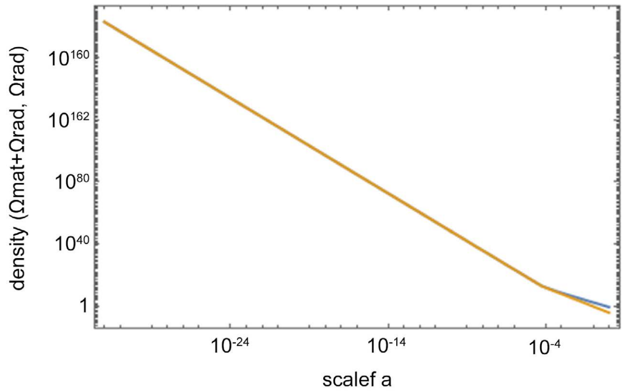


Figure 9. The density $\rho_r(a)$ (blue) and radiation density $\rho_{rad}(a)$ (orange), in dependence of scale factor *a*, analytic solution.

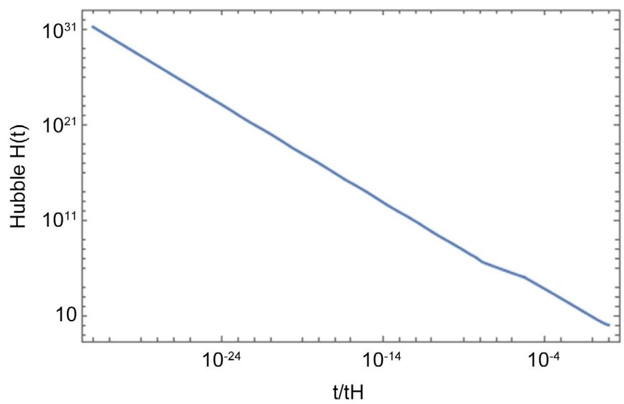


Figure 10. The Hubble parameter $H(x)$, in dependence of relative time $x = \frac{tc}{R_H}$, analytic solution.

The "naive" temperature $E_{th}^{(0)}(a)$ from (12a) is compared to the iterated temperature $E_{th}^{(1)}(a)$ calculated from the first analytic solution in (12a1) is shown in **Figure 11**. The point of deviation is $a_T \approx 0.5\times10^{-4}$, the corresponding time is $x_T \approx 10^{-8}$, thermal energy $E_{th} \approx 1\,\text{eV}$. This point coincides roughly with the critical point in density **Figure 6**.

Hubble parameter

Baryon pressure correction

Baryon pressure correction yields $t_{0c} = t_0/1.043t_0$, so $H_{0c} = 1.043H_0$, the corrected Planck-value is $H_{0Pc} = H_{0P}\times1.043 = 70.6\pm0.4$;

$H_{0R} = 69.8\pm1.7$ Red-Giants Freedmann 09/21;

$H_{0S} = 73.04\pm1.04$ Cepheids-SNIa SHOES 12/21;

$H_{0P} = 67.66\pm0.42$ Planck 07/18.

H_{0R} Red-Giants is in agreement with corrected Planck within error margin.

Assessed correction of the Cepheids-SNIa-measurement

Cepheids-SNIa-measurement based on time-brightness calibration for small redshift *z*, peak power $P_{\max} \sim T(t_{cr}) \sim \bar{m}_b$, with average nucleus mass $\bar{m}_b$ percentage of higher-mass nuclei at present: $r(O)=1.04\%$, $r(C)=0.46\%$, so $\frac{P_{\max}(z\gg1)}{P_{\max}(z\ll1)} \approx (1+r(O)+r(C)) = 1.015$ so z-corrected Cepheids-SNIa becomes 73.04/1.015 = 72. $H_{0Sc} = H_{0S}/1.015 = 72.\pm1.$, which is at error margin.

6. Relativistic Perturbations and the Perturbed Lambda-CDM Model

The Lambda-CDM model is locally homogeneous, but during inflation the quantum fluctuations are "blown-up", and the universe becomes inhomogeneous on small (galactic) scales and remains homogeneous on large scales. These local inhomogeneities generate structure, which we observe today.

In order to reproduce these local inhomogeneities in the *perturbed Lambda-CDM model*, we introduce small perturbations in the metric and in the density distribution. These perturbations are functions of conformal time η (defined by $\mathrm{d}\eta = \frac{\mathrm{d}t}{a}$), and space location vector x^i, and are not random variables. The randomness is introduced by initial conditions for perturbations (see chap. 8).

We introduce metric perturbations A, B_i, E_{ij} in the RW-metric [2] [3] [4]

$$\mathrm{d}s^2 = a^2(\eta)\left(-(1+2A)\mathrm{d}\eta^2 + 2B_i\mathrm{d}x^i\mathrm{d}\eta + (\delta_{ij}+2E_{ij})\mathrm{d}x^i\mathrm{d}x^j\right) \tag{13}$$

and split-up in scalar, vector, tensor parts:

scalar A

$B_i = \partial_i B + \hat{B}_i$, scalar B, vector $\hat{B}_i$

$E_{ij} = C\delta_{ij} + \partial_i\partial_j E + (\partial_i\hat{E}_j - \partial_j\hat{E}_i) + \hat{E}_{ij}$, scalar C E, vector $\hat{E}_i$, tensor $\hat{E}_{ij}$, where $\sum_i E_i^i = 3C$

Furthermore, we form the **gauge-invariant Bardeen variables** with 8 = 1scalar (A) + 3vector (B_i) + 4tensor (E_{ij}) degrees-of-freedom (dof's)

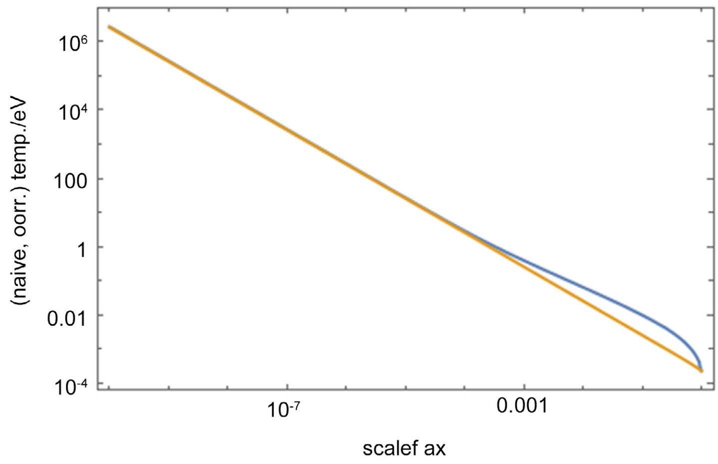


Figure 11. The naive temperature $E_{th}^{(0)}(a)$ compared to the iterated temperature $E_{th}^{(1)}(a)$, in dependence of scale factor *a*, analytic solution.

$$\Psi = A + H(B - E') + (B - E')', \quad \Phi = -C + \frac{1}{3}\nabla^2 E - H(B - E'),$$

$$\hat{\Phi}_i = \hat{B}_i - \hat{E}_i', \quad \hat{E}_{ij}$$

Since we have 6 Einstein equations, we can remove the 8 − 6 = 2 dof's by **gauge-fixing**.

- Newtonian gauge $B = E = 0$

$$ds^2 = a^2(\eta)\left(-(1+2\Psi)d\eta^2 + (1-2\Phi)\delta_{ij}dx^i dx^j\right)$$

$$A = \Psi, \quad C = -\Phi \tag{6.30}$$

- Spatially flat gauge $C = E = 0$
- Synchronous gauge $A = B = 0$

From now on, **we use the Newtonian gauge**.

We get for the energy-density tensor

$$T_0^0 = -(\bar{\rho} + \delta\rho)$$

$$T_0^i = -(\bar{\rho} + \bar{P})v^i$$

$$T_j^i = -(\bar{P} + \delta P)\delta_j^i + \Pi_j^i, \quad \Pi_i^i = 0 \ \forall i \tag{14}$$

The relativistic Euler equation is

$$\left(\frac{\rho c^2}{\sqrt{1-(v/c)^2}} + p\right)\frac{1}{c^2\sqrt{1-(v/c)^2}}\frac{d}{dt}\left(\frac{v_i}{\sqrt{1-(v/c)^2}}\right) + \partial_i p + \frac{1}{c^2\sqrt{1-(v/c)^2}}\frac{dp}{dt}v_i = 0,$$

The Euler equation in the RW metric becomes

$$v_i' = -\left(H + \frac{\bar{P}'}{\bar{P} + \bar{\rho}}\right)v_i - \frac{1}{\bar{P} + \bar{\rho}}\left(\partial_i \delta P + \partial^j \Pi_{ij}\right) - \partial_i \Psi \tag{6.76}$$

where Π_{ij} is the anisotropic stress with the decomposition

$$\Pi_{ij} = \partial_i \partial_j \Pi + \left(\partial_i \hat{\Pi}_j - \partial_j \hat{\Pi}_i\right) + \hat{\Pi}_{ij} \tag{6.39}$$

Finally, we get 10 fundamental equations:

6 Einstein equations

[4]

$$\nabla^2\Phi - 3H(\Phi' + H\Psi) = \pi G a^2 \delta\rho$$

$$\Phi' + H\Psi = \pi G a^2 \frac{a''}{a'H}$$

$$\partial_i\partial_j(\Phi - \Psi) = 8\pi G a^2 \Pi_{ij},\ i < j$$

$$\Phi'' + H\Psi' + 2H\Phi' + \frac{1}{3}\nabla^2(\Phi - \Psi) + (2H' + H^2)\Psi = \pi G a^2 \delta P \qquad \text{(15a-d)}$$

4 conservation equations: continuity +Euler

[4]

$$\delta' = -\left(1 + \frac{\bar{P}}{\bar{\rho}}\right)(\partial^i v_i - 3\Phi') - 3H\left(\frac{\delta P}{\delta\rho} - \frac{\bar{P}}{\bar{\rho}}\right)\delta$$

$$v_i' = -\left(H + \frac{\bar{P}'}{\bar{P} + \bar{\rho}}\right)v_i - \frac{1}{\bar{P} + \bar{\rho}}\left(\partial_i \delta P + \partial^j \Pi_{ij}\right) - \partial_i\Psi \qquad \text{(15ef)}$$

$$q^i = (\bar{\rho} + \bar{P})v^i,\quad \delta = \frac{\delta\rho}{\bar{\rho}}\quad \text{decelaration conformal}\quad q = -\frac{a''}{a'\mathcal{H}},\quad T_i^0 = \partial_i q,$$

for **10 variables** 4 scalar $\Phi, \Psi, \delta, \delta P$, 3 vector v^i, 3 tensor Π^i_j;

initial conditions 6

Φ 2c, Ψ 1c, v^i 3c, $(\delta, \delta P)$ 0c;

background parameters

$$\mathcal{H} = \frac{a'}{a},\quad q = -\frac{a''}{a'\mathcal{H}},\quad a,\quad \bar{\rho},\quad \bar{P}.$$

Fundamental equations in k-space ([14] Ma)

In the following, we transform the fundamental equations via Fourier-transform into k-space.

We use Newtonian gauge, conformal time η, $a' = \frac{\mathrm{d}a}{\mathrm{d}\eta}$, the metric in Newtonian gauge reduces to

$$\mathrm{d}s^2 = a(\eta)\left(-(1 + 2\Psi)\mathrm{d}\eta^2 + (1 - 2\Phi)\mathrm{d}x^i \mathrm{d}x_i\right)$$

We get 4 Einstein equations in k-space

$$k^2\Phi - 3H(\Phi' + H\Psi) = \pi G a^2 \delta\rho$$

$$k^2(\Phi' + H\Psi) = \pi G a^2 (\bar{P} + \bar{\rho})\theta$$

$$k^2(\Phi - \Psi) = 12\pi G a^2 (\bar{P} + \bar{\rho})\sigma$$

$$\Phi'' + H(\Psi' + 2\Phi') + \frac{1}{3}k^2(\Phi - \Psi) + (2H' + H^2)\Psi = 4\pi G a^2 \delta P \qquad \text{(16a-d)}$$

and 2 continuity-Euler equs in k-space

$$\delta' = -\left(1 + \frac{\bar{P}}{\bar{\rho}}\right)(\theta - 3\Phi') - 3H\left(\frac{\delta P}{\bar{\rho}\delta} - \frac{\bar{P}}{\bar{\rho}}\right)\delta \quad \text{density equ}$$

$$\theta' = -\left(H + \frac{\bar{P}'}{\bar{P}+\bar{\rho}}\right)\theta - \frac{\delta P}{\bar{P}+\bar{\rho}}k^2 - k^2\sigma + k^2\Psi \quad \text{velocity equ} \tag{16ef}$$

with the definitions

$$\delta = \frac{\delta\rho}{\bar{\rho}}, \quad \theta = ik^j v^j, \quad \sigma = -\frac{\left(\hat{k}_i\hat{k}_j - \frac{\delta_{ij}}{3}\right)\Pi_{ij}}{\bar{P}+\bar{\rho}},$$

where $\hat{k} = \frac{\vec{k}}{k}$ is the k-unit-vector, Π^i_j anisotropic stress

and the relations

$$T^0_0 = -(\bar{\rho}+\delta\rho), \quad T^0_i = (\bar{\rho}+\bar{P})v_i, \quad T^i_j = (\bar{P}+\delta P)\delta^i_j + \Pi^i_j, \quad \delta = \frac{\delta\rho}{\bar{\rho}} = -\frac{T^0_0}{\bar{\rho}}$$

$$\Pi^i_i = 0, i = 1,2,3, \quad \Pi^i_j \equiv T^i_j - T^k_k\delta^i_j$$

$$\theta = ik^j v_j, \quad (\bar{\rho}+\bar{P})\theta = ik^j\delta T^0_j, \quad (\bar{\rho}+\bar{P})\sigma = -\left(\hat{k}^i\hat{k}^j - \frac{1}{3}\delta_{ij}\right)\Pi^i_j.$$

We have here 6 variables $\Phi,\Psi,\theta,\sigma,\delta,\delta P$, $\delta P = \delta T^i_i$, $\delta\rho = \delta T^0_0$, which are functions of (k,η).

7. Evolution of Distribution Momenta

We introduce here density distribution momenta for density components radiation γ, neutrinos ν, electrons e, baryons b, cold-dark-matter d. The densities acquire their random nature from random initial conditions, and have therefore a (Gaussian) probability distribution. These distribution momenta are used in the calculation of CMB spectrum in chap. 10.

Evolution of distribution function momenta (Ma [14])

We have for Newtonian gauge, conformal time η, $a' = \frac{\mathrm{d}a}{\mathrm{d}\eta}$

$$\mathrm{d}s^2 = a(\eta)\left(-(1+2\Psi)\mathrm{d}\eta^2 + (1-2\Phi)\mathrm{d}x^i\mathrm{d}x_i\right).$$

Phase space distribution

With phase space element $\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3\mathrm{d}P_1\mathrm{d}P_2\mathrm{d}P_3$

$\mathrm{d}N = f(x^i, P_j, \eta)\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3\mathrm{d}P_1\mathrm{d}P_2\mathrm{d}P_3$ particle number in element (32)

$P_i = a(1-\Phi)p_i$ co-moving disturbed momentum

density distribution for matter fermions (Fermi-Dirac distribution +), density distribution for radiation bosons (Bose-Einstein distribution -)

$$f_0(\varepsilon,T) = \frac{g_s}{h^3}\frac{1}{\exp\left(\frac{\varepsilon}{k_B T}\right) \pm 1} \tag{17}$$

energy $\varepsilon = a\sqrt{p^2+m^2} = \sqrt{P^2+a^2m^2}$, temperature T, today temperature T_0.

We change variables: $x^i P_j$ to $x^i q_j$, and get the expressions:

scaled momentum $q_j = ap_j = qn_j$, unit momentum vector $\hat{n}$ with $n^i n_i = 1$

energy $\varepsilon = \sqrt{q^2 + a^2m^2}$;

change distribution $f\left(x^i, P_j, \eta\right)$ to $f\left(x^i, q, n_j, \eta\right)$.

Finally we get for the neutrino distribution perturbation function $\psi\left(x^i, q, n_j, \eta\right)$ (not equal to the metric perturbation Ψ)

$$f\left(x^i, P_j, \eta\right) = f_0\left(\varepsilon, T\right)\left(1+\psi\left(x^i, q, n_j, \eta\right)\right) \quad (35)$$

for the distribution of energy tensor

$$T_0^0 = a^{-4}\int \mathrm{d}q\mathrm{d}\Omega\, q^2\varepsilon f_0\left(\varepsilon, T\right)\left(1+\psi\right)$$
$$T_i^0 = a^{-4}\int \mathrm{d}q\mathrm{d}\Omega\, q n_i f_0\left(\varepsilon, T\right)\left(1+\psi\right)$$
$$T_j^i = a^{-4}\int \mathrm{d}q\mathrm{d}\Omega \frac{n_i n_j q^2}{\varepsilon} f_0\left(\varepsilon, T\right)\left(1+\psi\right)$$

Boltzmann equation in $\left(x^i, q, n_j, \eta\right)$, with collision term $\frac{\partial f_C}{\partial \eta}$ becomes

$$\frac{Df}{d\eta} = \frac{\partial f}{\partial \eta} + \frac{\partial x^i}{\partial \eta}\frac{\partial f}{\partial x^i} + \frac{\partial q}{\partial \eta}\frac{\partial f}{\partial q} + \frac{\partial n_i}{\partial \eta}\frac{\partial f}{\partial n_i} = \frac{\partial f_C}{\partial \eta}$$

GR geodesic equation $P^0\frac{\mathrm{d}P^\mu}{\mathrm{d}\eta} + \Gamma^\mu_{\alpha\beta}P^\alpha P^\beta = 0$ gives

$$\frac{\mathrm{d}q}{\mathrm{d}\eta} = q\dot{\Phi} - \varepsilon\left(q, \eta\right) n_i \partial_i \Psi \quad (39)$$

and **Boltzmann equation** becomes

$$\frac{\partial \psi}{\partial \eta} + i\frac{q}{\varepsilon}\left(\vec{k}\cdot\hat{n}\right)\psi + \frac{\mathrm{d}\ln f_0}{\mathrm{d}\ln q}\left(\dot{\Phi} - i\frac{\varepsilon}{q}\left(\vec{k}\cdot\hat{n}\right)\Psi\right) = \frac{1}{f_0}\frac{\partial f_C}{\partial \eta} \quad (18)$$

with **fluid equations cdm**

$$\delta_c{}' = -\theta_c + 3\Phi', \quad \theta_c{}' = -\frac{a'}{a}\theta_c + k^2\Psi \quad (19a)$$

Component evolution equations

In the following we present the evolution equations for *l*-momenta in *k*-space for important components.

Evolution equations massive neutrinos

We have for (average) background density, pressure

$$\bar{\rho}_h = a^{-4}\int \mathrm{d}q\mathrm{d}\Omega\, q^2\varepsilon f_0\left(\varepsilon, T\right), \quad \bar{P}_h = \frac{1}{3}a^{-4}\int \mathrm{d}q\mathrm{d}\Omega\, q^2\frac{q^2}{\varepsilon} f_0\left(\varepsilon, T\right)$$

the perturbations

$$\delta\rho_h = a^{-4}\int \mathrm{d}q\mathrm{d}\Omega\, q^2\varepsilon f_0\left(\varepsilon, T\right)\psi, \quad \delta P_h = \frac{1}{3}a^{-4}\int \mathrm{d}q\mathrm{d}\Omega\, q^2\frac{q^2}{\varepsilon} f_0\left(\varepsilon, T\right)\psi$$
$$\delta T_h{}^0{}_i = a^{-4}\int \mathrm{d}q\mathrm{d}\Omega\, q n_i f_0\left(\varepsilon, T\right)\psi,$$
$$\delta \Pi_h{}^0{}_i = \frac{1}{3}a^{-4}\int \mathrm{d}q\mathrm{d}\Omega\, q^2\frac{q^2}{\varepsilon}\left(n_i n_j - \frac{1}{3}\delta_{ij}\right) f_0\left(\varepsilon, T\right)\psi$$

distribution perturbation function are developed in Legendre polynomials of the angle $\left(\hat{k}\cdot\hat{n}\right)$

$$\psi\left(\vec{k}, \hat{n}, q, \eta\right) = \sum_{l=0}^{\infty}\left(-i\right)^l\left(2l+1\right)\psi_l\left(\vec{k}, q, \eta\right)P_l\left(\hat{k}\cdot\hat{n}\right) \quad (54)$$

$$\delta\rho_h = 4\pi a^{-4}\int \mathrm{d}q\, q^2 \varepsilon f_0(\varepsilon,T)\psi_0\,,\quad \delta P_h = \frac{4\pi}{3}a^{-4}\int \mathrm{d}q\, q^2 \frac{q^2}{\varepsilon} f_0(\varepsilon,T)\psi_0$$

$$(\bar\rho_h + \bar P_h)\theta_h = 4\pi k a^{-4}\int \mathrm{d}q\, q^3 f_0(\varepsilon,T)\psi_1\,,$$

$$(\bar\rho_h + \bar P_h)\sigma_h = \frac{4\pi}{3}a^{-4}\int \mathrm{d}q\, q^2 \frac{q^2}{\varepsilon} f_0(\varepsilon,T)\psi_0\,.$$

Boltzmann equation yields for evolution of perturbation momenta

$$\psi_0{}' = -\frac{qk}{\varepsilon}\psi_1 - \Phi'\frac{\mathrm{d}\ln f_0}{\mathrm{d}\ln q}\,,\quad \psi_1{}' = \frac{qk}{3\varepsilon}(\psi_0 - 2\psi_2) - \frac{\varepsilon k}{3q}\Psi\frac{\mathrm{d}\ln f_0}{\mathrm{d}\ln q}$$

$$\psi_l{}' = \frac{qk}{(2l+1)\varepsilon}\left(l\psi_{l-1} - (l+1)\psi_{l+1}\right),\quad l \ge 2 \qquad (19b)$$

truncating order $l_{\max}$

$$\psi_{l_{\max}+1} = \frac{(2l_{\max}+1)\varepsilon}{qk\eta}\psi_{l_{\max}} - \psi_{l_{\max}-1}\,.$$

Evolution equations photons

We assume $\gamma - e$ Thomson scattering with the Thomson cross-section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = 3\sigma_T\frac{1+\cos^2\theta}{16\pi}\,,\quad \sigma_T = 0.665\times10^{-24}\,\mathrm{cm}^2$$

with $F_\gamma(k,\hat n,\eta)$ distribution total intensity

with $G_\gamma(k,\vec n,\eta)$ distribution difference polarization components

with collision terms

$$\left(\frac{\partial F_\gamma}{\partial\eta}\right)_C = a n_e\sigma_T\left(-F_\gamma + F_{\gamma0} + 4(\hat n\cdot\vec v_e) - (F_{\gamma2} + G_{\gamma0} + G_{\gamma2})P_2\right)$$

$$\left(\frac{\partial G_\gamma}{\partial\eta}\right)_C = a n_e\sigma_T\left(-G_\gamma + \frac{1}{2}(F_{\gamma2} + G_{\gamma0} + G_{\gamma2})(1-P_2)\right)$$

with expansion

$$\left(\frac{\partial F_\gamma}{\partial\eta}\right)_C = a n_e\sigma_T\left(\frac{4i}{k}(\theta_\gamma - \theta_b)P_1 + \left(9\sigma_\gamma - \frac{1}{2}G_{\gamma0} - \frac{1}{2}G_{\gamma2}\right)P_2 - \sum_{l=3}^{\infty}(-i)^l(2l+1)F_{\gamma l}P_l\right)$$

$$\left(\frac{\partial G_\gamma}{\partial\eta}\right)_C = a n_e\sigma_T\left(\frac{1}{2}(F_{\gamma2} + G_{\gamma0} + G_{\gamma2})(1-P)_2 - \sum_{l=0}^{\infty}(-i)^l(2l+1)G_{\gamma l}P_l\right).$$

Resulting fluid equations are then

$$\delta_\gamma{}' = -\frac{4}{3}\theta_\gamma + 4\Phi'\,,\quad \theta_\gamma{}' = k^2\left(\frac{1}{4}\delta_\gamma - \sigma_\gamma\right) + k^2\Psi + a n_e\sigma_T(\theta_b - \theta_\gamma) \qquad (19c1)$$

and momenta evolution becomes

$$F_{\gamma2}{}' = 2\sigma_\gamma{}' = \frac{8}{15}\theta_\gamma - \frac{3}{5}kF_{\gamma3} - \frac{9}{5}a n_e\sigma_T\sigma_\gamma(\theta_\gamma - \theta_b)$$
$$+\frac{1}{10}a n_e\sigma_T(\theta_\gamma - \theta_b)(G_{\gamma0} + G_{\gamma2})$$

$$F_{\gamma l}{}' = \frac{k}{2l+1}\left(lF_{\gamma(l-1)} - (l+1)F_{\gamma(l+1)}\right) - an_e\sigma_T F_{\gamma l}, \quad l \ge 3 \tag{19c2}$$

$$G_{\gamma l}{}' = \frac{k}{2l+1}\left(lG_{\gamma(l-1)} - (l+1)G_{\gamma(l+1)}\right) + an_e\sigma_T\left(-G_{\gamma l} + \frac{1}{2}\left(F_{\gamma 2} + G_{\gamma 0} + G_{\gamma 2}\right)\left(\delta_{l0} + \frac{\delta_{l2}}{5}\right)\right) \tag{19c3}$$

Evolution equations baryons

We have the fluid equations

$$\delta_b{}' = -\theta_b + 3\Phi', \quad \theta_b{}' = -\frac{a'}{a}\theta_b + c_s^2 k^2\delta_b - \frac{4\bar{\rho}_\gamma}{3\bar{\rho}_b}an_e\sigma_T\left(\theta_b - \theta_\gamma\right) + k^2\Psi \tag{19d1}$$

with sound speed $c_s^2 = \frac{k_B T_b}{\mu}\left(1 - \frac{1}{3}\frac{\mathrm{d}\ln T_b}{\mathrm{d}\ln a}\right)$, μ mean baryon mass.

The temperature equation becomes

$$T_b{}' = -2\frac{a'}{a}T_b + \frac{8}{3}\frac{\mu}{m_e}\frac{\bar{\rho}_\gamma}{\bar{\rho}_b}an_e\sigma_T\left(T_\gamma - T_b\right)$$

Before recombination tight-coupling $\gamma - b$, we have

$$\theta_b - \theta_\gamma = \tau_c\left(\theta_\gamma{}' - k^2\left(\frac{1}{4}\delta_\gamma - \sigma_\gamma\right) - k^2\Psi\right) \tag{19d2}$$

$$\sigma_\gamma = \frac{\tau_c}{9}\left(\frac{8}{3}\theta_\gamma - 10\sigma_\gamma{}' - 3kF_{\gamma 3}\right) \tag{19d3}$$

$$\theta_\gamma{}' = -\frac{3}{4}\frac{\bar{\rho}_b}{\bar{\rho}_\gamma}\left(\theta_b{}' + \frac{a'}{a}\theta_b - c_s^2k^2\delta_b\right) + k^2\left(\frac{1}{4}\delta_\gamma - \sigma_\gamma\right) + \left(1 + \frac{3}{4}\frac{\bar{\rho}_b}{\bar{\rho}_\gamma}\right)k^2\Psi \tag{19d4}$$

8. Initial Conditions

Initial conditions in k-space for density components (radiation *γ*, neutrinos ν, electrons *e*, baryons *b*, cold-dark-matter *c)* and metric perturbations Ψ, Φ generate the random (Gaussian distributed) inhomogeneities required for structure formation.

Initial conditions k-space

For Newtonian gauge in conformal time η, initial conditions are chosen in such a way, that only the largest order in $k\eta$ is present (Ma [14])

$$\delta_\gamma = -\frac{40C}{3\left(\bar{P} + \bar{\rho}\right)} = -2\Psi$$

$$\delta_c = \delta_b = \frac{3}{4}\delta_\nu = \frac{3}{4}\delta_\gamma$$

$$\theta_\gamma = \theta_\nu = \theta_b = \theta_c = \frac{10C}{15 + 4R_\nu}\left(k^2\eta\right) = \frac{k^2\eta}{2}\Psi$$

$$\sigma_\nu = \frac{4C}{3\left(15 + 4R_\nu\right)}\left(k\eta\right)^2 = \frac{\left(k\eta\right)^2}{15}\Psi$$

$$\Psi = \frac{20C}{15+4R_\nu}, \quad \Phi = \left(1+\frac{2}{5}R_\nu\right)\Psi$$

with neutrino density ratio $R_\nu = \dfrac{\bar{\rho}_\nu}{\bar{\rho}_\gamma + \bar{\rho}_\nu}$

9. Structure Formation

In the following, we present in concise form cross sections, reaction rates and densities for important cosmological particle processes [2] [3] [4] [11] [23]. They are used in the background eos equations in chap. 2, and in the evolution equations of density distribution momenta in chap. 7.

Cosmic neutrino background

The reaction is $\nu_e + \bar{\nu}_e \leftrightarrow e^+ + e^-$, $e^- + \bar{\nu}_e \leftrightarrow e^- + \bar{\nu}_e$

with reaction rate $\Gamma = n\sigma v \approx G_F^2 T^5$, $G_F \approx 1.2\times 10^{-5}\,\text{GeV}^{-2}$ (3.58)

and corresponding Hubbble rate $H \approx \dfrac{T^2}{M_{Pl}}$, $\dfrac{\Gamma}{H} \approx \left(\dfrac{T}{1\,\text{MeV}}\right)^3$,

neutrinos decouple at $T_{\nu,d} = 1\,\text{MeV}$, $t_{\nu,d} = 1\,\text{s}$,

the number density $n_\nu \propto a^{-3}\int \mathrm{d}^3 q \dfrac{1}{\exp\left(\dfrac{q}{aT_\nu}+1\right)}$,

with $T_\nu \propto a^{-1}$ for $T_\nu > T_{\nu,d}$.

Gamma pair production

The gamma-pair production reaction is $\gamma + A \to e^+ + e^- + A$ [24] [25]

with the cross-section $\sigma = \alpha r_e^2 Z^2 P(E,Z)$, where Z = atomic number of material A, $k = \dfrac{E_\gamma}{E_e}$, α fine-structure-constant, and

$$P(E,Z) \approx \frac{2\pi}{3}\left(\frac{|k-2|}{k}\right)^3, \quad 2<k<4,$$

$$P(E,Z) \approx \frac{28}{9}\ln(2k) - \frac{218}{27} = 3.11\ln\left(2\frac{E_\gamma}{E_e}\right) - 8.07, \quad k>4,$$

wih reaction rate $\Gamma = n\sigma c$.

Electron-positron annihilation

The ep-annihilation reaction is $e^+ + e^- \to \gamma + \gamma$ shown in Figure 12.

wih the cross-section

$$\sigma_{e^+e^-}(\omega_0) = \left(1+\frac{\pi\alpha}{v}\right)\sigma_0(\beta) - \frac{2\alpha}{\pi}\left(-\frac{1+\beta^2}{2\beta}\log\left(\frac{1+\beta}{1-\beta}\right)-1\right)\log\left(\frac{\sqrt{s}}{2\omega_0}\right)\sigma_0(\beta) \quad [24]$$

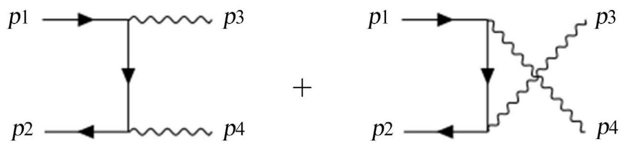


Figure 12. e-p annihilation.

where $\sigma_0(\beta)=\frac{\pi\alpha^2}{s\beta}\left(-\frac{3-\beta^4}{\beta}\log\left(\frac{1+\beta}{1-\beta}\right)-2\left(2-\beta^2\right)\right)$ Born cross-section, and Mandelstamm variables $s=(p_1+p_2)^2$, $t=(p_1-p_3)^2$, $u=(p_1-p_4)^2$, where

$$\beta=\sqrt{1-4\left(mc^2\right)^2/s},\quad z=\frac{1+\beta}{1-\beta}$$

ω_0 soft cut-off, $v=\frac{2\beta}{1+\beta^2}$ relative velocity, dof number

$g_S=\begin{cases}2+\frac{7}{8}\times4=\frac{11}{2} & T\ge m_e\\ 2 & T<m_e\end{cases}$ with photons decoupling at $T_{e,d}=0.5\,\text{MeV}$,

$t_{e,d}=6\,\text{s}$, duration $\Delta t_{e,d}=\frac{\alpha^2}{m_e}=10^{-18}\,\text{s}$

$T_\nu=\left(\frac{4}{11}\right)^{1/3}T_\gamma$, $t>t_{e,d}$ after ep-annihilation, so $T_{\gamma,0}=2.73\,\text{K}$, $T_{\nu,0}=1.95\,\text{K}$.

Planck data yield $\sum_i m_{\nu i}<0.13\,\text{eV}$, $\Omega_\nu<0.003$.

General photon eos

For $T>T_{an}$ in pair-production regime, we have in equilibrium (relativistic)

$$\sigma_0(\beta)=\frac{2\pi\alpha^2}{s\beta},\quad \beta=\frac{v_e}{c}$$

$$\Gamma_{ee\gamma}=2n_{e+}v\sigma\approx2n_{e+}\beta c\frac{2\pi\alpha^2\hbar^2c^2}{s\beta}\left(1+\frac{\pi\alpha}{\beta}\right)$$

$$\Gamma_{\gamma ee}=2n_\gamma c\sigma\approx2n_\gamma c\alpha r_e^2Z_{ef}^2\left(3.1\ln\left(\frac{E_\gamma}{E_e}\right)-8.1\right)\quad\text{with}\quad Z_{ef}=1\frac{n_b}{n_\gamma},\quad s=4E_{th}^2$$

$$\Gamma_{ee\gamma}=\Gamma_{\gamma ee}\quad\text{results}\quad n_\gamma=\frac{n_b^2}{n_{e+}}E_{th}^2\frac{r_e^2\,3.1\ln\left(\frac{E_\gamma}{E_e}\right)}{4\pi\alpha\hbar^2c^2\left(1+\frac{\pi\alpha c}{v_e}\right)},\quad \textit{i.e.}\quad n_\gamma\sim\frac{n_b}{n_{e+}}n_bE_{th}^2\sim E_{th}^4,$$

with thermal energy $E_{th}=k_BT$.

In the black-body regime we have the Stefan-Boltzmann relation $n_\gamma=a_{SB}E_{th}^4$.

The positron density n_{e+} results from equality of both n_γ from pair-production-annihilation and Stefan-Boltzmann

$$n_{e+}\approx\frac{n_b^2}{n_\gamma}0.17\alpha\left(\frac{E_{th}}{m_ec^2}\right)^2=\frac{n_b^2}{n_\gamma}\left(\frac{E_{th}}{m_ec^2}\right)^2 1.2\times10^{-3}.$$

Thomson scattering ([26] Hu)

We get density of free electrons

$n_e=\left(1-\frac{Y_p}{2}\right)X_en_b\approx\Omega_bh^2(1+z)^3\times10^{-5}\,\text{cm}^{-3}$, ionization fraction $X_e\approx1$,

where $Y_p\approx0.24$ Helium mass fraction.

The optical depth τ results from the Thomson equation $\frac{d\tau}{d\eta}=n_e\sigma_Ta$, where $\sigma_T = \frac{8\pi\alpha^2}{3m_e^2} = 6.65\times10^{-25}\,\text{cm}^2$ is the Thomson cross-section in photon-electron scattering.

Photons and neutrinos

After photon decoupling we have the relation for neutrino and photon temperature

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma \tag{3.62}$$

Hydrogen recombination ([4], chap. 2)

For hydrogen recombination we have the reaction $e^- + p^+ \to H + \gamma$,

and number density $\left(\frac{n_H}{n_e^2}\right) = \left(\frac{2\pi}{m_e T}\right)^{3/2} \exp\left(\frac{E_{ion}}{T}\right)$,

with ionization energy $E_{ion} = m_p + m_e - m_H = 13.6\,\text{eV}$, $E_{H,re} = 13.6\,\text{eV}$

and free electron fraction $X_e \equiv \frac{n_e}{n_p + n_H} = \frac{n_e}{n_b}$.

The free electron fraction obeys Saha equation

$$\frac{1 - X_e}{X_e^2} = \frac{2\zeta(3)}{\pi^2}\left(\frac{2\pi}{m_e T}\right)^{3/2} \eta \exp\left(\frac{E_{ion}}{T}\right) \quad (3.78) \quad \zeta(3) = 1.202$$

where $\frac{n_b}{n_\gamma} = \frac{n_{b,0}}{n_{\gamma,0}} = \frac{0.242\,\text{m}^{-3}}{0.41\times10^9\,\text{m}^{-3}} = 0.59\times10^{-9}$, and baryon-photon ratio $\eta \approx 6\times10^{-10}$.

The solution is $X_e = \frac{-1 + \sqrt{1 + 4 f(E_{th})}}{2 f(E_{th})}$,

$$f(E_{th}) = 4\zeta(3)\sqrt{\frac{2}{\pi}}\eta\left(\frac{E_{th}}{m_e c^2}\right)^{3/2} \exp\left(\frac{E_{H,re}}{E_{th}}\right) = 2.26\times10^{-9}\left(\frac{E_{th}}{m_e c^2}\right)^{3/2} \exp\left(\frac{E_{H,re}}{E_{th}}\right),$$

with limits

$$f \gg 1,\quad X_e \approx \frac{1}{\sqrt{f(E_{th})}},\quad n_e = n_b,\quad \frac{n_b}{n_H} \ll 1$$

$$f \ll 1,\quad X_e \approx 1,\quad n_e = n_b,\quad n_H = 0,$$

and recombination temperature $T_{rec} \approx 0.32\,\text{eV} = 3760\,\text{K}$, $t_{rec} \approx 290\,\text{ky}$.

Photon decoupling

The photon decoupling reaction is $e^- + \gamma \leftrightarrow e^- + \gamma$, with reaction rate $\Gamma_\gamma \approx n_e \sigma_T$, $\sigma_T \approx 2\times10^{-3}\,\text{MeV}^{-2}$, and decoupling temperature

$$\Gamma_\gamma(T_{dec}) \approx H(T_{dec}),\quad X_e(T_{dec})T_{dec}^{3/2} \approx \frac{\pi^2}{2\zeta(3)}\frac{H_0\sqrt{\Omega_m}}{\eta\sigma_T T_0^{3/2}},\quad T_{dec} \approx 0.25\,\text{eV} = 2970\,\text{K}$$

for $t_{dec} \approx 370\,\text{ky}$.

The Boltzmann equation is $\frac{\partial f}{\partial t} + \frac{\vec{p}}{m}\nabla f + \vec{F}\cdot\frac{\partial f}{\partial \vec{p}} = C(f)$, for reaction

$1+2 \leftrightarrow 3+4$ collision term is $C_i\left[\left\{n_j\right\}\right] = -\alpha_c n_1 n_2 + \alpha_c \beta_c n_3 n_4$, where $\alpha_c = \langle \sigma v \rangle$ thermally averaged cross-section, $\beta_c = \left(\frac{n_1 n_2}{n_3 n_4}\right)_{eq}$ detailed balanced coefficient.

From this follows **cosmic Boltzmann equation** with collision term

$$\frac{1}{a^3}\frac{\mathrm{d}\left(n_i a^3\right)}{\mathrm{d}t} = -\langle \sigma v \rangle \left(n_1 n_2 - \beta_c n_3 n_4\right) \tag{3.96}$$

where the particle number is $N_i \equiv \frac{n_i}{s} \propto n_i a^3$, $\frac{\mathrm{d}\left(\log N_1\right)}{\mathrm{d}\left(\log a\right)} = -\frac{\Gamma_1}{H}\left(1 - \left(\frac{N_1 N_2}{N_3 N_4}\right)_{eq} \frac{N_3 N_4}{N_1 N_2}\right)$, where $\Gamma_1 \equiv n_2 \langle \sigma v \rangle$ (1,2) interaction rate.

Dark matter cdm decoupling

The reaction for cdm particle *X*, light particle *l*: $X + \bar{X} \leftrightarrow l + \bar{l}$ with Boltzmann equation $\frac{1}{a^3}\frac{\mathrm{d}\left(n_X a^3\right)}{\mathrm{d}t} = -\langle \sigma v \rangle \left(n_X^2 - \left(n_X\right)_{eq}^2\right)$, with $Y_X \equiv \frac{n_X}{T^3}$ particles in co-moving volume, and reduced mass $x \equiv \frac{M_X}{T}$, $\frac{\mathrm{d}x}{\mathrm{d}t} = Hx$.

Using $\lambda \equiv \frac{\Gamma\left(M_X\right)}{H\left(M_X\right)} = \frac{M_X^3 \langle \sigma v \rangle}{H\left(M_X\right)}$, we get the Riccati equation $\frac{\mathrm{d}Y_X}{\mathrm{d}x} = -\frac{\lambda}{x^2}\left(Y_X^2 - \left(Y_X\right)_{eq}^2\right)$.

The asympotic value is $Y_{X,\infty} \approx \frac{x_f}{\lambda}$ with x_f reduced mass at freeze-out.

The cdm density is $\Omega_X \sim 0.1 \frac{x_f}{\sqrt{g_s\left(M_X\right)}} \frac{10^{-8}\,\mathrm{GeV}^{-2}}{\langle \sigma v \rangle}$ with reaction rate $\sqrt{\langle \sigma v \rangle} \sim 10^{-8}\,\mathrm{GeV}^{-2} \sim 0.1\sqrt{G_F}$ (≈weak interaction).

Baryo-genesis

In the following we present important cosmological processes of nuclei, with density evolution equation, cross-section, and charasteristic (freeze-out) time.

Neutron-proton decay

The reaction here is $n + \nu_e \leftrightarrow p^+ + e^-$, $n + e^+ \leftrightarrow p^+ + \bar{\nu}_e$ with density ratio $\left(\frac{n_n}{n_p}\right)_{eq} = \exp\left(-\frac{E_{np}}{k_B T}\right)$, $E_{np} = \left(m_n - m_p\right)c^2 = 1.30\,\mathrm{MeV}$, and with $X_n \equiv \frac{n_n}{n_n + n_p}$ relative n-abundance.

For X_n we get the equation

$$\frac{\mathrm{d}X_n}{\mathrm{d}t} = -\Gamma_n\left(x\right)\left(X_n - \left(1 - X_n\right)\exp\left(-\frac{E_{np}}{k_B T}\right)\right)$$

where

$$\Gamma_n\left(x\right) = \frac{255}{\tau_n}\frac{12 + 6x + x^2}{x^5}, \quad x = \frac{E_{np}}{k_B T}, \quad \tau_n = 886.7 \pm 0.8\,\mathrm{s} \text{ neutron lifetime.}$$

With freeze-out abundance $X_{n,\infty} = 0.15$ it becomes $X_n\left(t\right) = X_{n,\infty}\exp\left(-\frac{t}{\tau_n}\right)$.

Deuterium

The density ratio is $\left(\frac{n_D}{n_p}\right)_{eq} = \frac{3}{4} n_{n,eq} \left(\frac{4\pi\hbar^2 c^2}{m_p c^2 k_B T}\right)^{3/2} \exp\left(\frac{E_{npD}}{k_B T}\right)$, with $E_{npD} = \left(m_n + m_p - m_D\right)c^2 = 2.22\ \text{MeV}$ and temperature $T_{nuc} = 0.06\ \text{MeV}$ at $\left(\frac{n_D}{n_p}\right)_{eq} \left(T = T_{nuc}\right) = 1$, the corresponding time is $t_{nuc} = \left(\frac{0.1\ \text{MeV}}{T_{nuc}}\right)^2 120\ \text{s} \approx 330\ \text{s}$.

Helium

The reactions are

$$D + p^+ \leftrightarrow He^3 + \gamma,\quad H^3 + p^+ \leftrightarrow He^3 + n$$

$$D + D \leftrightarrow H^3 + p^+,\quad H^3 + D \leftrightarrow He^4 + p^+$$

$$D + D \leftrightarrow He^3 + n,\quad He^3 + D \leftrightarrow He^4 + p^+$$

helium-hydrogen ratio is then

$Y_P = \frac{4n_{He}}{n_H} = \frac{4n_{He}}{n_p} \approx \frac{2X_n\left(t_{nuc}\right)}{1 - X_n\left(t_{nuc}\right)} \sim 0.25$, which is observed.

Lithium beryllium

The reactions are

$Be^7 + n \leftrightarrow Li^7 + p^+,\quad Li^7 + p^+ \leftrightarrow He^4 + He^4,\quad Be^7 + e^- \leftrightarrow Li^7 + \nu_e$

$He^3 + He^4 \leftrightarrow Be^7 + \gamma,\quad H^3 + He^4 \leftrightarrow Li^7 + \gamma$.

Hydrogen recombination

The process of hydrogen recombination is shown in Figure 13.

We have the Peebles equation for free electron density X_e with an improved calculation in redshift z [27]

$$\frac{\mathrm{d}X_e}{\mathrm{d}z} = -\frac{C_r(T)}{H(z)(1+z)}\left(\left(\frac{m_e c^2 k_B T}{2\pi}\right)^{1/2}\left(1 - X_e\right)\exp\left(-\frac{E_I}{k_B T}\right) - \alpha(T)\frac{n_b}{n_\gamma}\frac{2\zeta(3)}{\pi^2}\left(k_B T\right)^3 X_e^2\right) \tag{20}$$

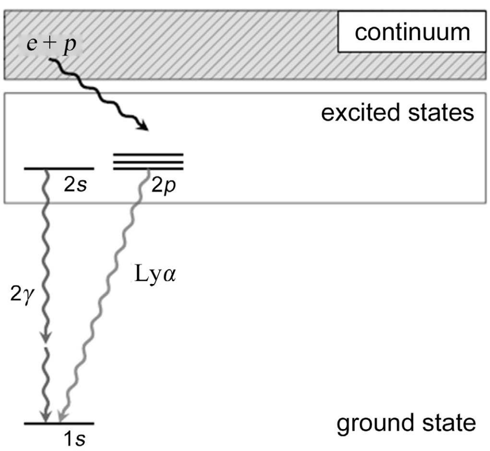


Figure 13. Hydrogen recombination state diagram [4].

with

$$C_r(T) \equiv \frac{\Lambda_{2\gamma} + \Lambda_\alpha}{\Lambda_{2\gamma} + \Lambda_\alpha + \beta_\alpha},$$

$$\Lambda_\alpha = \frac{27}{128\zeta(3)} \frac{H(T)}{(1 - X_e)(n_b/n_\gamma)(k_B T/E_I)^3},$$

$$\Lambda_{2\gamma} = 8.227\ \mathrm{s}^{-1},$$

$$\lambda_\alpha = \frac{8\pi\hbar c}{3E_I}\ \text{Lyman wavelength},\quad \beta_\alpha = \beta(T)\exp\left(\frac{3E_I}{4k_B T}\right),$$

$$\alpha(T) \approx 9.8 \frac{\alpha^2}{(m_e c^2)^2}\left(\frac{E_I}{k_B T}\right)^{1/2} \log\left(\frac{E_I}{k_B T}\right),$$

$$H(z) = \sqrt{\Omega_m} H_0 (1+z)^{3/2}\left(1 + \frac{1+z}{1+z_{eq}}\right),$$

$$H_0 \approx 1.5\times 10^{-33}\ \mathrm{eV},\quad T = (1+z)\,0.235\ \mathrm{eV}.$$

10. CMB Spectrum

In this chapter, we present first in concise way the contributions to the temperature anisotropy of the cosmic microwave background CMB.

Then we describe the scheme for the calculation of the CMB spectrum coefficients C_l.

The schematic of the calculation is shown in chap. 11.

Finally, we present the self-calculated results and a comparison with data.

10.1. CMB Spectrum Theory

CMB spectrum today

CMB as measured today has the parameters [28]:

temperature $T_{\gamma,0} = 2.7255 \pm 0.0006\ \mathrm{K}$.

CMB dipole is around 3.3621 ± 0.0010 mK

relative density $\Omega_\gamma = 6\times 10^{-5}$

temperature anisotropy $\Delta T_{\gamma,0} \approx 30\ \mu\mathrm{K}$, so $\frac{\Delta T_{\gamma,0}}{T_{\gamma,0}} \approx \frac{30\ \mu\mathrm{K}}{2.72\ \mathrm{K}} = 1.1\times 10^{-5}$.

Temperature anisotropy

The temperature anisotropy of the CMB has the following contributions:

$$\frac{\delta T}{\bar{T}}(\hat{n}) = \left(\mathrm{SW} = \left(\frac{1}{4}\delta_\gamma + \Psi\right)_*\right) + \left(\mathrm{Dop} = -(\hat{n}\cdot\vec{v}_b)_*\right) + \left(\mathrm{ISW} = \int_{\eta_*}^{\eta_0} \mathrm{d}\eta\,(\Phi' + \Psi')\right) \quad (7.29)$$

at conformal time $\eta = \eta_* = \eta_{dec}$.

▪ **SW** The first term is the so-called Sachs–Wolfe term. It represents the intrinsic temperature fluctuations associated to the photon density fluctuations $\delta_\gamma/4$ and the metric perturbation Ψ at last scattering.

▪ **Doppler** The second term is the Doppler term $\hat{n}\cdot\vec{v}_b$ caused by local velocity, this contribution is small on large scales.

▪ **ISW** The last term describes the additional gravitational redshift $\int_{\eta_*}^{\eta_0} \mathrm{d}\eta\left(\Phi'+\Psi'\right)$ due to the evolution of the metric.

The temperature anisotropy has the form

$$\Theta(\hat{n}) \equiv \frac{\delta T}{\bar{T}}(\hat{n}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \exp\left(i\vec{k}\cdot\hat{n}\,ct(\eta_*)\right)\left(F\left(\eta_*,\vec{k}\right)+i\left(\vec{k}\cdot\hat{n}\right)G\left(\eta_*,\vec{k}\right)\right),$$

where $F\left(\eta_*,\vec{k}\right)=\left(\frac{1}{4}\delta_\gamma+\Psi\right)$, $G\left(\eta_*,\vec{k}\right)=v_b$, $F_*(k)=\dfrac{F\left(\eta_*,\vec{k}\right)}{R\left(\eta=0,\vec{k}\right)}$, $G_*(k)=\dfrac{G\left(\eta_*,\vec{k}\right)}{R\left(\eta=0,\vec{k}\right)}$ and $R\left(\eta=0,\vec{k}\right)$ are the initial curvature anisotropies.

We get for the anisotropy the series in Legendre polynomials

$$\Theta(\hat{n}) = \sum_l i^l(2l+1)\int\frac{\mathrm{d}^3 k}{(2\pi)^3}\Theta_l(k)R\left(0,\vec{k}\right)P_l\left(\vec{k}\cdot\hat{n}\right)$$

with the transfer function including ISW

$$\Theta_l(k)=\Theta_l(k)=\left(F_*(k)\,j_j(\chi_* k)-G_*(k)\,j_j'(\chi_* k)\right)+\int_{\eta_*}^{\eta_0}\mathrm{d}\eta\left(\Phi'+\Psi'\right)j_j\left(ct(\eta)k\right),$$

with $\chi_* = ct(\eta_*)$.

The two-point temperature correlation (scalar TT-correlation) spectrum measured in CMB is $C(\theta)=\left\langle\Theta(\hat{n})\Theta(\hat{n}')\right\rangle$, with directions $\hat{n},\hat{n}'$, angle $\cos\theta=\hat{n}\cdot\hat{n}'$, and the series in Legendre polynomials

$$C(\theta)=\sum_l\frac{2l+1}{4\pi}C_l P_l(\cos\theta)$$

with series coefficients C_l

$$C_l = 2\pi\int_{-1}^{1}\mathrm{d}(\cos\theta)C(\theta)P_l(\cos\theta)=4\pi\int\frac{\mathrm{d}k}{k}\Theta_l^2(k)\Delta_R^2(k) \tag{7.6}$$

where $\Delta_R^2(k)=A_s\left(\frac{k}{k_0}\right)^{n_s-1}$ is the power amplitude, and where sound horizon is $r_s=\int\frac{\mathrm{d}\eta}{\sqrt{3\left(1+R(\eta)\right)}}$, with curvature $R(\eta)$.

Weinberg semi-analytic solution [29]

Weinberg proposed a semi-analytic solution for photon density perturbations

$$\delta_\gamma=\frac{4}{5}R\left(\eta=0,\vec{k}\right)\left(\frac{S(k)}{\left(1+R\left(\eta,\vec{k}\right)\right)^{1/4}}\cos\left(kr_s+\theta(k)\right)-\left(1+3R\left(\eta,\vec{k}\right)\right)T(k)\right)$$

with Weinberg semi-analytic transfer functions for SW and Doppler with

$$F_*(k)=\frac{1}{5}\left(\exp\left(-\frac{k^2}{k_{D*}{}^2}\right)\frac{S(k)}{\left(1+R\left(\eta_*,\vec{k}\right)\right)^{1/4}}\cos\left(kr_{s*}+\theta(k)\right)-3R\left(\eta_*,\vec{k}\right)T(k)\right)$$

$$G_*(k) = -\frac{\sqrt{3}}{5}\exp\left(-\frac{k^2}{k_{D*}{}^2}\right)\frac{S(k)}{\left(1+R\left(\eta_*,\vec{k}\right)\right)^{1/4}}\sin\left(kr_{s*}+\theta(k)\right)$$ where

$k_{D*}^{-1} = 8.8\,\text{Mpc}$

and the resulting CMB power spectrum

$$\frac{l(l+1)}{2\pi}C_l = \int_1^\infty \frac{d\beta}{\beta^2\sqrt{\beta^2-1}}\left(F_*^2\left(\frac{l\beta}{\chi_*}\right)+\frac{\beta^2-1}{\beta^2}G_*^2\left(\frac{l\beta}{\chi_*}\right)\right)\Delta_R^2\left(\frac{l\beta}{\chi_*}\right)$$ with

$\chi_* = ct(\eta_*)$

where

$$S(\kappa) = \left(\frac{1+(1.209\kappa)^2+(0.5611\kappa)^4+\sqrt{5}(0.1567\kappa)^6}{1+(0.9459\kappa)^2+(0.4249\kappa)^4+(0.167\kappa)^6}\right)^2$$

$$T(\kappa) = \frac{\log\left(1+(0.124\kappa)^2\right)}{(0.124\kappa)^2}\left(\frac{1+(1.257\kappa)^2+(0.4452\kappa)^4+(0.2197\kappa)^6}{1+(1.606\kappa)^2+(0.8568\kappa)^4+(0.3927\kappa)^6}\right)^{1/2}$$

$$\theta(\kappa) = \left(\frac{(1.1547\kappa)^2+(0.5986\kappa)^4+\sqrt{5}(0.2578\kappa)^6}{1+(1.723\kappa)^2+(0.8707\kappa)^4+(0.4581\kappa)^6+(0.2204\kappa)^8}\right)^{1/2}.$$

Calculation of CMB spectrum coefficients C_l ([30] Hu)

The temperature and photon polarization Stokes parameters anisotropy are expanded in a series in angular momentum (l, m),

$$\Theta(\eta,\vec{x},\hat{n}) = \int\frac{d^3k}{(2\pi)^3}\sum_{l=0}^{\infty}\sum_{m=-2}^{2}\Theta_{lm}G_{lm} \tag{21a}$$

$$(Q\pm iU)(\eta,\vec{x},\hat{n}) = \int\frac{d^3k}{(2\pi)^3}\sum_{l=0}^{\infty}\sum_{m=-2}^{2}(E_{lm}\pm iB_{lm})G_{lm}$$

with temperature (l, m)-moments

$$\Theta_l^{(m)} = \int dn Y_{lm}^*(\vec{n})\Theta(\vec{n}) \tag{21b}$$

and with temperature basis functions

$$G_{lm} = (-i)^l\sqrt{\frac{4\pi}{2l+1}}Y_{lm}(\hat{n})\exp\left(i\vec{k}\cdot\vec{x}\right) = \sum_l(-i)^l\sqrt{4\pi(2l+1)}j_l(kr)Y_{l0}(\theta,\varphi),$$

$$G_{l'm} = \sum_l(-i)^l\sqrt{4\pi(2l+1)}j_{ll'm}(kr)Y_{lm}(\theta,\varphi),$$

where

$$\exp\left(i\vec{k}\cdot\vec{x}\right) = \sum_l(-i)^l\sqrt{4\pi(2l+1)}j_l(kr)Y_{l0}(\theta,\varphi).$$

In this representation, the spectrum coefficients C_l are

$$\left\langle\Theta_l^{(m)},\Theta_{l'}^{(m')}\right\rangle_\eta \equiv \int d\eta\,\Theta*_l^{(m)}\Theta_{l'}^{(m')} = \delta_{ll'}\delta_{mm'}C_l \tag{21c}$$

where the power spectrum on the angular momentum l is

$$\Delta_T^2(l) = \frac{l(l+1)}{2\pi}C_lT^2 \quad \text{in}\ \mu\text{K}^2 \tag{21d}$$

We use the variables:

averaged pressure $V(\eta',k) = -\frac{8\pi G}{ka^2}\int_0^{\eta'} \mathrm{d}\eta\, a^4 \delta P$, $V'(\eta,k) = -\frac{8\pi G}{k} a^2 \delta P$

optical depth $\tau(\eta') = \sigma_T \int_0^{\eta'} \mathrm{d}\eta\, n_e a$, $\tau'(\eta) = n_e \sigma_T a$.

The temperature (*l, m*)-moments are calculated from the evolution equations

$$\Theta'_{lm} = k\left(\frac{\kappa_{0l}^m}{2l-1}\Theta_{lm} - \frac{\kappa_{0l+1}^m}{2l+3}\Theta_{l+1m}\right) - \tau'\Theta_{lm} + S_{lm} \tag{21e}$$

with sources

$$S_{00} = \tau'\Theta_{00} - \Phi',\quad S_{10} = \tau' v_{b0} + k\Psi,\quad S_{11} = \tau' v_{b1} + V'$$

$$S_{20} = \frac{1}{10}\tau'\left(\Theta_{20} - \sqrt{6}E_{20}\right),\quad S_{21} = \frac{1}{10}\tau'\left(\Theta_{21} - \sqrt{6}E_{21}\right),$$

$$S_{22} = \frac{1}{10}\tau'\left(\Theta_{22} - \sqrt{6}E_{22}\right) - \Phi'$$

$$S_{20} = \frac{1}{10}\tau'\left(\Theta_{20} - \sqrt{6}E_{20}\right),\quad S_{21} = \frac{1}{10}\tau'\left(\Theta_{21} - \sqrt{6}E_{21}\right),$$

$$S_{22} = \frac{1}{10}\tau'\left(\Theta_{22} - \sqrt{6}E_{22}\right) - \Phi'$$

$$\frac{\Theta_{lm}(\eta_0,k)}{2l+1} = \int_0^{\eta_0} \mathrm{d}\eta \exp(-\tau) \sum_{l'} S_{l'm}(\eta)\, j_{ll'm}\left(k(\eta_0-\eta)\right)$$

and $j_{ll'm}$ are spherical Bessel functions

$$j_{l00}(x) = j_l(x),\quad j_{l10}(x) = j_l'(x),\quad j_{l20}(x) = \frac{1}{2}\left(3j_l''(x) + j_l(x)\right)$$

$$j_{l11}(x) = \sqrt{\frac{l(l+1)}{2}}\frac{j_l(x)}{x},\quad j_{l21}(x) = \sqrt{\frac{3l(l+1)}{2}}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{j_l(x)}{x}\right),$$

$$j_{l22}(x) = \sqrt{\frac{3}{8}\frac{(l+2)!}{(l-2)!}}\frac{j_l(x)}{x^2}.$$

10.2. CMB Calculation Results

The metric perturbations Ψ, Φ in k-space for $k = 5$ are shown in **Figure 14**, as a function of relative scale factor a/a_{eq}, where $a_{eq} = a_{dec} = 0.9 \times 10^{-3}$ at photon decoupling. Note the transition from high to low amplitude at decoupling.

Density fluctuations for baryons, radiation, cdm δ_b, δ_r, δ_c, for $k = 5$ are shown in **Figure 15**, as a function of relative scale factor a/a_{eq}. The matter fluctuations decay before or after decoupling, whereas radiation fluctuation stabilizes at a higher level.

The calculated normalized scalar TT-correlation power spectrum of CMB, $\Delta_T^2(l) = \frac{l(l+1)}{2\pi} C_l T^2$, is shown in **Figure 16**, in $\mu\mathrm{K}^2$ over multipole order *l*, calculated for the original Planck Hubble value $H_{0,P} = 67.74\ \mathrm{km}\cdot\mathrm{s}^{-1}\cdot\mathrm{Mpc}$. Note the characteristic decrease from the first to the second maximum and from the third to the following maxima.

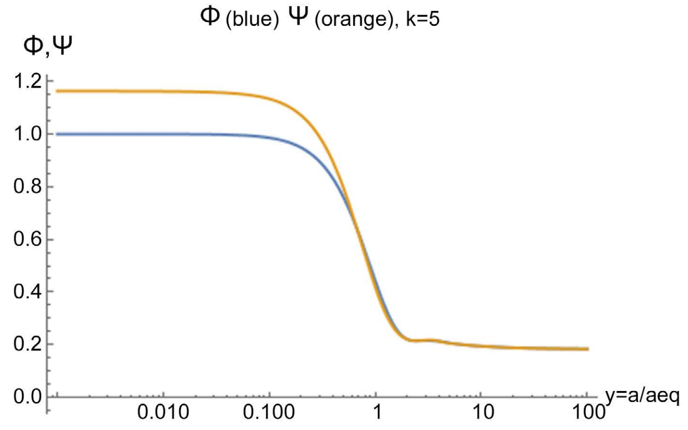


Figure 14. Metric perturbations, Ψ, $k = 5$ [31].

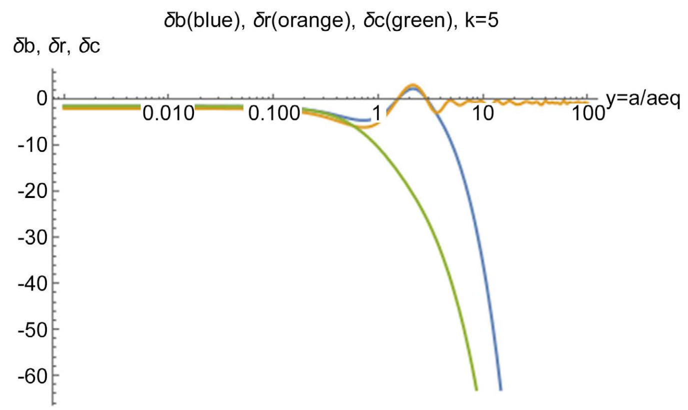


Figure 15. Density fluctuations δ_b, δ_r, δ_c, $k = 5$ [31], double logarithmic plot.

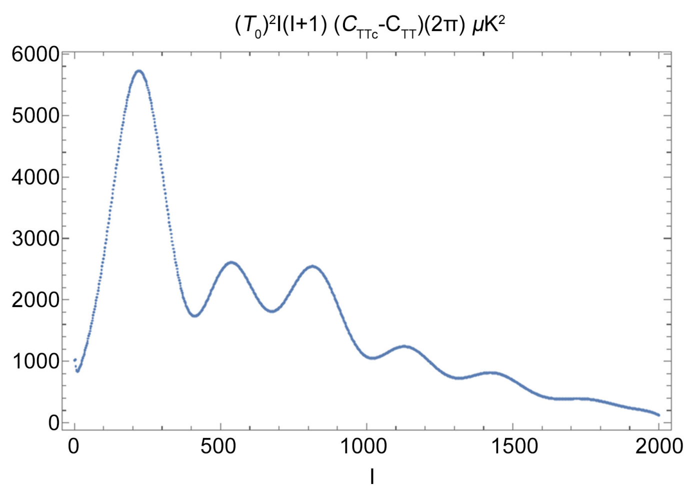


Figure 16. Temperature scalar TT-correlation spectrum $y = T^2 \frac{l(l+1)}{2\pi} C_l$, $[y] = \mu K^2$, $x = l$ [31].

The background Hubble parameter H_0 influences the CMB spectrum, but the deviation δ = 1.3% caused by the calculated correction from chap. 5 is within measurement error.

The plot in Figure 17 shows the difference between the power spectrum for Planck-Hubble-parameter $\Delta_T^2\left(l, H_{0,P}\right)=\frac{l(l+1)}{2\pi}C_l T^2$, and for the background-corrected Hubble-parameter $\Delta_T^2\left(l, H_{0,Pc}\right)=\frac{l(l+1)}{2\pi}C_l T^2$, where $H_{0,Pc}=H_{0,P}\times 1.043=70.6\pm 0.4$, with maximum deviation of δ = 1.3%.

In Figure 18 is shown the scalar TT-correlation power spectrum from Figure 16, together with measurement data and its error bars.

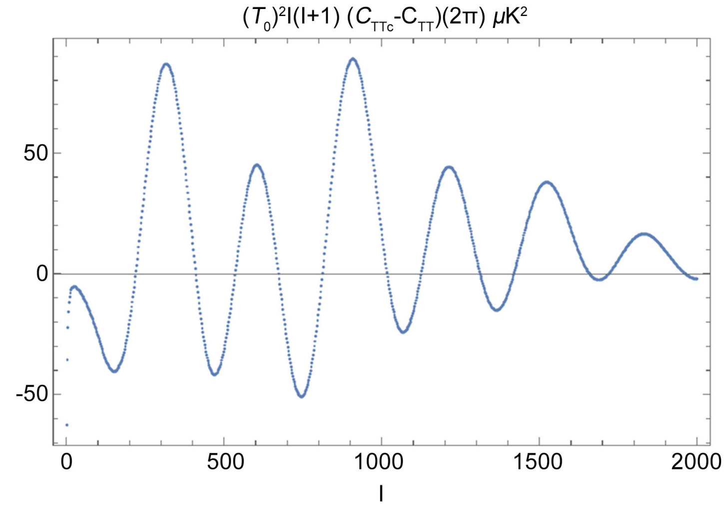


Figure 17. Power TT spectrum Hubble correction, max rel.dev. δ = 1.3% [31].

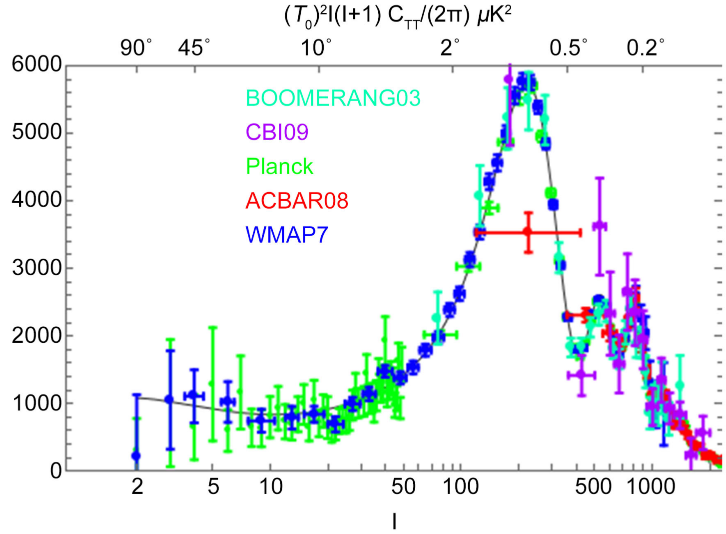
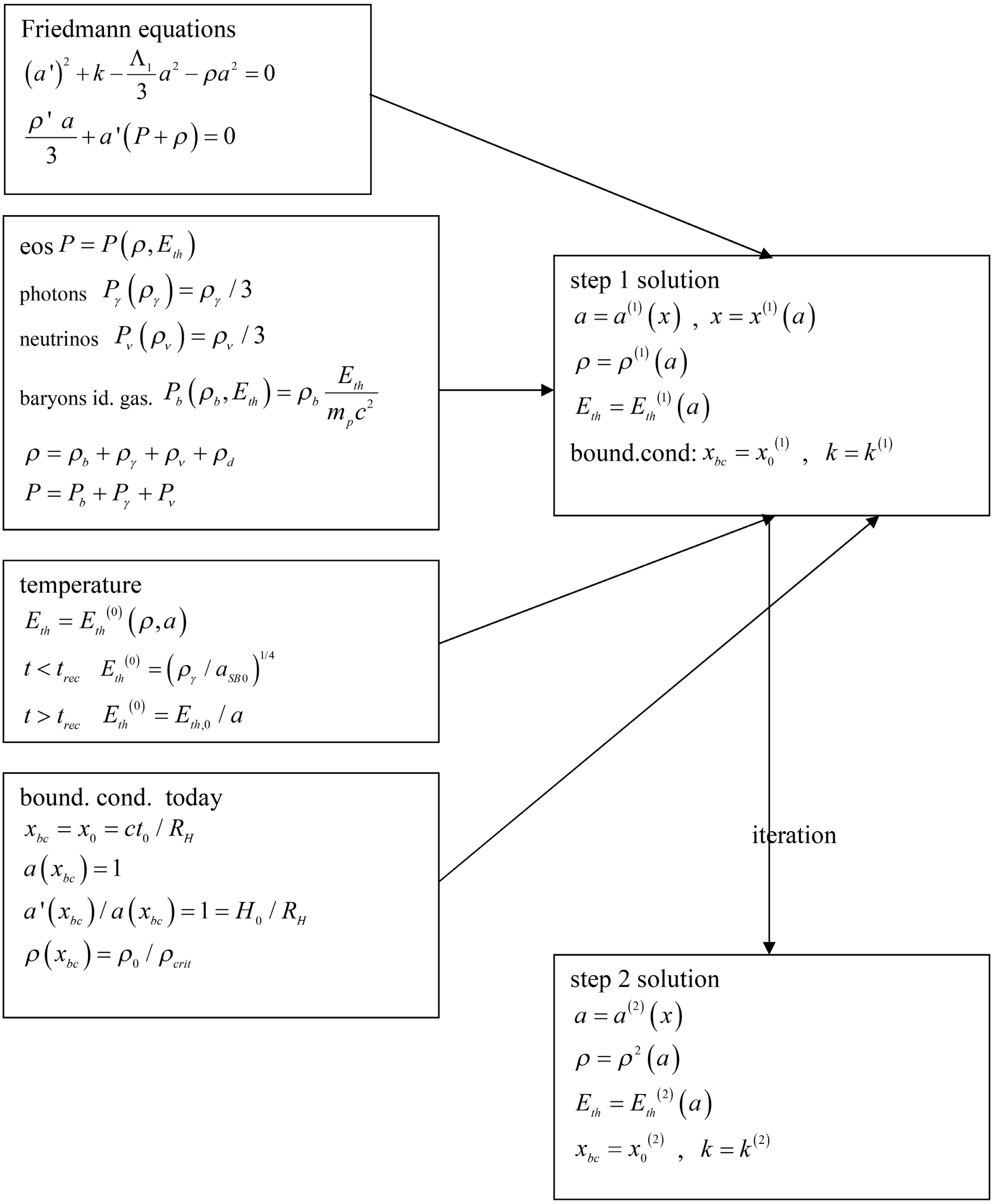


Figure 18. Temperature scalar TT-correlation power spectrum with measured data [22] [31], for measurements Planck, WMAP, ACBAR, CBI, and BOOMERANG.

11. Concise Presentation

In the following, we present the fundamental equations, the solution process and results in form of schematic diagrams for the background calculation and for the CMB calculation.

Lambda-CDM background calculation:



Lambda-CDM CMB calculation:

perturbations

$ds^2 = a(\eta)\left(-(1+2\Psi)d\eta^2 + (1-2\Phi)dx^i dx_i\right)$

$\Phi,\Psi,\theta,\sigma,\delta,\delta P$ perturbations

δP pressure

$\theta = ik^j v_j$ velocity

$\delta = \delta\rho / \bar{\rho}$ relative density

$\sigma = -\left(\hat{k}^i \hat{k}^j - \frac{1}{3}\delta_{ij}\right)\Pi^i{}_j / (\bar{\rho} + \bar{P})$ stress

$\bar{\rho}, \bar{P}, a, E_{th}, \tau$ background

initial conditions

$\Phi,\Psi,\theta,\sigma,\delta,\delta P$ = variables ξ_i

$\xi_i(a=0) = \xi_{i,1}$

Einstein equations k-space

$k^2\Phi - 3H(\Phi' + H\Psi) = \pi G a^2 \delta\rho$

$k^2(\Phi' + H\Psi) = \pi G a^2 (\bar{P} + \bar{\rho})\theta$

$k^2(\Phi - \Psi) = 12\pi G a^2 (\bar{P} + \bar{\rho})\sigma$

$\Phi'' + H(\Psi' + 2\Phi') + \frac{1}{3}k^2(\Phi - \Psi)$

$+(2H' + H^2)\Psi = 4\pi G a^2 \delta P$

thermodynamic: density+Euler

$\delta' = -\left(1 + \frac{\bar{P}}{\bar{\rho}}\right)(\theta - 3\Phi') - 3H\left(\frac{\delta P}{\bar{\rho}\,\delta} - \frac{\bar{P}}{\bar{\rho}}\right)\delta$

$\theta' = -\left(H + \frac{\bar{P}'}{\bar{P} + \bar{\rho}}\right)\theta - \frac{\delta P}{(\bar{P} + \bar{\rho})}k^2 - k^2\sigma + k^2\Psi$

CMB power spectrum coef C_l

$\Theta(\eta,\vec{x},\hat{n}) = \int \frac{d^3k}{(2\pi)^3} \sum_{l=0}^{\infty}\sum_{m=-2}^{2} \Theta_{lm} G_{lm}$

$\left\langle \Theta_l^{(m)}, \Theta_{l'}^{(m')} \right\rangle_\eta \equiv \int d\eta\, \Theta^{*(m)}_l \Theta_{l'}^{(m')} = \delta_{ll'}\delta_{mm'}C_l$

$\Theta_l^{(m)} = \int dn\, Y_{lm}^*(\vec{n})\Theta(\vec{n})$

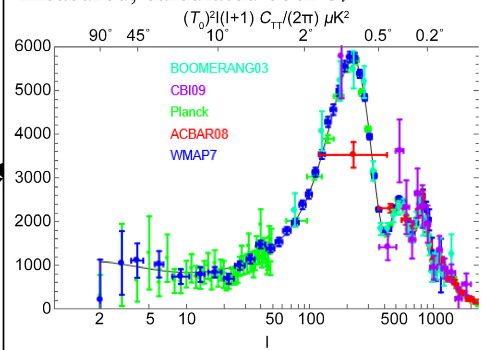
$\Theta'_{lm} = k\left(\frac{\kappa^m_{0l}}{2l-1}\Theta_{lm} - \frac{\kappa^m_{0l+1}}{2l+3}\Theta_{l+1m}\right) - \tau'\Theta_{lm} + S_{lm}$

measurement temperature correlations

$C(\theta) = \langle \Theta(\hat{n})\Theta(\hat{n}') \rangle$

$C(\theta) = \sum_l \frac{2l+1}{4\pi} C_l P_l(\cos\theta)$

measured, calculated coef C_l



13 fitted parameters

$\Omega_b, \Omega_c, \Omega_\Lambda, t_0, H_0, A_s, n_s, \tau, w, \Sigma m_\nu, N_\nu, r_t, \frac{dn_s}{dk}$

12. Conclusions

The results for the background part are presented in schematic form in chap. 11 Lambda-CDM background calculation.

We start with the Friedmann equations

$$(a')^2 + k - \frac{\Lambda_1}{3}a^2 - \rho a^2 = 0$$

$$\frac{\rho' a}{3} + a'(P+\rho) = 0$$

with the variables in dependence of the scale factor a (inverting the scalefactor-time relation $a = a(x)$,

$x(a)$ time,

$\rho_i(a)$ density of component *i*,

$E_{th}(a)$ temperature,

for components radiation γ, neutrinos ν, electrons *e*, protons *p*, neutrons *n*, cdm *d,* where the pressure $P_i(a)$ is eliminated using the component eos $P_i = P_i(\rho_i, E_{th})$.

In difference to the conventional ansatz,

-the temperature resp. **thermal energy is introduced as explicit function of time** $E_{th}(t)$;

-we use **the ideal gas eos for baryons**, instead of the usual setting $P_b = 0$ (dust eos).

As we show in chap. 5, this leads to a correction of 4.3% for the present value of Hubble parameter $H_{0c} = 1.043H_0$, which brings it into agreement with the measured Red-Giant-result, and within error margin with the Cepheids-SNIa-measurement.

We carry out an iterated calculation with two steps *i* = 1 and *i* = 2, the results are shown graphically in chap. 10.2.

Note the deviation of the temperature from the conventional linear behavior (brown) to the calculated first-iteration-value (blue) for later times. This produces also a slight "bump" for the Hubble parameter $H(a)$, and there is a slight "kink" in $x(a)$.

The results for the perturbation part are presented in schematic form in chap. 11 Lambda-CDM CMB calculation.

We start with the perturbed metric

$$\mathrm{d}s^2 = a(\eta)\left(-(1+2\Psi)\mathrm{d}\eta^2 + (1-2\Phi)\mathrm{d}x^i\mathrm{d}x_i\right)$$

perturbations $\Phi, \Psi, \theta, \sigma, \delta, \delta P$, where

δP pressure

$\theta = ik^j v_j$ velocity

$\delta = \delta\rho/\bar{\rho}$ relative density

$\sigma = -\left(\hat{k}^i\hat{k}^j - \frac{1}{3}\delta_{ij}\right)\Pi^i_j/(\bar{\rho}+\bar{P})$ stress

$\bar{\rho}, \bar{P}, a, E_{th}$ are background functions calculated already in the background part.

And τ = reionization optical depth is a parameter used for the CMB calculation.

The perturbations result from (random) initial conditions and represent the random nature of structure formation.

The resulting fundamental equations are transformed to k-space (*i.e.* Fourier transformed), and consist of two parts.

The Einstein equations in k-space resulting from the perturbed metric ansatz

$$k^2\Phi - 3H\left(\Phi' + H\Psi\right) = \pi G a^2 \delta\rho$$

$$k^2\left(\Phi' + H\Psi\right) = \pi G a^2\left(\bar{P} + \bar{\rho}\right)\theta$$

$$k^2\left(\Phi - \Psi\right) = 12\pi G a^2\left(\bar{P} + \bar{\rho}\right)\sigma$$

$$\Phi'' + H\left(\Psi' + 2\Phi'\right) + \frac{1}{3}k^2\left(\Phi - \Psi\right) + \left(2H' + H^2\right)\Psi = 4\pi G a^2 \delta P$$

and the thermodynamic: density and Euler (relativistic fluid) equation, resulting from the relativistic Boltzmann transport equation

$$\delta' = -\left(1 + \frac{\bar{P}}{\bar{\rho}}\right)\left(\theta - 3\Phi'\right) - 3H\left(\frac{\delta P}{\bar{\rho}\,\delta} - \frac{\bar{P}}{\bar{\rho}}\right)\delta$$

$$\theta' = -\left(H + \frac{\bar{P}'}{\bar{P} + \bar{\rho}}\right)\theta - \frac{\delta P}{\bar{P} + \bar{\rho}}k^2 - k^2\sigma + k^2\Psi$$

The CMB power spectrum coefficients C_l depend on the angular moments of temperature correlation Θ_{lm}, which obey the iterative differential equation in k-space

$$\Theta'_{lm} = k\left(\frac{\kappa_{0l}^m}{2l - 1}\Theta_{lm} - \frac{\kappa_{0l+1}^m}{2l + 3}\Theta_{l+1m}\right) - \tau'\Theta_{lm} + S_{lm}$$

with parameters, which are calculated from the fundamental equations.

The actual numerical calculation is performed in program [31], based on a function library from [22].

Then a fit is carried out between the calculated parameterized coefficients $C_l\left(p_i\right)$ and tthe measured values $C_{l,exp}$.

The 13 fitted parameters

$p_i = \left(\Omega_b, \Omega_c, \Omega_\Lambda, t_0, H_0, A_s, n_s, \tau, w, \Sigma m_\nu, N_\nu, r_t, \frac{\mathrm{d}n_s}{\mathrm{d}k}\right)$ are calculated by the Planck collaboration [32], and are not recalculated here.

The fitted [32] and measured coefficients C_l are shown in a plot.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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