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Effects of Compactification on Free Massive Scalar Fields in Five-Dimensional Space-Time with an Extra Time Dimension: Analysing Some Results

Sajjad Zahir

University of Lethbridge, Lethbridge, Canada
Email: zahir@uleth.ca

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Abstract

This paper deals with some aspects of two-time physics (*i.e.*, 2T + 3S five-dimensional space) for a Minkowski-like space with distinct speeds of causality for the time dimensions. Detailed calculations are provided to obtain results of Kaluza-Klein type compactification for free massive scalar fields and abelian free gauge fields. As already indicated in the literature, a tower of massive fields results from the compactification with mass terms having signs opposite to those of the ones appearing in other five-dimensional theories with an extra space dimension. We perform elaborate numerical calculations to highlight the magnitude of the imaginary masses and ask if we need to explore alternative compactification techniques.

Keywords

Two-Time Physics, Theory of Special Relativity, Kaluza-Klein Theory, Compactification, Klein-Gordon Equation, Tachyons, Abelian Gauge Theory

1. Introduction

Kaluza-Klein's (KK) theory [1] [2] is about a hundred years old and it considered an extra space dimension. Its goal was to unify electromagnetism with gravity. Recent string theory deals with many more dimensions (see Zwiebach [3] and references therein) at the quantum level to unify the standard model with gravity, as well. In this paper, we will not consider gravity—just a flat space-time. In many of these extra-dimensional theories, time remained one-dimensional. Some researchers considered a two-time world and developed theories for it creating a new type of physics called Two-time Physics (for example see Bars [4], Wesson [5], and Rizzo [6] and references therein). However, it is worth pointing out that [4] and [5] did not consider a flat space. For example, [4] developed the representation theory of the O(2, d) theory and [5] developed the General Relativity (curved space) in five dimensions (*i.e.*, 2 time and 3 space). Both papers are relevant in M-theory and supersymmetry and their objectives and interpretations are different.

Since the world we are familiar with at the current energy levels is four-dimensional (*i.e.*, a 1T + 3S dimensional Minkowski [7] space), the extra dimensions are compactified (*i.e.*, "curled up") into ultra-small circular topologies. Recently Zahir [8] considered a two-time (characterized by distinct speeds c_1 and c_2 of causality) and three space dimensional Minkowski [7] space (*i.e.*, a 2T + 3S dimensional space). He derived relativistic coordinate and velocity transformation formulas and expressions for a new effective speed limit $c_e = \sqrt{c_1^2 + c_2^2/k^2}$ where k is a scale factor connecting the two times t_1 and t_2. If dimensions of t_1, t_2, and any of the space dimensions are denoted by T, t, and L respectively, then dimensions of c_1, c_2, and k are LT^{-1}, Lt^{-1}, and Tt^{-1} respectively.

Extending the ideas of Einstein's Theory of Special Relativity (TSR) [9], concepts of five-velocity and five-momenta were introduced leading to an invariant five-momenta squared norm expression that conceptually incorporates a five-dimensional mass term (see the text [10] and references therein). Based on a non-relativistic limit, a two-time dependent Schrödinger-like equation was developed [11]. As an example, a two-time dependent infinite square-well potential problem was considered. After compactifying the extra time dimension on a closed loop topology with a period matching the Planck time, the solution generated interference of additional quantum states with ultra-small periods of oscillation, as well. In this paper, it is shown that by taking the five-dimensional (2T + 3S) relativistic momenta into consideration, a Klein-Gordon-type two-time quantum field theory for a massive scalar field is readily available. Researchers have considered similar five-dimensional scalar field theories in 1T + 4S dimensions [12] and noted that compactifying the extra space dimension using the KK technique, generated a tower of additional massive scalar fields. Such a tower is generated in a five-dimensional (2T + 3S) theory as well through KK-type compactification. However, the tower mass terms have a different signature, and the results are presented in this paper performing an elaborate numerical calculation to highlight the magnitude of the imaginary mass terms.

We start with five-dimensional energy-momentum representations as discussed by Zahir [8] in deriving a Klein-Gordon type equation [13]. Then, we solve it with compactification on a circular topology in Section 2. For the sake of completeness, the same formulations are extended to a free Abelian gauge field, and present the effect of Kaluza-Klein-type compactification in **Appendix A**. In Section 3, we analyze the results of compactification in light of some interesting observations (e.g., four-dimensional mass versus five-dimensional mass) made in Zahir [8]. To the best of the knowledge of the author, such problems have not been discussed by other researchers in the literature using a detailed numerical computation. It has been noted in many other works that the mass terms of the generated Kaluza-Klein tower of higher modes in the compactification of time-like extra dimension have negative signs in contrast with those found in the compactification of an extra spacelike dimension. In Section 4, we pointed out in detail how the resulting imaginary masses lead to the necessity for handling the case using various methodologies reported in the literature under the topic called tachyonic Klein-Gordon (TKG) theory (see [14] and references therein). In this paper, we provide detailed calculation procedures for two reasons: 1) to make it easier for those who are new in this field to grasp the subject and 2) to promote new ideas regarding the compactification technique other than the familiar infinite Fourier series expansion approach originally introduced in Kaluza-Klein theory more than a hundred years ago. Conclusions are presented in the final Section 5.

2. Revisiting the "Relativistic" Scaler Field Theory with Compactification of the Extra Time Dimension

Zahir [8] used distinct speeds of causality c_1 and c_2 for time t_1 and t_2 respectively. In the 2T + 3S space-time, the five coordinates are $(c_1t_1, c_2t_2, x, y, z)$ mapped with the metric signatures as

(+, +, −, −, −) such that the invariant norm s of a five-vector is given by,

$$s^2 = x_1^2 + x_2^2 - x_3^2 - x_4^2 - x_5^2 = (c_1t_1)^2 + (c_2t_2)^2 - x^2 - y^2 - z^2 \tag{1}$$

The invariant relativistic energy-momentum relation is given by (see Zahir [8] for definition of each term),

$$\left(\frac{(E_1)_1}{c_1}\right)^2 + \left(\frac{(E_1)_2}{c_2}\right)^2 - \bar{p}_1^2 = m_0^2 c_1^2 \tag{2}$$

m_0 is something like a five-dimensional mass. We use quantum operators for $(E_1)_1$, $(E_1)_2$, and $\bar{p}_1$ as follows,

$$\begin{aligned} (E_1)_1 &\rightarrow i\hbar\frac{\partial}{\partial t_1} \\ (E_1)_2 &\rightarrow i\hbar\frac{\partial}{\partial t_2} \\ \bar{p}_1 &\rightarrow -i\hbar\bar{\nabla} \end{aligned} \tag{3}$$

Let both sides of Equation (2) operate on field Φ. In this paper, we only consider real fields representing a zero-charge scalar field [13]. We get the Klein-Gordon type equation in 2T + 3S dimension,

$$\frac{\partial^2\Phi}{c_1^2\partial t_1^2} + \frac{\partial^2\Phi}{c_2^2\partial t_2^2} - \bar{\nabla}^2\Phi + \frac{m_0^2c_1^2}{\hbar^2}\Phi = 0 \tag{4}$$

We use index M with values 1, 2, 3, 4, 5 and index μ having values 1, 3, 4, 5 corresponding to variables $x_1 = c_1 t_1$, $x_2 = c_2 t_2$, $x_3 = x$, $x_4 = y$, $x_5 = z$ and $x_1 = c_1 t_1$, $x_3 = x$, $x_4 = y$, $x_5 = z$ respectively. In addition, we use metric tensors $g_{MN} = (+1,+1,-1,-1,-1)$ and $g_{\mu\nu} = (+1,-1,-1,-1)$. We rewrite Equation (4) as,

$$\left(\partial_M \partial^M + \frac{m_0^2 c_1^2}{\hbar^2}\right)\Phi = 0 \tag{5}$$

This equation can be derived also from the Hamiltonian principle of stationary action. To do that, we write down the five-dimensional Lagrangian expression for the 2T + 3S dimensional space-time. We can derive the Klein-Gordon type equation (in 2T + 3S dimension) in Equation (4) using the following action *S*.

$$S = \int \mathrm{d}^5 x \, \mathcal{L} \tag{6}$$

$\mathcal{L}$ is the Lagrangian density expressed in terms of the scalar field as follows,

$$\mathcal{L} = \frac{1}{2}\left(\partial_M \Phi \partial^M \Phi - \frac{m_0^2 c_1^2}{\hbar^2}\Phi^2\right) \tag{7}$$

Therefore,

$$S = \frac{1}{2}\int \mathrm{d}^5 x \left(\partial_M \Phi \partial^M \Phi - \frac{m_0^2 c_1^2}{\hbar^2}\Phi^2\right) \tag{8}$$

The corresponding Euler-Lagrangian equation gives the Klein-Gordon type equation (*i.e.*, as the one in Equation (5)) in 2T + 3S dimension as follows,

$$\left(\sum_{M=1}^{5} \frac{\partial}{\partial x^M} \frac{\partial}{\partial\left(\dfrac{\partial \Phi}{\partial x^M}\right)} - \frac{\partial}{\partial \Phi}\right)\mathcal{L} = 0 \tag{9}$$

$$\frac{\partial^2 \Phi}{c_1^2 \partial t_1^2} + \frac{\partial^2 \Phi}{c_2^2 \partial t_2^2} - \bar{\nabla}^2 \Phi + \frac{m_0^2 c_1^2}{\hbar^2}\Phi = 0; \tag{10}$$

3. Compactifying the Five-Dimensional Klein-Gordon Equation: Two Approaches

We can compactify the five-dimensional equation (see (A) below) or compactify the action and derive the compactified equations (see (B) below). We get the same results in the end.

(A) Compactify the Field Equation

We note that $\partial_M \partial^M = \partial_2^2 + \partial_\mu \partial^\mu$. Following the compactification ideas of the Kaluza-Klein theory, we explore Equation (5) while assuming that the variable x_2 is compactified on a circle of radius x_0 such that
$\Phi\left(x^M\right) = \Phi\left(x_2, x^\mu\right) = \Phi\left(x_2 + 2\pi x_0, x^\mu\right)$. Here, $x_0 = c_2 T_0$, and T_0 is related to Planck time (see below). We expand $\Phi\left(x^M\right)$ in an infinite Fourier series,

$$\Phi\left(x_2, x^\mu\right) = \frac{1}{\sqrt{2\pi x_0}} \sum_{q\in Z} \mathrm{e}^{\frac{iqx_2}{x_0}} \varphi^n\left(x^\mu\right) \tag{11}$$

Then, the resulting equation becomes,

$$\frac{1}{\sqrt{2\pi x_0}} \sum_{q\in Z} \left[\partial_\mu \partial^\mu \varphi^q\left(x^\mu\right) + \frac{m_0^2 c_1^2}{\hbar^2} \varphi^q\left(x^\mu\right) - \frac{q^2}{x_0^2} \varphi^q\left(x^\mu\right)\right] \mathrm{e}^{\frac{iqx_2}{x_0}} = 0 \tag{12}$$

We multiply the equation by $\frac{1}{\sqrt{2\pi x_0}} \mathrm{e}^{-\frac{inx_2}{x_0}}$ and integrate over dx_2 and get,

$$\frac{1}{2\pi x_0} \int \mathrm{e}^{\frac{i(n-q)x_2}{x_0}} \mathrm{d}x_2 = \delta_{nq} \tag{13}$$

The term on the right-hand side of Equation (8) is Kronecker Delta.

$\delta_{nq} = 1$ for $n = q$, $\delta_{nq} = 0$ for $n \neq q$.

Summing over q we get,

$$\partial_\mu \partial^\mu \varphi^n\left(x^\mu\right) + \left(\frac{m_0^2 c_1^2}{\hbar^2} - \frac{n^2}{x_0^2}\right) \varphi^n\left(x^\mu\right) = 0; n \in Z \tag{14}$$

In the literature, such a tower of massive scalar fields is derived in the case of 1T + 4S dimensional Kaluza-Klein theory after similarly compactifying the extra space dimension. The difference is the sign of the mass terms involving n^2.

(B) Compactify the Action

As is often done in the literature, we can also directly compactify the action in Equation (8) as was done to derive Equation (14) using Equation (11).

The action S is

$$\begin{aligned}
S &= \frac{1}{2} \int \mathrm{d}^5 x \left(\partial_M \Phi \partial^M \Phi - \frac{m_0^2 c_1^2}{\hbar^2} \Phi^2\right) \\
&= \frac{1}{2} \sum_{n\in Z} \sum_{m\in Z} \frac{1}{2\pi x_0} \int \mathrm{d}x_2 \mathrm{d}^4 x \left\{\partial_\mu \varphi^n\left(x^\mu\right) \partial^\mu \varphi^m\left(x^\mu\right) - \frac{nm}{x_0^2} \varphi^n\left(x^\mu\right) \varphi^m\left(x^\mu\right)\right. \\
&\quad \left. - \frac{m_0^2 c_1^2}{\hbar^2} \varphi^n\left(x^\mu\right) \varphi^m\left(x^\mu\right)\right\} \mathrm{e}^{\frac{i(n+m)x_2}{x_0}} \\
&= \frac{1}{2} \sum_{n\in Z} \sum_{m\in Z} \int \mathrm{d}^4 x \left\{\partial_\mu \varphi^n\left(x^\mu\right) \partial^\mu \varphi^m\left(x^\mu\right) - \frac{nm}{x_0^2} \varphi^n\left(x^\mu\right) \varphi^m\left(x^\mu\right)\right. \\
&\quad \left. - \frac{m_0^2 c_1^2}{\hbar^2} \varphi^n\left(x^\mu\right) \varphi^m\left(x^\mu\right)\right\} \delta(n+m) \\
&= \frac{1}{2} \sum_{n\in Z} \int \mathrm{d}^4 x \left\{\partial_\mu \varphi^n\left(x^\mu\right) \partial^\mu \varphi^{-n}\left(x^\mu\right) + \frac{n^2}{x_0^2} \varphi^n\left(x^\mu\right) \varphi^{-n}\left(x^\mu\right)\right. \\
&\quad \left. - \frac{m_0^2 c_1^2}{\hbar^2} \varphi^n\left(x^\mu\right) \varphi^{-n}\left(x^\mu\right)\right\}
\end{aligned} \tag{15}$$

We can break the summation into three ranges $-\infty$ to $-1, 0, 1$ to ∞ and get,

$$\begin{aligned} S &= \sum_{n=1}^{\infty}\int \mathrm{d}^4x\left[\partial_\mu\varphi^n\left(x^\mu\right)\partial^\mu\varphi^{-n}\left(x^\mu\right)+\left(\frac{n^2}{x_0^2}-\frac{m_0^2c_1^2}{\hbar^2}\right)\varphi^n\left(x^\mu\right)\varphi^{-n}\left(x^\mu\right)\right] \\ &\quad +\frac{1}{2}\int \mathrm{d}^4x\left[\partial_\mu\varphi^0\left(x^\mu\right)\partial^\mu\varphi^0\left(x^\mu\right)-\frac{m_0^2c_1^2}{\hbar^2}\varphi^0\left(x^\mu\right)^2\right] \\ &= \int \mathrm{d}^4x\,\mathcal{L}' \end{aligned} \tag{16}$$

where the four-dimensional action $\mathcal{L}'$ is given by,

$$\begin{aligned} \mathcal{L}' &= \sum_{n=1}^{\infty}\left[\partial_\mu\varphi^n\left(x^\mu\right)\partial^\mu\varphi^{-n}\left(x^\mu\right)+\left(\frac{n^2}{x_0^2}-\frac{m_0^2c_1^2}{\hbar^2}\right)\varphi^n\left(x^\mu\right)\varphi^{-n}\left(x^\mu\right)\right] \\ &\quad +\frac{1}{2}\left[\partial_\mu\varphi^0\left(x^\mu\right)\partial^\mu\varphi^0\left(x^\mu\right)-\frac{m_0^2c_1^2}{\hbar^2}\varphi^0\left(x^\mu\right)^2\right] \end{aligned} \tag{17}$$

The action in Equation (17) is a compactified form obtained by using the Fourier transform technique (*i.e.*, the KK approach). It is a four-dimensional action (in 1T + 3S space) as it is free of the x_2 variable and involves three types of independent fields $\varphi^0,\varphi^n,\varphi^{-n}$ with $1\le n\le\infty$ defined in four-dimensional 1T + 3S space. We can derive the field equations for each type using the Euler-Lagrangian equation of Equation (9). The Klein-Gordon type Equation (10) for ϕ is defined for five-dimensional 2T + 3S space. The equations for $\varphi^0,\varphi^n,\varphi^{-n}$ with $1\le n\le\infty$ are,

1)

$$\left(\sum_{\mu=1,3,4,5}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial\left(\frac{\partial\varphi^0}{\partial x^\mu}\right)}-\frac{\partial}{\partial\varphi^0}\right)\mathcal{L}'=0 \tag{18}$$

Or,

$$\partial_\mu\partial^\mu\varphi^0\left(x^\mu\right)+\frac{m_0^2c_1^2}{\hbar^2}\varphi^0\left(x^\mu\right)=0 \tag{19}$$

2)

$$\left(\sum_{\mu=1,3,4,5}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial\left(\frac{\partial\varphi^n}{\partial x^\mu}\right)}-\frac{\partial}{\partial\varphi^n}\right)\mathcal{L}'=0 \tag{20}$$

$$\partial_\mu\partial^\mu\varphi^n\left(x^\mu\right)+\left(\frac{m_0^2c_1^2}{\hbar^2}-\frac{n^2}{x_0^2}\right)\varphi^n\left(x^\mu\right)=0 \tag{21}$$

3)

$$\left(\sum_{\mu=1,3,4,5}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial\left(\frac{\partial\varphi^{-n}}{\partial x^\mu}\right)}-\frac{\partial}{\partial\varphi^{-n}}\right)\mathcal{L}'=0 \tag{22}$$

Or,

$$\partial_\mu \partial^\mu \varphi^{-n}\left(x^\mu\right)+\left(\frac{m_0^2 c_1^2}{\hbar^2}-\frac{n^2}{x_0^2}\right)\varphi^{-n}\left(x^\mu\right)=0 \tag{23}$$

The Equations (19), (21), and (23) are just the ones in Equation (14). So, we get the same tower of equations in both ways.

4. Analysis of the Compactification Results

Since the Equation (23) depends on n quadratically, we can take $\varphi^{n}\left(x^\mu\right)=\varphi^{-n}\left(x^\mu\right)$. From Equations (14), (19), (21), and (23) we get two sets of equations for the residual fields after the compactifications satisfying the two equations,

1) The zero-mode equation

$$\partial_\mu \partial^\mu \varphi^{0}\left(x^\mu\right)+\frac{m_0^2 c_1^2}{\hbar^2}\varphi^{0}\left(x^\mu\right)=0 \tag{24}$$

2) The $1 \le n \le \infty$ mode equation representing a tower of scalar fields,

$$\partial_\mu \partial^\mu \varphi^{n}\left(x^\mu\right)+\left(\frac{m_0^2 c_1^2}{\hbar^2}-\frac{n^2}{x_0^2}\right)\varphi^{n}\left(x^\mu\right)=0 \tag{25}$$

Dimension x_2 is compactified from a 2T + 3S to a 1T + 3S worldand index μ has values 1, 3, 4, and 5 with metric signatures (+, −, −, −). Other than the tower of scalar fields implied by the symbol n, what are the residuals from the five-dimensional space-time? m_0 is the "five-dimensional mass" and real, and $x_0 = c_2 T_0$, and T_0 is assumed to be related to Planck time. Zahir [8] has derived an expression relating "five-dimensional mass" m_0 to "four-dimensional mass" m as,

$$\begin{gathered} m_0 = m \cdot s(\lambda) \\ s(\lambda)=\sqrt{\frac{1+\lambda^2}{1+\frac{2\lambda}{1+\lambda^2}}} \end{gathered} \tag{26}$$

where $\lambda = c_2/(c_1 k)$ which is a dimensionless quantity. **Figure 1** is the plot of s against λ [8].

Therefore, the zero-mode equation is just like the normal Klein-Gordon equation with an effective real mass $M_0 = m.s$. The details of the solution of the normal classical and second-quantized KG equation are discussed in detail in the text [13] dealing with associated issues like negative energy and problems with a probabilistic interpretation of the wave function being reinterpreted as charge conservation equation [13]. In our case, we started with a real field and thus the charge is zero for the neutral scalar field.

Next, we focus on the higher mode Equation (25) whose effective mass term is $\left(\frac{m_0^2 c_1^2}{\hbar^2}-\frac{n^2}{x_0^2}\right)$. Let us define,

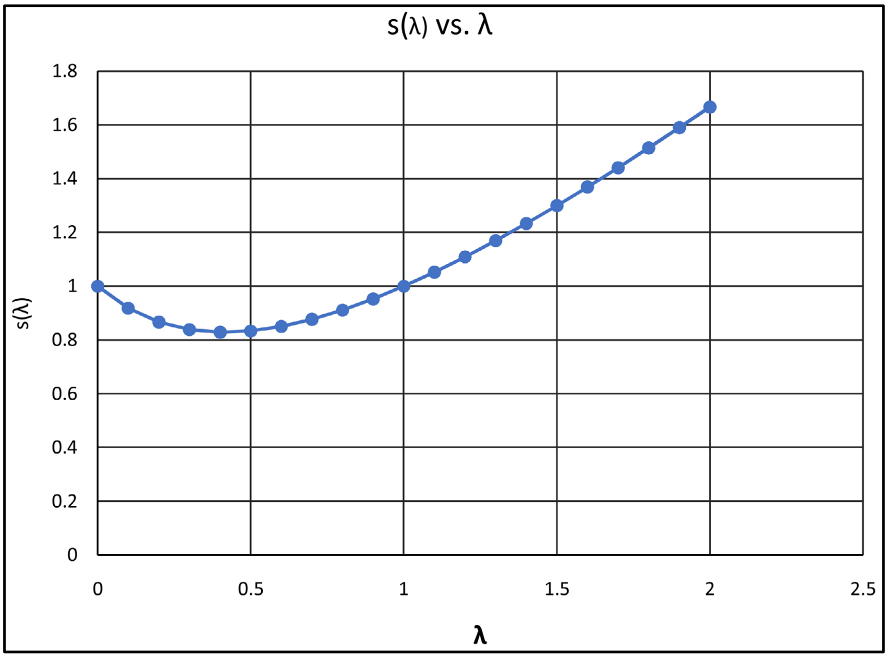

Figure 1. Plot of s vs. λ.

$$M_T^2 = \frac{m_0^2 c_1^2}{\hbar^2} - \frac{n^2}{x_0^2} = \frac{s^2 m^2 c_1^2}{\hbar^2} - \frac{n^2}{x_0^2} \tag{27}$$

Before we proceed, we need to recall and settle some of the dimensional matters. We compactified time t_2 which was tied to the speed of causality c_2, and we are left with t_1 which is tied to the speed of causality c_1 in the residual 1T + 3S dimensional space. So, we can safely assume that c_1 is like the speed of light and t_1 is the present-era time dimension. We mentioned in the beginning that if the dimensions of t_1, t_2, and any of the space dimensions are denoted by T, t, and L respectively, then the dimensions of c_1, c_2, and k are LT^{-1}, Lt^{-1}, and Tt^{-1} respectively. $x_0 = c_2 T_0$, and T_0 has the same dimension as t_2 and thus

$$\begin{aligned} x_0 &= c_2 T_0 = \frac{c_2}{k}(kT_0) = \frac{c_2}{k}(T_p) = \frac{c_2}{c_1 k}(c_1 T_p) = \lambda L_p \\ T_p &= kT_0;\ L_p = c_1 T_p \end{aligned} \tag{28}$$

where T_p is Planck time and L_p is Planck length. Therefore,

$$M_T^2 = \frac{s^2 m^2 c_1^2}{\hbar^2} - \frac{n^2}{x_0^2} = \frac{m^2 c_1^2}{\hbar^2}\left(s^2 - \frac{n^2}{\lambda^2}\frac{\lambda_c^2}{L_p^2}\right) \tag{29}$$

$\lambda_c = \dfrac{\hbar}{mc_1}$ = Reduced Compton wavelength of the particle of mass m.

The US National Institute of Standards and Technology (NIST) website (https://www.nist.gov) gives a value for proton Compton wavelength equal to 1.32×10^{-15} m. This gives the value for the reduced Compton wavelength = 2.1×10^{-16} m for a proton that has a mass of about 938 MeV. If we assume that the particle mass m = 100 MeV then we can easily calculate $\lambda_c = 1.97 \times 10^{-15}$ m. The same NIST source gives a value for Planck length $L_p = 1.62 \times 10^{-35}$ m. Therefore, we have,

$$\frac{\lambda_c}{L_p} = 1.22 \times 10^{20} \text{ and } \left(\frac{\lambda_c}{L_p}\right)^2 \approx 10^{40} \tag{30}$$

If we consider a "reasonable range" for λ and read from **Figure 1** a "reasonable" value for s, we see from Equation (29) that M_T^2 is negative (as the second term in the parenthesis is much bigger than the first term) with a very large absolute value. For the following estimate, we have taken the approximations $\lambda \approx 1$ and $s(\lambda) \ll \frac{n^2}{\lambda^2}\frac{\lambda_c^2}{L_p^2}$.

$$M_T^2 \simeq -\left(\frac{\lambda_c^2}{L_p^2} n^2 m^2\right)\frac{c_1^2}{\hbar^2} \approx -\left(mn10^{20}\right)^2 \frac{c_1^2}{\hbar^2} \tag{31}$$

Thus, with m = 100 MeV, the higher modes ($n \geq 1$) have imaginary masses having their absolute values about ($n \times 10^{20}$) times m. This estimate is the same as even when $m = 0$ since we ignored the first term in the parenthesis in Equation (29).

5. Meaning of Imaginary Masses: Tachyonic Klein-Gordon (TKG) Equation and Existing Literature

Klein-Gordon equation with imaginary masses has been discussed in the literature even until recently (for example see [14] and [15]). Such an equation is sometimes called a tachyonic Klein-Gordon (TKG) equation [14]. In the literature, a hypothetical particle moving faster than light is known as a tachyon (which has not been found experimentally) whereas a field having an imaginary mass is known as a tachyonic field. As we give examples of discussions below, many authors argue that having an imaginary mass does not mean that the related particle is superluminal. Two-time five-dimensional research in [4] and [5] do not postulate tachyon or ghost solutions. In the previous section, we noted that the $n > 1$ modes of the KK tower of fields have imaginary masses in agreement with many others. Nanni [15] (see discussions below) recently theoretically investigated subluminal particles with imaginary mass. We do not see any reason to find another solution here for imaginary mass; rather we overview some selected attempts (often emphasizing the mathematical techniques to solve a second-order differential equation touching on the Cauchy initial value problem) reported by many researchers over the last seven decades.

It all started [17] as an intriguing conceptual possibility resulting from the invariant energy-momentum relation from the Lorentz transformations formulations [9] as conceptually vindicated by Einstein's TSR. We try to outline the developments chronologically below.

A long list of articles dealing with imaginary mass and superluminal velocities have been reported in the literature since the matter was seriously discussed first in connection with TSR by Bilaniuk, Deshpande, and Sudarshan [17]. They considered relativistic kinematics of classical physics where we recall that formulas for the relativistic momentum ($mv\Upsilon$) and energy ($m\Upsilon c^2$) contain the famous Lorentz factor $\Upsilon = 1\Big/\sqrt{1-v^2/c^2}$. While we understand the problem for a particle starting with $v < c$ to gain a velocity reaching c and going beyond, the authors in [16] proposed "whether the existence of a class of particles, *created* with a velocity $v > c$, may be hypothesized" noting that it would imply imaginary "rest mass" for this class of particles. But they argued "Only energy and momentum, by virtue of their conservation in interactions, are measurable, therefore must be real. Thus, the imaginary result for the rest mass of the hypothetical 'meta' particles offends only the traditional way of thinking, and not observable physics." They even suggested experiments for detecting such meta particles in the context of the Cerenkov effect.

If we recall the well-known relativistic energy-momentum relation $E^2 - p^2c^2 = m^2c^4$, if m is imaginary (*i.e.*, $m = i\mu$, μ real), we have $p^2c^2 - E^2 = \mu^2c^4$. Therefore, $p^2c^2 > E^2$ making the relativistic four-momenta always spacelike.

The next important paper in this field in the literature is by Feinberg [17] who considered the possibility of describing, within the special theory of relativity, particles with spacelike four-momentum. It was Feinberg who first called these particles with imaginary masses and moving already faster than light as "tachyons". Therefore, to conform with TSR, "The limiting velocity is c, but a limit has two sides". He argued "The possibility of particles whose four-momenta are always spacelike, and whose velocities are therefore always greater than c is not in contradiction with special relativity, and such particles might be created in pairs without any necessity of accelerating ordinary particles through the "light barrier". He considered the classical Klein-Gordon field equation for free tachyons. However, he restricted the analysis only to those solutions having real energies in the Fourier spectrum. He found that tachyons cannot be localized, that the Cauchy data for the tachyon field equation cannot be freely assigned, and that the appropriate Green's function is not Lorentz-invariant. To investigate many-tachyon systems, Feinberg [18] also extended the classical scalar field theory to second-quantized quantum field theory. While exploring many particle systems, he listed five possible cases with initial/final states of the particle systems having various combinations of normal particles and tachyons with null, timelike, or spacelike total momenta.

Case A. State contains only normal particles, with timelike total momentum.

Case B. State contains normal particles and tachyons, with spacelike total momentum.

Case C. State contains normal particles and tachyons, with timelike or null total momentum.

Case D. State contains only tachyons, with spacelike total momentum.

Case E. State contains only tachyons, with timelike total momentum.

He concludes, after a thorough analysis, that "A description of such particles, called tachyons, by the formalism of relativistic quantum field theory is possible, at least for the case of spinless, noninteracting particles. The field theory constructed is explicitly Lorentz invariant. The particles described by this formalism have several peculiar properties. Among these are the following:

1) The spinless particle cannot be quantized by Bose statistics but can be quantized by Fermi statistics.

2) The vacuum state is not invariant under Lorentz transformations but rather changes into a state containing many tachyons." (see [18] p. 1100).

There was another specialty of Feinberg's [17] approach that required the introduction of creation and annihilation operators related to the sign of the energy leading to a noninvariant commutator function [18]. Arons and Sudarshan [19] subsequently criticized Feinberg's [18] method by proposing to construct tachyon fields belonging to $m^2 < 0$ particles using Wigner's irreducible representation in the Fock space as is done in the ordinary $m^2 > 0$ cases. Schroer [20] later attempted to construct a causal covariant quantum field theory for the classical Klein-Gordon equation with imaginary mass. Schrorer [20] comments that "the correct quantum theory" corresponding to the Klein-Gordon equation with imaginary mass is "quite different and much more complicated than the aforementioned proposals" of Feinberg [18] and Arons-Sudarshan [19]. He concludes, "We have demonstrated that the quantization of an $m^2 < 0$ field equation leads to causal fields." and "The validity of causality means that tachyons cannot be used for propagating signals with a speed faster than the speed of light, a fact that unfortunately makes the name an *abusus linguae*." Robinett [21] investigated the propagation of solutions of the Klein-Gordon equation with an arbitrary complex mass following Green's function analysis of the global Cauchy problem. He concluded "that an imaginary-mass particle (complex-mass particle, in general) may behave in a somewhat bizarre fashion, but it nevertheless does not travel faster than light. It is hoped that the venerable tachyon can now be gracefully retired."

Nanni [15] recently obtained a general solution of the TKG equation as a Fourier integral performed on a suitable path in the complex energy plane. In this work, it is proved that the solution does not contain any superluminal components once appropriate initial conditions have been set and the result is validated for the Chodos equation [22], as well. It is proved that the wave packet propagates in spacetime with subluminal group velocities and that it behaves as a localized wave for sufficiently small energies.

Lopez-Ruiz, Guerrero, and Aldaya [14] investigate mathematically the TKG equation and the conventional real-mass KG equation ($m^2 > 0$) in connection with the Helmholtz equation providing a unified framework for the scalar products of the three equations. This work further reduces the gap between real versus imaginary mass of free scalar field equations in the mathematical sense.

We must emphasize that all previous discussions relating to imaginary mass correspond to Minkowski space. Our work in this paper starts with an extra time dimension in a modified Minkowski space as well. Then, compactification of the extra time dimension using the KK technique [6] led to the emergence of a tower of imaginary mass scalar fields (called TKG fields as per the nomenclature used in the literature [14]). It is worth pointing out that extra timelike dimension has been considered in curved space involving a complex brane world scenario (see [16] and references therein), too. Iglesias and Kakushadze [16] constructed a solitonic 3-brane solution in the 5-dimensional Einstein-Hilbert-Gauss-Bonnet theory with the space-time signature (−, +, +, +, −). The direction transverse to the brane was the second time-like direction. They found no propagating tachyonic or negative norm states even though the extra dimension was time-like.

The objective of reviewing the selected research from the literature is to preclude the necessity for producing another solution for the TKG equations of the KK tower in Equation (29). One can follow some of the methodologies discussed above to find a solution. We focused on the absolute magnitude of the imaginary masses of the KK tower (*i.e.*, the mass term M_T in Equation (31)). To be specific, we pointed out that the absolute value of the imaginary masses can be greater than the Planck mass.

6. Conclusions

Based on Zahir's [8] framework of a 2T + 3S Minkowski space with distinct speeds of causality, we derived classical Klein-Gordon field equations in five dimensions. As the extra time dimension is compactified following Kaluza-Klein's Fourier expansion technique, we obtained a tower of higher modes of scalar fields in agreement with the findings of other researchers in this field. Because the extra dimension is timelike, we also found that the non-zero modes, as expected, have imaginary masses. We performed a detailed numerical calculation and determined how large the magnitudes of the imaginary masses are compared to the zero mode one (see Section 4, Equation (30) and the discussion below). The outcome can be attributed to the concept of compactifying the extra time dimension on an ultra-small circular topology matching the size of Plank time. We contrasted these results with several research works on TKG that suggested experimental routes for the detection of such particles called tachyons. Lacking any experimental success so far, we can question the validity of the Fourier expansion approach as the compactification technique although it has great intellectual appeal even in current string theory. We emphasize that the role of the extra dimension and its compactification has a different meaning. What we need, maybe, is an alternative approach to compactify a dimension. Let's consider a bold proposition that the extra time dimension was tied to the massless weak vector bosons traveling at a velocity c_2 (*i.e.*, already partially broken gauge symmetry) before the Higgs mechanism fully breaks the standard model gauge symmetry. We can pursue the idea that the Higgs field not only gave masses to weak bosons (and others except photons) but also played a role in compactifying the extra time dimension. Zahir [8] introduced this conceptual approach by noting that a scalar field called inflaton has been responsible for metric expansion during the inflationary phase [23]. Then it is unclear if it is too far-fetched or not to consider the Higgs field responsible for compactifying the extra time dimension [24]. Only further research can tell what the theoretical predictions will be.

In this paper, we considered a scalar field in a modified Minkowski space-time (*i.e.*, a 2T + 3S dimensional space) and obtained a tower of scalers with imaginary masses after compactifying the extra time dimension. However, we have not addressed if such a modified theory can impact CPT symmetries. However, Salesi [25] investigated free spin-1/2 particles which behave like tachyons in the momentum space ($p^2 = -m^2$) but behave like subluminal particles ($v < c$) in the ordinary space. He proposed to call them Pseudotachyons (PT's). He straightforwardly extended the standard Dirac theory and investigated a quantum mechanical wave equation describing free spinning particles. He also showed that his proposed theory about PTs to be separately invariant under the C, P, T transformations and the covariance under Lorentz transformations was also proved. Further work is needed to extend his work to scalar fields and particles being considered in this paper.

Finally, we would like to suggest future perspectives of this work that can be extended to other fields in the context of quantum mechanics, for example, in geometric phases [26], scattering [27], non-inertial effects [28], and solutions of bound states [29] [30] [31] [32].

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendix A: Revisiting the Abelian Gauge Field Theory

Next, we consider the action S of a U(1) gauge field in 2T + 3S dimension and explore the effect of compactification of the extra time dimension on a circle of radius $x_0 = c_2 T_0$.

$$\boxed{\begin{aligned} &S=\int \mathrm{d}^5 x\left[-\frac{1}{4}F_{MN}F^{MN}\right]\\ &F_{MN}=\partial_M A_N-\partial_N A_M;\, M,N=1,2,3,4,5\\ &F_{MN}F^{MN}=F_{\mu\nu}F^{\mu\nu}+\left(\partial_2 A_\mu-\partial_\mu A_2\right)\left(\partial^2 A^\mu-\partial^\mu A^2\right)\\ &\qquad\qquad+\left(\partial_\mu A_2-\partial_2 A_\mu\right)\left(\partial^\mu A^2-\partial^2 A^\mu\right)\\ &\qquad\quad=F_{\mu\nu}F^{\mu\nu}+2\left(\partial_\mu A_2-\partial_2 A_\mu\right)\left(\partial^\mu A^2-\partial^2 A^\mu\right)\end{aligned}} \tag{A-1}$$

The coordinate 2 is $x_2 = c_2 t_2$. We assume that the fields are compact on a circle of radius x_0 such that,

$$A_M\left(x^M\right)=A_M\left(x^\mu,x_2\right)=A_M\left(x^\mu,x_2+2\pi x_0\right) \tag{A-2}$$

We expand the fields using the Fourier series as,

$$A_M\left(x^\mu,x_2\right)=\frac{1}{\sqrt{2\pi x_0}}\sum_{n\in Z}\mathrm{e}^{in\frac{x_2}{x_0}}A_M^{(n)}\left(x^\mu\right) \tag{A-3}$$

x^μ corresponds to 1, 3, 4, 5 components, *i.e.*, $x_1 = c_1 t_1$ and 3, 4, 5 components are x, y, z respectively. Similarly, A^μ corresponds to 1, 3, 4, 5 components and

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu \tag{A-4}$$

This gives,

$$\boxed{\begin{aligned} &\partial_2 A_\mu=\frac{1}{\sqrt{2\pi x_0}}\sum_{n\in Z}\frac{in}{x_0}A_\mu^{(n)}\left(x^\mu\right)\mathrm{e}^{in\frac{x_2}{x_0}}\\ &\partial_\mu A_2=\frac{1}{\sqrt{2\pi x_0}}\sum_{n\in Z}\partial_\mu A_2^{(n)}\left(x^\mu\right)\mathrm{e}^{in\frac{x_2}{x_0}}\end{aligned}} \tag{A-5}$$

Then the action of Equation (16) becomes,

$$\begin{aligned} S=\frac{1}{2\pi x_0}\int \mathrm{d}^4x\mathrm{d}x_2\sum_{n\in Z}\sum_{m\in Z}-\Bigg[&\frac{1}{4}F_{\mu\nu}^{(m)}F^{(n)\mu\nu}\mathrm{e}^{i(m+n)\frac{x_2}{x_0}}\\ &+\frac{1}{2}\left(\partial_\mu A_2^{(m)}-\frac{im}{x_0}A_\mu^{(m)}\right)\left(\partial^\mu A^{2(n)}-\frac{in}{x_0}A^{\mu(n)}\right)\mathrm{e}^{i(m+n)\frac{x_2}{x_0}}\Bigg]\end{aligned} \tag{A-6}$$

and

$$F_{\mu\nu}^{(n)}=\partial_\mu A_\nu^{(n)}-\partial_\nu A_\mu^{(n)} \tag{A-7}$$

Using

$$\frac{1}{2\pi x_0}\int \mathrm{e}^{i(m+n)\frac{x_2}{x_0}}\mathrm{d}x_2=\delta_{m,-n} \tag{A-8}$$

And summing over index m, we get

$$S=\int \mathrm{d}^4x\sum_{n\in Z}\left[-\frac{1}{4}F_{\mu\nu}^{(-n)}\left(x^\mu\right)F^{(n)\mu\nu}\left(x^\mu\right)\right.$$
$$\left.-\frac{1}{2}\left(\partial_\mu A_2^{(-n)}\left(x^\mu\right)+\frac{in}{x_0}A_\mu^{(-n)}\left(x^\mu\right)\right)\left(\partial^\mu A^{2(n)}\left(x^\mu\right)-\frac{in}{x_0}A^{\mu(n)}\left(x^\mu\right)\right)\right] \tag{A-8}$$

Next, we separate the 0th mode and change the summation over $n\geq 1$. We get,

$$S=\int \mathrm{d}^4x\left\{-\frac{1}{4}F_{\mu\nu}^{(0)}\left(x^\mu\right)F^{(0)\mu\nu}\left(x^\mu\right)-\frac{1}{2}\partial_\mu A_2^{(0)}\left(x^\mu\right)\partial^\mu A_2^{(0)}\left(x^\mu\right)\right.$$
$$+\sum_{n\geq 1}2\left[-\frac{1}{4}F_{\mu\nu}^{(-n)}\left(x^\mu\right)F^{(n)\mu\nu}\left(x^\mu\right)\right. \tag{A-9}$$
$$\left.\left.-\frac{1}{2}\left(\partial_\mu A_2^{(-n)}\left(x^\mu\right)+\frac{in}{x_0}A_\mu^{(-n)}\left(x^\mu\right)\right)\left(\partial^\mu A^{2(n)}\left(x^\mu\right)-\frac{in}{x_0}A^{\mu(n)}\left(x^\mu\right)\right)\right]\right\}$$

Now, we make gauge transformations,

$$\boxed{\begin{aligned}A_\mu^{(n)}\left(x^\mu\right)&=A_\mu'^{(n)}\left(x^\mu\right)-\frac{i}{n/x_0}\partial_\mu A_2^{(n)}\left(x^\mu\right)\\ A_\mu'^{(-n)}\left(x^\mu\right)&=A_\mu'^{(-n)}(x^\mu)-\frac{i}{-n/x_0}\partial_\mu A_2^{(-n)}\left(x^\mu\right)\end{aligned}} \tag{A-10}$$

We finally get,

$$S=\int \mathrm{d}^4x\left\{-\frac{1}{4}F_{\mu\nu}^{(0)}\left(x^\mu\right)F^{(0)\mu\nu}\left(x^\mu\right)-\frac{1}{2}\partial_\mu A_2^{(0)}\left(x^\mu\right)\partial^\mu A_2^{(0)}\left(x^\mu\right)\right.$$
$$\left.+\sum_{n\geq 1}2\left[-\frac{1}{4}F_{\mu\nu}^{(-n)}\left(x^\mu\right)F^{(n)\mu\nu}\left(x^\mu\right)-\frac{1}{2}\frac{n^2}{x_0^2}A'^{(n)\mu}\left(x^\mu\right)A_\mu'^{(-n)}\left(x^\mu\right)\right]\right\} \tag{A-11}$$

Deriving the field equations and further analysis of the compactification is left as a possible further extension of this work.