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Logical Conflict between Bell's Locality and Probability Theory

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Abstract

Based on "locality" considerations, John Stuart Bell and his followers have derived inequalities and theorems that, when taken together with actual experiments that have been performed by Aspect and others, appear to contradict physical reality as defined by Einstein. However, their specifically applied concept of locality is in conflict with the Fundamental Model of probability theory and the set theoretic definition of conditional probabilities. Bell-type inequalities are, therefore, not adequate to decide ponderous questions regarding physical reality.

Keywords

Bell's Inequalities, Quantum Entanglement, EPR Experiments

1. Introduction

The well-known inequalities and corresponding theorems of Bell [1] and Clauser-Horne-Shimony-Holt (CHSH) [2] have been based on mathematical reasoning involving probability spaces and physical reasoning involving Einstein's separation principle. The mathematical reasoning has been shown to involve a number of issues with respect to the premises used in the Bell-CHSH proofs [3] [4]. These mathematical issues should have been sufficient to relegate the Bell-CHSH theorems to a place of lesser importance, if Bell would not have reformulated his theorem purely in terms of the attributes of "local" and "not-local" [5]:

> "But if [a hidden variable theory] is local it will not agree with quantum mechanics, and if it agrees with quantum mechanics it will not be local."

The notion of "hidden variable theory" relates to "elements of physical reality" as defined by Einstein, Podolsky and Rosen (EPR) [6]. They stated:

> "If without in any way disturbing a system, we can predict with certainty… the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity."

The "hidden variables" that have taken center stage in the well-known Einstein-Bohr debate, are just mathematical representations of these elements of physical reality that Einstein presented as alternative to "spooky" influences at a distance.

Key to the EPR arguments are Gedanken-experiments performed in two spatially distant stations. Aspect and others [7] have performed such measurements, using photon-pairs emitted from a source and subsequently passing polarizers and being detected at certain times determined by synchronized clocks. The photon pairs are prepared at the source S in a special way and said to be entangled when so prepared.

The locality-postulates of Bell and followers are usually based on the fact that the entangled pairs may not in any way be related to or depend on the configuration of the measurement equipment including polarizer-angles and detector outcomes in the respective stations. To show this fact, Aspect and others have involved very fast switching of the polarizer angles just before the pair-measurement. As soon as one accepts Einstein's separation principle [6] that is based on the limitation of all velocities by the velocity of light c in vacuum, the fast switching proves indeed that the photon-pair properties must be independent of the polarizer angles just before measurement. Unfortunately, and incorrectly, any such dependence is chastised in the well-known Alice-Bob tutorials and labeled "spooky", even after the sorting by the measurements, using Einstein's own words against his elements of physical reality.

I will show, however, that the evaluation of Einstein's elements of physical reality by polarizers and detector clicks leads to a separation of these elements into distinct sub-sets even for trivially local computer models. This fact contradicts Bell's definition of locality. I will further show that other definitions of locality that have been used by Bell, Aspect, CHSH and many researchers in papers and text-books, are in direct conflict with set-theoretic probability theory and fail, therefore, as well.

2. Einstein- and Bell-Type Models of Quantum Entities

The nature of experiments involving quantum entities is complicated by the well-known facts: 1) That these entities are not directly accessible to our senses, 2) That they are possibly disturbed (meaning changed with respect to their physical properties) by any instrument that we use to experimentally investigate them and 3) That the physical properties associated with them may only be defined in connection with the measuring instruments and the procedure of measurement. We discuss these factors specifically for models related to Aspect-type experiments.

2.1. Einstein's Gedanken-Experiments

Naturally, one cannot speak about a photon-polarization such as "horizontal", as long as one has not defined "horizontal" by the configuration of a polarizer in ordinary space and performed an evaluation of a photon with that polarizer. Bohr's postulate that no property of quantum entities exists before measurement may be interpreted that way. Einstein constructed his Gedanken-experiments in a way that acknowledged the three points of the previous paragraph. He involved two related distant measurements of especially prepared correlated (procedurally entangled) quantum entities that are generated at a common source. Each entity is measured only once and any involved disturbance is, therefore, of no direct consequence. The pair-measurement is important, because one measurement evaluates a physical property of one quantum-entity and the second measurement recognizes that property of the correlated entity by a confirming measurement outcome. If that outcome can be predicted with certainty, then EPR postulate that an element of physical reality must have transferred the information, while otherwise we would need to accept "spooky" influences; instantaneous influences at a distance.

In the theory of relativity, evaluations of distant elements of physical reality by use of local instruments and photons are defined only *relative* to each other. The nonlocality that is inherent in the word relative is acceptable within Einstein's methods and space-time system. Interestingly, the Bell-CHSH inequalities are also derived by only considering measurement-outcomes *relative* to each other and are, therefore, (as will be shown below in detail) subject to a more subtle interpretation of the concepts "local" and "not-local".

2.2. Einstein's vs Bell's "Locality" and Aspects "Nonlocality"

The derivation of the Bell-CHSH inequalities deals with model-function values $A = \pm 1$ in station S_A and $B \pm 1$ in S_B, respectively, for given polarizer angles (e.g. $j = a$ and $j' = b$). It is, however, not the values of the functions by themselves that matter but only the fraction of $A = B$ versus the fraction of $A \neq B$, as determined by the measurement. Because the functions only possible values are ±1 and because Bell's derivations involve only the products $A \cdot B$, the Bell-CHSH procedure is entirely equivalent to the asking for the outcomes of B relative to those of A (and vice versa).

The $A \cdot B$ products also encompass the notion that both A and B (seen as model events) occur, which is in set theoretic probability denoted by $A \cap B$. Any such statement involves necessarily both distant stations and, therefore, a nonlocality in the broadest sense of the word. Bell's theorem (as stated in the introduction) is, thus, trivially true as soon as we deal with such products.

Bell-CHSH and their followers have, however, narrowed their definition of "local" by two further postulates that are used for the derivation of their inequalities. A model is Bell-CHSH-local (**BC-local**):

1) If and only if the properties of the photon-pairs and their corresponding mathematical variable that Bell denoted by λ neither depend on the polarizer angles nor the function-outcomes corresponding to detector clicks.

And

2) If and only if the conditional probability of B to assume a certain value in station S_B does not depend on the fact that A assumes any value in S_A and vice versa.

I will prove in two separate sections below that (1) is incorrect, because (as mentioned) a careful distinction needs to be made between the elements of reality before and after their evaluations by polarizers and detectors. The distinction is necessary, in general, even if the act of measurement does not alter the elements of physical reality. Einstein's hypothesis is that the photon pairs do have properties that are recognized by the measurement instruments. The assumption of Bell and followers that their variable λ (and the sets of values λ_s that the variable exhibits under different circumstances) may *never* show any relation to the measurement instruments, denies Einstein's hypothesis from the start and without physical or mathematical reason.

Furthermore, I will show that (2) is incorrect because it entails, even for trivially local models, serious mathematical problems with respect to the definition of conditional probabilities in the Kolmogorov framework. The Bell-CHSH definition of "local" is, therefore, not acceptable from a mathematical point of view.

As an alternative, I propose a definition of "local" that is free of these contradictions and denote it by the term ST-local. We denote the measurement time by t_s (which may also stand for two different times t'_s, t''_s [4]). The subscript $s = 1, 2, 3, \cdots, N$ labels a particular pair-measurement and N is a very large number. Furthermore, I denote the properties of the photons that determine the detector clicks after passing the polarizers by the mathematical model value λ_s.

A model of the Aspect-type experiments is defined as **ST-local:**

3) If and only if the function value of B is deduced entirely from facts available in S_B, yet relative to the value of A that also must be derived in S_B from the value of λ_s for any imaginable polarizer angle in S_A at the time t_s.

The idea of **ST-local** is the following. The theorist, relying on space-time, may derive in station S_B a law of nature for the product $A \cdot B$ by imagining all possible equipment configurations in S_A. After all the measurements are done, the actual equipment configurations of the measurements can be determined from the registered clock-times t_s. The theory is then judged from the consistency with a large number of measurements. This underlying idea is probably as old as the existence of time-measurement. The amazing feature is just that it is applicable to random photon-pair emanations.

Finally, we define the extreme opposites. A model is trivially Einstein-local (**TE-local**):

4) If and only if the objects sent out from the source are at all times accessible to our senses and identified as the direct cause for the outcomes in the distant stations.

A model is **spooky:**

5) If and only if the (model) measurement of B with given polarizer angle at time t_s, alters the model value of the function A for that same measurement-time. (The definition is also valid with A and B exchanged).

A majority of physicists, including Aspect himself have, unfortunately, accepted that the true nature of Aspect-type experiments is spooky as defined by (5), while they found Bell's (1) and (2) acceptable and did not consider (3).

3. The Fundamental Model of Probability Theory and the Quantum Result

These rather abstract definitions are best explained by a specific example of the Fundamental Model of probability theory. I use and follow the notation of Bell-CHSH. However, in contrast to them, I strictly distinguish Bell's variable λ from the specific mathematical model values λ_s that (together with the polarizer angles at the time t_s) determine the detector outcomes and, thus, the values of A, B. There are two detectors related to any given polarizer angles. One for detecting "horizontal" in station S_A, whose clicks we model by the function-outcomes $A(j,\lambda_s)=+1$, while "horizontal" in station S_B is denoted by the function $B(j',\lambda_s)=+1$. Clicks of the second detector are modeled by the same functions, with a value of -1 instead and called "vertical". Bell-CHSH considered mostly two possible polarizer angles in each station: $j=a$ or a' in station S_A and $j'=b$ or b' in S_B and also equal polarizer angles in both stations. While Bell's variable λ must not depend in any way on the instrument configurations, the values λ_s may be clearly linked to both detector outcomes and polarizer angles, as described in the following mathematical model (computer experiment).

The Fundamental model represents the λ_s by a real number, randomly and uniformly chosen from the interval $\Omega=[0,1]$ for each separate model-measurement (see [4] and the explanations in Williams' textbook "Weighing the Odds" [8]). Following Williams, we also may introduce a probability measure x corresponding to the statement "a chosen number is less than or equal to x". We may use that probability measure to determine approximate data averages as well as corresponding theoretical expectation values for $N\to\infty$. The particular choice of λ_s out of [0, 1] permits great flexibility in modeling or simulating properties of the elements of physical reality and, in particular, permits us to develop an infinite variety of mathematical models, one of which may hopefully fit the physics of the problem.

Consider first the following **TE-local** computer-model with both polarizer angles fixed and equal a. Let the λ_s be random numbers created by an excellent random number generator from the interval [0, 1] and let them be sent from the source to the two computers. We consider now a mathematical model with the following features. The "mathematical" polarizers evaluate only a certain digit of λ_s in its binary representation, depending on the time t_s. For reasons of simplicity, we discuss the example that at time t_s it is the digit of number (s + 10) that is being evaluated (it may be a different digit for a different polarizer angle).

Consider now the measurement number $s=10$. Then it is the 20th digit of λ_{10} that will be evaluated. If that random number features a 0 in the 20th digit of its binary representation, we denote λ_s by λ_{10}^0 and evaluate it such that $A\left(a,\lambda_{10}^0\right)=+1$ as well as $B\left(a,\lambda_{10}^0\right)=-1$. Had the 20th digit been 1, we would have denoted it by λ_{10}^1 and evaluated it such that $A\left(a,\lambda_{10}^1\right)=-1$ and $B\left(a,\lambda_{10}^1\right)=+1$. Bell's reasoning that deterministic outcomes lead (among other factors) to his inequalities can, thus, be turned against him: determinism leads, even for trivially local (**TE-local**) models, to a dependence of the properties of sets of λ_s (as for λ_{10} above) on polarizer angles and function outcomes, which invalidates postulate (1).

We are also able to construct a more general **ST-local** model. Two separated computers evaluate the functions A and B with the only external input of λ_s. We keep the polarizer angle fixed to a for the computer in station S_A, while rotating the angle of the polarizer in S_B during random times t_k toward the angle j', where j' may have any value (and even be switched between any number of values). For simplicity we discuss only $j'=b$. We evaluate B locally in station S_B relative to the value of the function A and conditional to a fixed and arbitrary polarizer angle a in S_A as follows: analog to the procedure above, we introduce (locally in station S_B) the Lebesgue probability measure x for the outcome-product such that $A\left(a,\lambda_k\right)\cdot B\left(b,\lambda_k\right)=-1$ if and only if $\lambda_k \le x=\cos^2\left(b-a\right)$ and +1 otherwise. This choice of probability measure for B relative to A, represents a Malus-type natural law.

We obtain for N model measurements:

$$A\cdot B=-1 \text{ for } \approx N\cos^2\left(b-a\right) \tag{1}$$

while the number of positive product outcomes is about:

$$A\cdot B=+1 \text{ for } \approx N\sin^2\left(b-a\right) \tag{2}$$

For the single outcomes we may choose $A\left(a,\lambda_k^0\right)=+1$ and $A\left(a,\lambda_k^1\right)=-1$. The marginals ($A$ and B separately) are then in good approximation randomly equal to ±1 by the definitions of the Fundamental Model. The probabilities yield, therefore, the results of quantum mechanics for both the averages of the marginals and for the product averages and may also be used to model actual experiments.

A standard Bell-CHSH (Alice and Bob) objection is that I keep the polarizer angle in station S_A fixed and that my choices in S_B are conditional to that fixed angle in S_A, while Bob "cannot know" the polarizer angle of Alice at the time of measurement t_k. Bob can, however, explore a law of nature conditional to the instrument configurations in S_A and is justified to do so if he is only interested in his outcomes relative to those induced in S_A by λ_k at the time t_k. In fact, the experimenters collect, after all is done, the outcomes for 4 pairs of fixed polarizer angles and determine the averages of N measurements for each pair. The polarizer angles and measurement outcomes that form pairs are identified by the clock-times t_k. The switching of polarizer angles in between the registered measurements has no effect [7].

Another objection arises from the claim of Bell-CHSH that the experimenters have the freedom to choose any polarizer angle (and angle pair) to measure the properties of one given photon pair [4]. For the Fundamental Model, Alice and Bob have no freedom at all to choose the property that is to be evaluated, such as λ_{10}^{0}. That property is entirely determined by the random choice of λ_s and the assumed law of nature and will be different for any other measurement. There exists no freedom to investigate λ_{10}^{0} by using different polarizer angles.

Why is it then that the Bell-CHSH locality conditions are so important for a majority of physicists. One reason has been discussed in [4]: it is simply not possible to find four outcome products $A \cdot B$ for the four CHSH polarizer-angle pairs consistent with a probabilistic Malus type law and for the same λ_s, which has nothing to do with locality considerations. To avoid this problem, we have fixed the angle in station S_A and need, therefore, at any measurement-time t_s only one outcome-pair consistent with Malus. There exists, however, one more argument in Bell's definition of "local", which is postulate (2) that appears to clearly show why my Equations (1) and (2) above must be "not-local". In fact, however, it is postulate (2) which contains a deep-seated mathematical problem and can, therefore, not be applied.

4. Error of the Second Bell-CHSH Locality Postulate

As explained in (2) above, Bell and his followers have postulated that, assuming locality, the probability of $B(j', \lambda_s) = +1$ must not depend on whether $A = +1$ or $A = -1$ for all λ_s and that, therefore, Equations (1) and (2) must be non-local. Gisin [9] appears to clearly proof this fact. Following Bell, Gisin uses the following conditional probabilities (given in my notation):

$$P_{AB}^{cd}(A, B \mid j, j', \lambda_s) = P_A^{cd}(A \mid j, \lambda_s) P_B^{cd}(B \mid j', A, \lambda_s) \tag{3}$$

The vertical line | indicates that A and B assume certain values conditional to the values that the symbols after the vertical line assume. Gisin continues that, owing to locality-postulate (2), we must have in addition:

$$P_B^{cd}(B \mid j', A, \lambda_s) = P_B^{cd}(B \mid j', \lambda_s) \tag{4}$$

This equation is crucial and leads to:

$$P_{AB}^{cd}(A, B \mid j, j', \lambda_s) = P_A^{cd}(A \mid j, \lambda_s) P_B^{cd}(B \mid j', \lambda_s), \tag{5}$$

from which Gisin derives the Bell-CHSH-type inequalities.

However, The Bell-Gisin locality-argument (2) (which results in the crucial Equation (4)) is in conflict with the *set theoretic definition* of conditional probabilities P_B^{cd} of B given A, which is [8]:

$$P_B^{cd}(B \mid A) := \frac{P(A \cap B)}{P(A)} \tag{6}$$

If and only if $P(A) \neq 0$.

To appreciate the necessity of $P(A) \neq 0$ that Bell and Gisin did not include into their derivations, consider the following completely **TE-local** model. Just send out, randomly and exclusively, actual numbers of opposite sign (+1, −1 or −1, +1) toward the two distant stations. Here is the problem: The conditional probability on the left-hand side of Equation (4) is only defined if we have $A \cdot B = -1$ and is ill defined otherwise, namely a 0/0 (zero over zero) problem, while the right-hand side of Equation (4) is always equal to 0 or 1. Gisin's other derivations depend, in addition on BC-locality condition (1) that was also shown to be invalid.

In my **ST-local** computer example, the probability $P\left(A(a, \lambda_s) = +1\right)$ vanishes for all subsets λ_s^1 according to expression (2). Gisin's Equation (4) involves, thus, for our **ST-local** model and an infinite number of trivial **TE-local** models, conditioning to events with probability 0 which is equivalent to dividing by 0. Because the range of the functions is restricted to $A, B = \pm 1$, Equation (4) is incorrect for a large fraction of the λ_s (50% in the above examples) and, therefore, must not be used to derive the Bell-CHSH inequalities.

Based on these findings, the original experiments on entangled photons by Kocher and Commins [10] assume new importance and appear to support the existence of Einstein's elements of physical reality.

Perhaps it is useful to discuss, by another example, why even respectable researchers and their writings go awry for any number of reasons, when discussing Bell's theorem. Take the two-computer model of Susskind and Friedman [11]. They let the experimenters press a button at each of two computers, in order to "measure" at any moment. However, there is no guarantee that a correlated pair is available at these moments. Of course, such a guarantee may be provided by their "instantaneous cable" that is installed between the computers. But why not using clocks as the experimenters do? Their experimenters (Alice & Bob) may also choose any arbitrary equipment configurations such as the polarizer angles for any given photon pair (λ_s). This, however, is not possible, because the λ_s are randomly chosen and evaluated (e.g. $\lambda_{10}^0, \lambda_{10}^1$) for each new event in the Fundamental Model.

Furthermore, Susskind and Friedman do not discuss that, from all the possible choices of such random equipment configurations, only four pairs are used to derive all averages and the corresponding inequalities of Bell-CHSH. The only randomness of the actually used measurements is that of the sequence of these four pairs, and it is known from the actual experiments that the sequence of the measurements plays no role. Why, then, not accept a model for which the sequence plays no role in the first place. Susskind did emphasize, however, that the locality concepts of Einstein and Bell differ greatly.

5. Conclusions

It is this author's conviction that classical set-theoretic probability theory is indeed applicable to the physics of the Einstein-Bohr debate in contrast to the claims of Bell-CHSH and followers such as Aspect, Gisin, Clauser and many others. However, great care is required when defining conditional probabilities based on physical concepts that have no precise meaning within the Kolmogorov framework. Great care is also required for the physical definitions of "local" and "spooky".

It has been an unfortunate coincidence of factors that made the incorrect Bell-CHSH interpretations prevail for more than half a century: There exists indeed a powerful mathematical theory [3] that leads to Bell-CHSH type theorems, just not for the precise Bell-CHSH premises. Also, the disagreement of Bell-CHSH with experiments was for many a welcome confirmation of Bohr's conviction that the quantum world requires a completely new conceptual net, different from all classical thinking. The very wish to confirm Bohr's notions and to prove Einstein is wrong has led a large number of researchers to throw all caution in the wind and to abandon not only classical thinking but also the classical calculus.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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