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The Significance of Generalized Gauge Transformation across Fundamental Interactions

Bi Qiao

Wuhan University of Technology, Wuhan, China
Email: biqiao@gmail.com

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Abstract

The author of this paper has put forward a unified program of gauge field from the mathematical and physical picture of the principal associated bundles: thinking that our universe may have more fundamental interactions than the four fundamental interactions, and these basic interaction gauge fields are only the projection components to the base manifold, that is our universe, from a unified gauge potential or connection of the principal associated bundle manifold on the base manifold. These components can satisfy the transformation of gauge potential, and can even be transformed from one basic interaction gauge potential to another basic interaction gauge potential, and can be summarized into a unified equation, that is, the generalized gauge Equation (GGE), but the gauge potential or connection on the principal bundle is invariant, corresponding to the invariance of gauge transformation [1]. In this paper, we will continue to discuss this aspect concretely, and specifically construct a spatiotemporal model with the frame bundle as the principal bundle, and the tensor bundle as the associated bundle, so that the four fundamental interactions, especially the electromagnetic interaction and the gravitational interaction, can be reflected in the bottom manifold, that is, the regional distributions in our universe. Furthermore, this paper studies the existence of gauge transformation across basic interactions by establishing a model of gauge transformation of basic interaction field; it is found that the unified expression formula is GGE and the expression relation on the curvature of space-time. Therefore, the author discusses the feasibility of the generalized gauge transformation across the basic electromagnetic interaction and the basic gravitational interaction, and on this basis, specifically determines a method or way to find the generalized gauge transformation, so as to try to realize the last step of the "unification" of the four fundamental interactions in physics, that is, the "unification" of electromagnetism and gravity.

Keywords

Generalized Gauge Transformation, Unification of Fundamental Interactions, Principal Bundle, Connection and Curvature

1. Introduction

Following the spirit of Einstein's unified field and Yang Mills' gauge field theory [2], many scholars tried to expand the gauge "quantum" field theory to the category of gravity, hoping to establish a grand unified theory of four fundamental interactions such as gravity and electromagnetic forces [3] [4] [5] [6], but until now, gravity has not been unified with the other three basic interactions; the quantization theory of gravitational field has always been inconsistent with the microscopic quantum field theory [7] [8] [9], which has also become an exciting point for the creation of various theoretical hypotheses such as superstring and loop quantum gravity [10] [11] [12]. These still inspire us to constantly think about a question today, that is, considering the experimental fact that gravity is so weak in the elemental particle region, can we say with certainty that gravity can be quantized?

The second question is whether there are more than four basic interactions in nature? There seems to be no principle that can limit the basic interactions of nature to four kinds, namely gravity, electromagnetic, weak and strong interactions. Dark matter and dark energy have put forward an interpretation of this question from the perspective of astrophysics or cosmic scale framework [13] [14] [15] [16]. Is dark matter and dark energy the real existence or the representation of some unknown basic interactions? Suppose that there are only four basic interactions in nature, so far, by constructing a very specific product form of structural group $U(1)\times SU(2)\times SU(3)$, electromagnetic, weak and strong interactions correspond to a standard model of gauge unified field theory, which has been basically completed [17] [18] [19] [20], so the difficulty of unified field theory of four interactions is the unification of gravitational interaction and electromagnetic interaction. Therefore, in this paper, the author uses the combination of the principal fiber bundle theory and the physical concept of gauge field [1] [21] [22] [23] to establish a model of gauge transformation of four fundamental interaction fields. Specifically reveal the physical meaning of gauge transformation and GGE across four basic interaction gauge fields, especially between electromagnetic field and gravitational field, as well as the significance of connection with space-time region, so as to try to realize the last step of the unification of the four fundamental interactions in physics, namely a "unity" of electromagnetism and gravity.

2. Basic Point of View

1) The four basic interactions of gravity, electromagnetic force and strong and

weak are forces related to the connection and curvature of space-time regions. Why is it so difficult to unify gravity and electromagnetic force? One possible reason is that gravity is very weak in the space-time region of quantum distribution, so its quantization is not worth and may not exist.

2) Similarly, the gravity of long-range interaction can reach a very strong vast area, and the short-range interaction such as strong and weak cannot reach these space-time areas. Although the electromagnetic force may be relative weak, it is still the long-range interaction, so it can intersect with gravity, which gives a foundation for the unification of electromagnetic force and gravitational force, namely the specification transformation across the basic interaction on the intersection of electromagnetic interaction and gravitational interaction. Therefore, the basis of unification can only be based on their space-time characteristics, which is represented by the invariance of gauge transformation, that is, from a mathematical and physical point of view, they are the projection components in the universal bottom manifold from the connection or curvature of the high-dimensional space-time manifold of the principal associated bundles of the universe.

3) The connection of the higher-dimensional space-time manifold of the principal bundle is the gauge potential, the curvature is the gauge field strength, and the connection of the associated bundle is the gauge field, which will not change with the gauge transformation. The gauge transformation is only the transformation between components where the base manifold (our universe) has an intersection domain of the interactions, and the meaning of the gauge transformation across the basic interactions is the transformation between the basic interaction components that are projected at the intersection domain, for example, the transformation of gauge potentials between electromagnetic interaction and gravitational interaction.

4) The four basic fields of the universe (gravity, electromagnetic force, weak force, strong force) are unified in one cosmic space-time gauge potential $\tilde{\omega}$ (corresponding to a cosmic space-time gauge field). The corresponding gauge field and the mutual transformation between the four fundamental gauge fields can be expressed by a generalized gauge potential transformation Equation (GGE for short): its concise expression under certain conditions is the curvature similarity equation:

$$\mathbf{\Omega}_V = g_{UV}^{-1}\mathbf{\Omega}_U g_{UV}$$

here, $\mathbf{\Omega}_V$ and $\mathbf{\Omega}_U$ is the projected component of the curvature $\tilde{\mathbf{\Omega}}$ of the principal bundle on the bottom manifold region V and U respectively, g_{UV} is the conversion function of these region components, which is associated with a generalized gauge transformation. It shows that all different gauge potentials or curvatures are just the components of the unified connection or curvature of the spatiotemporal manifold of the principal associated bundle of the universe in different regions of the base manifold; based on this equation, we can determine a method or approach to find the generalized gauge transformation.

3. On the Construction of Principal Associated Bundles

3.1. Concept

The principal fiber bundle $P(M,G)$ is composed of a bundle manifold $P = G \times M$. The bottom manifold (which can represent our universe) M is composed of the structure group as a Lie group G; in order to meet the requirements of general relativity, M is matched with metric, G is a Lie group including 4 subgroups and their product $SO(1,3) \times U(1) \times SU(2) \times SU(3)$, and also contain the group element field that can include the conversion function between the four basic interactions (*i.e.* gravity, electromagnetic, weak and strong interactions). For the requirements of associated bundle, it is important to select the typical fiber F: F is required to include the basic interaction fields such as gravitational field and electromagnetic field.

3.2. The Relations between Projection Mapping of Principal Bundle and Associated Bundle

1) $\hat{\tau}: P \times F \to Q$, defined as $\hat{\tau}(p, f) := (p, f) = p \cdot f \in Q$. That is, $\forall p \in P$, $\hat{\tau}_p : F \to Q$; from this, one can define the topology of Q so that $\phi \subset Q$ is open if and only if $\hat{\tau}^{-1}[\phi] \subset P \times F$ is open, then Q is topological space, $\hat{\tau}$ is continuous mapping. Not only that, we can also prove that Q is a manifold.

2) $\hat{\pi}: Q \to M$, defined as: $\hat{\pi}(q) := \pi(p) \in M, \forall q = p \cdot f \in Q$. $\therefore$ more accurately, $\hat{\tau}_p : F \to \hat{\pi}^{-1}[x]$, $x \equiv \pi(p)$; $R_p : G \to \pi^{-1}[x]$, $x \equiv \pi(p)$, and $\hat{\tau}_p$, R_p are all differential homeomorphic maps. In other words, $\hat{\tau}_p$, R_p respectively brings the manifold structure of F or G into the fiber $\hat{\pi}^{-1}[x]$ of the associated bundle Q or the fiber $\pi^{-1}[x]$ of the principal bundle P.

3) $\tau: P \times F \to P, \tau(p, f) := p, \forall p \in P, f \in F$.

4) $\pi: P \to M$, and meet: $\pi^{-1}[\pi(p)] = \{pg | g \in G\}, \forall p \in P$.

Here, the relevant definitions in (3) and (4) have been given by [1] [23].

If every $x \in M$ has an open neighborhood $\mathcal{M} \subset M$, whose inverse image $\pi^{-1}[\mathcal{M}]$ and product manifold $\mathcal{M} \times G$ is differential homeomorphism, *i.e.* $\pi^{-1}[\mathcal{M}] = \mathcal{M} \times G$, then the corresponding $T_{\mathcal{M}}$ is local trivial, and the corresponding principal bundle is local trivial, where $\mathcal{M}$ may correspond to four regions, namely the gravitational interaction region V, electromagnetic interaction area U, strong interaction region W_1 and weak action area W_2. If $\mathcal{M} = M$, then $\pi^{-1}[\mathcal{M}] = P = M \times G$ is called as globe trivial. In general, it can be said that any principal bundle is local trivial, so one can extend the local trivialization to the principal associated bundle diagram of universe as see **Figure 1**.

Therefore, through the analysis of the requirements for the structure group G, manifold M and typical fiber F mentioned above, we can consider selecting the frame bundle as the principal bundle and the tensor bundle as its associated bundle to form a cosmic principal associated bundle structure.

3.3. The Frame Bundle as Principal Bundle

M is supposed as the n-dimensional manifold, $P \equiv \left\{x, \{e_\mu\} \middle| x \in M\right\}$, $\{e_\mu\}$ is a

$$
\begin{array}{ccccc}
 & & \pi^{-1}[\mathcal{M}] \times F & & \\
 & \tau \downarrow & & \downarrow \hat{\tau} & \\
\mathcal{M} \times G \overset{T_{\mathcal{M}}}{\longleftarrow} & \pi^{-1}[\mathcal{M}] & & \hat{\pi}^{-1}[\mathcal{M}] & \overset{\hat{T}_{\mathcal{M}}}{\longrightarrow} \mathcal{M} \times F \\
 & \pi \downarrow & & \downarrow \hat{\pi} & \\
 & \mathcal{M} & = & \mathcal{M} &
\end{array}
$$

Figure 1. A more specific structure of the principal associated bundle diagram of the universe, $\pi^{-1}[\mathcal{M}]$ and $\hat{\pi}^{-1}[\mathcal{M}]$ represent the principal fiber bundle and associated bundle on $\mathcal{M}$ respectively; here $\mathcal{M}$ represents the overall trivial or locally trivial region on *M*, which may correspond to four regions, namely, the region of gravitational interaction *V*; the region of electromagnetic interaction *U*; the region of strong interaction W_1 and the region of weak interaction W_2.

basis of T_xM, abbreviated as $e_\mu T_x$ represents the tangent space of $x \in M$. Then *P* can be proved to be $n+n^2$ dimensional manifold. Now choose $GL(n)$ as the structure group *G*, which is large enough to contain the subgroups $SO(1,3)$, $U(1)$, $SU(2)$, $SU(3)$, and $SO(1,3)\times U(1)\times SU(2)\times SU(3)$, then a frame bundle can be constructed by the following three steps:

1) Define the right action of the matrix group $GL(n)$ on *P*, $R: P\times GL(n)\to P$ as $R_g\left(x, e_\nu g^\nu_\mu\right)$, where g^ν_μ represents *g* matrix elements.

2) Define the projection map $\pi: P\to M$, that is, $\pi\left(x, e_\mu\right) := x, \forall\left(x, e_\mu\right)\in P$.

3) Define local trivial $T_U: \pi^{-1}[U]\to U\times G$, $T_U\left(x, e_\mu\right) := (x, h)$, where $h \equiv S_U\left(x, e_\mu\right)\in G$, $\left.\frac{\partial}{\partial x^\nu}\right|_x h^\nu_\mu = e_\mu$, and $S_U(pg) = S_U(p)g, \forall g\in G$. So T_U is differential homeomorphism.

The principal bundle $P\left(M, GL(n)\right)$ constructed by the above three steps is called the frame bundle and is recorded as *FM*.

3.4. Tensor Bundle as Associated Bundle

On the basis of *FM*, take manifold $F=\mathbb{R}^n$, then *F* is vector space, $f\in F$ can be expressed as a column matrix of *n* real numbers, namely $\left(f^1,\cdots,f^n\right)$; so we can define left action $\chi: G\times F\to F$ is $\left(\chi_g(f)\right)^\mu := g^\mu_\nu f^\nu$, $\forall g\in GL(n)$, $f\in F$; by right and left actions one can determine $\xi: (P\times F)\times G\to P\times F$, $\xi_g: P\times F\to P\times F$. Specifically $\xi_g(p,f) = \left(pg, g^{-1}f\right) \Rightarrow$ $\xi_g\left(x, e_\mu; f^\rho\right) = \left(x, e_\nu g^\nu_\mu; \left(g^{-1}\right)^\rho_\sigma f^\sigma\right)$. Here $\left(x, e_\mu; f^\rho\right)\in P\times F$ can produce $v\equiv e_\mu f^\mu \in T_xM$, and on the same orbit $v = e_\mu f^\mu = v' = e'_\mu f'^\mu$; that is to say, every $q\in\hat{\pi}^{-1}[x]$ point (representing a orbit) 1-1 corresponds to vector *v* in T_xM, all different *v* in T_xM correspond to different *q* above to form a tangent bundle $\hat{\pi}^{-1}[x]$, namely $\hat{\pi}^{-1}[x] \xleftrightarrow{\text{1-1}} T_xM$; so tangent bundle $Q = P\times F/\sim$ (here, ~representing equivalence relationship) is the associated bundle of *FM*. Further, *Q* can be regarded as the tangent bundle *TM* on *M*, $Q = TM$, so that the cross-section of any region $\hat{\sigma}[U]: U\to Q$ (because 1-1 corresponds to the vector of tangent space on *U*) is a tangent field on $U\subset M$. Since it is a vector field, at least preliminary description of the cross-section $\hat{\sigma}: U\to Q$ is related to the regional distribution. Different cross-sections correspond to different regional distributions, and there is a transformation relationship of the transfer function between the cross-sections.

More than this, on the basis of *FM*, if the manifold $F=\left(\mathbb{R}^n\right)^*=\mathcal{T}_{\mathbb{R}^n}(0,1)$, $f=(f_1,\cdots,f_n)\in F$, $\left(\chi_g(f)\right)_\mu:=\left(g^{-1}\right)^\nu_\mu f_\nu$, then giving any point $\left(x,e_\mu;f_\rho\right)\in P\times F$, it can produce: $\beta\equiv e^\mu f_\mu\in T_x^*M$ (the dual space of T_xM), and there is $\beta=\beta'$ on the same orbital; all the different β in T_x^*M correspond to the different q above it to form a cotangent bundle $\hat{\pi}^{-1}[x]$, that is, $\hat{\pi}^{-1}[x]\xleftrightarrow{1\text{-}1}T_x^*M$; so the cotangent bundle $Q=P\times F/\sim$ is also an associated bundle of *FM*. Any of its section $\hat{\sigma}:U\to Q$ is a covector field (dual vector field) on $U\subset M$.

Further, if $P=FM$, $G=GL(n)$, $F=\mathcal{T}_{\mathbb{R}^n}(1,1)$, $f=\left(f^\mu_\nu\right)\in F$, then choose: 1) $\chi:G\times F\to F$; 2) $\hat{G}\equiv\left\{\chi_g:F\to F\middle|g\in G\right\}$ is a Lie transformation group, which is the homomorphic mapping or realization or representation of *G*, and *F* is the realization space; then the left action can be defined:
$\left(\chi_g(f)\right)^\mu_\nu:=g^\mu_\alpha\left(g^{-1}\right)^\beta_\nu f^\alpha_\beta,\forall g\in GL(n),f\in F$. So any given point
$\left(x,e_\mu;f^\rho_\sigma\right)\in P\times F\leftrightarrow T^a_b:=\left(e_\mu\right)^a\left(e^\nu\right)_b f^\mu_\nu$, that is, T^a_b is a tensor of type (1, 1) in point *x*. The necessary and sufficient condition of $T^a_b=T'^a_b$ is that the given points are on the same orbit. Then we get a tensor bundle of type (1, 1) on the bottom manifold *M*, which is also an associated bundle of *FM*. Any of its section $\hat{\sigma}:U\to Q$ is a (1, 1) type tensor field on $U\subset M$.

After consideration, the author boldly believes that one of the more universal possible structures of the principal associated bundles of the universe is the frame bundle plus (k,l) tensor bundle as the associated vector bundle. According to the previous analysis and requirements, it can be considered that the principal associated bundle structure of the universe can accommodate the universal gauge fields and the corresponding four basic interactions. The main reasons are as follows:

1) Its structure group $GL(n)$ is a general linear transformation matrix group, which is sufficient to contain the subgroups $SO(1,3)$, $U(1)$, $SU(2)$, $SU(3)$ or subgroup product $G=SO(1,3)\times U(1)\times SU(2)\times SU(3)$ corresponding to the gauge transformation of the basic interactions required by the principal bundle sections transformation.

2) The (1, 3) tensor bundle, as the structure of the associated vector bundle, may be sufficient to contain all kinds of gravitation-related tensor fields, electromagnetic force gauge fields, etc. However, the relatively simple structure of the principal associated bundle of the universe may still be the frame bundle plus tangent bundle, *FM*+ *TM*.

4. Principal Associated Bundles and Gauge Field

4.1. Gauge Selection and Section

Definition: Let $P(M,G)$ be the principal bundle, *U* be the open subset of *M*, C^∞ mapping $\sigma:U\to P$ is called a local section, if $\pi\left(\sigma(x)\right)=x,\forall x\in U$. Here if $U = M$, then $\sigma : M \to P$, which is called the globe section. In the case of a local cross-section, we further explore the physical meaning of the cross-section: let U be the open subset of the bottom manifold *M*, and G be the structural group to construct a non-trivial principal bundle $P = U \times G$, where the free right-hand action of G on P is: $R : (U \times G) \times G \to U \times G$, that is, $\forall g_1 \in G$, define $R_{g_1} : U \times G \to U \times G$ as: $R_{g_1}(x, g_2) := (x, g_2 g_1)$, $\forall (x, g_1) \in U \times G$. Let $\sigma : U \to P$ and $\sigma' : U \to P$ be the local section of P respectively, then $\forall x \in U$ has a unique group element field $g : U \to G$ such that: $\forall g(x) \in G, x \in U$, $\sigma'(x) = \sigma(x) g(x)^{-1}$. Therefore, there exists a representation group element such that $U(x) \equiv \rho(g(x)) \in \hat{G}$, so that a local gauge transformation can be constructed to act on the local gauge field $\phi(x)$: $\phi'(x) = U(x)\phi(x) \equiv \rho(g(x))\phi(x), \forall \phi(x) \in \mathcal{V}$, where $\mathcal{V}$ is the representation space of $\hat{G}$, and $\hat{G}$ is a representation of *G*. At this time, $\phi(x)$ is actually a column matrix, and $\rho(g(x))$ is a square matrix, *i.e.* $\rho(g(x)) : G \to \hat{G}$. In addition, if the tangent bundle *TM* is selected as the associated bundle of *FM*, then there is naturally: $F = \mathcal{V}$ (the representation space of ρ), through the left action $\chi : G \times F \to F$ as $\forall g_1 \in G$, $\chi_{g_1} : F \to F$, $\chi_{g_1}(f_1) := \rho(g_1)(f_1), \forall f_1 \in F$, then there is an associated bundle

$$\Phi(x) \equiv q = p \cdot f = \sigma(x) \cdot f(x) \in \hat{\pi}^{-1}[x] \subset Q,$$

where $f : U \to F = \mathcal{V}, \forall f(x) \in F = \mathcal{V}$. So $\Phi(x)$ is determined by the cross-section σ and *f*. In addition, $g(x)$ can generate: 1) $\sigma'(x) = \sigma(x) g(x)^{-1}$, 2) $f'(x) = \chi_{g(x)} f(x) = \rho(g(x)) f(x) = U(x) f(x)$ (*i.e.* gauge transformation), which is equivalent to

$$\begin{aligned}\Phi'(x) &= \sigma'(x) \cdot f'(x) = \sigma(x) g(x)^{-1} \cdot g(x) f(x) \\ &= \sigma(x) \cdot g(x)^{-1} g(x) f(x) = \sigma(x) \cdot f(x) = \Phi(x) \in \hat{\pi}^{-1}[x]\end{aligned}.$$

It can be seen from the above that the so-called local (global) gauge transformation is actually the transformation section $\sigma(x) \to \sigma'(x)$, which is equivalent to the transformation of the frame and the transformation of the component of the physical field under the internal frame field, namely, $f(x) = \phi(x) \to f'(x) = \phi'(x)$, but the total physical field (internal vector $\Phi(x)$) is constant, *i.e.* $\Phi'(x) = \Phi(x)$. The so-called gauge selection is to select different cross-section while one cross-section on the associated bundle $\hat{\sigma}$ is exactly the invariant physical field $\Phi(x)$! In short, the change of cross-section on the principal bundle is the change of internal frame. If the internal frame change, it is equivalent to changing a gauge. Therefore, selecting a cross-section of the principal bundle is to select a gauge, as shown in **Figure 2**.

4.2. Construction of Generalized Gauge Transformation

The above discussion (including the generalization of the Yang-Mills potential [1] [2] [22] [23]) shows that in the very general principal associated bundle structure, that is, in $\phi'(x) = U(\vec{\theta}(x))\phi(x)$ equation, one can choose to define $U(\vec{\theta}(x)) \equiv \rho(g(x)) \in \hat{G}$, then one can construct a local gauge transformation:

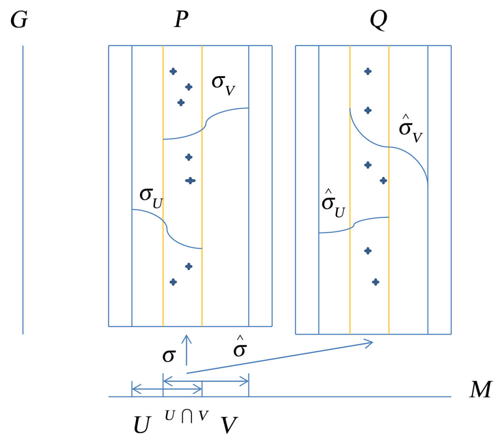

Figure 2. The cross-section of the principal bundle is the choice of gauge; the cross-section of the associated bundle is the gauge field $\Phi(x)$. $\forall x \in U \cap V$, then $\sigma_V(x) = \sigma_U(x) g_{UV}(x)$ represents the gauge transformation, where the conversion function $g_{UV}: U \cap V \to G$, $g_{UV}(x)$ is the group element field. Besides, W_1, W_2, $W_1 \cap W_2 \subset U$.

$$\phi'(x) = U(x)\phi(x) \equiv \rho(g(x))\phi(x), \forall \phi(x) \in \mathcal{V},$$

where $\mathcal{V}$ is the representation space of $\hat{G}$. Then choose $F = \mathcal{V}$, $\forall f(x) \in F = \mathcal{V}$, define

$$\Phi(x) \equiv \sigma(x) \cdot f(x) \in \hat{\pi}^{-1}[x] \subset Q,$$

where $f(x) \in F$, F is a typical fiber, one can deduce $\Phi'(x) = \Phi(x)$. In addition, for the principal bundle *FM* and associated bundle *TM*, $q = \sigma(x) \cdot f^{\mu} = (x, e_{\mu}) f^{\mu} = e_{\mu} f^{\mu} = e'_{\mu} f'^{\mu} \equiv v \sim \Phi(x)$, v is called a space-time vector (representing a tangent vector of point *x*), and $\Phi(x)$ can be called an internal vector of point *x*; $\sigma(x)$ is called the internal frame of point *x*, and $f^{\mu}(x)$ is called the component of the internal vector expanded by the internal frame. But if $\sigma(x) g(x)^{-1} = \sigma'(x)$, there are also internal vectors that are invariant under the gauge transformation: $\Phi(x) = \Phi'(x)$, changing only its component $\phi(x) \to \phi'(x)$. In the discussion in [1] [22] [23] one also saw that in order to ensure the invariance of the total Lagrangian density $\mathcal{L}$ under local gauge transformation, that is, equivalent to cross-sectional transformation of the principal bundle *P*, the (electromagnetic) gauge potential $A_{\mu}(x)$ must be introduced, and make its across basic interaction gauge transformation (that is, cross section transformation $\sigma'(x) = \sigma(x) g(x)^{-1}$, where $g(x)$ is a group element field) to become (gravitational) gauge potential $A'_{\mu}(x)$, then $A_{\mu}(x) \to A'_{\mu}(x)$ (that is, the gauge potential transformation that satisfies GGE [1], and see Equation (2) below) also corresponds to an absolute invariance, that is, the connection $\tilde{\omega}$ on the principal bundle, or the connection on the principal bundle is the gauge potential, which is invariant under the gauge potential transformation (it corresponds to the internal vector is invariant under gauge transformation), what change is only its component on the bottom manifold. A connection on the principal bundle $P(M,G)$ is to the local triviality $T_U : \pi^{-1}[U] \to U \times G$ specifies a 1-form field $\boldsymbol{\omega}_U$ of C^∞ $\mathcal{G}$ value on *U*, that is, $\boldsymbol{\omega}_U$ is a connection on the bottom manifold region $U \subset M$. At this time, if $T_V : \pi^{-1}[V] \to V \times G$ is another local triviality, that is, $U \cap V \neq \varnothing$, and the transition function from T_U to T_V is g_{UV}, then the transformation between $\boldsymbol{\omega}_U$ and $\boldsymbol{\omega}_V$ is given by GGE description:

$$\boldsymbol{\omega}_V(Y) = \mathcal{A}d_{g_{UV}(x)^{-1}}\boldsymbol{\omega}_U(Y) + L^{-1}_{g_{UV}(x)*}g_{UV*}(Y), \forall x \in U \cap V, Y \in T_x M \tag{1}$$

where $L^{-1}_{g_{UV}(x)}$ is the inverse mapping of left translation $L_{g_{UV}(x)}$ generated by $g_{UV}(x) \in G$, $L^{-1}_{g_{UV}(x)*} \equiv \left(L^{-1}_{g_{UV}(x)}\right)_*$.

Or for general cases, define: $-iL_r \equiv \rho_* e_r \in \hat{\mathcal{G}}$, here $\hat{\mathcal{G}}$ is the representation of Lie algebra of *G*, or Lie algebra of $\hat{G}$, e_r is a basis vector of Lie algebra $\hat{\mathcal{G}}$, ρ_* is the push forward mapping of ρ, then one can define $\hat{A}_\mu(x) \equiv -i\vec{L} \cdot \vec{A}_\mu(x) = -iL_r A^r_\mu(x) \in \hat{\mathcal{G}}$, therefore, the Formula (1) becomes the Formula (2) below, and it can be proved that the right side of the Formula (2) also belongs to $\hat{\mathcal{G}}$, namely

$$\hat{A}'_\mu(x) = U\left(\vec{\theta}(x)\right)\hat{A}_\mu(x)U\left(\vec{\theta}(x)\right)^{-1} - k^{-1}\partial_\mu U\left(\vec{\theta}(x)\right)U\left(\vec{\theta}(x)\right)^{-1} \tag{2}$$

Here the transfer function $U\left(\vec{\theta}(x)\right)$ for the gauge potential transformation across the basic interactions can be determined by the chosen the cross-sections.

For example, if one take the general gauge potential on the bottom manifold (that is, it is not limited to the electromagnetic gauge potential, but also includes the gravitational gauge potential) as: $A^r_\mu(x) \to A'^r_\mu(x)$ (1 form field of real or complex value), then there are: $e_r A^r_a \in \Lambda_U(1,\mathcal{G})$, where e_r is the basis in Lie algebra $\mathcal{G}$. In addition,

$$\boldsymbol{\omega} = \sigma^* \tilde{\boldsymbol{\omega}} \to \boldsymbol{\omega}' = \sigma'^* \tilde{\boldsymbol{\omega}}, \forall \boldsymbol{\omega}, \boldsymbol{\omega}' \in \Lambda_M(1,\mathcal{G}),$$

where $\Lambda_M(1,\mathcal{G})$ is the set of 1-form fields of the valued Lie algebra $\mathcal{G}$ taken from *M*. So one can define: $\omega_a \equiv ke_r A^r_a \in \Lambda_M(1,\mathcal{G})$, or $\boldsymbol{\omega} \equiv ke_r \boldsymbol{A}^r \equiv k\boldsymbol{A} \in \Lambda_M(1,\mathcal{G})$, note that here $U \in M$, M is a general base manifold which is suitable to satisfy the local trivial condition, that is, the manifold of our "universe" which can be equipped with a suitable metric. The gauge potential of the so-called basic interaction respectively corresponds to the gauge potential of electromagnetism, gravitation, weak interaction, and strong interaction in the bottom manifold $U \cap V$, or W_1, W_2, $W_1 \cap W_2 \subset U$ as well as respectively corresponds to the relevant subgroups or subgroup product $SO(1,3) \times U(1) \times SU(2) \times SU(3) \in GL(n)$, etc.

Now we want to discuss that the $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$ defined in this way satisfy the transformation relation (1), $\boldsymbol{\omega}_V(Y) \to \boldsymbol{\omega}_U(Y)$, of course this is a cross basic interaction gauge potential, if it is the transformation between gravitational gauge potential and electromagnetic gauge potential, then its corresponding structure group may be the subgroup product $SO(1,3) \times U(1) \in GL(n)$.

For example, if $\omega_\mu \equiv ke_r A^r_\mu \in \Lambda_U(0,\mathcal{G})$, $\omega'_\mu \equiv ke_{r'} A^{r'}_\mu \in \Lambda_V(0,\mathcal{G})$, then it is uniformly written as $\omega_\mu(x) \equiv ke_r A^r_\mu(x) \in \mathcal{G}$, now let the gauge transformation be

$$\sigma_V(x) = \sigma_U(x) g_{UV}(x) = \sigma'(x) = \sigma(x) g^{-1}(x) \tag{3}$$

Essentially it is possible to define the transition function as

$$g_{UV}(x) \equiv \sigma_U(x)^{-1} \sigma_V(x) \tag{4}$$

For the transformation between gravitational gauge potential and electromagnetic gauge potential, the establishment of Formula (1) seems to be no problem, but the most important thing to determine in Formula (1) is $g^{-1}(x)$ or the "choice" of $g_{UV}(x)$, *i.e.* what exactly does it equal? The "selection" of $g^{-1}(x)$ or $g_{UV}(x)$ is related to $\sigma_V(x)$ and $\sigma_U(x)$, that is, to the gauge transformation (3). In addition, $g_{UV}(x)$ is related to U and V area, and through σ_V, σ_U induce the cross sections of the associated bundle $\hat{\sigma}_V = \hat{\sigma}_U \Rightarrow \Phi'(x) = \Phi(x)$ and then determine the components of the gauge field $\Phi(x)$, *i.e.* ϕ_V, ϕ_U, in the base manifold *M*. This is just one of the mysteries of gauge transformation. Here the selection of the gauge with respect to ϕ_V gravitational field and ϕ_U electromagnetic field is determined by the V area corresponding to the gravitational area and U area corresponding to the electromagnetic force area in the "universal" base manifold, or it is determined by the "boundary conditions" and the intersection domain $U \cap V$ of the gravitational and electromagnetic effects respectively. The other two basic interactions, namely the strong interaction and the weak interaction, are basically considered to have no area intersection with the gravitational interaction, so Formulas (3) or (4) determines that $g_{UV} \equiv g^{-1}$ is the "group element" of gauge transformation, $g_{UV}: U \cap V \to G$, which can transform the electromagnetic field ϕ_U into the gravitational field ϕ_V. So this kind of gauge transformation across the basic interaction field can exist, and no "restriction" is found from the theoretical point of view of the principal associated bundles above.

Furthermore, if assume $\sigma_U : U \to P$ and $\sigma_V \to V$ are two local cross-sections of *P*, then there is a unique $g(x) \in G$, $\forall x \in U \cap V$ so that Equation (3) holds. It shows that a section transformation $\sigma_U \to \sigma_V$ of the principal bundle gives a group element field $g_{UV} \equiv g^{-1}$ on $x \in U \cap V$, and thus a local gauge transformation constructed with group element $g(x)$ can be determined, *i.e.*

$$U(x) \equiv \rho(g(x)) \in \hat{G} \tag{5}$$

Using $U(x) \equiv U(\boldsymbol{\theta}(x))$ to act on the gauge field $\Phi(x)$ (column matrix) one can get the local gauge transformation (between electromagnetism and gravity),

$$\phi_V(x) = U(x)\phi_U(x) = \rho(g(x))\phi_U(x) \tag{6}$$

That is, one can define:

$$U(x) \equiv \phi_V(x)\phi_U(x)^{-1} \tag{7}$$

Here, ρ is again defined as a homomorphic mapping: $G \to \hat{G}$, $\hat{G}$ is a representation of G or a Lie transformation group. Therefore $\rho(g(x))$ is a representation of G (for example, it is possible $SO(1,3)\times U(1)$), that is, $U(x)$ is a group element field of Lie transformation G, and by the product of column matrix $\phi_V(x)$ and row matrix $\phi_U(x)^{-1}$ is defined as a matrix.

The question now is why it is said that $U(x)\equiv\phi_V(x)\phi_U(x)^{-1}$ represents the gauge transformation from the electromagnetic field to the gravitational field, rather than a gauge transformation between other gauge fields? That is, why is $\phi_U(x)$ an electromagnetic field, and $\phi_V(x)$ represents a gravitational field? The answer is: 1) the subgroup $SO(1,3)$ in the structural group G on our model (see **Figure 2**) corresponds to the gravitational field, and $U(1)$ corresponds to the electromagnetic field; 2) The area U in the region corresponds to the boundary condition of electromagnetic interaction, and V corresponds to the boundary condition of gravitational interaction in the bottom manifold of the principal and associated bundle of the universe, so the section of the principal bundle on the area U corresponds to electromagnetic interaction, and the section of principal bundle on the area V corresponds to gravitational interaction. Zone is a spacetime! Reflecting the introduced ($T_U\to T_V$) transition function $g_{UV}:U\cap V\to G$ is connected with the transformation of space-time "features", because in essence, both the gravitational gauge potential and the electromagnetic gauge potential are related to the regions. So they are also related to the connection properties of time and space, and are the projection of the "unchanged" principal bundle connection and the pull-back mapping of the related cross-sections in "our world", reflecting the different properties of time and space connection. Therefore, it is appropriate to introduce the definition of g_{UV} into Formula (4), so as to determine Formula (7). Hence there are also related to the determination of Formula (4) as following:

$$\boldsymbol{\omega}_U=\sigma_U^*\tilde{\boldsymbol{\omega}},\ \boldsymbol{\omega}_V=\sigma_V^*\tilde{\boldsymbol{\omega}},\ \forall\boldsymbol{\omega}_U\in\Lambda_U(1,\mathcal{G}),\ \boldsymbol{\omega}_V\in\Lambda_V(1,\mathcal{G}) \tag{8}$$

$\boldsymbol{\omega}_U$ under the pullback mapping of the section σ_U^* in the above formula corresponds to the electromagnetic gauge potential on the bottom manifold U, while $\boldsymbol{\omega}_V$ under the pullback mapping of the section σ_V^* corresponds to the gravitational gauge potential on the bottom manifold V, both are components of connection $\tilde{\boldsymbol{\omega}}$ of the principal bundle. This cross-basic interaction can be further explained by the cosmic principal associated bundles structure about the basic interactions that we constructed in **Figure 2**.

That is, on the bottom manifold, let the W_1 and W_2 areas represent the strong interaction area and the weak interaction area, and they have an intersecting area, that is, $W_1\cap W_2$; U represents the electromagnetic interaction area, and V represents the gravitational interaction area. Electromagnetic interaction is equivalent to "intermediary", it has intersections with W_1, W_2 and V, but considering that the strength of gravity is extremely small in the area of strong and weak interactions, it can be considered that V has no intersections with W_1, W_2! Therefore, from the point of view of physical experiment observation, at $x\in U\cap V$, based on the cross-sections σ_U^* and σ_V^*, only two basic interactions of electromagnetism and gravitation can be observed, which correspond to two kinds of space-time connections $\boldsymbol{\omega}_U$ and $\boldsymbol{\omega}_V$ respectively, and these are the two components of the space-time connection $\tilde{\boldsymbol{\omega}}$ of the principal bundle, which is established on the basis of our universe. The unique group field $g_{UV}(x) \in G$ can be determined from the transformation relationship between $\boldsymbol{\omega}_U$ and $\boldsymbol{\omega}_V$ in Equation (4), where G is the structural group. Further, Formula (7) can be deduced to determine the gauge transformation $U(x)$ with more physical meaning, that is, $U(x)$ is a Lie transformation group element field of G, and it is composed of the product of column matrix $\phi_V(x)$ and row matrix $\phi_U(x)^{-1}$ as a matrix. These "requirements" are fed back to the structure group G. Fortunately, the structure group we choose for the principal associated bundle is the general linear matrix group $GL(n)$, which is large enough to meet the requirements of the matrix group for gauge transformations between basic interactions. Although the author is not yet able to say what the specific structure of those necessary subgroups is, $GL(n)$ certainly include $SO(1,3) \times U(1) \times SU(2) \times SU(3)$ as its subgroup. Here $\phi_V(x)$ or $\phi_U(x)$ belongs to the typical fiber F, namely $\phi : U \cap V \to F$.

For the case of the region $U \cap V$ where both the gravitational gauge potential and the electromagnetic gauge potential exist, it can be proved that both sides of the Equation (2) belong to the representation $\hat{\mathcal{G}}$ of Lie algebra, which is a kind of matrix expression equation. If Formula (7) is given, Formula (2) can be calculated in principle. Further, by introducing the generalized Yang-Mills field strength, that is, introducing R gauge potentials $A_\mu^r(x)$, then there should be R gauge field strengths $F_{\mu\nu}^r(r = 1, \cdots, R)$ correspondingly, they can be expressed as:

$$F_{\mu\nu}^r(x) = \partial_\mu A_\nu^r - \partial_\nu A_\mu^r + k \sum_{s,t=1}^{R} C_{st}^r A_\mu^s(x) A_\nu^t(x), (r = 1, \cdots, R) \tag{9}$$

Here C_{st}^r represents the structural constant of the Lie algebra $\hat{\mathcal{G}}$ of G under the basis $\{e_r\}$; the metric $g^{\mu\alpha}$, $g^{\nu\beta}$ can be used to improve the index of $F_{\alpha\beta}^r$: $F^{r\mu\nu} = g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta}^r$.

Then introduce the simplified notation: $\hat{F}_{\mu\nu}(x) \equiv -iL_r F_{\mu\nu}^r(x) \in \hat{\mathcal{G}}$, the similar Formula (9) can be changed to

$$\hat{F}_{\mu\nu}(x) = \partial_\mu \hat{A}_\nu(x) - \partial_\nu \hat{A}_\mu(x) + k\left[\hat{A}_\mu(x), \hat{A}_\nu(x)\right] \tag{10}$$

where $\left[\hat{A}_\mu(x), \hat{A}_\nu(x)\right]$ is the Lie bracket of the Lie algebra element $\hat{A}_\mu(x)$ and $\hat{A}_\nu(x)$. Among them, k is defined as the coupling constant, $k = e \leftrightarrow$ Electromagnetic gauge field; $k = -1 \leftrightarrow$ Gravitational gauge field; then consider Equation (10) and gauge transformation formula, an important mutual transformation matrix expression between electromagnetic intensity and gravitational intensity can be given by

$$\hat{F}'_{\mu\nu}(x) = U(\boldsymbol{\theta}(x)) \hat{F}_{\mu\nu}(x) U(\boldsymbol{\theta}(x))^{-1} \tag{11}$$

4.3. Existence of Generalized Gauge Transformation

The origin of the above Formula (11) can also be explained more clearly from the curvature transformation relation ($\boldsymbol{\Omega} \to \boldsymbol{\Omega}'$). In fact, under the cross-section transformation σ, the transformation relationship of $\boldsymbol{\omega} \to \boldsymbol{\omega}'$ on the bottom manifold is Formula (1), but the transformation relationship of $\boldsymbol{\Omega} \to \boldsymbol{\Omega}'$ needs to prove the following theorems to get [1] [23]:

Theorem 1: $\boldsymbol{\Omega} \Leftrightarrow F_{\mu\nu}^{r}(x)$, *i.e.*

$$\begin{cases} \boldsymbol{\Omega} = d\boldsymbol{\omega} + \frac{1}{2}[\boldsymbol{\omega}, \boldsymbol{\omega}] \\ \Updownarrow \\ F_{\mu\nu}^{r}(x) = \partial_{\mu} A_{\nu}^{r} - \partial_{\nu} A_{\mu}^{r} + k \sum_{s,t=1}^{R} C_{st}^{r} A_{\mu}^{s}(x) A_{\nu}^{t}(x), (r = 1, \cdots, R) \end{cases} \tag{12}$$

Proof

Using Cartan's second structural equation, one can get

$$\begin{aligned} \boldsymbol{\Omega} = d\boldsymbol{\omega} + \frac{1}{2}[\boldsymbol{\omega}, \boldsymbol{\omega}] &= d\left(k e_r A_{\mu}^{r} \mathrm{d}x^{\mu}\right) + \frac{1}{2}\left[k e_s A_{\mu}^{s} \mathrm{d}x^{\mu}, k e_t A_{\nu}^{t} \mathrm{d}x^{\nu}\right] \\ &= k e_r \mathrm{d}A_{\mu}^{r} \wedge \mathrm{d}x^{\mu} + \frac{1}{2} k^2 \left[e_s, e_t\right] A_{\mu}^{s} A_{\nu}^{t} \mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu} \\ &= \frac{1}{2} k e_r \left(\partial_{\mu} A_{\nu}^{r} - \partial_{\nu} A_{\mu}^{r}\right) \mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu} + \frac{1}{2} k^2 C_{st}^{r} e_r A_{\mu}^{s} A_{\nu}^{t} \mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu} \\ &= \frac{1}{2} k e_r \left(\partial_{\mu} A_{\nu}^{r} - \partial_{\nu} A_{\mu}^{r} + k C_{st}^{r} A_{\mu}^{s} A_{\nu}^{t}\right) \mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu} \\ &= \frac{1}{2} k e_r F_{\mu\nu}^{r} \mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu} = k\boldsymbol{F} \end{aligned} \tag{13}$$

Here, note:

$$\left[e_s, e_t\right] = C_{st}^{r} e_r, \tag{14}$$

as well as

$$\begin{aligned} \mathrm{d}A_{\mu}^{r} \wedge \mathrm{d}x^{\mu} &= \frac{\partial A_{\mu}^{r}}{\partial x^{\nu}} \mathrm{d}x^{\nu} \wedge \mathrm{d}x^{\mu} = \left(\partial_{\nu} A_{\mu}^{r}\right) \mathrm{d}x^{\nu} \wedge \mathrm{d}x^{\mu} \\ &= \left(\partial_{[\mu} A_{\nu]}^{r}\right) \mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu} = \frac{1}{2}\left(\partial_{\mu} A_{\nu}^{r} - \partial_{\nu} A_{\mu}^{r}\right) \mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu} \end{aligned} \tag{15}$$

So from the Formula (13) one can have

$$\begin{cases} \boldsymbol{\Omega} = \frac{1}{2} k e_r \left(\partial_{\mu} A_{\nu}^{r} - \partial_{\nu} A_{\mu}^{r} + C_{st}^{r} e_r k^2 A_{\mu}^{s} A_{\nu}^{t}\right) \mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu} = \frac{1}{2} k e_r F_{\mu\nu}^{r} \mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu} \\ F_{\mu\nu}^{r} = \partial_{\mu} A_{\nu}^{r} - \partial_{\nu} A_{\mu}^{r} + C_{st}^{r} e_r k^2 A_{\mu}^{s} A_{\nu}^{t} \end{cases} \tag{16}$$

q.e.d.

Theorem 2: If the structural group is a matrix group, the GGE can be expressed by curvature transformation [1] [22] [23], $\boldsymbol{\Omega}_V = \mathcal{A}d_{g_{UV}^{-1}} \boldsymbol{\Omega}_U$, then it can also be expressed in a similar transformation form:

$$\boldsymbol{\Omega}_V = g_{UV}^{-1} \boldsymbol{\Omega}_U g_{UV} \tag{17}$$

Proof

1) According to Theorem 2, let $g_{UV}: U \cap V \to G$ be the local trivial transition function from T_U to T_V, then on $U \cap V$ of the bottom manifold, $\boldsymbol{\Omega}_V = \mathcal{A}d_{g_{UV}^{-1}} \boldsymbol{\Omega}_U$ can be established;

2) Suppose G is a matrix Lie group, because $\forall \mathbf{\Omega}_U \in \mathcal{G}$, $g_{UV}^{-1} \in G$, so one can obtain

$$\begin{aligned} \mathcal{A}d_{g_{UV}^{-1}} \mathbf{\Omega}_U &= I_{g_{UV}^{-1}*} \mathbf{\Omega}_U = I_{g_{UV}^{-1}*} \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} Exp\left(t\mathbf{\Omega}_U\right) \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} I_{g_{UV}^{-1}}\left(Exp\left(t\mathbf{\Omega}_U\right)\right) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} g_{UV}^{-1}\left(Exp\left(tA\right)\right) g_{UV} \end{aligned} \tag{18}$$

Also because

$$\begin{aligned} & g_{UV}^{-1}\left(Exp\left(t\mathbf{\Omega}_U\right)\right) g_{UV} = g_{UV}^{-1}\left(I + t\mathbf{\Omega}_U + \frac{1}{2!}t^2\mathbf{\Omega}_U^2 + \frac{1}{3!}t^3\mathbf{\Omega}_U^3 + \cdots\right) g_{UV} \\ &= I + t g_{UV}^{-1}\mathbf{\Omega}_U g_{UV} + \frac{1}{2!}t^2 g_{UV}^{-1}\mathbf{\Omega}_U g_{UV} g_{UV}^{-1}\mathbf{\Omega}_U g_{UV} + \frac{1}{3!}t^3\left(g_{UV}^{-1}\mathbf{\Omega}_U g_{UV}\right)^3 + \cdots \\ &= Exp\left(t g_{UV}^{-1}\mathbf{\Omega}_U g_{UV}\right) \end{aligned} \tag{19}$$

hence Formula (18) becomes:

$$\begin{aligned} \mathcal{A}d_{g_{UV}^{-1}} \mathbf{\Omega}_U &= I_{g_{UV}^{-1}*} \mathbf{\Omega}_U = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(Exp\left(t g_{UV}^{-1}\mathbf{\Omega}_U g_{UV}\right)\right) \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(Exp\left(t I_{g_{UV}^{-1}}\left(\mathbf{\Omega}_U\right)\right)\right) = I_{g_{UV}^{-1}}\left(\mathbf{\Omega}_U\right) = g_{UV}^{-1}\mathbf{\Omega}_U g_{UV} \end{aligned} \tag{20}$$

So Formula (17) can be gotten by

$$\mathbf{\Omega}_V = \mathcal{A}d_{g_{UV}^{-1}} \mathbf{\Omega}_U = g_{UV}^{-1}\mathbf{\Omega}_U g_{UV}$$

Note that G is a matrix Lie group at this time. If $\forall \mathbf{\Omega}_U \in \mathcal{G}$, $g_{UV}^{-1} \in G$, then there is $\mathcal{A}d_{g_{UV}^{-1}} \mathbf{\Omega}_U = g_{UV}^{-1}\mathbf{\Omega}_U g_{UV}$, that is, $\mathcal{A}d_{g_{UV}^{-1}} \mathbf{\Omega}_U$ is equal to the product of three matrices, but if G is not a matrix Lie group, then $g_{UV}^{-1}\mathbf{\Omega}_U g_{UV}$ is meaningless, because at this time the product of the Lie group element g_{UV}^{-1} and its Lie algebra element $\mathbf{\Omega}_U$ is meaningless.

q.e.d.

Theorem 3: The following similar transformations are equivalent:

$$\mathbf{\Omega}_V = g_{UV}^{-1}\mathbf{\Omega}_U g_{UV} \Leftrightarrow \hat{F}'_{\mu\nu} = U\hat{F}_{\mu\nu}U^{-1} \tag{21}$$

In fact, from the above Formula (17), and then use the basis $\{e_r\}$ to expand the connection $\boldsymbol{\omega}$ and the curvature $\mathbf{\Omega}$ on the bottom manifold as $\boldsymbol{\omega} = e_r\boldsymbol{\omega}^r$ and $\mathbf{\Omega} = e_r\mathbf{\Omega}^r$ respectively, then $\boldsymbol{\omega}^r$ and $\mathbf{\Omega}^r$ are the (real-valued) 1-form and 2-form fields on the region U, respectively. Then ω_μ^r and $\Omega_{\mu\nu}^r$ represent the components of $\boldsymbol{\omega}^r$ and $\mathbf{\Omega}^r$ in the coordinate basis $\left\{\frac{\partial}{\partial x^\mu}\right\}$ in turn:

$$\omega_\mu^r = \boldsymbol{\omega}^r\left(\frac{\partial}{\partial x^\mu}\right) \tag{22}$$

$$\Omega_{\mu\nu}^r = \mathbf{\Omega}^r\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right) \tag{23}$$

Then one can find:

$$\begin{cases} \omega_\mu^r = kA_\mu^r \\ \Omega_{\mu\nu}^r = kF_{\mu\nu}^r \end{cases} \tag{24}$$

That is, ω_μ^r and $\Omega_{\mu\nu}^r$ is k times of the gauge potential A_μ^r and the gauge field strength $F_{\mu\nu}^r$ respectively, so physically, the connection $\boldsymbol{\omega}$ and the curvature $\boldsymbol{\Omega}$ on the bottom manifold can represent the gauge potential and the gauge field strength respectively, so the Formula (24) can be deduced which proves that Formula (21) is correct, and vice versa.

q.e.d.

Now use Formula (21), $\hat{F}'_{\mu\nu} = U\hat{F}_{\mu\nu}U^{-1}$, one can determine the matrix representation of the transformation function as follows:

First of all, consider that there are many matrix expressions of gravitational intensity, which are diagonal matrix expressions of second-order covariant tensor [22] [23], namely

$$\{g_{jj}\} \equiv \begin{pmatrix} g_{00} & 0 & 0 & 0 \\ 0 & g_{11} & 0 & 0 \\ 0 & 0 & g_{22} & 0 \\ 0 & 0 & 0 & g_{33} \end{pmatrix} \tag{25}$$

and also consider the matrix expression of electromagnetic field strength is also a kind of matrix representation of second-order anti-symmetric covariant tensor $\hat{F}_{\mu\nu}(x)$, for example

$$\{\hat{F}_{\mu\nu}(x)\} \equiv \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} \tag{26}$$

we find that the gravitational intensity can be a certain diagonal matrix expression of the second order metric tensor corresponding to $\{\hat{F}_{\mu\nu}(x)\}$. So we can always find such matrix similar transformation $\{U(\boldsymbol{\theta}(x))\}$ to diagonalize $\{\hat{F}_{\mu\nu}(x)\}$ as the matrix representation of the electromagnetic tensor for obtaining a matrix expression of the gravitational strength, that is the expression of the diagonal matrix of the metric $\{g_{jj}\}$, such as the Schwarzschild vacuum solution, etc. [24] [25]. The similar diagonalization expression is presented as follows:

$$\begin{aligned} &\begin{pmatrix} u_{00} & u_{01} & u_{02} & u_{03} \\ u_{10} & u_{11} & u_{12} & u_{13} \\ u_{20} & u_{21} & u_{22} & u_{23} \\ u_{30} & u_{31} & u_{32} & u_{33} \end{pmatrix} \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} \begin{pmatrix} u_{00} & u_{01} & u_{02} & u_{03} \\ u_{10} & u_{11} & u_{12} & u_{13} \\ u_{20} & u_{21} & u_{22} & u_{23} \\ u_{30} & u_{31} & u_{32} & u_{33} \end{pmatrix}^{-1} \\ &= \begin{pmatrix} g_{00} & 0 & 0 & 0 \\ 0 & g_{11} & 0 & 0 \\ 0 & 0 & g_{22} & 0 \\ 0 & 0 & 0 & g_{33} \end{pmatrix} \end{aligned} \tag{27}$$

Here, defining

$$\{U(\boldsymbol{\theta}(x))\} \equiv \begin{pmatrix} u_{00} & u_{01} & u_{02} & u_{03} \\ u_{10} & u_{11} & u_{12} & u_{13} \\ u_{20} & u_{21} & u_{22} & u_{23} \\ u_{30} & u_{31} & u_{32} & u_{33} \end{pmatrix} = \begin{pmatrix} U_0 & U_1 & U_2 & U_3 \end{pmatrix} \tag{28}$$

Then one has

$$\left\{\hat{F}_{\mu\nu}(x)\right\}\left\{U(\boldsymbol{\theta}(x))\right\}^{-1}=\left\{U(\boldsymbol{\theta}(x))\right\}^{-1}\left\{g_{jj}\right\} \tag{29}$$

$$\left\{\hat{F}_{\mu\nu}(x)\right\}(U_0U_1U_2U_3)=(g_{00}U_0g_{11}U_1g_{22}U_2g_{33}U_3) \tag{29}$$

$$\left\{\hat{F}_{\mu\nu}(x)\right\}U_i=g_{ii}U_i, i=0,1,2,3 \tag{30}$$

which allows one to obtain

$$\begin{cases} g_{00}=-E_1u_{10}-E_2u_{20}-E_3u_{30} \\ g_{11}=E_1u_{01}+B_3u_{21}-B_2u_{31} \\ g_{22}=E_2u_{02}-B_3u_{12}+B_1u_{32} \\ g_{33}=E_3u_{03}+B_2u_{13}-B_1u_{23} \end{cases} \tag{31}$$

Now one can associate the eigenvalues and eigenvectors of the matrix, that is, g_{ii} is the ith eigenvalue of the matrix $\left\{\hat{F}_{\mu\nu}(x)\right\}$, and U_i is the eigenvector corresponding to g_{ii}. $\left\{U(\boldsymbol{\theta}(x))\right\}^{-1}$ needs to be an invertible matrix, namely the eigenvectors of $\left\{\hat{F}_{\mu\nu}(x)\right\}$ need to be linearly independent, that is, the necessary and sufficient condition for the order n square matrix $\left\{\hat{F}_{\mu\nu}(x)\right\}$ similar to the diagonal matrix $\left\{g_{jj}\right\}$ is that $\left\{\hat{F}_{\mu\nu}(x)\right\}$ has n linearly independent eigenvectors $(U_0 \quad U_1 \quad U_2 \quad U_3)$. Certainly, these conditions for $\left\{\hat{F}_{\mu\nu}(x)\right\}$ can be satisfied. For example, from Equation (31), by taking $u_{20}=u_{30}=1$, $u_{01}=u_{31}=1$, $u_{02}=u_{12}=1=u_{23}=u_{13}$, one can get 4 linearly independent eigenvectors of $\left\{\hat{F}_{\mu\nu}(x)\right\}$ as

$$U_0=\begin{pmatrix} 1 \\ -\dfrac{g_{00}+E_2+-E_3}{E_1} \\ 1 \\ 1 \end{pmatrix} \tag{32}$$

$$U_1=\begin{pmatrix} 1 \\ 1 \\ \dfrac{g_{11}+B_2-E_1}{B_3} \\ 1 \end{pmatrix} \tag{33}$$

$$U_2=\begin{pmatrix} 1 \\ 1 \\ 1 \\ \dfrac{g_{22}-E_2+B_3}{B_1} \end{pmatrix} \tag{34}$$

$$U_3=\begin{pmatrix} \dfrac{g_{33}-B_1+B_3}{E_2} \\ 1 \\ 1 \\ 1 \end{pmatrix} \tag{35}$$

which proves that the most important conclusion in this paper: **it is existed that the (generalized) gauge transformation** $\boldsymbol{U(\theta(x))}$ **across fundamental interactions**.

5. Conclusions and Prospects

1) On the basis of the program of the grand unification of physics proposed in [1], this paper concretely constructs a space-time model with the frame bundle as the principal bundle, and the tensor bundle as the associated bundle, so that the four basic interactions, especially the electromagnetic and gravitational interactions, can be reflected in the base manifold, that is, the regional distribution of our universe. Gravitation is basically zero in the region of strong and weak interaction, and can have an intersection domain with electromagnetic interaction. This shows that the basic interaction is related to the "characteristics" of regional space-time, or they are the connection or curvature of space-time, while in the path of unification of four basic functions, whether gravity needs "quantization" is not a key or necessary issue.

2) This paper studies the existence and feasibility of generalized gauge transformation across basic interactions; it is found that the unified expression formula is the generalized gauge equation GGE and its expression relationship on the space-time curvature. Therefore, the author discusses the existence and feasibility of the generalized gauge transformation across the electromagnetic interaction and the gravitational interaction throughout the paper, and on this basis, specifically determines a method or way to find the generalized gauge transformation, so as to try to realize the last step of the "unification" of the four basic interactions in physics, that is, the "unification" of electromagnetism and gravity.

3) This paper once again affirms this key point: all interactions in the world are unified in the gauge potential or curvature of the principal bundle in the universal picture, while the four basic interactions on the bottom manifold are only the components representation of the gauge potential or curvature of the principal bundle, and they can be transformed from one basic interaction to another basic interaction according to the GGE formulation.

4) Outlook: a) The basic interaction may transform with each other. The basic equation of transformation is GGE or the similar transformation expression of the curvature matrix. b) Finding the structure group which can express more gauge field components; simplifying and solving the GGE so that it can concretely express the transformation relationship between any two gauge field components, especially the transformation relationship between electromagnetic force and gravity, which is extremely important for solving human aerospace dynamics, will be an important task of the future physics research on the grand unification.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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