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Mathematical Aspects of $SU(2)$ and $SO(3,\mathbb{R})$ Derived from Two-Mode Realization in Coordinate-Invariant Form

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Abstract

Some mathematical aspects of the Lie groups $SU(2)$ and $SO(3,\mathbb{R})$ in realization by two pairs of boson annihilation and creation operators and in the parametrization by the vector parameter $\boldsymbol{\varphi}$ instead of the Euler angles (α,β,γ) and the vector parameter $\boldsymbol{c}$ of Fyodorov are developed. The one-dimensional root scheme of $SU(2)$ is embedded in two-dimensional root schemes of some higher Lie groups, in particular, in inhomogeneous Lie groups and is represented in text and figures. The two-dimensional fundamental representation $D^{\left(\frac{1}{2}\right)}$ of $SU(2)$ is calculated and from it the composition law for the product of two transformations and the most important decompositions of general transformations in special ones are derived. Then the transition from representation $D^{\left(\frac{1}{2}\right)}$ of $SU(2)$ to $D^{(1)}$ of $SO(3,\mathbb{R})$ is made where in addition to the parametrization by vector $\boldsymbol{\varphi}$ the convenient parametrization by vector $\boldsymbol{c}$ is considered and the connections are established. The measures for invariant integration are derived for $SO(3,\mathbb{R})$ and for $SU(2)$. The relations between 3D-rotations of a unit sphere to fractional linear transformations of a plane by stereographic projection are discussed. All derivations and representations are tried to make in coordinate-invariant way.

Keywords

Boson Operators, Lie Algebra, Root Diagram, Invariant Integration, Hamilton-Cayley Identity, Cayley-Gibbs-Fyodorov Parametrization, Composition Law, Quaternion, Stereographic Projection, Fractional Linear Transformation

Remarks about Notations

Occasionally, we use for explanations of coordinate-invariant expression with three-dimensional tensors an index form including the Levi-Civita or fully anti-symmetric symbols ε_{ijk} defined by

$$\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij} = -\varepsilon_{ikj} = -\varepsilon_{kji} = -\varepsilon_{jik} \equiv \varepsilon^{ijk}, \quad \varepsilon_{123} = \varepsilon^{123} = 1. \tag{1}$$

Our most important notations deviating from an essential part of English literature are the following very rational ones for scalar product, vector product and dyadic product of two vectors $\boldsymbol{a}$ and $\tilde{\boldsymbol{b}}$

$$\tilde{\boldsymbol{a}}\boldsymbol{b}, \quad [\boldsymbol{a},\boldsymbol{b}], \quad \boldsymbol{b}\cdot\boldsymbol{a}, \quad \text{or} \quad \tilde{\boldsymbol{a}}\boldsymbol{b} \equiv \tilde{a}_i b^i, \quad [\boldsymbol{a},\boldsymbol{b}]_i \equiv \varepsilon_{ijk} a^j b^k, \quad (\boldsymbol{b}\cdot\boldsymbol{a})^i_j \equiv b^i \tilde{a}_j, \tag{2}$$

and for the volume product of three vectors $\boldsymbol{a},\boldsymbol{b}$ and $\boldsymbol{c}$

$$[\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}] = [\boldsymbol{a},\boldsymbol{b}]\boldsymbol{c} = \boldsymbol{a}[\boldsymbol{b},\boldsymbol{c}] = \boldsymbol{a}[\boldsymbol{b}]\boldsymbol{c} \equiv \varepsilon_{ijk} a^i b^j c^k. \tag{3}$$

They agree fully or partially with many weighty sources, in particular, most completely with Rosenfel'd [1] but also widely with, e.g., [2] [3] [4] [5] [6] and many weighty others.

The action of operators $\mathsf{A},\mathsf{B},\cdots$ onto vectors $\boldsymbol{x},\boldsymbol{y},\cdots$ from the left and co-vectors $\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}},\cdots$ from the right we denote by

$$\boldsymbol{y} = \mathsf{A}\boldsymbol{x}, \leftrightarrow y^i = A^i_j x^j, \quad \tilde{\boldsymbol{z}} = \tilde{\boldsymbol{y}}\mathsf{A}, \leftrightarrow \tilde{z}_j = \tilde{y}_i A^i_j. \tag{4}$$

A bilinear form $\tilde{\boldsymbol{y}}\mathsf{A}\boldsymbol{x}$ in vectors $\boldsymbol{x}$ and co-vectors $\tilde{\boldsymbol{y}}$ is then

$$\tilde{\boldsymbol{y}}\mathsf{A}\boldsymbol{x} = \tilde{y}_i A^i_j x^j. \tag{5}$$

Operators in arbitrary n-dimensional spaces obey a Hamilton-Cayley identity which in cases of $n = 1,2,3$ takes on the forms (e.g., [6] [7])

$$\begin{aligned} &\text{1-dim.:} \quad 0 = \mathsf{A} - \langle\mathsf{A}\rangle \mathrm{I}, \\ &\text{2-dim.:} \quad 0 = \mathsf{A}^2 - \langle\mathsf{A}\rangle\mathsf{A} + [\mathsf{A}]\mathrm{I}, \\ &\text{3-dim.:} \quad 0 = \mathsf{A}^3 - \langle\mathsf{A}\rangle\mathsf{A}^2 + [\mathsf{A}]\mathsf{A} - |\mathsf{A}|\mathrm{I}, \end{aligned} \tag{6}$$

where $\mathrm{I} \equiv \mathsf{A}^0$ is the corresponding identity operator in considered dimension. Unspecifically for dimension, we introduced for the first three invariants with respect to similarity transformation $\mathsf{A}' = \mathsf{B}\mathsf{A}\mathsf{B}^{-1}$ notations by special symbols according to (other notations, e.g., Fyodorov [6] [7] and Lagally [8])

$$\begin{aligned} &\langle\mathsf{A}\rangle \equiv A^i_i, \quad (\text{trace}), \\ &[\mathsf{A}] \equiv \frac{1}{2}\left(\langle\mathsf{A}\rangle^2 - \langle\mathsf{A}^2\rangle\right), \quad (\text{second invariant}), \\ &|\mathsf{A}| \equiv \frac{1}{6}\left(\langle\mathsf{A}\rangle^3 - 3\langle\mathsf{A}\rangle\langle\mathsf{A}^2\rangle + 2\langle\mathsf{A}^3\rangle\right), \quad (\text{determinant}). \end{aligned} \tag{7}$$

The second invariant in the form (7) is at once the determinant of two-dimensional operators. By introduction of new higher operator invariants this series may be continued to higher dimension but in this case it is difficult to invent for them new bracket symbols (but $\|\mathsf{A}\|$ for determinant seems to be acceptable to add for 4D-case and $|\mathsf{A}|$ becomes then third invariant and inversely $\|\mathsf{A}\|$ vanishes in 3D-case) and an index numbering for invariants is a possible solution. The Hamilton-Cayley identities allow to introduce specifically for dimension complementary operators $\overline{\mathsf{A}}$ to A and with their help the corresponding inverse operators A^{-1} in the cases $n=1,2,3$ are

$$\mathsf{A}^{-1}=\frac{\mathrm{I}}{\langle\mathsf{A}\rangle}\equiv\frac{\overline{\mathsf{A}}}{\langle\mathsf{A}\rangle},\quad \mathsf{A}^{-1}=\frac{\langle\mathsf{A}\rangle\mathrm{I}-\mathsf{A}}{[\mathsf{A}]}\equiv\frac{\overline{\mathsf{A}}}{[\mathsf{A}]},\quad \mathsf{A}^{-1}=\frac{[\mathsf{A}]\mathrm{I}-\langle\mathsf{A}\rangle\mathsf{A}+\mathsf{A}^2}{|\mathsf{A}|}\equiv\frac{\overline{\mathsf{A}}}{|\mathsf{A}|}. \tag{8}$$

Symmetric and antisymmetric parts of an operators A can be determined in spaces with a given invariant symmetric scalar product (Euclidean or pseudo-Euclidean ones with a nonsingular metric tensor $g_{ij}=g_{ji}$, $|\boldsymbol{g}|\neq 0$) from $A_{ik}\equiv g_{ij}A^j_k$ in coordinate form $A_{ik}=B_{ik}+C_{ik}$ with $B_{ik}=\frac{1}{2}(A_{ik}+A_{ki})=B_{ki}$, $C_{ik}=\frac{1}{2}(A_{ik}-A_{ki})=-C_{ki}$ or in index form (upper index T means transposition $A^{\mathsf{T}}_{ik}\equiv A_{ki}$)

$$\mathsf{A}=\mathsf{B}+\mathsf{C},\quad \mathsf{B}=\frac{1}{2}\left(\mathsf{A}+\mathsf{A}^{\mathsf{T}}\right)=\mathsf{B}^{\mathsf{T}},\quad \mathsf{C}=\frac{1}{2}\left(\mathsf{A}-\mathsf{A}^{\mathsf{T}}\right)=-\mathsf{C}^{\mathsf{T}}. \tag{9}$$

Specifically, three-dimensional antisymmetric second-rank tensors $C_{ik}=-C_{ki}$ can be mapped in unique way onto (axial) vectors c^j according to

$$\mathsf{C}=-\mathsf{C}^{\mathsf{T}}:\quad \mathsf{C}\equiv[\boldsymbol{c}],\quad \text{or}\quad C_{ik}=\varepsilon_{ijk}c^j\Leftrightarrow c^j=\frac{1}{2}\varepsilon^{ijk}C_{ik}, \tag{10}$$

with $[\boldsymbol{c}]$ built from vector $\boldsymbol{c}$ as alternative notation for the antisymmetric operator C with correspondences

$$\mathsf{C}\boldsymbol{x}=[\boldsymbol{c}]\boldsymbol{x}=[\boldsymbol{c},\boldsymbol{x}],\quad \boldsymbol{y}\mathsf{C}=\boldsymbol{y}[\boldsymbol{c}]=[\boldsymbol{y},\boldsymbol{c}],\quad \boldsymbol{y}\mathsf{C}\boldsymbol{x}=\boldsymbol{y}[\boldsymbol{c}]\boldsymbol{x}=[\boldsymbol{y},\boldsymbol{c},\boldsymbol{x}]. \tag{11}$$

In considered spaces we can relate in unique way covariant vectors $\tilde{\boldsymbol{y}}$ with contravariant vectors $\boldsymbol{y}$, in coordinate form by $\tilde{y}_i=g_{ij}y^j$, and may define quadratic forms $\tilde{\boldsymbol{x}}\mathsf{B}\boldsymbol{x}\equiv\boldsymbol{x}\mathsf{B}\boldsymbol{x}$ according to

$$\tilde{\boldsymbol{x}}\mathsf{B}\boldsymbol{x}=x^i g_{ij}B^j_k x^k=\tilde{x}_j B^j_k x^k=x^i B_{ik}x^k. \tag{12}$$

The explained notations are convenient for two- and three-dimensional coordinate-invariant calculations[1].

1. Introduction

The aim of present article is to review and to continue to develop mathematical

[1]In the 5-th edition of the remarkable monograph of Lagally [8] (my first book to this topics) revised by the Editor W. Franz he changed older notations in favor of the notations of Gibbs (mostly used in English literature) that was also in the sense of the late Lagally. This is said in the Preface. Now, I find some of the older or some alternative notations more suited for coordinate-invariant calculation, for example, scalar products as well as application of operators onto vectors without a point between the factors but dyadic products with a separating point. It is favorable to denote vector products by square brackets such as the "products" in Lie algebras due to some (not incidentally) analogous relations. In multiple vector products or in combination with other products due to non-associativity brackets are often needed and are then already present and, by experience, written complicated expressions become mostly shorter. Quantum mechanics also does not use a point in scalar products and in the action of operators onto state vectors (e.g., Dirac's notations). My additional notations for operator invariants make the formulae easier to read. In physical texts with respect to the kind of letters I do not apply very strong rules.

aspects for the classical and quantum-mechanical description of the polarization of two-mode quasiplane quasimonochromatic light beams by means of the Lie group $SU(2)$. In a possible continuation it is intended to apply this to the investigation of the transformation of polarized light beams in reflection and refraction problems and to discuss the quantum-mechanical conditions for unpolarized light beams. Classically and for vacuum which we only consider, it is mostly sufficient to investigate a two-dimensional polarization matrix composed from the electric field and with components perpendicular to beam propagation from which can be determined the Stokes parameters which vanish for unpolarized light beams. Quantum-statistically this can be done from a density operator for, at least, two mutually orthogonal boson modes and to determine their expectation values for the polarization matrix.

There are two well-known realizations of the Lie group $SU(2)$, first, the translation of the classical formula for the angular momentum into a vector operator according to the rules of transition to canonical quantum mechanics and, second, the two-mode realization by the two-dimensional fundamental representation of $SU(2)$ acting onto two quantum-mechanical states which correspond to two orthogonal polarization vectors in the operator representation of the electric field.

The quantum-statistical description of two-mode light polarization rests on the Jordan-Schwinger realization of the Lie algebra operators to $SU(2)$ by quadratic combinations of the boson annihilation and creation operators of the two modes (Jordan 1935 [9], and of Schwinger 1952 [10], the last republished in [11]) in the specific application to polarization. This description was developed in short form by Jauch and Rohrlich [12] in 1955. The monograph of Peřina [13] from 1971 contains a chapter about the polarization properties of light in which the polarization matrix (called coherence matrix) is quantum-statistically defined. In the same year 1971, Prakash and Chandra [14] determined the general form of two-mode unpolarized light beams by its definition of $SU(2)$ invariance and Agarwal [15] determined the quasiprobabilities of such light beams. After some time of stagnation the rigorous quantum-statistical description of light polarization was further developed by many authors and author groups and was reviewed in the article of Luis and Sánchez-Soto [16]. V. Peřinová, A. Lukš and J. Peřina [17] consider and refer the application of $SU(2)$ operators to atomic coherent states. The best-known realization of $SU(2)$ not considered here is the application to quantum-mechanical angular momentum (e.g., [3] [18] [19]).

With coordinate-invariant methods initiated in second half of last century mainly by F.I. Fyodorov [4] [5] [6] we develop some mathematical aspects of the groups $SU(2)$ and for $SO(3,\mathbb{R})$ in parametrization by the three-dimensional vector $\boldsymbol{\varphi}$ instead of the Euler angles (α,β,γ) that is rarely to find compared with the huge "group" literature from which we cite the early works of Lyubarski [20], Gürsey, [21], Behrends, Dreitlein, Fronsdal and Lee, [22], Wybourne [23], Gilmore [24], Barut and Rączka [25] and Hamermesh [26]. The group $SO(3,\mathbb{R})$

can be with advantage also parameterized by a three-dimensional vector $\boldsymbol{c}$ which simplifies mainly the composition law for two and more transformations and was developed with a few predecessors also mainly by Fyodorov. This is discussed here in details and applied for the calculation of the invariant (Haar) measure in both mentioned parametrization (Section 16). In Section 5 it is shown how one can determine representation matrices if one knows a complete set of basis operators, in our case for the fundamental two-dimensional representation of $SU(2)$ with the vector parameter $\boldsymbol{\varphi}$. Some important decompositions (disentanglements) of these matrices are calculated in Section 10 and Section 11 considers shortly the representation of $SU(2)$ by quaternions. A comparison of the two mentioned parametrizations by vector $\boldsymbol{\varphi}$ and Euler angles (α,β,γ) is made in **Appendix A**.

2. Short Quantum Description of Two-Mode Light Polarization of Beams by $SU(2)$ Transformations

The group $SU(2)$ can be applied in quantum optics for the theory of polarized and unpolarized light beams and for the lossless beamsplitter, in each case when two amplitudes can be transformed into each other or into two other amplitudes.

First we consider a light beam in an isotropic medium (here vacuum) which may be composed of two partial beams with two possible polarizations described by normalized, in general, complex-valued polarization vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$. The electric field $\boldsymbol{E}(\boldsymbol{r},t)$ of such a beam can be represented in the following way

$$\boldsymbol{E}(\boldsymbol{r},t)=(\boldsymbol{e}_1A_1+\boldsymbol{e}_2A_2)\mathrm{e}^{\mathrm{i}(\boldsymbol{k}_0\boldsymbol{r}-\omega_0t)}+(\boldsymbol{e}_1^*A_1^*+\boldsymbol{e}_2^*A_2^*)\mathrm{e}^{-\mathrm{i}(\boldsymbol{k}_0\boldsymbol{r}-\omega_0t)},\quad \omega_0=c|\boldsymbol{k}_0|. \tag{2.1}$$

The mean value of the wave vector of the beam is denoted by $\boldsymbol{k}_0$ and its mean frequency by ω_0. Both are assumed here to be real-valued. The complex-valued amplitudes to two possible (mean) polarizations $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are denoted by A_1 and A_2 and they depend slowly from position and time $(\boldsymbol{r},t)$ that we do not explicitly write since it does not play a role in our further considerations. In quantum-mechanical context the electric field $\boldsymbol{E}(\boldsymbol{r},t)$ becomes an operator and, correspondingly, the amplitudes $((A_\mu,A_\mu^*),\mu=1,2)$ too proportional to pairs of boson annihilation creation operators $((a_\mu,a_\mu^\dagger),\mu=1,2)$. The polarization vectors are orthogonal to the wave vector $\boldsymbol{k}_0=\boldsymbol{k}_0^*$ and should satisfy the orthonormalization conditions

$$\boldsymbol{k}_0\boldsymbol{e}_\mu=0,\quad \boldsymbol{e}_\mu\boldsymbol{e}_\nu^*=\delta_{\mu\nu},\quad (\mu,\nu=1,2). \tag{2.2}$$

The description of the beam polarization by vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ is not obligatory and we can use also other two polarization vectors $\boldsymbol{e}_1'$ and $\boldsymbol{e}_2'$ which are connected with the primary ones and satisfying the relations (2.2) by

$$(\boldsymbol{e}_1',\boldsymbol{e}_2')=(\boldsymbol{e}_1,\boldsymbol{e}_2)\mathsf{S}=(\boldsymbol{e}_1,\boldsymbol{e}_2)\begin{pmatrix}S_{11} & S_{12}\\ S_{21} & S_{22}\end{pmatrix}=(\boldsymbol{e}_1S_{11}+\boldsymbol{e}_2S_{21},\boldsymbol{e}_1S_{12}+\boldsymbol{e}_2S_{22}), \tag{2.3}$$

The general transformation S of this kind proves to be a two-dimensional unitary unimodular transformation. Then the new amplitudes A_1' and A_2' to polarization vectors $\boldsymbol{e}_1'$ and $\boldsymbol{e}_2'$ in the beam are connected with the primary ones by

$$\left(\boldsymbol{e}_1',\boldsymbol{e}_2'\right)\begin{pmatrix} A_1' \\ A_2' \end{pmatrix} = \left(\boldsymbol{e}_1,\boldsymbol{e}_2\right)\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}\begin{pmatrix} A_1' \\ A_2' \end{pmatrix} = \left(\boldsymbol{e}_1,\boldsymbol{e}_2\right)\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, \tag{2.4}$$

and the primary amplitudes are connected with the new amplitudes by the same matrix S but in the way

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \mathsf{S}\begin{pmatrix} A_1' \\ A_2' \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}\begin{pmatrix} A_1' \\ A_2' \end{pmatrix} = \begin{pmatrix} S_{11}A_1' + S_{12}A_2' \\ S_{21}A_1' + S_{22}A_2' \end{pmatrix}. \tag{2.5}$$

The two complex polarization vectors $\left(\boldsymbol{e}_1,\boldsymbol{e}_2\right)$ contain 4 real components and by conditions (2.2) they are restricted to 3 independent real components. The general state of polarization of a light beam is a partially polarized state. An unpolarized state is a state which remains invariant with respect to all transformations S of the $SU(2)$-group. From Section 5 on we consider a parametrization of the unitary unimodular matrices S by a three-dimensional vector $\boldsymbol{\varphi}$.

Another device where the $SU(2)$ transformations may be applied with advantage is a beamsplitter between two isotropic media. In this case we have an incident wave A_1^{i} from medium "1" and (or) an incident wave A_2^{i} from medium "2" of coupled modes from which result a reflected or refracted wave A_1^{r} and A_2^{r} in media "1" and "2" in dependence from which side we see the incident wave. In this case it is interesting to consider, in general, polarized incident waves and to calculate from them the reflected and refracted waves. The transformations of the beam are here actively made but inner details of the action of beamsplitter (e.g., a layer) are not involved. The correspondences to a singular beam with two possible polarizations are here

$$\left(A_1,A_2\right) \leftrightarrow \left(A_1^{\mathrm{i}},A_2^{\mathrm{i}}\right), \quad \left(A_1',A_2'\right) \leftrightarrow \left(A_1^{\mathrm{r}},A_2^{\mathrm{r}}\right). \tag{2.6}$$

One may, however, consider in case of a beamsplitter only one incident wave from medium "1" or medium "2" with two possible polarizations but in this case we have a reflected or refracted partially polarized wave in both media that means, at least, 6 possible polarizations (or 8 in case of incident waves from both sides) and the consideration are not in full analogy to a simple beam.

3. The Group $SU(2)$ Embedded into the Symplectic Group $Sp(4,\mathbb{R})$ of All Quadratic Combinations of Two Pairs of Boson Operators

Lie algebras were created as the tool to describe the local properties of Lie groups in the neighborhood of the identity element. It seems that pairs of boson annihilation and creation operators in quadratic combinations can be considered as elementary building stones of series of Lie algebras, in particular $A_n = SU(n+1)$, $B_n = SO(2n+1,\mathbb{R})$, $C_n = Sp(2n,\mathbb{R})$, $D_n = SO(2n,\mathbb{R})$ (e.g., [23] [24] [25] and many others). For a more general insight into the Lie group of polarization transformations of two modes which leads to a realization of the two-dimensional unitary unimodular group $SU(2)$ it is favorable to embed them into the group of all possible transformations made by quadratic combinations of two pairs of annihilation and creation operators of a two-mode system.

Pairs of boson annihilation boson annihilation and creation operators $\left(a_\mu, a_\mu^\dagger\right), (\mu = 1, \cdots, n)$ comprise $2n$ operators which obey the following commutation relations (I identity operator in (Hilbert) representation space)

$$\left[a_\mu, a_\nu^\dagger\right] = \delta_{\mu\nu} I, \quad \left[a_\mu, a_\nu\right] = \left[a_\mu^\dagger, a_\nu^\dagger\right] = 0, \quad (\mu, \nu = 1, \cdots, n). \tag{3.1}$$

These boson annihilation and creation operators result from (Hermitean) canonical operators Q_μ and P_μ by the definitions

$$a_\mu \equiv \frac{Q_\mu + \mathrm{i}P_\mu}{\sqrt{2\hbar}}, \quad a_\mu^\dagger \equiv \frac{Q_\mu - \mathrm{i}P_\mu}{\sqrt{2\hbar}}, \quad \left(Q_\mu, P_\mu\right) = \left(Q_\mu^\dagger, P_\mu^\dagger\right), \tag{3.2}$$

with the commutation relations

$$\left[Q_\mu, P_\nu\right] = \mathrm{i}\hbar I \delta_{\mu\nu}, \quad (\mu, \nu = 1, 2, \cdots, n). \tag{3.3}$$

The relations (3.1), (3.2) and (3.3) form the Heisenberg-Weyl algebra of an n-mode system. As important partial set of quadratic combinations of n pairs of boson operators may be considered the number operators N_i defined by

$$N_\mu \equiv a_\mu^\dagger a_\mu = a_\mu a_\mu^\dagger - I, \quad (\mu = 1, \cdots, n), \quad N \equiv N_1 + N_2 + \cdots + N_n, \tag{3.4}$$

The operator N is the total number operator and is the sum of all partial number operators.

Due to our main interest here for the description of polarized and unpolarized light beams we consider now two pairs of boson annihilation and creation operators $\left(a_1, a_1^\dagger\right)$ and $\left(a_2, a_2^\dagger\right)$. We introduce three Hermitean operators (J_1, J_2, J_3) by the definitions (e.g., Schwinger [10], p. 545, Jauch and Rohrlich [12], pp. 40-49, Messiah [19], Chap. XIII)

$$\begin{aligned} J_1 &\equiv \frac{1}{2}\left(a_1 a_2^\dagger + a_1^\dagger a_2\right) = \frac{1}{2\hbar}\left(Q_1 Q_2 + P_1 P_2\right) = J_1^\dagger, \\ J_2 &\equiv \frac{i}{2}\left(a_1 a_2^\dagger - a_1^\dagger a_2\right) = \frac{1}{2\hbar}\left(Q_1 P_2 - P_1 Q_2\right) = J_2^\dagger, \\ J_3 &\equiv \frac{1}{2}\left(a_1 a_1^\dagger - a_2 a_2^\dagger\right) = \frac{1}{4\hbar}\left(Q_1^2 + Q_2^2 - P_1^2 - P_2^2\right) \equiv \frac{1}{2}\left(N_1 - N_2\right) = J_3^\dagger. \end{aligned} \tag{3.5}$$

They satisfy the abstract commutation relations for a Lie algebra $su(2)$

$$\left[J_2, J_3\right] = \mathrm{i}J_1, \quad \left[J_3, J_1\right] = \mathrm{i}J_2, \quad \left[J_1, J_2\right] = \mathrm{i}J_3, \quad \Leftrightarrow \quad \left[J_i, J_j\right] = \mathrm{i}\varepsilon_{ijk} J_k, \tag{3.6}$$

where ε_{ijk} is the Levi-Civita symbol (or Levi-Civita pseudo-tensor) and here ad once they are the structure coefficients of the Lie algebra $su(2)$. From (3.6) follows

$$J_k = -\frac{\mathrm{i}}{2}\varepsilon_{ijk}\left[J_i, J_j\right] = -\frac{\mathrm{i}}{2}\varepsilon_{kij}\left(J_i J_j - J_j J_i\right) = -\mathrm{i}\varepsilon_{kij} J_i J_j, \tag{3.7}$$

and therefore (the operator $\varepsilon_{ijk} J_i J_j J_k$ is analogous to the fully antisymmetric volume product)

$$\boldsymbol{J}^2 \equiv J_k J_k = -\mathrm{i}\varepsilon_{ijk} J_i J_j J_k. \tag{3.8}$$

In addition to (3.5) we introduce raising and lowering operators J_- and J_+ as usual by

$$J_- \equiv J_1 - \mathrm{i}J_2 = a_1 a_2^\dagger, \quad J_+ \equiv J_1 + \mathrm{i}J_2 = a_1^\dagger a_2, \quad J_\pm = J_\mp^\dagger, \tag{3.9}$$

which satisfy the commutation relations

$$[J_3, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = 2J_3. \tag{3.10}$$

The operator C defined by (see also (3.8))

$$C \equiv \boldsymbol{J}^2 = (J_1)^2 + (J_2)^2 + (J_3)^2 = \frac{1}{2}(J_+J_- + J_-J_+) + (J_3)^2, \tag{3.11}$$

commutes with all operators of the Lie algebra $su(2)$

$$[\boldsymbol{J}^2, J_k] = 0, \quad \Rightarrow \quad [\boldsymbol{J}^2, J_\pm] = 0, \quad [\boldsymbol{J}^2, J_3] = 0, \tag{3.12}$$

and is usually taken as the Casimir operator to the Lie algebra $su(2)$. It can be represented after substitution of (J_1, J_2, J_3) according to (3.5) using the number operators defined in (3.4) in the following way

$$C \equiv \boldsymbol{J}^2 = \frac{N}{2}\left(\frac{N}{2} + I\right). \tag{3.13}$$

For the finite-dimensional irreducible representations which are parameterized by a discrete parameter j it is proportional to the identity operator I of the corresponding representation space of dimension $n = 2j+1$ and for these representations it can be specialized to

$$\boldsymbol{J}^2 = j(j+1)I, \quad \left(j = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots\right). \tag{3.14}$$

The Casimir operator C does not belong to operators of the Lie algebra $su(2)$. The number operator $N \equiv N_1 + N_2$ also commutes with all operators of the Lie algebra $su(2)$

$$[N, J_k] = 0, \tag{3.15}$$

and can be taken in addition to the operators of $su(2)$ forming the Lie algebra $u(2)$ to unitary transformations of the group $U(2)$.

With the 4 annihilation and creation operators $(a_1, a_2, a_1^\dagger, a_2^\dagger)$ one may form 16 quadratic combination from which only 10 are linearly independent. These are the 4 squared operators ($a_1^2 \equiv 2K_{1,-}$, $a_2^2 \equiv 2K_{2,-}$, $a_1^{\dagger 2} \equiv 2K_{1,+}$, $a_2^{\dagger 2} \equiv 2K_{2,+}$ and the remaining 12 squared operators reduce to 6 independent operators due to the mutual commutation relations that means totally to 10 independent operators.

From the quadratic combinations we may separate the Lie algebra of the two-mode squeezing group determined by the following three basic Hermitean operators (K_1, K_2, K_0)

$$K_1 \equiv \frac{1}{2}\left(a_1 a_2 + a_1^\dagger a_2^\dagger\right) = \frac{1}{2\hbar}(Q_1 Q_2 - P_1 P_2) = K_1^\dagger,$$

$$K_2 \equiv \frac{\mathrm{i}}{2}\left(a_1 a_2 - a_1^\dagger a_2^\dagger\right) = -\frac{1}{2\hbar}\left(Q_1 P_2 + P_1 Q_2\right) = K_2^\dagger,$$
$$K_0 \equiv \frac{1}{2}\left(a_1 a_1^\dagger + a_2^\dagger a_2\right) = \frac{1}{2}\left(a_1^\dagger a_1 + a_2 a_2^\dagger\right) \tag{3.16}$$
$$= \frac{1}{4\hbar}\left(Q_1^2 + P_1^2 + Q_2^2 + P_2^2\right) = \frac{1}{2}\left(N_1 + N_2 + I\right) = K_0^\dagger.$$

They satisfy the abstract commutation relations for a Lie algebra $su(1,1)$

$$[K_1, K_2] = -\mathrm{i}K_0, \quad [K_2, K_0] = \mathrm{i}K_1, \quad [K_0, K_1] = \mathrm{i}K_2, \tag{3.17}$$

In addition to the three operators (K_1, K_2, K_0) we introduce in analogy to $su(2)$ the operators K_- and K_+ by linear combinations

$$K_- \equiv K_1 - \mathrm{i}K_2 = a_1 a_2, \quad K_+ \equiv K_1 + \mathrm{i}K_2 = a_1^\dagger a_2^\dagger, \quad K_\pm = K_\mp^\dagger, \tag{3.18}$$

which satisfy the commutation relations

$$[K_0, K_\pm] = \pm K_\pm, \quad [K_-, K_+] = 2K_0. \tag{3.19}$$

By the formal substitutions

$$J_1 \leftrightarrow \mathrm{i}K_1, \quad J_2 \leftrightarrow \mathrm{i}K_2, \quad J_3 \leftrightarrow K_0, \quad J_+ \leftrightarrow \mathrm{i}K_+, \quad J_- \leftrightarrow \mathrm{i}K_-, \tag{3.20}$$

they make the transition to the commutation relations (3.6) for a Lie algebra $su(2)$. The operator C' defined by

$$C' \equiv \boldsymbol{K}^2 = K_0^2 - K_1^2 - K_2^2 = K_0^2 - \frac{1}{2}\left(K_- K_+ + K_+ K_-\right), \tag{3.21}$$

commutes with all operators K_k of the Lie algebra $su(1,1)$

$$\left[\boldsymbol{K}^2, K_k\right] = 0, \quad \Rightarrow \quad \left[\boldsymbol{K}^2, K_\pm\right] = 0, \quad \left[\boldsymbol{K}^2, K_0\right] = 0, \tag{3.22}$$

and is usually taken as the Casimir operator of the Lie algebra $su(1,1)$. For irreducible representations it is proportional to the identity operator I of the representation space and is signified by a parameter k (Bargmann index) according to

$$C' \equiv \boldsymbol{K}^2 = k(k-1)I. \tag{3.23}$$

In the realization of $su(1,1)$ by (3.16) the operator C' takes on the form

$$C' = \frac{1}{4}\left\{\left(N_1 - N_2\right)^2 - I\right\} = \left(\frac{|N_1 - N_2| + I}{2}\right)\left(\frac{|N_1 - N_2| - I}{2}\right), \tag{3.24}$$

which using (3.5) can be also represented

$$C = \left(J_3 + \frac{1}{2}I\right)\left(J_3 - \frac{1}{2}I\right). \tag{3.25}$$

In comparison, the Casimir operator C of $su(2)$ in (3.13) can be represented by

$$C' = \left(K_0 - \frac{1}{2}I\right)\left(K_0 + \frac{1}{2}I\right). \tag{3.26}$$

The operator J_3 in the realization (3.5) commutes with all operators K_k of the Lie algebra $su(1,1)$ in the realizations (3.16) and the operator K_0 with all operators J_l of the Lie algebra $su(2)$ and we have

$$[J_3, K_k] = 0, \quad [K_0, J_l] = 0. \tag{3.27}$$

The two sets of 4 operators (J_1, J_2, J_3, K_0) and (K_1, K_2, K_0, J_3) form the extended Lie algebras $u(2)$ and $u(1,1)$ of only unitary but not necessarily unimodular operators, respectively.

The 6 operators $(J_-, J_+, J_3, K_-, K_+, K_0)$ are not closed as a Lie algebra since the commutators between the $su(2)$ operators (J_-, J_+, J_3) and the $su(1,1)$ operators (K_-, K_+, K_0) lead to new operators according to

$$\begin{aligned} [J_-, K_-] &= -a_1^2 \equiv -2K_{1,-}, \quad [J_+, K_+] = a_1^{\dagger 2} \equiv 2K_{1,+}, \\ [J_+, K_-] &= -a_2^2 \equiv -2K_{2,-}, \quad [J_-, K_+] = a_2^{\dagger 2} \equiv 2K_{2,+}. \end{aligned} \tag{3.28}$$

which cannot be represented as linear combinations of these 6 operators. Closing them to a new Lie algebra can be obtained by adding two new groups of operators $(K_{\mu,1}, K_{\mu,2}, K_{\mu,0}), (\mu = 1,2)$, formed from the pairs of annihilation and creation operators $(a_\mu, a_\mu^\dagger), (\mu = 1,2)$,

$$K_{\mu,1} = \frac{1}{4}\left(a_\mu^2 + a_\mu^{\dagger 2}\right), \quad K_{\mu,2} = \frac{\mathrm{i}}{4}\left(a_\mu^2 - a_\mu^{\dagger 2}\right), \quad K_{\mu,0} = \frac{1}{4}\left(a_\mu a_\mu^\dagger + a_\mu^\dagger a_\mu\right). \tag{3.29}$$

From (3.29) we combine the operators $K_{\mu,1}$ and $K_{\mu,2}$ to new operators $K_{\mu,-}$ and $K_{\mu,+}$ according to ($\mu = 1,2$)

$$K_{\mu,-} \equiv K_{\mu,1} - \mathrm{i}K_{\mu,2}, \quad K_{\mu,+} \equiv K_{\mu,1} + \mathrm{i}K_{\mu,2}, \tag{3.30}$$

leading explicitly to

$$\begin{aligned} K_{1,-} &\equiv \frac{1}{2}a_1^2, \quad K_{1,+} \equiv \frac{1}{2}a_1^{\dagger 2}, \quad K_{1,0} \equiv \frac{1}{4}\left(a_1 a_1^\dagger + a_1^\dagger a_1\right) = \frac{1}{2}\left(N_1 + \frac{1}{2}I\right) \equiv \frac{1}{2}N_1', \\ K_{2,-} &\equiv \frac{1}{2}a_2^2, \quad K_{2,+} \equiv \frac{1}{2}a_2^{\dagger 2}, \quad K_{2,0} \equiv \frac{1}{4}\left(a_2 a_2^\dagger + a_2^\dagger a_2\right) = \frac{1}{2}\left(N_2 + \frac{1}{2}I\right) \equiv \frac{1}{2}N_2'. \end{aligned} \tag{3.31}$$

They satisfy the commutation relations of an $su(1,1)$-algebra in analogy to (3.19)

$$[K_{\mu,0}, K_{\mu,\pm}] = \pm K_{\mu,\pm}, \quad [K_{\mu,-}, K_{\mu,+}] = 2K_{\mu,0}, \quad (\mu = 1,2). \tag{3.32}$$

For completeness there remains to calculate the commutation relation of $(K_{\mu,0}, K_{\mu,+}, K_{\mu,-})$ with the operators (K_0, K_+, K_-) and with (J_+, J_-, J_3) for which we find

$$\begin{aligned} &[K_0, K_{\mu,0}] = 0, \quad [K_0, K_{\mu,\pm}] = \pm K_{\mu,\pm}, \\ &[K_+, K_{\mu,0}] = -\frac{1}{2}K_+, \quad [K_+, K_{\mu,+}] = 0, \quad [K_+, K_{1,-}] = -J_-, \\ &[K_+, K_{2,-}] = -J_+, \quad [K_-, K_{\mu,0}] = +\frac{1}{2}K_-, \quad [K_-, K_{\mu,-}] = 0, \\ &[K_-, K_{1,+}] = +J_+, \quad [K_-, K_{2,+}] = +J_-. \end{aligned} \tag{3.33}$$

and

$$\begin{aligned} &[J_3, K_{\mu,0}] = 0, \quad [J_3, K_{1,\pm}] = \pm K_{1,\pm}, \quad [J_3, K_{2,\pm}] = \mp K_{2,\pm}, \\ &[J_\mp, K_{1,0}] = \pm\frac{1}{2}J_\mp, \quad [J_\mp, K_{2,0}] = \mp\frac{1}{2}J_\mp. \end{aligned} \tag{3.34}$$

The operators $K_{1,0}$ and $K_{2,0}$ are connected with J_3 and K_0 by

$$J_3 = K_{1,0} - K_{2,0} = \frac{1}{2}(N_1 - N_2), \quad K_0 = K_{1,0} + K_{2,0} = \frac{1}{2}(N_1 + N_2 + I). \tag{3.35}$$

Thus the operators $K_{1,0}$ and $K_{2,0}$ are already contained as linear combinations of the operators J_3 and K_0 or N_1 and N_2 and must not separately be taken into account for closing the Lie algebra of the above mentioned 6 operators. The operators $(K_{\mu,-}, K_{\mu,+}, K_{\mu,0}),(\mu = 1,2)$ form two bases of two Lie algebras $su(1,1)$ of squeezing operators of the two modes with indices "1" and "2" separately.

The 10 operators $(J_-, J_+, J_3, K_-, K_+, K_0, K_{1,-}, K_{1,+}, K_{2,-}, K_{2,+})$ form a possible basis of the 10-parameter Lie algebra $sp(4,\mathbb{R})$ with $\frac{10\times 9}{2} = 45$ independent commutation relations. An informative overview about the structure of the Lie algebra with its commutation relations give the root diagrams that we investigate in the next Section.

4. The Root Diagrams for the Homogeneous and Inhomogeneous Symplectic Group $Sp(4,\mathbb{R})$ of Two Pairs of Boson Operators

For a first overview about the structure of a Lie group it is very useful to consider the root diagram (e.g., [21] [22] [23] [24] [25]) of its Lie algebra which describes the neighborhood of the identity operator I of the Lie group. We suppose that the compact part of the Lie group in the neighborhood of the identity element can be parameterized by vectors $\boldsymbol{\xi} \equiv (\xi^1, \cdots, \xi^d)$ with d independent components where d is called the dimension of the Lie group and we require that the vector $\boldsymbol{\xi} = \mathbf{0}$ describes the identity element. We denote the group operators by $A(\boldsymbol{\xi})$. In the neighborhood of the identity element $A(\mathbf{0}) = I$ an arbitrary group operator $A(\boldsymbol{\xi})$ can be expanded in a Taylor series according to

$$A(\boldsymbol{\xi}) = A(\mathbf{0}) + \frac{\partial A}{\partial \xi^i}(\mathbf{0})\xi^i + \cdots \equiv I + X_i \xi^i + \cdots, \quad (i = 1, \cdots, d). \tag{4.1}$$

The operators $X_i, (i = 1, \cdots, d)$ are infinitesimal operators of the Lie group and the set of all possible linear combinations of these operators forms the Lie algebra of dimension d. We give here some definitions and a few basic results to the general theory of Lie algebras and discuss later the root diagrams from Equation (4.16) (refrootdef) on.

The commutator $[X,Y]$ of two arbitrary elements X and Y takes on the role of multiplication in the Lie algebra and the result Z has to be an element of the Lie algebra for closing it ($\langle A \rangle$ denotes the trace of an operator A)

$$Z = [X,Y] \equiv XY - YX, \quad \Rightarrow \quad \langle Z \rangle = 0. \tag{4.2}$$

From the definition of the commutator follows immediately

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0, \tag{4.3}$$

which is called the Jacobi identity. It takes on the role of the associative law for, e.g., the algebra of real and complex numbers or the algebra of matrices. Now we choose a basis of d linearly independent operators E_i in such way that arbitrary operators of the Lie algebra can be represented as

$$X = E_i x^i, \quad Y = E_i y^i, \quad Z = E_i z^i, \quad \left(i = 1, \cdots, d\right), \tag{4.4}$$

where $x^i, y^i, z^i, \cdots$ are vector components (in analogy to vectors $\boldsymbol{x} = \boldsymbol{e}_i x^i$ and later with operators A to $\boldsymbol{y} = \mathsf{A}\boldsymbol{x} = \mathsf{A}\boldsymbol{e}_i x^i = \boldsymbol{e}_k A_i^k x^i = \boldsymbol{e}_k y^k$). Then from (4.2) we find for the commutator $Z = \left[X, Y\right]$

$$E_k z^k = Z = \left[X, Y\right] = \left[E_i x^i, E_j y^j\right] = \left[E_i, E_j\right] x^i y^j \equiv E_k c_{ij}^k x^i y^j, \tag{4.5}$$

where we have introduced coefficients c_{ij}^k by definition

$$\left[E_i, E_j\right] \equiv E_k c_{ij}^k, \quad \Rightarrow \quad c_{ij}^k = -c_{ji}^k, \tag{4.6}$$

which are called the structure coefficients of the Lie algebra with respect to the chosen basis E_k. From (4.6) follows

$$E_k z^k = Z = \left[X, Y\right] = \left[E_i, E_j\right] x^i y^j = E_k c_{ij}^k x^i y^j, \quad \Rightarrow \quad z^k = c_{ij}^k x^i y^j. \tag{4.7}$$

From the Jacobi identity (4.3) follows then for arbitrary three basis operators E_i, E_j, E_l follows then

$$\begin{aligned} 0 &= \left[\left[E_i, E_j\right], E_l\right] + \left[\left[E_j, E_l\right], E_i\right] + \left[\left[E_l, E_i\right], E_j\right] \\ &= \left[E_m c_{ij}^m, E_l\right] + \left[E_m c_{jl}^m, E_i\right] + \left[E_m c_{li}^m, E_j\right] \\ &= E_n \left(c_{ij}^m c_{ml}^n + c_{jl}^m c_{mi}^n + c_{li}^m c_{mj}^n\right), \end{aligned} \tag{4.8}$$

and therefore

$$c_{ij}^m c_{ml}^n + c_{jl}^m c_{mi}^n + c_{li}^m c_{mj}^n = 0. \tag{4.9}$$

By contraction over the indices $j = n$ and then interchanging the free indices i and l follow the two equations

$$\begin{aligned} c_{in}^m c_{ml}^n + c_{nl}^m c_{mi}^n + c_{li}^m c_{mn}^n &= 0, \\ c_{\ln}^m c_{mi}^n + c_{ni}^m c_{ml}^n + c_{il}^m c_{mn}^n &= 0, \end{aligned} \tag{4.10}$$

where we used that from the antisymmetry of the structure coefficients in the lower indices follows for the sum terms in (4.10)

$$c_{in}^m c_{ml}^n = c_{lm}^n c_{ni}^m = c_{ln}^m c_{mi}^n, \quad c_{nl}^m c_{mi}^n = c_{ml}^n c_{ni}^m = c_{ni}^m c_{ml}^n, \quad c_{li}^m c_{mn}^n = -c_{il}^m c_{mn}^n. \tag{4.11}$$

Forming the difference of the Equations (4.10) and using the symmetry (4.11) the first two sum terms cancel and from the third sum terms using again (4.11) results

$$c_{li}^m c_{mn}^n = c_{il}^m c_{mn}^n = -c_{li}^m c_{mn}^n = 0, \tag{4.12}$$

and each of the equations (10) simplifies to

$$c_{in}^m c_{ml}^n + c_{nl}^m c_{mi}^n = c_{in}^m c_{ml}^n - c_{ln}^m c_{mi}^n = 0, \tag{4.13}$$

from which follows

$$\gamma_{il} \equiv c_{ni}^m c_{ml}^n = c_{nl}^m c_{mi}^n \equiv \gamma_{li}. \tag{4.14}$$

The symmetric tensor $\gamma_{il} = \gamma_{li}$ is called the Killing form and with its help one may define a bilinear symmetric scalar product of two operators $X = E_i x^i$ and $Y = E_l y^l$ written (XY) as follows

$$\begin{aligned} &(XY) \equiv \left(E_i x^i E_l y^l\right) = \left(E_i E_l\right) x^i y^l \equiv \gamma_{il} x^i y^l = (YX), \\ &\gamma_{il} \equiv \left(E_i E_l\right) = \left(E_l E_i\right) = \gamma_{li}. \end{aligned} \tag{4.15}$$

The second-rank symmetric tensor γ_{il} is a kind of metric tensor for the Lie algebra and plays an important role for the distinction of different kinds of Lie algebras (e.g., Levy-Maltsev theorem, [23] [25]).

Now comes into play the Cartan subalgebra of the Lie algebra which is the linear space of the maximum of commuting operators of the Lie algebra with operators usually denoted by $\boldsymbol{H} \equiv \left(H_1, \cdots, H_r\right)$. The number r of independent operators of the Cartan subalgebra is called the rank of the Lie algebra. From linear combinations of the Lie-algebra operators X_i one may select by linear combinations d operators $E_{\boldsymbol{\alpha}}$ which are eigenvectors of the operator of the Cartan subalgebra in the sense

$$\left[\boldsymbol{H}, E_{\boldsymbol{\alpha}}\right] = \boldsymbol{\alpha} E_{\boldsymbol{\alpha}}, \quad \left(\text{or} \quad \left[H_i, E_{\boldsymbol{\alpha}}\right] = \alpha_i E_{\boldsymbol{\alpha}}\right), \quad \left(\boldsymbol{\alpha} \equiv \left(\alpha_1, \cdots, \alpha_r\right)\right). \tag{4.16}$$

The vectorial eigenvalues $\boldsymbol{\alpha}$ are called the root vectors of the Lie algebra and their dimensionality is equal to the rank of the Lie algebra or the dimension r of the Cartan subalgebra. Only the vectorial eigenvalue $\boldsymbol{\alpha} = \mathbf{0}$ is r-fold degenerate and their root vectors are linear combinations of the operators H_i. The root diagram represents the $d - r$ root vectors in the r-dimensional space plus the r operators H_i of the Cartan subalgebra in the center. The basis operators of the Cartan subalgebra are not uniquely determined and can be defined in different variants ways of the theory. The commutating operators H_i of the Cartan subalgebra plus the root vectors determine already a certain amount of all commutation relations. For the remaining commutation relations of the periphery the theory of Lie algebras derives relations from the Jacobi identity which restrict their possibilities. This is only a minimum of the many well-known relations for Lie algebras (see, e.g., [23] [24] [25] and many others).

For some generalizations we extend the mainly here considered Lie group $SU(2)$. In **Figure 1** we represent the two-dimensional root vectors of the Lie algebra $sp(4,\mathbb{R})$ to the symplectic group $Sp(4,\mathbb{R})$ as arrows in two bases of the Cartan subalgebra $\left(J_3, K_0\right)$ and $\left(N_1, N_2\right)$. For example, the arrow in the left-hand partial picture to the operator J_- means the commutator relation $\left[\left(J_3, K_0\right), J_-\right] = (-1,0) J_-$ and the right-hand partial picture the commutator relation $\left[\left(N_1, N_2\right), J_-\right] = (-1,1) J_-$.

Each pair of boson annihilation and creation vectors $\left(a, a^\dagger\right)$ or corresponding canonical operators (Q, P) with the commutation relations $\left[a, a^\dagger\right] = I$ or $[Q, P] = \mathrm{i}\hbar I$ forms also a Lie algebra called Heisenberg-Weyl algebra which is of dimension zero and reduces to a point. However, if we take in addition to pairs of boson operators of the Heisenberg-Weyl algebra the corresponding number operators $N_i \equiv a_i^\dagger a_i, (i = 1, \cdots, n)$ then due to commutation relations

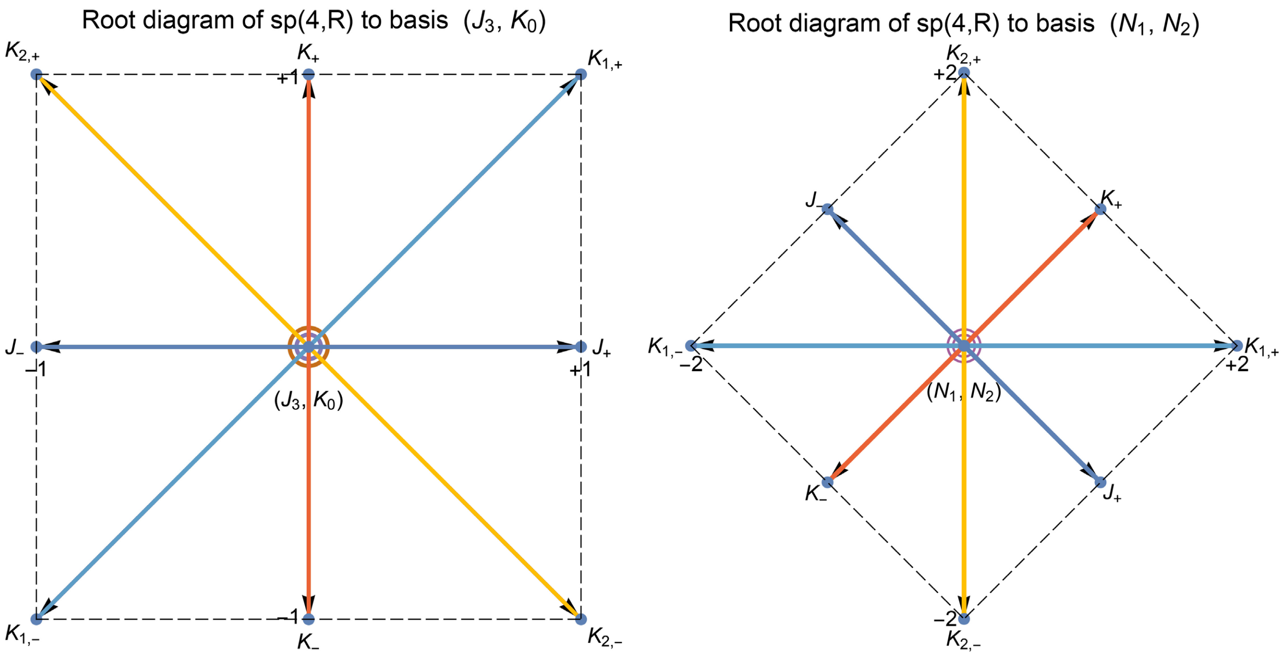

Figure 1. Root diagrams of Lie algebra to the homogeneous symplectic group $Sp(4,\mathbb{R})$ in different bases. It is 10-dimensional and therefore a basis possesses 10 operators. In first basis of the Cartan subalgebra we have $\left(J_3=\frac{1}{2}(N_1-N_2), K_0=\frac{1}{2}(N_1+N_2+I)\sim\frac{1}{2}(N_1+N_2)\right)$ and in the second basis $\left(N_1\equiv a_1^\dagger a_1, N_2\equiv a_2^\dagger a_2\right)$ where N_1 and N_2 are the number operators to the two modes. The identity operator I in the Cartan subalgebra in the center of the diagram does not play a role since it commutes with all operators of the Lie group and does not provide a contribution to the roots.

$\left[N_i,a_j\right]=-\delta_{ij}a_j$, $\left[N_i,a_j^\dagger\right]=+\delta_{ij}a_j^\dagger$, $\left[N_i,N_j\right]=0$ the Heisenberg-Weyl algebras taken together with the number operators also form Lie algebras of corresponding rank. The same is, for example, by combination of Lie algebras $sp(2n,\mathbb{R})$ or $su(n+1)$ representable by pairs of annihilation and creation operators we find new algebras which we call inhomogeneous Lie algebras $i.sp(2n,\mathbb{R})$ or $i.su(n+1)$, respectively.

In **Figure 2** we represent the root diagrams of $i.sp(4,\mathbb{R})$ in different basis systems. Besides the operators of the Cartan subalgebra in the center they contain there the identity operator I for closing them. In **Figure 3** we represent the root diagram of a subalgebra of $i.sp(4,\mathbb{R})$ in **Figure 2** and make from it the transition to the root diagram of the Lie algebra $su(3)$. In **Figure 4** the root diagram for $SU(3)$ of the right-hand **Figure 3** is stretched in direction of the ordinate to the canonical form in a way that it takes on the maximal symmetry of a regular hexagon. One may check that operator $N\equiv N_1+N_2+N_3$, $N_1\equiv a_1^\dagger a_1$, $N_2\equiv a_2^\dagger a_2$, $N_1\equiv a_3^\dagger a_3$, commutes with all operators $J_\pm^{(12)}, J_\pm^{(13)}, J_\pm^{(23)}$ to the Lie algebra $su(3)$ constructed from the inhomogeneous group $I.Sp(4,\mathbb{R})$ in described way and illustrated in **Figure 4**

$$\left[N,J_\pm^{(12)}\right]=\left[N,J_\pm^{(13)}\right]=\left[N,J_\pm^{(23)}\right]=0,\quad\left(N\equiv N_1+N_2+N_3\right).\tag{4.17}$$

Therefore one may add to the operator $K_0=\frac{1}{2}\left(N_1+N_2+I\right)$ a multiple of

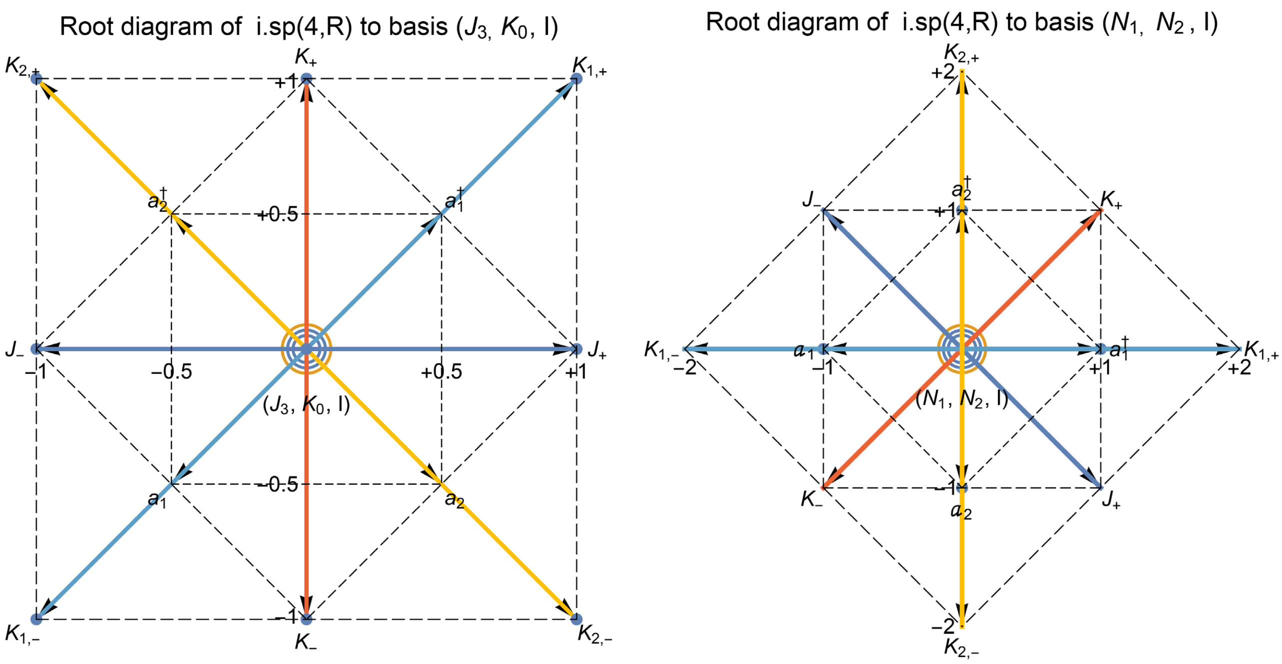

Figure 2. Root diagram of Lie algebra to the inhomogeneous symplectic group $I.Sp(4,\mathbb{R})$. These root diagrams contain in addition to the homogeneous symplectic group $Sp(4,\mathbb{R})$ the pairs of operators $(a_1, a_1^\dagger)$ and $(a_1, a_1^\dagger)$ with commutation relations $[a_1, a_1^\dagger] = [a_2, a_2^\dagger] = I$. Therefore, the identity operator I belongs to the Lie algebra and since it commutes with all operators it belongs to the Cartan subalgebra in the center of the diagrams. The diagram is quasi two-dimensional since no root operator possesses a component in direction of I perpendicular to the paper plane.

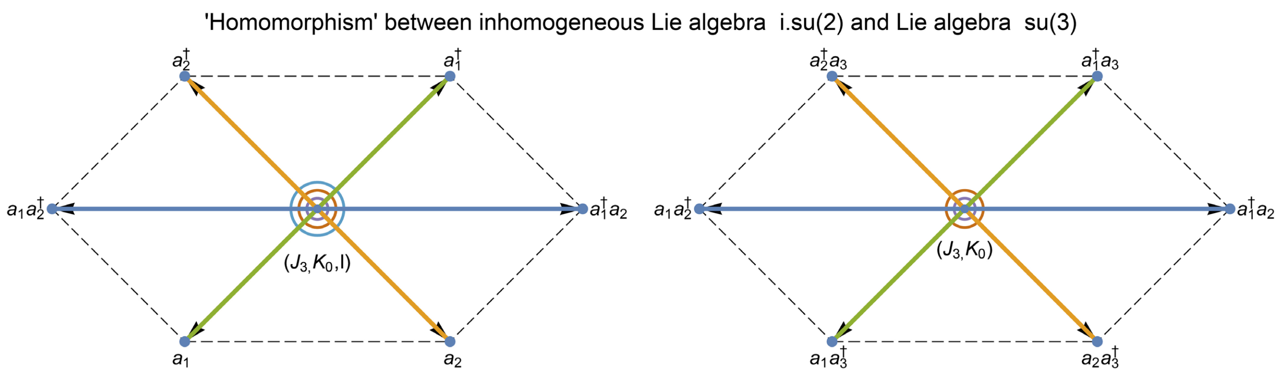

Figure 3. I somorphism of root diagram to Lie subalgebra of $i.sp(4,\mathbb{R})$ to the Lie algebra $su(3)$ of homogeneous unitary group $SU(3)$. The root diagram of this Lie algebra on the left-hand side is part of the root diagram in Figure 2. The operators in the center are defined by $J_3 \equiv \frac{1}{2}(N_1 - N_2)$, $K_0 \equiv \frac{1}{2}(N_1 + N_2)$ with $N_1 \equiv a_1^\dagger a_1$, $N_2 \equiv a_2^\dagger a_2$. Since the operators J_3 and K_0 commute with the operators $(a_3, a_3^\dagger)$ both root diagrams are equivalent.

$N = N_1 + N_2 + N_3$ and for the same reason a multiple of the identity operator I without changing the root diagram of $su(3)$ that is represented on the right-hand picture of Figure 4. If we add to this scheme now the annihilation and creation operators $(a_i, a_i^\dagger), (i = 1,2,3)$ which do not commute with the operator $N = N_1 + N_2 + N_3$ then we may obtain the root diagram for the Lie algebra to

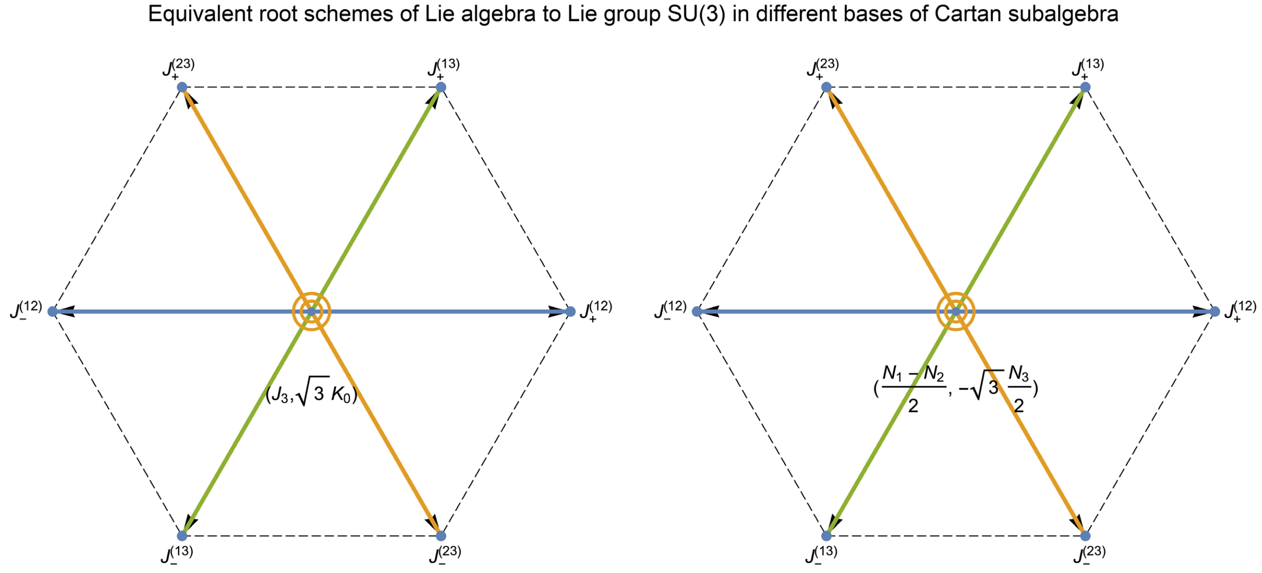

Figure 4. Root diagram of Lie algebra to the unitary unimodular group $SU(3)$ in two equivalent bases. The operators in the center are defined by $J_3 \equiv \frac{1}{2}(N_1 - N_2)$, $K_0 \equiv \frac{1}{2}(N_1 + N_2 + I)$ with $N_1 \equiv a_1^\dagger a_1$, $N_2 \equiv a_2^\dagger a_2$ and additionally $N_3 \equiv a_3^\dagger a_3$. The two schemes are equivalent because the number operator $N \equiv N_1 + N_2 + N_3$ commutes with all operators of the Lie algebra to $SU(3)$ in considered realization and thus the operators of the Cartan subalgebra in the center can be substituted using the given identity. These diagrams correspond to the decontorted right-hand diagram of Figure 3 in perpendicular direction which leads to highest symmetry of the root diagram for $SU(3)$ that becomes clear if we take into account $J_-^{(12)} = a_1 a_2^\dagger$, $J_+^{(12)} = a_1^\dagger a_2$, $J_-^{(13)} = a_1 a_3^\dagger$, $J_+^{(13)} = a_1^\dagger a_3$, $J_-^{(23)} = a_2 a_3^\dagger$, $J_+^{(23)} = a_2^\dagger a_3$.

the inhomogeneous group $I.SU(3)$ represented in Figure 5. It contains 15 operators, 3 of the Cartan subalgebra in the center and 12 in the periphery and is very similar to the root diagram of the Lie algebra g_2 to the exceptional Lie group G_2 with 14 basis operators (e.g., [23]). The root scheme to the next unitary unimodular group $SU(4)$ is three-dimensional and thus of rank 3. It possesses 3 operators in the center belonging to the Cartan subalgebra and 12 in the periphery which with their tips form in the most symmetric way the 12 corners of a cuboctahedron which is a semi-regular polyhedron.

In cases when the identity operator I belongs to the Cartan subalgebra in addition to $2n$ annihilation and creation operators (inhomogeneous groups) the root scheme, nevertheless, remains quasi of rank n since operator I commutes with all operators and does not provide a contribution to a higher rank $n+1$.

A conclusion is that by transition from known series of Lie groups to their inhomogeneous partner groups some problems emerge with their root diagrams and algebraic properties (traces, see also **Appendix B**).

5. Fundamental Representation $D^{\left(\frac{1}{2}\right)}$ of $SU(2)$ in Parametrization by a Three-Dimensional Vector φ

We now derive the fundamental representation of $SU(2)$ in the basis of the

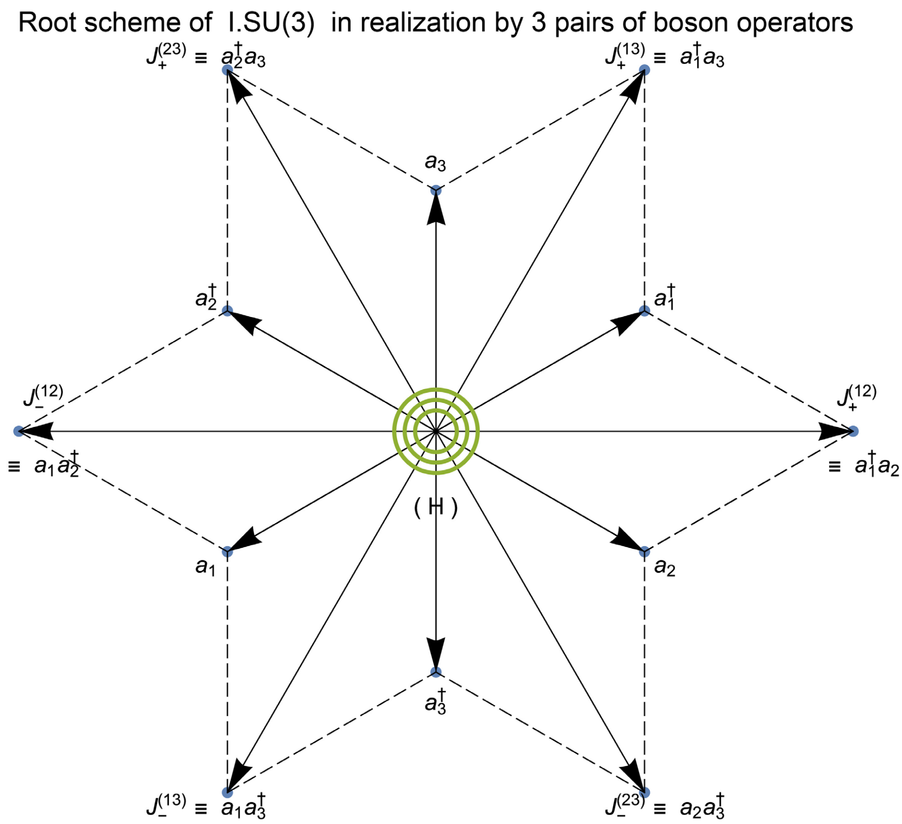

Figure 5. Root scheme of Lie algebra of inhomogeneous group $I.SU(3)$ in realization by 3 pairs of boson annihilation and creation operators. The 3 operators (H) of the Cartan subalgebra are defined by $(H) = \left(J \equiv \frac{1}{2}(N_1 - N_2), K \equiv \frac{\sqrt{3}}{6}(N_1 + N_2 - 2N_3), I \right)$ with $N_1 \equiv a_1^\dagger a_1$, $N_2 \equiv a_2^\dagger a_2$, $N_3 \equiv a_3^\dagger a_3$. The distance from the center to the tips of the star is equal to 1. The diagram is very similar to that of the exceptional group G_2 and becomes the same if we omit the operator I in the center since it does not contribute to a third dimension. On the other side all commutators should belong to the diagram.

operators $\left(a_1^\dagger, a_2^\dagger\right)$ and clarify in this way the transformation properties of the involved basic quantities such as boson operators and polarization vectors. For this purpose, we use a real three-dimensional vector parameter $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \varphi_3)$ and represent the general element x of the Lie algebra $su(2)$ in the following way

$$\begin{aligned} x &\equiv \boldsymbol{J\varphi} = J_1\varphi_1 + J_2\varphi_2 + J_3\varphi_3 = \frac{1}{2}\left(J_+\varphi_- + J_-\varphi_+\right) + J_3\varphi_3, \\ \varphi_k &= \varphi_k^*, \quad x = x^\dagger, \quad \varphi_+ \equiv \varphi_1 + \mathrm{i}\varphi_2, \quad \varphi_- \equiv \varphi_1 - \mathrm{i}\varphi_2. \end{aligned} \tag{5.1}$$

In **Appendix A**, we establish the connection to the Euler angles as parameters. The transition from the Lie algebra $su(2)$ to the Lie group $SU(2)$ and its inversion is made by the exponential mapping

$$x = \boldsymbol{J\varphi} \leftrightarrow X \equiv \mathrm{e}^{\mathrm{i}x} = \exp\left(\mathrm{i}\boldsymbol{J\varphi}\right), \quad \Leftrightarrow \quad x = x^\dagger \leftrightarrow X^{-1} = X^\dagger. \tag{5.2}$$

The construction of the fundamental representation of $SU(2)$ in the basis of the operators $\left(a_1^\dagger, a_2^\dagger\right)$ requires to consider a part of the Lie algebra of the inhomogeneous unitary unimodular group $I.SU(2)$ with the operators $\left(J_1, J_2, J_3, a_1, a_2, a_1^\dagger, a_2^\dagger, I\right)$ and their commutation relations as a possible set of basis operators. Instead of writing down all commutation relations, we use the necessary ones in the following mapping of $\left(J_1, J_2, J_3\right)$ onto matrices $\left(\mathsf{s}_1, \mathsf{s}_2, \mathsf{s}_3\right)$ in the two-dimensional fundamental representation of $su(2)$

$$\begin{aligned} \left[J_1, \left(a_1^\dagger, a_2^\dagger\right)\right] &= \left(a_1^\dagger, a_2^\dagger\right)\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \left(a_1^\dagger, a_2^\dagger\right)\mathsf{s}_1, \quad \mathsf{s}_1 = \frac{1}{2}\sigma_1, \\ \left[J_2, \left(a_1^\dagger, a_2^\dagger\right)\right] &= \left(a_1^\dagger, a_2^\dagger\right)\frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} = \left(a_1^\dagger, a_2^\dagger\right)\mathsf{s}_2, \quad \mathsf{s}_2 = \frac{1}{2}\sigma_2, \\ \left[J_3, \left(a_1^\dagger, a_2^\dagger\right)\right] &= \left(a_1^\dagger, a_2^\dagger\right)\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \left(a_1^\dagger, a_2^\dagger\right)\mathsf{s}_3, \quad \mathsf{s}_3 = \frac{1}{2}\sigma_3. \end{aligned} \tag{5.3}$$

The matrices s_k are essentially (multiplied by factor 2) the Pauli spin matrices σ_k which possess the properties

$$\sigma_k\sigma_l = \delta_{kl}\mathrm{I} + \mathrm{i}\varepsilon_{jkl}\sigma_j, \quad \Rightarrow \quad \sigma_k\sigma_l + \sigma_l\sigma_k = 2\delta_{kl}\mathrm{I}, \quad \left[\sigma_k, \sigma_l\right] = \mathrm{i}2\varepsilon_{jkl}\sigma_j. \tag{5.4}$$

The most direct relation to the Pauli spin matrices is one reason that we construct the fundamental representation of the Lie algebra $su(2)$ in the basis $\left(a_1^\dagger, a_2^\dagger\right)$ and not in the adjoint basis $\left(a_1, a_2\right)$.

From (5.3) follows for the representation of the Lie algebra operators $x = \boldsymbol{J}\boldsymbol{\varphi}$

$$\left[x, \left(a_1^\dagger, a_2^\dagger\right)\right] = \left(a_1^\dagger, a_2^\dagger\right)\frac{1}{2}\begin{pmatrix} \varphi_3 & \varphi_- \\ \varphi_+ & -\varphi_3 \end{pmatrix} \tag{5.5}$$

This is the mapping $x \to \mathsf{x}$ of the operators x on two-dimensional matrices x according to

$$x \to \mathsf{x} = \frac{1}{2}\begin{pmatrix} \varphi_3 & \varphi_- \\ \varphi_+ & -\varphi_3 \end{pmatrix}, \quad x^2 \to \mathsf{x}^2 = \frac{1}{4}|\boldsymbol{\varphi}|^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{4}|\boldsymbol{\varphi}|^2\,\mathrm{I}, \tag{5.6}$$

By means of the well-known operator expansion (e.g., [19])

$$\mathrm{e}^{\mathsf{A}}\mathsf{B}\mathrm{e}^{-\mathsf{A}} = \sum_{n=0}^{\infty}\frac{1}{n!}\underbrace{\left[\mathsf{A},\left[\mathsf{A},\cdots\left[\mathsf{A}\right.\right.\right.}_{n\times},\mathsf{B}\underbrace{\left.\left.\left.\right]\cdots\right]\right]}_{n\times} = B + \left[\mathsf{A},\mathsf{B}\right] + \frac{1}{2!}\left[\mathsf{A},\left[\mathsf{A},\mathsf{B}\right]\right] + \cdots. \tag{5.7}$$

and using the Hamilton-Cayley identity for two-dimensional operators A in (5.6) we find the corresponding mapping $\mathrm{e}^{\mathrm{i}x} \to \mathrm{e}^{\mathrm{i}\mathsf{x}}$ into the Lie group (note the difference between x and x)

$$\mathrm{e}^{\mathrm{i}x} = \sum_{n=0}^{\infty}\frac{\mathrm{i}^n}{n!}x^n \to \mathrm{e}^{\mathrm{i}\mathsf{x}} = \sum_{n=0}^{\infty}\frac{\mathrm{i}^n}{n!}\mathsf{x}^n, \quad \mathrm{e}^{\mathrm{i}x}\left(a_1^\dagger, a_2^\dagger\right)\mathrm{e}^{-\mathrm{i}x} = \left(a_1^\dagger, a_2^\dagger\right)\mathrm{e}^{\mathrm{i}\mathsf{x}}. \tag{5.8}$$

In described way we obtain from (5.3) the two-dimensional fundamental representation $D^{\left(\frac{1}{2}\right)}$ of $SU(2)$ by unitary unimodular matrices $\mathsf{S} \equiv \mathsf{S}(\boldsymbol{\varphi})$ in the basis of creation operators $\left(a_1^\dagger, a_2^\dagger\right)$

$$\begin{aligned} \left(a_1'^\dagger, a_2'^\dagger\right) &\equiv \exp\left(\mathrm{i}\boldsymbol{J}\boldsymbol{\varphi}\right)\left(a_1^\dagger, a_2^\dagger\right)\exp\left(-\mathrm{i}\boldsymbol{J}\boldsymbol{\varphi}\right) \equiv \left(a_1^\dagger, a_2^\dagger\right)\mathsf{S} = \left(a_1^\dagger, a_2^\dagger\right)\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \\ &= \left(a_1^\dagger S_{11} + a_2^\dagger S_{21}, a_1^\dagger S_{12} + a_2^\dagger S_{22}\right), \quad \left[\mathsf{S}\right] = 1. \end{aligned} \tag{5.9}$$

In analogy we find from (5.9) for the transformation of annihilation operators

$$\begin{aligned}\begin{pmatrix} a_1' \\ a_2' \end{pmatrix} &\equiv \exp(\mathrm{i}\boldsymbol{J}\boldsymbol{\varphi})\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\exp(-\mathrm{i}\boldsymbol{J}\boldsymbol{\varphi}) = \mathsf{S}^{\dagger}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} S_{22} & -S_{12} \\ -S_{21} & S_{11} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \\ &= \begin{pmatrix} S_{22}a_1 - S_{12}a_2 \\ -S_{21}a_1 + S_{11}a_2 \end{pmatrix}.\end{aligned} \tag{5.10}$$

According to

$$N_1' + N_2' \equiv a_1'^{\dagger}a_1' + a_2'^{\dagger}a_2' = \left(a_1'^{\dagger}, a_2'^{\dagger}\right)\begin{pmatrix} a_1' \\ a_2' \end{pmatrix} = a_1^{\dagger}a_1 + a_2^{\dagger}a_2 \equiv N_1 + N_2, \tag{5.11}$$

these transformations possess the total number operator $N \equiv N_1 + N_2$ as basic invariant.

The form of the matrix S is

$$\mathsf{S} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = \exp(\mathrm{i}\mathbf{s}\boldsymbol{\varphi}), \quad (\mathbf{s}\boldsymbol{\varphi} \equiv \mathsf{s}_1\varphi_1 + \mathsf{s}_2\varphi_2 + \mathsf{s}_3\varphi_3), \tag{5.12}$$

and is explicitly found in described way using the Hamilton-Cayley identity and with abbreviations φ_+ and φ_- (compare, e.g., Gilmore [24], p. 150, Equation (6.4[1]))

$$\begin{aligned}\mathsf{S} \equiv \mathsf{S}(\boldsymbol{\varphi}) &= \begin{pmatrix} \cos\left(\frac{|\boldsymbol{\varphi}|}{2}\right) + \mathrm{i}\frac{\varphi_3}{|\boldsymbol{\varphi}|}\sin\left(\frac{|\boldsymbol{\varphi}|}{2}\right) & \mathrm{i}\frac{\varphi_-}{|\boldsymbol{\varphi}|}\sin\left(\frac{|\boldsymbol{\varphi}|}{2}\right) \\ \mathrm{i}\frac{\varphi_+}{|\boldsymbol{\varphi}|}\sin\left(\frac{|\boldsymbol{\varphi}|}{2}\right) & \cos\left(\frac{|\boldsymbol{\varphi}|}{2}\right) - \mathrm{i}\frac{\varphi_3}{|\boldsymbol{\varphi}|}\sin\left(\frac{|\boldsymbol{\varphi}|}{2}\right) \end{pmatrix}, \quad \mathsf{S}\mathsf{S}^{\dagger} = \mathrm{I}, \\ |\boldsymbol{\varphi}| &\equiv \sqrt{\varphi_+\varphi_- + \varphi_3^2} = \sqrt{\varphi_1^2 + \varphi_2^2 + \varphi_3^2}, \quad \varphi_{\pm} \equiv \varphi_1 \pm \mathrm{i}\varphi_2 = \varphi_{\mp}^{*}.\end{aligned} \tag{5.13}$$

The relations of unitarity $\mathsf{S}\mathsf{S}^{\dagger} = \mathrm{I}$ and of unimodularity $[\mathsf{S}] = 1$ in addition are more explicitly ($[\mathsf{S}]$ denotes determinant of two-dimensional matrix or operator S)

$$\begin{aligned}&\mathsf{S}^{-1} = \mathsf{S}^{\dagger}, \quad \Leftrightarrow \quad S_{11} = S_{22}^{*}, \quad S_{12} = -S_{21}^{*}, \\ &[\mathsf{S}] = S_{11}S_{22} - S_{12}S_{21} = S_{11}S_{11}^{*} + S_{12}S_{12}^{*} = 1.\end{aligned} \tag{5.14}$$

The character of the representation or trace of the representation matrices is ($\langle\mathsf{S}\rangle$ denotes trace of a matrix or operator S)

$$\langle\mathsf{S}\rangle \equiv S_{11} + S_{22} = 2\cos\left(\frac{|\boldsymbol{\varphi}|}{2}\right) = \frac{\sin(|\boldsymbol{\varphi}|)}{\sin\left(\frac{|\boldsymbol{\varphi}|}{2}\right)}, \tag{5.15}$$

with $\sin\left(\frac{|\boldsymbol{\varphi}|}{2}\right)$ expressed by the matrix elements using the unimodularity and unitarity (5.14) of S

$$\sin\left(\frac{|\boldsymbol{\varphi}|}{2}\right) = \pm\sqrt{1 - \left(\frac{S_{11} + S_{22}}{2}\right)^2} = \pm\sqrt{S_{12}S_{12}^{*} - \left(\frac{S_{11} - S_{11}^{*}}{2}\right)^2}. \tag{5.16}$$

We mention still that using

$$\begin{pmatrix} \varphi_3 & \varphi_- \\ \varphi_+ & -\varphi_3 \end{pmatrix}^2 = \begin{pmatrix} \varphi_3^2 + \varphi_-\varphi_+ & 0 \\ 0 & \varphi_+\varphi_- + \varphi_3^2 \end{pmatrix} = |\boldsymbol{\varphi}|^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{5.17}$$

and additionally using the identity (5.7) the matrix (5.13) together with the chosen basis $\left(a_1^\dagger, a_2^\dagger\right)$ can be straightforwardly derived also as follows

$$\begin{aligned} \mathrm{e}^{\mathrm{i}\boldsymbol{J\varphi}}\left(a_1^\dagger, a_2^\dagger\right)\mathrm{e}^{-\mathrm{i}\boldsymbol{J\varphi}} &= \left(a_1^\dagger, a_2^\dagger\right)\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\mathrm{i}}{2}\right)^n \begin{pmatrix} \varphi_3 & \varphi_- \\ \varphi_+ & -\varphi_3 \end{pmatrix}^n \\ &= \left(a_1^\dagger, a_2^\dagger\right)\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m)!}\left(\frac{|\boldsymbol{\varphi}|}{2}\right)^{2m}\right. \\ &\quad \left. + \mathrm{i}\begin{pmatrix} \varphi_3 & \varphi_- \\ \varphi_+ & -\varphi_3 \end{pmatrix}\frac{1}{|\boldsymbol{\varphi}|}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)!}\left(\frac{|\boldsymbol{\varphi}|}{2}\right)^{2m+1}\right\} \\ &= \left(a_1^\dagger, a_2^\dagger\right)\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\left(\frac{|\boldsymbol{\varphi}|}{2}\right) + \mathrm{i}\begin{pmatrix} \varphi_3 & \varphi_- \\ \varphi_+ & -\varphi_3 \end{pmatrix}\frac{1}{|\boldsymbol{\varphi}|}\sin\left(\frac{|\boldsymbol{\varphi}|}{2}\right)\right\}, \end{aligned} \tag{5.18}$$

that is identical with (5.13). The components of $\boldsymbol{J}$ and their relations to annihilation and creation operators are explained in (3.5) and (3.9) and furthermore $\varphi_\pm \equiv \varphi_1 \pm \mathrm{i}\varphi_2$ is used.

Other bases, for example, (a_1, a_2), lead only to a reordering with changing signs in the matrices in (5.13). In explained way one may calculate also other irreducible and reducible representations even with other dimension if one possesses a suited basis.

6. Determination of a Basic Range of Vector Parameter $\boldsymbol{\varphi}$ for $SU(2)$

We now determine a basic range for the vector parameter $\boldsymbol{\varphi}$. Two vector parameters $\boldsymbol{\varphi}$ and $\boldsymbol{\varphi}'$ which lead to the same explicitly given matrix S in (5.13) are called equivalent and this is denoted by $\boldsymbol{\varphi} \sim \boldsymbol{\varphi}'$. We show that the transformations of the parameter $\boldsymbol{\varphi}$ according to

$$\boldsymbol{\varphi} \to \boldsymbol{\varphi}' = \boldsymbol{\varphi} + n4\pi\frac{\boldsymbol{\varphi}}{|\boldsymbol{\varphi}|} = \left(|\boldsymbol{\varphi}| + n4\pi\right)\frac{\boldsymbol{\varphi}}{|\boldsymbol{\varphi}|} \sim \boldsymbol{\varphi}, \quad (n = 0, \pm 1, \pm 2, \cdots), \tag{6.1}$$

leave the operators S in (5.13) unchanged. From (6.1) follows for modulus and direction of $\boldsymbol{\varphi}'$

$$|\boldsymbol{\varphi}'| = \left||\boldsymbol{\varphi}| + n4\pi\right| = \pm\left(|\boldsymbol{\varphi}| + n4\pi\right), \quad \Leftrightarrow \quad \frac{\boldsymbol{\varphi}'}{|\boldsymbol{\varphi}'|} = \frac{|\boldsymbol{\varphi}| + n4\pi}{\left||\boldsymbol{\varphi}| + n4\pi\right|}\frac{\boldsymbol{\varphi}}{|\boldsymbol{\varphi}|} = \pm\frac{\boldsymbol{\varphi}}{|\boldsymbol{\varphi}|}, \tag{6.2}$$

with correlated signs. In connection with $\cos\left(\frac{|\boldsymbol{\varphi}'|}{2}\right) = \cos\left(\frac{|\boldsymbol{\varphi}|}{2}\right)$ and $\sin\left(\frac{|\boldsymbol{\varphi}'|}{2}\right) = \pm\sin\left(\frac{|\boldsymbol{\varphi}|}{2}\right)$ we see that the transformation (6.1) preserves the matrices (5.13). The minimal difference of two different equivalent points is obtained if we set $n = \pm 1$ in (6.1) and it is

$$\boldsymbol{\varphi}'-\boldsymbol{\varphi}=\pm 4\pi\frac{\boldsymbol{\varphi}}{|\boldsymbol{\varphi}|},\quad |\boldsymbol{\varphi}'-\boldsymbol{\varphi}|=4\pi,\quad \Rightarrow\quad \mathsf{S}'=\mathsf{S}. \tag{6.3}$$

Such equivalent points possess opposite directions considered from the center. Therefore, one may choose as basic range of inequivalent parameters $\boldsymbol{\varphi}$ a three-dimensional ball[2] of radius 2π

$$0\le|\boldsymbol{\varphi}|\le 2\pi. \tag{6.4}$$

Inner points of this three-dimensional ball possess equivalent points only outside it. The whole surface (sphere $\mathbb{S}^2$) of this three-dimensional ball $\boldsymbol{\varphi}'=2\pi\frac{\boldsymbol{\varphi}}{|\boldsymbol{\varphi}|}$, independently on the direction of $\frac{\boldsymbol{\varphi}}{|\boldsymbol{\varphi}|}$, corresponds to the negatively taken identity matrix $\mathsf{S}=-\mathsf{I}$ that means topologically to one point of the group manifold.

We consider now the transformation

$$\boldsymbol{\varphi}\to\boldsymbol{\varphi}'=\boldsymbol{\varphi}+(2n+1)2\pi\frac{\boldsymbol{\varphi}}{|\boldsymbol{\varphi}|}\sim\boldsymbol{\varphi}+2\pi\frac{\boldsymbol{\varphi}}{|\boldsymbol{\varphi}|}=\left(|\boldsymbol{\varphi}|+2\pi\right)\frac{\boldsymbol{\varphi}}{|\boldsymbol{\varphi}|}, \tag{6.5}$$

It is only necessary to investigate the case $n=0$ since the remaining part is identical with the already considered transformation (6.1). From

$$|\boldsymbol{\varphi}'|=|\boldsymbol{\varphi}|+2\pi,\quad \frac{\boldsymbol{\varphi}'}{|\boldsymbol{\varphi}'|}=\frac{\boldsymbol{\varphi}}{|\boldsymbol{\varphi}|}, \tag{6.6}$$

follows

$$\cos\left(\frac{|\boldsymbol{\varphi}'|}{2}\right)=-\cos\left(\frac{|\boldsymbol{\varphi}|}{2}\right),\quad \sin\left(\frac{|\boldsymbol{\varphi}'|}{2}\right)=-\sin\left(\frac{|\boldsymbol{\varphi}|}{2}\right),\quad \Rightarrow\quad \mathsf{S}'=-\mathsf{S}. \tag{6.7}$$

Thus the transformation (6.5) is the transition from the matrices (5.13) to their negative matrices.

The inversion of the matrices is made by the transformation

$$\boldsymbol{\varphi}\to\boldsymbol{\varphi}'=-\boldsymbol{\varphi},\quad \Rightarrow\quad \mathsf{S}'=\mathsf{S}^{-1}=\mathsf{S}^{\dagger}. \tag{6.8}$$

In applications the vector parameter $\boldsymbol{\varphi}$ does not possess the same weight (or Haar measure) independently on the modulus $|\boldsymbol{\varphi}|$ and therefore not the same topology as a usual three-dimensional sphere with equal weight for all $\boldsymbol{\varphi}$ and with volume $V=\frac{4\pi}{3}R^3$ for radius R. As a second possible fundamental range of inequivalent parameters $|\boldsymbol{\varphi}|$ one may also choose

$$2\pi\le|\boldsymbol{\varphi}|<4\pi. \tag{6.9}$$

Therefore, the invariant measure for the described basic ranges of parameters $|\boldsymbol{\varphi}|$ should be vanishing for all $|\boldsymbol{\varphi}|=n2\pi,(n=0,1,2,\cdots)$. We come back to this important problem in Section 16 when we discuss the derivation of an invariant

[2]We distinguish between a three-dimensional ball $\mathbb{B}$ with its two-dimensional surface $\mathbb{S}^2$ called sphere. In two-dimensional case the analogous notions are a (round) disc $\mathbb{D}$ with its one-dimensional border, the circle $\mathbb{S}^1$.

measure over the group $SU(2)$. This invariant measure has to become vanishing for $|\boldsymbol{\varphi}|=2\pi$ and $|\boldsymbol{\varphi}|=4\pi$ such as for the center $|\boldsymbol{\varphi}|=0$.

7. Inversion of the Mapping $\boldsymbol{\varphi} \rightleftharpoons \mathbf{S}(\boldsymbol{\varphi})$

From the matrix S one may determine the parameter $\boldsymbol{\varphi}=(\varphi_1,\varphi_2,\varphi_3)$ up to the equivalence $\boldsymbol{\varphi}\sim\boldsymbol{\varphi}'$ given in (6.1). First from (5.15) follows

$$\frac{|\boldsymbol{\varphi}|}{2}=\arccos\left(\frac{\langle\mathsf{S}\rangle}{2}\right)=\arccos\left(\frac{S_{11}+S_{22}}{2}\right)=\pm\arcsin\left(\sqrt{1-\left(\frac{S_{11}+S_{22}}{2}\right)^2}\right),$$
$$\Rightarrow\frac{\frac{\boldsymbol{\varphi}}{2}}{\sin\left(\frac{\boldsymbol{\varphi}}{2}\right)}=\frac{\arcsin\left(\sqrt{1-\left(\frac{S_{11}+S_{22}}{2}\right)^2}\right)}{\sqrt{1-\left(\frac{S_{11}+S_{22}}{2}\right)^2}}. \tag{7.1}$$

Therefore

$$\varphi_1=-\mathrm{i}\frac{\arcsin\left(\sqrt{1-\left(\frac{S_{11}+S_{22}}{2}\right)^2}\right)}{\sqrt{1-\left(\frac{S_{11}+S_{22}}{2}\right)^2}}\left(S_{12}+S_{21}\right),$$
$$\varphi_2=\frac{\arcsin\left(\sqrt{1-\left(\frac{S_{11}+S_{22}}{2}\right)^2}\right)}{\sqrt{1-\left(\frac{S_{11}+S_{22}}{2}\right)^2}}\left(S_{12}-S_{21}\right), \tag{7.2}$$
$$\varphi_3=-\mathrm{i}\frac{\arcsin\left(\sqrt{1-\left(\frac{S_{11}+S_{22}}{2}\right)^2}\right)}{\sqrt{1-\left(\frac{S_{11}+S_{22}}{2}\right)^2}}\left(S_{11}-S_{22}\right),$$

or for $\varphi_-\equiv\varphi_1-\mathrm{i}\varphi_2$ and $\varphi_+\equiv\varphi_1+\mathrm{i}\varphi_2$ instead of φ_1 and φ_2

$$\varphi_-=-\mathrm{i}2\frac{\arcsin\left(\sqrt{1-\left(\frac{S_{11}+S_{22}}{2}\right)^2}\right)}{\sqrt{1-\left(\frac{S_{11}+S_{22}}{2}\right)^2}}S_{12},$$
$$\varphi_+=-\mathrm{i}2\frac{\arcsin\left(\sqrt{1-\left(\frac{S_{11}+S_{22}}{2}\right)^2}\right)}{\sqrt{1-\left(\frac{S_{11}+S_{22}}{2}\right)^2}}S_{21}. \tag{7.3}$$

Thus from the matrix (5.13) the vector parameter $\boldsymbol{\varphi}$ can be determined with the indeterminacy of $\boldsymbol{\varphi}$ described by (6.1).

8. Eigenvalues and Eigenvectors of $SU(2)$ in the Two-Dimensional Fundamental Representation with Vector Parameter $\boldsymbol{\varphi}$

In this Section we determine the right-hand and left-hand eigenvectors (spinors) $\boldsymbol{x}$ and $\tilde{\boldsymbol{x}}$ of the matrix S in (5.13) to the well-known eigenvalues $\lambda = \exp\left(\pm \mathrm{i}\frac{|\boldsymbol{\varphi}|}{2}\right)$ according to

$$\mathsf{S}\boldsymbol{x}_1 = \lambda_1 \boldsymbol{x}_1, \quad \tilde{\boldsymbol{x}}_1 \mathsf{S} = \lambda_1 \tilde{\boldsymbol{x}}_1,$$
$$\mathsf{S}\boldsymbol{x}_2 = \lambda_2 \boldsymbol{x}_2, \quad \tilde{\boldsymbol{x}}_2 \mathsf{S} = \lambda_2 \tilde{\boldsymbol{x}}_2, \tag{8.1}$$

We consider $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ as column vectors and $\tilde{\boldsymbol{x}}_1$ and $\tilde{\boldsymbol{x}}_2$ as row vectors. From scalar multiplication of the first equation with $\tilde{\boldsymbol{x}}_2$ and of the third equation with $\boldsymbol{x}_1$ and forming the difference of the obtained equations follows for $\lambda_1 \neq \lambda_2$ the orthogonality of $\boldsymbol{x}_1$ and $\tilde{\boldsymbol{x}}_2$ and similarly of $\boldsymbol{x}_2$ and $\tilde{\boldsymbol{x}}_1$. This leads to the following representation of S

$$\mathsf{S} = \lambda_1 \frac{\boldsymbol{x}_1 \cdot \tilde{\boldsymbol{x}}_1}{\tilde{\boldsymbol{x}}_1 \boldsymbol{x}_1} + \lambda_2 \frac{\boldsymbol{x}_2 \cdot \tilde{\boldsymbol{x}}_2}{\tilde{\boldsymbol{x}}_2 \boldsymbol{x}_2} \equiv \lambda_1 \Pi_1 + \lambda_2 \Pi_2, \quad \tilde{\boldsymbol{x}}_2 \boldsymbol{x}_1 = \tilde{\boldsymbol{x}}_1 \boldsymbol{x}_2 = 0. \tag{8.2}$$

The operators Π_1 and Π_2 are one-dimensional projection operators with the properties

$$\Pi_1^2 = \Pi_2^2 = \mathrm{I}, \quad \Pi_1 \Pi_2 = \Pi_2 \Pi_1 = 0, \quad \langle \Pi_1 \rangle = \langle \Pi_2 \rangle = 1. \tag{8.3}$$

Since S are unitary unimodular matrices their eigenvalues λ are complex numbers on the unit circle in the complex plane. The unimodularity of the matrix S makes the product of its eigenvalues equal to 1 and thus complex conjugate in two-dimensional case. Concretely, one finds from the eigenvalue equation

$$\lambda^2 - \langle \mathsf{S} \rangle \lambda + [\mathsf{S}] = \lambda^2 - 2\cos\left(\frac{|\boldsymbol{\varphi}|}{2}\right)\lambda + 1 = 0, \tag{8.4}$$

the following well-known solutions for the eigenvalues

$$\lambda_1 = \exp\left(+\mathrm{i}\frac{|\boldsymbol{\varphi}|}{2}\right), \quad \lambda_2 = \exp\left(-\mathrm{i}\frac{|\boldsymbol{\varphi}|}{2}\right), \tag{8.5}$$

as already mentioned. Inserting these eigenvalues into (8.1) in the concrete representation (5.13) one obtains equations with the following solutions for the two-dimensional eigenvectors in a non-normalized form

$$\boldsymbol{x}_1 = \begin{pmatrix} \varphi_- \\ |\boldsymbol{\varphi}| - \varphi_3 \end{pmatrix}, \quad \tilde{\boldsymbol{x}}_1 = \left(\varphi_+, |\boldsymbol{\varphi}| - \varphi_3\right), \quad \tilde{\boldsymbol{x}}_1 \boldsymbol{x}_1 = 2|\boldsymbol{\varphi}|\left(|\boldsymbol{\varphi}| - \varphi_3\right), \quad \tilde{\boldsymbol{x}}_1 \boldsymbol{x}_2 = 0,$$
$$\boldsymbol{x}_2 = \begin{pmatrix} -\varphi_- \\ |\boldsymbol{\varphi}| + \varphi_3 \end{pmatrix}, \quad \tilde{\boldsymbol{x}}_2 = \left(-\varphi_+, |\boldsymbol{\varphi}| + \varphi_3\right), \quad \tilde{\boldsymbol{x}}_2 \boldsymbol{x}_2 = 2|\boldsymbol{\varphi}|\left(|\boldsymbol{\varphi}| + \varphi_3\right), \quad \tilde{\boldsymbol{x}}_2 \boldsymbol{x}_1 = 0. \tag{8.6}$$

In general, they are complex vectors. There are possibilities to represent these eigenvector in another way and to choose other proportionality factors. As a more symmetrical and normalized form of the eigenvectors (8.6) one may choose

$$\boldsymbol{u}_1 \equiv \frac{\boldsymbol{x}_1}{\sqrt{\tilde{\boldsymbol{x}}_1\boldsymbol{x}_1}}, \quad \boldsymbol{u}_2 \equiv \frac{\boldsymbol{x}_2}{\sqrt{\tilde{\boldsymbol{x}}_2\boldsymbol{x}_2}}, \quad \tilde{\boldsymbol{u}}_1 \equiv \frac{\tilde{\boldsymbol{x}}_1}{\sqrt{\tilde{\boldsymbol{x}}_1\boldsymbol{x}_1}}, \quad \tilde{\boldsymbol{u}}_2 \equiv \frac{\tilde{\boldsymbol{x}}_2}{\sqrt{\tilde{\boldsymbol{x}}_2\boldsymbol{x}_2}}, \tag{8.7}$$

which leads to

$$\begin{aligned}
\boldsymbol{u}_1 &= \begin{pmatrix} \sqrt{\frac{1}{2}\sqrt{\frac{\varphi_-}{\varphi_+}}\left(1+\frac{\varphi_3}{|\boldsymbol{\varphi}|}\right)} \\ \sqrt{\frac{1}{2}\sqrt{\frac{\varphi_+}{\varphi_-}}\left(1-\frac{\varphi_3}{|\boldsymbol{\varphi}|}\right)} \end{pmatrix}, \quad \tilde{\boldsymbol{u}}_1 = \left(\sqrt{\frac{1}{2}\sqrt{\frac{\varphi_+}{\varphi_-}}\left(1+\frac{\varphi_3}{|\boldsymbol{\varphi}|}\right)}, \sqrt{\frac{1}{2}\sqrt{\frac{\varphi_-}{\varphi_+}}\left(1-\frac{\varphi_3}{|\boldsymbol{\varphi}|}\right)}\right), \\
\boldsymbol{u}_2 &= \begin{pmatrix} \sqrt{\frac{1}{2}\sqrt{\frac{\varphi_-}{\varphi_+}}\left(1-\frac{\varphi_3}{|\boldsymbol{\varphi}|}\right)} \\ -\sqrt{\frac{1}{2}\sqrt{\frac{\varphi_+}{\varphi_-}}\left(1+\frac{\varphi_3}{|\boldsymbol{\varphi}|}\right)} \end{pmatrix}, \quad \tilde{\boldsymbol{u}}_2 = \left(\sqrt{\frac{1}{2}\sqrt{\frac{\varphi_+}{\varphi_-}}\left(1-\frac{\varphi_3}{|\boldsymbol{\varphi}|}\right)}, -\sqrt{\frac{1}{2}\sqrt{\frac{\varphi_+}{\varphi_-}}\left(1+\frac{\varphi_3}{|\boldsymbol{\varphi}|}\right)}\right),
\end{aligned} \tag{8.8}$$

with the orthonormality relations

$$\tilde{\boldsymbol{u}}_1\boldsymbol{u}_1 = 1, \quad \tilde{\boldsymbol{u}}_2\boldsymbol{u}_2 = 1, \quad \tilde{\boldsymbol{u}}_1\boldsymbol{u}_2 = 0, \quad \tilde{\boldsymbol{u}}_2\boldsymbol{u}_1 = 0. \tag{8.9}$$

In degenerate case $|\boldsymbol{\varphi}| = 0$ one finds $\lambda_1 = \lambda_2 = 1$ and thus $S = I$ and in degenerate case $|\varphi| = 2\pi$ one has $\lambda_1 = \lambda_2 = -1$ corresponding to $\mathsf{S} = -\mathsf{I}$ that means only to one group transformation.

9. Composition Law for Vector Parameters $\boldsymbol{\varphi}$ Corresponding to Products of $SU(2)$ Transformations

The group $SU(2)$ is described by the vector parameter $\boldsymbol{\varphi}$, for example, in the fundamental representation by the matrix S given in (5.13) with respect to a basis discussed in Section 5. We now consider the composition of two such matrices S_1 and S_2 with the vector parameters $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$ to the product matrix $\mathsf{S} = \mathsf{S}_1\mathsf{S}_2$ and ask how the vector parameter $\boldsymbol{\varphi}$ to the matrix S is connected with the vector parameters $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$ according to the correspondences

$$\mathsf{S} = \mathsf{S}_1\mathsf{S}_2 \leftrightarrow \boldsymbol{\varphi}, \quad \mathsf{S}_1 \leftrightarrow \boldsymbol{\varphi}_1, \quad \mathsf{S}_2 \leftrightarrow \boldsymbol{\varphi}_2. \tag{9.1}$$

From the multiplication of two such matrices of the form (5.13) and reorganization of the obtained terms one easily finds the following scalar and vector equation ($|\boldsymbol{\varphi}_1| < 4\pi, |\boldsymbol{\varphi}_2| < 4\pi, |\boldsymbol{\varphi}| < 4\pi$)

$$\begin{aligned}
\cos\left(\frac{|\boldsymbol{\varphi}|}{2}\right) &= \cos\left(\frac{|\boldsymbol{\varphi}_1|}{2}\right)\cos\left(\frac{|\boldsymbol{\varphi}_2|}{2}\right) - \frac{\boldsymbol{\varphi}_1\boldsymbol{\varphi}_2}{|\boldsymbol{\varphi}_1||\boldsymbol{\varphi}_2|}\sin\left(\frac{|\boldsymbol{\varphi}_1|}{2}\right)\sin\left(\frac{|\boldsymbol{\varphi}_2|}{2}\right), \\
\frac{\boldsymbol{\varphi}}{|\boldsymbol{\varphi}|}\sin\left(\frac{|\boldsymbol{\varphi}|}{2}\right) &= \frac{\boldsymbol{\varphi}_1}{|\boldsymbol{\varphi}_1|}\sin\left(\frac{|\boldsymbol{\varphi}_1|}{2}\right)\cos\left(\frac{|\boldsymbol{\varphi}_2|}{2}\right) + \frac{\boldsymbol{\varphi}_2}{|\boldsymbol{\varphi}_2|}\cos\left(\frac{|\boldsymbol{\varphi}_1|}{2}\right)\sin\left(\frac{|\boldsymbol{\varphi}_2|}{2}\right) \\
&\quad - \frac{[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2]}{|\boldsymbol{\varphi}_1||\boldsymbol{\varphi}_2|}\sin\left(\frac{|\boldsymbol{\varphi}_1|}{2}\right)\sin\left(\frac{|\boldsymbol{\varphi}_2|}{2}\right),
\end{aligned} \tag{9.2}$$

where $[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2]$ denotes the vector product of the vectors $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$. These two equations can be resolved with respect to the vector $\boldsymbol{\varphi}$ in unique way that provides the following formula (notation $\langle , \rangle$ see below)

$$\frac{\boldsymbol{\varphi}}{2} = \frac{\frac{\boldsymbol{\varphi}_1}{|\boldsymbol{\varphi}_1|}\sin\left(\frac{|\boldsymbol{\varphi}_1|}{2}\right)\cos\left(\frac{|\boldsymbol{\varphi}_2|}{2}\right) + \frac{\boldsymbol{\varphi}_2}{|\boldsymbol{\varphi}_2|}\cos\left(\frac{|\boldsymbol{\varphi}_1|}{2}\right)\sin\left(\frac{|\boldsymbol{\varphi}_2|}{2}\right) - \frac{[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2]}{|\boldsymbol{\varphi}_1||\boldsymbol{\varphi}_2|}\sin\left(\frac{|\boldsymbol{\varphi}_1|}{2}\right)\sin\left(\frac{|\boldsymbol{\varphi}_2|}{2}\right)}{\left|\frac{\boldsymbol{\varphi}_1}{|\boldsymbol{\varphi}_1|}\sin\left(\frac{|\boldsymbol{\varphi}_1|}{2}\right)\cos\left(\frac{|\boldsymbol{\varphi}_2|}{2}\right) + \frac{\boldsymbol{\varphi}_2}{|\boldsymbol{\varphi}_2|}\cos\left(\frac{|\boldsymbol{\varphi}_1|}{2}\right)\sin\left(\frac{|\boldsymbol{\varphi}_2|}{2}\right) - \frac{[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2]}{|\boldsymbol{\varphi}_1||\boldsymbol{\varphi}_2|}\sin\left(\frac{|\boldsymbol{\varphi}_1|}{2}\right)\sin\left(\frac{|\boldsymbol{\varphi}_2|}{2}\right)\right|} \quad (9.3)$$

$$\cdot \arccos\left(\cos\left(\frac{|\boldsymbol{\varphi}_1|}{2}\right)\cos\left(\frac{|\boldsymbol{\varphi}_2|}{2}\right) - \frac{\boldsymbol{\varphi}_1\boldsymbol{\varphi}_2}{|\boldsymbol{\varphi}_1||\boldsymbol{\varphi}_2|}\sin\left(\frac{|\boldsymbol{\varphi}_1|}{2}\right)\sin\left(\frac{|\boldsymbol{\varphi}_2|}{2}\right)\right) \equiv \left\langle \frac{\boldsymbol{\varphi}_1}{2}, \frac{\boldsymbol{\varphi}_2}{2} \right\rangle.$$

One may call this formula the composition law for the chosen vector parameter of the group $SU(2)$. The factor on first line of the right-hand gives the direction of the new vector $\boldsymbol{\varphi}$ which as we have seen in Section 4 possesses another meaning for $SU(2)$ in comparison to the rotation group and the factor on the second line the modulus of $\frac{\boldsymbol{\varphi}}{2}$. For the new parameter $\frac{\boldsymbol{\varphi}}{|\boldsymbol{\varphi}|}$ we find

$$\frac{\boldsymbol{\varphi}}{|\boldsymbol{\varphi}|} = \frac{\frac{\boldsymbol{\varphi}_1}{|\boldsymbol{\varphi}_1|}\sin\left(\frac{|\boldsymbol{\varphi}_1|}{2}\right)\cos\left(\frac{|\boldsymbol{\varphi}_2|}{2}\right) + \frac{\boldsymbol{\varphi}_2}{|\boldsymbol{\varphi}_2|}\cos\left(\frac{|\boldsymbol{\varphi}_1|}{2}\right)\sin\left(\frac{|\boldsymbol{\varphi}_2|}{2}\right) - \frac{[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2]}{|\boldsymbol{\varphi}_1||\boldsymbol{\varphi}_2|}\sin\left(\frac{|\boldsymbol{\varphi}_1|}{2}\right)\sin\left(\frac{|\boldsymbol{\varphi}_2|}{2}\right)}{\sqrt{1 - \left(\cos\left(\frac{|\boldsymbol{\varphi}_1|}{2}\right)\cos\left(\frac{|\boldsymbol{\varphi}_2|}{2}\right) - \frac{\boldsymbol{\varphi}_1\boldsymbol{\varphi}_2}{|\boldsymbol{\varphi}_1||\boldsymbol{\varphi}_2|}\sin\left(\frac{|\boldsymbol{\varphi}_1|}{2}\right)\sin\left(\frac{|\boldsymbol{\varphi}_2|}{2}\right)\right)^2}}. \quad (9.4)$$

The above composition formulae are somehow similar (but not identical) to formulae for spherical trigonometry of the surface (sphere) of a three-dimensional ball but here play a role also the inner points. In addition, in applications our three-dimensional ball possesses different weights of its points in dependence on the modulus $|\boldsymbol{\varphi}|$ and thus another topology as a usual three-dimensional ball with equal weight measure for all its inner points. For example, as discussed the whole surface (sphere) of our three-dimensional ball with $|\boldsymbol{\varphi}|$ corresponds to the operator $-I$ that means to only one point and therefore the (Haar) measure has to vanish for $|\boldsymbol{\varphi}|$. In Section 14-15, we consider another parametrization where the composition law for $SO(3,\mathbb{R})$ takes on a simpler form.

We mention that Fyodorov [5] (§. 29, p. 447) and [6] (§. 3, p. 18) introduced a special notation $\langle , \rangle$ for the composition of parameters corresponding to the product of two group elements, in our case of $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$ to $\boldsymbol{\varphi}$ according to

$$\boldsymbol{\varphi} = \langle \boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2 \rangle, \quad \langle \boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2 \rangle \neq \langle \boldsymbol{\varphi}_2, \boldsymbol{\varphi}_1 \rangle, \quad (\text{in general}). \quad (9.5)$$

For the composition of three parameters holds an associative law

$$\langle \langle \boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2 \rangle, \boldsymbol{\varphi}_3 \rangle = \langle \boldsymbol{\varphi}_1, \langle \boldsymbol{\varphi}_2, \boldsymbol{\varphi}_3 \rangle \rangle \equiv \langle \boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \boldsymbol{\varphi}_3 \rangle, \quad (9.6)$$

corresponding to the associative law for the multiplication of group elements. Therefore, one may omit the inner brackets. The introduced symbols are very convenient and hardly come into conflict with other generally used symbols (only Hermitean scalar products are sometimes denoted by such brackets).

10. Decompositions of $S(\varphi)$ Matrices or Disentanglement Relations for the Group $SU(2)$

Beside the composition it is sometimes useful to decompose the general matrix $\mathsf{S}(\boldsymbol{\varphi})$ in (5.13) into products of simpler matrices which is called disentanglement. The obtained decompositions are then true for arbitrary irreducible representations. With the two-dimensional fundamental representation of $SU(2)$ in representation by the vector parameter $\boldsymbol{\varphi}$ in Section 5 we have developed at once the mathematical means for the disentanglement of $SU(2)$ group operators that we present here. The method is the same as used, for example, in [27] [28] for $SU(1,1)$. We have to decompose the general matrices S of the fundamental representation derived in (5.13) into products of simpler matrices and have to look for the corresponding decompositions of the general group operators $\exp(\mathrm{i}\boldsymbol{J\varphi})$ into products of special group operators. Another method is the derivation of differential equations for the exponents in the decomposition formulae by introduction of an additional parameter and differentiation with respect to this parameter and then to solve the obtained differential equations that is made in a paper of Ban [29] based on Lie algebra methods (see also, e.g., [30] [31]).

The following more special matrices of (5.13) are mainly of interest

$$\begin{aligned}
&\exp\left(\frac{\mathrm{i}}{2}J_{+}\varphi_{-}\right)\rightarrow\mathsf{S}(\varphi_{-},0,0)=\begin{pmatrix}1 & \mathrm{i}\dfrac{\varphi_{-}}{2}\\ 0 & 1\end{pmatrix},\\
&\exp\left(\frac{\mathrm{i}}{2}J_{-}\varphi_{+}\right)\rightarrow\mathsf{S}(0,\varphi_{+},0)=\begin{pmatrix}1 & 0\\ \mathrm{i}\dfrac{\varphi_{+}}{2} & 1\end{pmatrix},\\
&\exp(\mathrm{i}J_{3}\varphi_{3})\rightarrow\mathsf{S}(0,0,\varphi_{3})=\begin{pmatrix}\exp\left(\mathrm{i}\dfrac{\varphi_{3}}{2}\right) & 0\\ 0 & \exp\left(-\mathrm{i}\dfrac{\varphi_{3}}{2}\right)\end{pmatrix}.
\end{aligned}\tag{10.1}$$

The first two are oppositely triangular matrices and the third is a diagonal matrix.

If we take into account the most interesting decompositions of the general operator of $SU(2)$ into products of special operators with J_{+}, J_{-} and J_{3} separately in the arguments of the exponentials, we can make the following 6 product decompositions of the 2D matrix S using its unimodularity $S_{11}S_{22}-S_{12}S_{21}=1$

$$\begin{aligned}
\begin{pmatrix}S_{11}, & S_{12}\\ S_{21}, & S_{22}\end{pmatrix}&=\begin{pmatrix}1 & 0\\ \dfrac{S_{21}}{S_{11}} & 1\end{pmatrix}\begin{pmatrix}1 & S_{12}S_{11}\\ 0 & 1\end{pmatrix}\begin{pmatrix}S_{11} & 0\\ 0 & \dfrac{1}{S_{11}}\end{pmatrix}=\begin{pmatrix}1 & 0\\ \dfrac{S_{21}}{S_{11}} & 1\end{pmatrix}\begin{pmatrix}S_{11} & 0\\ 0 & \dfrac{1}{S_{11}}\end{pmatrix}\begin{pmatrix}1 & \dfrac{S_{12}}{S_{11}}\\ 0 & 1\end{pmatrix}\\
&=\begin{pmatrix}S_{11} & 0\\ 0 & \dfrac{1}{S_{11}}\end{pmatrix}\begin{pmatrix}1 & 0\\ S_{21}S_{11} & 1\end{pmatrix}\begin{pmatrix}1 & \dfrac{S_{12}}{S_{11}}\\ 0 & 1\end{pmatrix}=\begin{pmatrix}\dfrac{1}{S_{22}} & 0\\ 0 & S_{22}\end{pmatrix}\begin{pmatrix}1 & S_{12}S_{22}\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \dfrac{S_{21}}{S_{22}} & 1\end{pmatrix}
\end{aligned}$$

$$= \begin{pmatrix} 1 & \frac{S_{12}}{S_{22}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{S_{22}} & 0 \\ 0 & S_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{S_{21}}{S_{22}} & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{S_{12}}{S_{22}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ S_{21}S_{22} & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{S_{22}} & 0 \\ 0 & S_{22} \end{pmatrix}. \quad (10.2)$$

Evidently, these decompositions are not specific for $SU(2)$ and are applicable for all two-dimensional unimodular matrices of the Special linear group $SL(2,\mathbb{C})$, for example, also similar for the two-dimensional non-unitary fundamental representation of $SU(1,1)$ and decompositions of matrices including triangular matrices were already known to Gauss [32]. The obtained disentanglement relations in the corresponding 6 considered orderings of group operators of $SU(2)$ are

$$\begin{aligned}
&\exp\left\{\mathrm{i}\left(J_+\frac{\varphi_-}{2}+J_-\frac{\varphi_+}{2}+J_3\varphi_3\right)\right\}\\
&=\exp\left(J_-\frac{S_{21}}{S_{11}}\right)\exp\left(J_+S_{12}S_{11}\right)\exp\left(2J_3\log\left(S_{11}\right)\right)\\
&=\exp\left(J_-\frac{S_{21}}{S_{11}}\right)\exp\left(2J_3\log\left(S_{11}\right)\right)\exp\left(J_+\frac{S_{12}}{S_{11}}\right)\\
&=\exp\left(2J_3\log\left(S_{11}\right)\right)\exp\left(J_-S_{21}S_{11}\right)\exp\left(J_+\frac{S_{12}}{S_{11}}\right)\\
&=\exp\left(-2J_3\log\left(S_{22}\right)\right)\exp\left(J_+S_{12}S_{22}\right)\exp\left(J_-\frac{S_{21}}{S_{22}}\right)\\
&=\exp\left(J_+\frac{S_{12}}{S_{22}}\right)\exp\left(-2J_3\log\left(S_{22}\right)\right)\exp\left(J_-\frac{S_{21}}{S_{22}}\right) \qquad (10.3)\\
&=\exp\left(J_+\frac{S_{12}}{S_{22}}\right)\exp\left(J_-S_{21}S_{22}\right)\exp\left(-2J_3\log\left(S_{22}\right)\right),
\end{aligned}$$

with the explicit form of the elements of the two-dimensional matrix S (see (5.13))

$$\begin{aligned}
&S_{11}=\cos\left(\frac{|\boldsymbol{\varphi}|}{2}\right)+\mathrm{i}\frac{\varphi_3}{|\boldsymbol{\varphi}|}\sin\left(\frac{|\boldsymbol{\varphi}|}{2}\right),\quad S_{12}=\mathrm{i}\frac{\varphi_-}{|\boldsymbol{\varphi}|}\sin\left(\frac{|\boldsymbol{\varphi}|}{2}\right),\\
&S_{21}=\mathrm{i}\frac{\varphi_+}{|\boldsymbol{\varphi}|}\sin\left(\frac{|\boldsymbol{\varphi}|}{2}\right),\quad S_{22}=\cos\left(\frac{|\boldsymbol{\varphi}|}{2}\right)-\mathrm{i}\frac{\varphi_3}{|\boldsymbol{\varphi}|}\sin\left(\frac{|\boldsymbol{\varphi}|}{2}\right), \qquad (10.4)\\
&\varphi_\pm\equiv\varphi_1\pm\mathrm{i}\varphi_2,\quad |\boldsymbol{\varphi}|\equiv\sqrt{\varphi_+\varphi_-+(\varphi_3)^2}.
\end{aligned}$$

In the special case $\varphi_3 = 0$ corresponding to

$$\exp\left(\frac{\mathrm{i}}{2}\left(J_+\varphi_-+J_-\varphi_+\right)\right)\to S(\varphi_-,\varphi_+,0)=\begin{pmatrix}\cos\left(\frac{|\boldsymbol{\varphi}|}{2}\right) & \mathrm{i}\frac{\varphi_-}{|\boldsymbol{\varphi}|}\sin\left(\frac{|\boldsymbol{\varphi}|}{2}\right)\\ \mathrm{i}\frac{\varphi_+}{|\boldsymbol{\varphi}|}\sin\left(\frac{|\boldsymbol{\varphi}|}{2}\right) & \cos\left(\frac{|\boldsymbol{\varphi}|}{2}\right)\end{pmatrix}, \qquad (10.5)$$

$$|\boldsymbol{\varphi}|\equiv\sqrt{\varphi_+\varphi_-}=\sqrt{\varphi_1^2+\varphi_2^2},$$

we obtain from (10.3) and (10.4) the disentanglement relations

$$
\begin{aligned}
&\exp\left\{\frac{\mathrm{i}}{2}\left(J_+\varphi_- + J_-\varphi_+\right)\right\} \\
&= \exp\left(\mathrm{i}J_-\frac{\varphi_+}{|\varphi|}\mathrm{tg}\left(\frac{|\varphi|}{2}\right)\right)\exp\left(\mathrm{i}J_+\frac{\varphi_-\sin\left(|\varphi|\right)}{2|\varphi|}\right)\left(\cos\left(\frac{|\varphi|}{2}\right)\right)^{2J_3} \\
&= \exp\left(\mathrm{i}J_-\frac{\varphi_+}{|\varphi|}\mathrm{tg}\left(\frac{|\varphi|}{2}\right)\right)\left(\cos\left(\frac{|\varphi|}{2}\right)\right)^{2J_3}\exp\left(\mathrm{i}J_+\frac{\varphi_-}{|\varphi|}\mathrm{tg}\left(\frac{|\varphi|}{2}\right)\right) \\
&= \left(\cos\left(\frac{|\varphi|}{2}\right)\right)^{2J_3}\exp\left(\mathrm{i}J_-\frac{\varphi_+\sin\left(|\varphi|\right)}{2|\varphi|}\right)\exp\left(\mathrm{i}J_+\frac{\varphi_-}{|\varphi|}\mathrm{tg}\left(\frac{|\varphi|}{2}\right)\right) \\
&= \left(\cos\left(\frac{|\varphi|}{2}\right)\right)^{-2J_3}\exp\left(\mathrm{i}J_+\frac{\varphi_-\sin\left(|\varphi|\right)}{2|\varphi|}\right)\exp\left(\mathrm{i}J_-\frac{\varphi_+}{|\varphi|}\mathrm{tg}\left(\frac{|\varphi|}{2}\right)\right) \\
&= \exp\left(\mathrm{i}J_+\frac{\varphi_-}{|\varphi|}\mathrm{tg}\left(\frac{|\varphi|}{2}\right)\right)\left(\cos\left(\frac{|\varphi|}{2}\right)\right)^{-2J_3}\exp\left(\mathrm{i}J_-\frac{\varphi_+}{|\varphi|}\mathrm{tg}\left(\frac{|\varphi|}{2}\right)\right) \\
&= \exp\left(\mathrm{i}J_+\frac{\varphi_-}{|\varphi|}\mathrm{tg}\left(\frac{|\varphi|}{2}\right)\right)\exp\left(\mathrm{i}J_-\frac{\varphi_+\sin\left(|\varphi|\right)}{2|\varphi|}\right)\left(\cos\left(\frac{|\varphi|}{2}\right)\right)^{-2J_3}, \quad |\varphi| \equiv \sqrt{\varphi_+\varphi_-},
\end{aligned}
\tag{10.6}
$$

where on the right-hand side the operator J_3 appears although it is not on the left-hand side. The matrices $S(\varphi_-,\varphi_+,0)$ in (10.5) themselves do not form a group.

These formulae are important, for example, for the derivation of $SU(2)$ group-coherent states in the sense of Perelomov [32] and for their representation.

We consider now the following decompositions of the unimodular 2D matrix S into products of two unimodular matrices

$$
\begin{aligned}
\begin{pmatrix} S_{11}, & S_{12} \\ S_{21}, & S_{22} \end{pmatrix} &= \begin{pmatrix} \sqrt{S_{11}S_{22}} & \sqrt{\frac{S_{11}}{S_{22}}}S_{12} \\ \sqrt{\frac{S_{22}}{S_{11}}}S_{21} & \sqrt{S_{11}S_{22}} \end{pmatrix}\begin{pmatrix} \sqrt{\frac{S_{11}}{S_{22}}} & 0 \\ 0 & \sqrt{\frac{S_{22}}{S_{11}}} \end{pmatrix} \\
&= \begin{pmatrix} \sqrt{\frac{S_{11}}{S_{22}}} & 0 \\ 0 & \sqrt{\frac{S_{22}}{S_{11}}} \end{pmatrix}\begin{pmatrix} \sqrt{S_{11}S_{22}} & \sqrt{\frac{S_{22}}{S_{11}}}S_{12} \\ \sqrt{\frac{S_{11}}{S_{22}}}S_{21} & \sqrt{S_{11}S_{22}} \end{pmatrix}.
\end{aligned}
\tag{10.7}
$$

They correspond to the following disentanglement of group operators with J_3 and with $J_\pm$

$$
\begin{aligned}
\exp\left\{\mathrm{i}\left(J_+\frac{\varphi_-}{2} + J_-\frac{\varphi_+}{2} + J_3\varphi_3\right)\right\} &= \exp\left\{\mathrm{i}\left(J_+\frac{\varphi'_-}{2} + J_-\frac{\varphi'_+}{2}\right)\right\}\exp\left(\mathrm{i}J_3\varphi'_3\right) \\
&= \exp\left(\mathrm{i}J_3\varphi'_3\right)\exp\left\{\mathrm{i}\left(J_+\frac{\varphi''_-}{2} + J_-\frac{\varphi''_+}{2}\right)\right\},
\end{aligned}
\tag{10.8}
$$

with the relations

$$\frac{\varphi'_\pm}{2} = \exp\left(\mp\mathrm{i}\frac{\varphi'_3}{2}\right)\frac{\varphi_\pm}{\sqrt{\varphi_+\varphi_-}}\arcsin\left(\frac{\sqrt{\varphi_+\varphi_-}}{|\boldsymbol{\varphi}|}\sin\left(\frac{|\boldsymbol{\varphi}|}{2}\right)\right), \quad \exp\left(\pm\mathrm{i}\frac{\varphi'_3}{2}\right)\frac{\varphi'_\pm}{\sqrt{\varphi'_+\varphi'_-}} = \frac{\varphi_\pm}{\sqrt{\varphi_+\varphi_-}},$$
$$\frac{\varphi''_\pm}{2} = \exp\left(\pm\mathrm{i}\frac{\varphi'_3}{2}\right)\frac{\varphi_\pm}{\sqrt{\varphi_+\varphi_-}}\arcsin\left(\frac{\sqrt{\varphi_+\varphi_-}}{|\boldsymbol{\varphi}|}\sin\left(\frac{|\boldsymbol{\varphi}|}{2}\right)\right), \quad \exp\left(\mp\mathrm{i}\frac{\varphi'_3}{2}\right)\frac{\varphi''_\pm}{\sqrt{\varphi''_+\varphi''_-}} = \frac{\varphi_\pm}{\sqrt{\varphi_+\varphi_-}}, \quad (10.9)$$
$$\exp\left(\mathrm{i}\frac{\varphi'_3}{2}\right) = \sqrt{\frac{\cos\left(\frac{|\boldsymbol{\varphi}|}{2}\right)+\mathrm{i}\frac{\varphi_3}{|\boldsymbol{\varphi}|}\sin\left(\frac{|\boldsymbol{\varphi}|}{2}\right)}{\cos\left(\frac{|\boldsymbol{\varphi}|}{2}\right)-\mathrm{i}\frac{\varphi_3}{|\boldsymbol{\varphi}|}\sin\left(\frac{|\boldsymbol{\varphi}|}{2}\right)}}, \quad \mathrm{tg}\left(\frac{\varphi'_3}{2}\right) = \frac{\varphi_3}{|\boldsymbol{\varphi}|}\mathrm{tg}\left(\frac{|\boldsymbol{\varphi}|}{2}\right).$$

From these relations follows

$$\cos\left(\frac{\sqrt{\varphi'_+\varphi'_-}}{2}\right) = \cos\left(\frac{\sqrt{\varphi''_+\varphi''_-}}{2}\right) = \sqrt{\cos^2\left(\frac{|\boldsymbol{\varphi}|}{2}\right)+\frac{(\varphi_3)^2}{|\boldsymbol{\varphi}|^2}\sin^2\left(\frac{|\boldsymbol{\varphi}|}{2}\right)},$$
$$\sin\left(\frac{\sqrt{\varphi'_+\varphi'_-}}{2}\right) = \sin\left(\frac{\sqrt{\varphi''_+\varphi''_-}}{2}\right) = \frac{\sqrt{\varphi_+\varphi_-}}{|\boldsymbol{\varphi}|}\sin\left(\frac{|\boldsymbol{\varphi}|}{2}\right). \quad (10.10)$$

The stable part in the decompositions (10.8) is the factor $\exp(\mathrm{i}J_3\varphi'_3)$. The inversion of (10.9) and (10.10) can be found using

$$\cos\left(\frac{|\boldsymbol{\varphi}|}{2}\right) = \cos\left(\frac{\sqrt{\varphi'_+\varphi'_-}}{2}\right)\cos\left(\frac{\varphi'_3}{2}\right) = \cos\left(\frac{\sqrt{\varphi''_+\varphi''_-}}{2}\right)\cos\left(\frac{\varphi'_3}{2}\right), \quad (10.11)$$

which follows from combination of the relation for $\cos\left(\frac{\varphi'_3}{2}\right)$ in (10.9) with the relation for $\cos\left(\frac{\sqrt{\varphi'_+\varphi'_-}}{2}\right)$ in (10.10).

11. Parametrization of $SU(2)$ by Quaternions

For some completeness we will shortly consider the parametrization of $SU(2)$ by quaternions which were introduced by W.R. Hamilton in the middle of the 19th century after searching for more general number systems than complex numbers (e.g. [33] [34] [35]).

A quaternion $r = (r_0, \boldsymbol{r})$ consists of a scalar part r_0 and of a vectorial part $\boldsymbol{r} \equiv (r_1, r_2, r_3)$ which both (by definition) are real in case of real quaternions. The associative but not commutative multiplication law in the quaternion algebra $\mathbb{H}$ for two quaternions $r = (r_0, \boldsymbol{r})$ and $s = (s_0, \boldsymbol{s})$ is

$$rs \equiv (r_0, \boldsymbol{r})(s_0, \boldsymbol{s}) = (r_0 s_0 - \boldsymbol{rs}, s_0\boldsymbol{r} + r_0\boldsymbol{s} + [\boldsymbol{r}, \boldsymbol{s}]),$$
$$sr \equiv (s_0, \boldsymbol{s})(r_0, \boldsymbol{r}) = (r_0 s_0 - \boldsymbol{rs}, s_0\boldsymbol{r} + r_0\boldsymbol{s} - [\boldsymbol{r}, \boldsymbol{s}]), \quad (11.1)$$

where $\boldsymbol{rs}$ denotes the scalar product and $[\boldsymbol{r}, \boldsymbol{s}]$ the vector product of two vectors $\boldsymbol{r}$ and $\boldsymbol{s}$. From (11.1) follows

$$\frac{rs+sr}{2} = (r_0 s_0 - \boldsymbol{rs}, s_0\boldsymbol{r} + r_0\boldsymbol{s}), \quad \frac{rs-sr}{2} = (0, [\boldsymbol{r}, \boldsymbol{s}]), \quad (11.2)$$

The quaternion $\bar{r} = \overline{(r_0, \boldsymbol{r})} \equiv (r_0, -\boldsymbol{r})$ is called the conjugate quaternion to $r = (r_0, \boldsymbol{r})$ and the product $r\bar{r} = \bar{r}r = (r_0^2 + \boldsymbol{r}^2, \boldsymbol{0})$ is proportional to the identical quaternion $(1, \boldsymbol{0})$ and therefore $r^{-1} \equiv \dfrac{\bar{r}}{r_0^2 + \boldsymbol{r}^2}$ is the reciprocal quaternion to r. The nonnegative number $|r| \equiv \sqrt{r_0^2 + \boldsymbol{r}^2}$ is the modulus or norm of a quaternion. The multiplication law (11.1) can be realized by matrix multiplication, in lowest-dimensional case by special 2D matrices, for example, by the following correspondence

$$(r_0, \boldsymbol{r}) \equiv (r_0, (r_1, r_2, r_3)) \leftrightarrow \begin{pmatrix} r_0 - \mathrm{i}r_3 & \mathrm{i}(r_1 - \mathrm{i}r_2) \\ \mathrm{i}(r_1 + \mathrm{i}r_2) & r_0 + \mathrm{i}r_3 \end{pmatrix} \equiv \mathsf{S}, \tag{11.3}$$

with the additive decomposition of S as follows[3]

$$\begin{aligned} \mathsf{S} &= r_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \mathrm{i}r_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \mathrm{i}r_2 \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} - \mathrm{i}r_3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= r_0 \mathrm{I} + \mathrm{i}r_1\sigma_1 + \mathrm{i}r_2\sigma_2 - \mathrm{i}r_3\sigma_3, \end{aligned} \tag{11.4}$$

where $(\sigma_1, \sigma_2, \sigma_3)$ are the three Pauli spin matrices σ_k explicitly given in (5.3). By comparison with (5.13), one finds the following correspondences between $SU(2)$ matrices S and real quaternions $(r_0, \boldsymbol{r})$

$$\mathsf{S} \leftrightarrow (r_0, (r_1, r_2, r_3)) = \left(\cos\left(\frac{|\boldsymbol{\varphi}|}{2}\right), \frac{(\varphi_1, \varphi_2, -\varphi_3)}{|\boldsymbol{\varphi}|} \sin\left(\frac{|\boldsymbol{\varphi}|}{2}\right) \right). \tag{11.5}$$

This means that the squared modulus of the quaternion is the determinant $[\mathsf{S}]$ of the two-dimensional matrix S which due to unimodularity is equal to 1 and the scalar part is half the trace $\langle \mathsf{S} \rangle$ of the matrix S. Therefore, $SU(2)$ matrices correspond to real unit quaternions (modulus equal to 1 by definition) with 3 independent real parameters. The multiplication of matrices (5.13) or the quaternion multiplication (11.1) allow to establish the composition law of two $SU(2)$ transformations. Clearly, the noncommutative matrix multiplication is the more generally applicable operation in comparison to quaternion multiplication.

12. Regular Representation $D^{(1)}$ of $SU(2)$ as Basic Representation of $SO(3, \mathbb{R})$

In this Section we construct the regular (or adjoint) representation of $SU(2)$ which provides the group of inner automorphisms of $SU(2)$ and thus the transformation of the vector operator $\boldsymbol{J}$. It uses the operators of the abstract Lie algebra themselves as a basis and, therefore, is three-dimensional. If we use (J_+, J_3, J_-) as basis, we can construct the three-dimensional representation matrices to J_k from the commutators $[J_k, (J_+, J_3, J_-)]$ in analogy to (5.3) that

[3]This formula shows a blemish of beauty, the negative sign in front of σ_3 compared with the positive signs in front of σ_1 and σ_2. We did not find a way to remove it and, likely, this is impossible. If we change, for example, $r_3 \to -r_3$ in (11.3) then the matrix multiplication is no more compatible with the definition (11.1) of the quaternion product.

we do not write down explicitly. From this realization we find in analogy to (5.12) the regular representation of $SU(2)$ in the form

$$\left(J'_+, J'_3, J'_-\right) \equiv \exp(\mathrm{i}\boldsymbol{J}\boldsymbol{\varphi})\left(J_+, J_3, J_-\right)\exp(-\mathrm{i}\boldsymbol{J}\boldsymbol{\varphi}) = \left(J_+, J_3, J_-\right)\mathsf{R}, \qquad (12.1)$$

with the following three-dimensional matrix $\mathsf{R} = \mathsf{R}(\boldsymbol{\varphi})$ with particularly simple structure

$$\mathsf{R} = \begin{pmatrix} S_{11}S_{11} & -S_{11}S_{12} & -S_{12}S_{12} \\ -2S_{11}S_{21} & S_{11}S_{22} + S_{12}S_{21} & 2S_{12}S_{22} \\ -S_{21}S_{21} & S_{21}S_{22} & S_{22}S_{22} \end{pmatrix},$$
$$\langle \mathsf{R} \rangle = 1 + 2\cos(|\boldsymbol{\varphi}|) = \frac{\sin\left(3\frac{|\boldsymbol{\varphi}|}{2}\right)}{\sin\left(\frac{|\boldsymbol{\varphi}|}{2}\right)}, \quad [\mathsf{R}] = \langle \mathsf{R} \rangle, \quad |\mathsf{R}| = 1, \qquad (12.2)$$

where $\langle \mathsf{R} \rangle$ denotes the trace, $[\mathsf{R}]$ the second invariant and $|\mathsf{R}|$ the determinant of the operator R (see also below) and where we took into account $S_{11}S_{22} - S_{12}S_{21} = 1$. It is a special complex unimodular matrix which is equivalent to a real orthogonal matrix as we will now show. Inserting (5.13) for S, we obtain an explicit form of this matrix which we do not write down. It is now straightforward to get the matrix R of the mapping $\exp(\mathrm{i}\boldsymbol{J}\boldsymbol{\varphi}) \to \mathsf{R}$ in the basis of the operators (J_1, J_2, J_3) instead of (J_+, J_3, J_-) that can be represented in the vector form[4]

$$\boldsymbol{J}' \equiv \exp(\mathrm{i}\boldsymbol{J}\boldsymbol{\varphi})\boldsymbol{J}\exp(-\mathrm{i}\boldsymbol{J}\boldsymbol{\varphi}) = \boldsymbol{J}\mathsf{R}, \quad \Leftrightarrow \quad J'_l = J_k R_{kl}, \qquad (12.3)$$

and explicitly in representation by vector components (sum convention)

$$R_{kl} \equiv \frac{\varphi_k\varphi_l}{|\boldsymbol{\varphi}|^2} + \left(\delta_{kl} - \frac{\varphi_k\varphi_l}{|\boldsymbol{\varphi}|^2}\right)\cos(|\boldsymbol{\varphi}|) + \varepsilon_{klj}\frac{\varphi_j}{|\boldsymbol{\varphi}|^2}\sin(|\boldsymbol{\varphi}|), \qquad (12.4)$$

or written in coordinate-invariant form

$$\mathsf{R} = \frac{\boldsymbol{\varphi}\cdot\boldsymbol{\varphi}}{|\boldsymbol{\varphi}|^2} + \left(\mathrm{I} - \frac{\boldsymbol{\varphi}\cdot\boldsymbol{\varphi}}{|\boldsymbol{\varphi}|^2}\right)\cos(|\boldsymbol{\varphi}|) - \left[\frac{\boldsymbol{\varphi}}{|\boldsymbol{\varphi}|}\right]\sin(|\boldsymbol{\varphi}|),$$
$$|\boldsymbol{\varphi}| \equiv \sqrt{\varphi_1^2 + \varphi_2^2 + \varphi_3^2} \le 2\pi. \qquad (12.5)$$

The matrix $\mathsf{R} = \mathsf{R}(\boldsymbol{\varphi})$ in the representation R_{kl} corresponding to the basis (J_1, J_2, J_3) is a three-dimensional real orthogonal matrix with determinant equal to +1 (*i.e.* a proper rotation) that means

$$\langle \mathsf{R} \rangle = [\mathsf{R}] = 1 + 2\cos(|\boldsymbol{\varphi}|), \quad |\mathsf{R}| = 1, \quad \mathsf{R}\mathsf{R}^{\mathrm{T}} = \mathrm{I}, \quad \left(R_{kj}R_{lj} = \delta_{kl}\right). \qquad (12.6)$$

The rotation axis is described by the unit axial vector $\frac{\boldsymbol{\varphi}}{|\boldsymbol{\varphi}|}$ and the rotation angle is $|\boldsymbol{\varphi}|$ or, equivalently, by the unit vector $-\frac{\boldsymbol{\varphi}}{|\boldsymbol{\varphi}|}$ and the rotation angle $-|\boldsymbol{\varphi}|$.

[4]We do not introduce a new notation for R in comparison to (12.2) because we consider it as the same operator in matrix representation with respect to different basis vectors.

In the form $J_k R_{kl}$ it describes by convention an anti-clockwise rotation of the vector J_k and in the form $R_{kl} r_l$ a clockwise rotation of the vector r_l about an angle $\varphi \equiv |\boldsymbol{\varphi}|$ if the vector $\frac{\boldsymbol{\varphi}}{|\boldsymbol{\varphi}|}$ is perpendicular to the clock plane.

Rotations with the parameters $\boldsymbol{\varphi}$ and $\boldsymbol{\varphi}+n2\pi\frac{\boldsymbol{\varphi}}{|\boldsymbol{\varphi}|},(n=\pm1,\pm2,\cdots)$ lead to the same matrix R. We consider the following transformations of the vector parameter φ

$$\boldsymbol{\varphi} \to \boldsymbol{\varphi}' = \boldsymbol{\varphi}+n2\pi\frac{\boldsymbol{\varphi}}{|\boldsymbol{\varphi}|} = \left(|\boldsymbol{\varphi}|+n2\pi\right)\frac{\boldsymbol{\varphi}}{|\boldsymbol{\varphi}|} \sim \boldsymbol{\varphi}, \quad (n=0,\pm1,\pm2,\cdots), \tag{12.7}$$

from which follows for modulus and direction of $\boldsymbol{\varphi}'$

$$|\boldsymbol{\varphi}'| = \big||\boldsymbol{\varphi}|+n2\pi\big| = \pm\left(|\boldsymbol{\varphi}|+n2\pi\right), \quad \Leftrightarrow \quad \frac{\boldsymbol{\varphi}'}{|\boldsymbol{\varphi}'|} = \frac{|\boldsymbol{\varphi}|+n2\pi}{\big||\boldsymbol{\varphi}|+n2\pi\big|}\frac{\boldsymbol{\varphi}}{|\boldsymbol{\varphi}|} = \pm\frac{\boldsymbol{\varphi}}{|\boldsymbol{\varphi}|}. \tag{12.8}$$

Inserted in (12.5) it leaves the rotation R unchanged.

Two other special cases are, in particular, the special case $n=-1$ in (6.1) for which follows

$$\boldsymbol{\varphi} \to \boldsymbol{\varphi}' = \boldsymbol{\varphi}-2\pi\frac{\boldsymbol{\varphi}}{|\boldsymbol{\varphi}|}, \quad \Rightarrow \quad |\boldsymbol{\varphi}| \to 2\pi-|\boldsymbol{\varphi}|, \quad \frac{\boldsymbol{\varphi}}{|\boldsymbol{\varphi}|} \to -\frac{\boldsymbol{\varphi}}{|\boldsymbol{\varphi}|}, \quad \mathsf{R} \to \mathsf{R}. \tag{12.9}$$

In comparison, for the transformation $\boldsymbol{\varphi} \to \boldsymbol{\varphi}-\pi\frac{\boldsymbol{\varphi}}{|\boldsymbol{\varphi}|}$ we find

$$\boldsymbol{\varphi} \to \boldsymbol{\varphi}' = \boldsymbol{\varphi}-\pi\frac{\boldsymbol{\varphi}}{|\boldsymbol{\varphi}|}, \quad \Rightarrow \quad |\boldsymbol{\varphi}| \to \pm\left(\pi-|\boldsymbol{\varphi}|\right), \quad \frac{\boldsymbol{\varphi}}{|\boldsymbol{\varphi}|} \to \pm\frac{\boldsymbol{\varphi}}{|\boldsymbol{\varphi}|}, \quad \mathsf{R} \to \mathsf{R}^{-1}. \tag{12.10}$$

This allows to restrict a fundamental region of the parameters $\boldsymbol{\varphi}=(\varphi_1,\varphi_2,\varphi_3)$ to the three-dimensional sphere $|\boldsymbol{\varphi}| \le \pi$ with identification of opposite points on the surface of the two-dimensional sphere $\mathbb{S}^2$ with $|\boldsymbol{\varphi}|=\pi$ as its boundary. The very direct relation to covariant quantities of $SO(3,\mathbb{R})$ is an advantage of using the vector parameter $\boldsymbol{\varphi}=(\varphi_1,\varphi_2,\varphi_3)$ in comparison to the Euler angles (α,β,γ).

We can look to relation (12.3) also in another way. For this purpose we take the vector parameter $\boldsymbol{\varphi}_0$ for an arbitrary element $x_0 \equiv \boldsymbol{J}\boldsymbol{\varphi}_0$ of the Lie algebra to $SU(2)$ and consider its transformation

$$\boldsymbol{J}'\boldsymbol{\varphi}_0 \equiv \exp(\mathrm{i}\boldsymbol{J}\boldsymbol{\varphi})(\boldsymbol{J}\boldsymbol{\varphi}_0)\exp(-\mathrm{i}\boldsymbol{J}\boldsymbol{\varphi}) = \boldsymbol{J}\mathsf{R}\boldsymbol{\varphi}_0 \equiv \boldsymbol{J}\boldsymbol{\varphi}_0', \tag{12.11}$$

from which follows

$$\boldsymbol{\varphi}_0' = \mathsf{R}\boldsymbol{\varphi}_0, \quad \left(0 \le |\boldsymbol{\varphi}_0| < 2\pi\right). \tag{12.12}$$

This means that all elements of $SU(2)$ with vector parameters $\boldsymbol{\varphi}_0'$ where $\boldsymbol{\varphi}_0' = \mathsf{R}\boldsymbol{\varphi}_0$ is obtained from $\boldsymbol{\varphi}_0$ by an arbitrary rotation of the three-dimensional rotation group $SO(3,\mathrm{R})$ are equivalent and form one class within $SU(2)$. This transformation changes only the axis direction $\boldsymbol{n}_0 \equiv \frac{\boldsymbol{\varphi}_0}{|\boldsymbol{\varphi}_0|}$ and all elements of

$SU(2)$ with the same $|\boldsymbol{\varphi}_0|$ are equivalent.

The relation between the three-dimensional matrix R_{kl} and the two-dimensional matrix S and its inversion in covariant form is (e.g., see [18] (Equations (2.16) and (2.32)), [25] (pp. 42-43) and [36] (Equation (2.2.15))[5])

$$R_{kl} = \frac{1}{2}\langle \sigma_k \mathsf{S} \sigma_l \mathsf{S}^\dagger \rangle, \quad \mathsf{S} = \pm \frac{\mathrm{I} + \sigma_k R_{kl} \sigma_l}{\sqrt{[\mathrm{I} + \sigma_k R_{kl} \sigma_l]}}, \tag{12.13}$$

where $\langle \ldots \rangle$ and $[\ldots]$ denote here the trace and determinant of two-dimensional matrices, respectively. Matrices $+\mathsf{S}$ and $-\mathsf{S}$ lead to the same R. Thus we have constructed the known 2-1 homomorphism of $SU(2)$ to $SO(3,\mathrm{R}) = SU(2)/Z_2$ where Z_2 is the center of $SU(2)$ consisting of two elements I and $-\mathrm{I}$. We can look to Equations (1)-(6) as to the inner automorphisms of the unitary unimodular group $SU(2)$ that means to the inner transformations of the operators (J_1, J_2, J_3) which leave unchanged the commutation relations. This is important for the coordinate-invariant interpretation.

13. Parametrization of Rotation Group $SO(3,\mathbb{R})$ by Vector $\boldsymbol{\varphi}$ in Coordinate-Invariant Description

The three-dimensional rotation operator R can be represented in the following exponential form

$$\mathsf{R} \equiv \mathrm{e}^{\Phi}, \quad \Rightarrow \quad \mathrm{e}^{-\Phi} = \mathsf{R}^{-1} = \mathsf{R}^{\mathsf{T}} = \mathrm{e}^{\Phi^{\mathsf{T}}}, \quad \Rightarrow \quad \Phi = -\Phi^{\mathsf{T}}, \tag{13.1}$$

which defines a real three-dimensional antisymmetric operator Φ. Three-dimensional anti-symmetric operators $\Phi = -\Phi^{\mathsf{T}}$ can be mapped onto three-dimensional axial vectors $\boldsymbol{\varphi}$ according to

$$\Phi \leftrightarrow \Phi_{kl} = \varepsilon_{kjl}\varphi_j = -\Phi_{lk}, \quad \boldsymbol{\varphi} \leftrightarrow \varphi_j = \frac{1}{2}\varepsilon_{kjl}\Phi_{kl}, \tag{13.2}$$

from which follows

$$\langle \Phi \rangle = 0, \quad |\Phi| = 0, \quad [\Phi] \equiv \frac{1}{2}\left(\langle \Phi \rangle^2 - \langle \Phi^2 \rangle\right) = -\frac{1}{2}\langle \Phi^2 \rangle. \tag{13.3}$$

Only the second invariant $[\Phi]$ is in general non-vanishing whereas trace and determinant vanish and the Hamilton-Cayley identity for Φ reduces to

$$\Phi^3 + [\Phi]\Phi = 0, \tag{13.4}$$

with the consequence that all powers of Φ reduce to powers of Φ and Φ^2 multiplied by factors

$$\Phi^{2n+1} = (-[\Phi])^n \Phi, \quad \Phi^{2n+2} = (-[\Phi])^n \Phi^2, \quad (n = 0,1,2,\cdots). \tag{13.5}$$

From the Taylor series $\mathsf{R} = \mathrm{e}^{\Phi} = \sum_{n=0}^{\infty} \frac{1}{n!}\Phi^n$ and analogously for $\mathsf{R}^{-1} = \mathrm{e}^{-\Phi} = \mathsf{R}^{\mathsf{T}}$

[5]In representations of quantum field theory such as in [36], it is mostly given in more general form for the Lorentz group $SO(3,1) \sim SL(2,\mathbb{R}) \supset SO(3,\mathbb{R})$. The inversion of the relation analogously to second equation in (12.13) which is often absent can be found, e.g., in [25].

by substitution $\Phi \to -\Phi$ follows

$$\mathrm{R} = \mathrm{I} + \frac{\sin\left(\sqrt{[\Phi]}\right)}{\sqrt{[\Phi]}}\Phi + \frac{1-\cos\left(\sqrt{[\Phi]}\right)}{[\Phi]}\Phi^2. \tag{13.6}$$

Due to relation $|\mathrm{B}| = \left|e^{\mathrm{A}}\right| = e^{\langle \mathrm{A}\rangle}$ for an exponential operator $\mathrm{B} \equiv e^{\mathrm{A}}$ we check for determinant $|\mathrm{R}|$

$$|\mathrm{R}| = \exp\left(\langle\Phi\rangle\right) = \exp(0) = 1. \tag{13.7}$$

Using $|\mathrm{R}| = 1$ with consequence $\mathrm{R}^{\mathrm{T}} = \mathrm{R}^{-1} = \overline{\mathrm{R}}$ and in addition $\langle\Phi^2\rangle = -2[\Phi]$ according to (13.3) taking into account $\langle\Phi\rangle = 0$ follows explicitly from (13.6) for the trace and the second invariant of R

$$\langle\mathrm{R}\rangle = [\mathrm{R}] = 1 + 2\cos\left(\sqrt{[\Phi]}\right). \tag{13.8}$$

The equality $\langle\mathrm{R}\rangle = [\mathrm{R}]$ of first and second invariant of R is due to $\langle\mathrm{R}^{\mathrm{T}}\rangle = \langle\overline{\mathrm{R}}\rangle$ combined with the general identity $\langle\overline{\mathrm{A}}\rangle = [\mathrm{A}]$ for general three-dimensional operators A.

Our next problem is the transition from the antisymmetric operator Φ in the above formulae to a representation by the vector parameter $\boldsymbol{\varphi}$ by means of the formulae (13.2). First we make the transition of Φ^2 to a representation by $\boldsymbol{\varphi}$

$$\Phi_{ik}\Phi_{km} = \varepsilon_{ijk}\varepsilon_{klm}\varphi_j\varphi_l = \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)\varphi_j\varphi_l = \varphi_i\varphi_m - |\boldsymbol{\varphi}|^2\delta_{im}, \tag{13.9}$$

which in coordinate-invariant representation is

$$\Phi^2 = \boldsymbol{\varphi}\cdot\boldsymbol{\varphi} - |\boldsymbol{\varphi}|^2\mathrm{I}, \quad \Rightarrow \quad \langle\Phi^2\rangle = -2|\boldsymbol{\varphi}|^2 = -2[\Phi], \quad [\Phi] = |\boldsymbol{\varphi}|^2. \tag{13.10}$$

Inserting (13.10) into (13.6) for R follows

$$\mathrm{R} = \boldsymbol{n}\cdot\boldsymbol{n} + \sin\left(|\boldsymbol{\varphi}|\right)[\boldsymbol{n}] + \cos\left(|\boldsymbol{\varphi}|\right)\left(\mathrm{I} - \boldsymbol{n}\cdot\boldsymbol{n}\right), \quad \left(\boldsymbol{n} \equiv \frac{\boldsymbol{\varphi}}{|\boldsymbol{\varphi}|}, \boldsymbol{n}^2 = 1\right). \tag{13.11}$$

The abbreviation $\boldsymbol{n}$ is a unit vector in direction of the rotation axis and $|\boldsymbol{\varphi}|$ the rotation angle counter-clockwise taken. The application of R onto an arbitrary vector $\boldsymbol{x}$ leads to

$$\mathrm{R}\boldsymbol{x} = (\boldsymbol{n}\boldsymbol{x})\boldsymbol{n} + \sin\left(|\boldsymbol{\varphi}|\right)[\boldsymbol{n},\boldsymbol{x}] + \cos\left(|\boldsymbol{\varphi}|\right)\left[[\boldsymbol{n},\boldsymbol{x}],\boldsymbol{n}\right]. \tag{13.12}$$

By comparison of $\mathrm{R} = e^{\Phi}$ with the general form $e^{\mathrm{i}\boldsymbol{J}\boldsymbol{\varphi}}$ of group elements in representations of $SU(2)$ in (5.2) which in specialization to the vector basis becomes $R_{kl} = e^{\Phi_{kl}} = e^{\varepsilon_{kjl}\varphi_j} = e^{-\mathrm{i}(J_j)_{kl}\varphi_j}$ we find that the operators $J_j, (j = 1,2,3)$ are represented in the three-dimensional regular representation by the following matrices closely related to the Levi-Civita symbol

$$J_j \to \left(J_j\right)_{kl} = \mathrm{i}\varepsilon_{kjl} = -\mathrm{i}\varepsilon_{jkl}. \tag{13.13}$$

One may check that this three-dimensional matrix representation of the operators $J_j, (j = 1,2,3)$ satisfies the commutation relations (3.10) and may be taken as alternative starting point for the construction of the three-dimensional representation of $SU(2)$.

14. Cayley-Gibbs-Fyodorov Parametrization of Rotation Group $SO(3,\mathbb{R})$ by Vector Parameter c in Coordinate-Invariant Description

There is yet another very interesting parametrization of the three-dimensional rotation group obtained by specialization from the Cayley representation (e.g., [7]) of proper orthogonal operators R of arbitrary dimension (*i.e.*, operators satisfying $\mathsf{R}^{-1}=\mathsf{R}^{\mathrm{T}}$ with determinant $|\mathsf{R}|=+1$) by antisymmetric operators C which is possible in general *n*-dimensional case as follows[6]

$$\mathsf{R}\equiv\frac{\mathrm{I}+\mathsf{C}}{\mathrm{I}-\mathsf{C}}=\exp\left(2\mathrm{Arth}(\mathsf{C})\right),\quad \mathsf{C}=\frac{\mathsf{R}-\mathrm{I}}{\mathsf{R}+\mathrm{I}}=\mathrm{th}\left(\frac{1}{2}\log(\mathsf{R})\right)=-\mathsf{C}^{\mathrm{T}}, \tag{14.1}$$

where the second relation of antisymmetry of C follows from

$$\mathsf{C}^{\mathrm{T}}=\frac{\mathsf{R}^{\mathrm{T}}-\mathrm{I}}{\mathsf{R}^{\mathrm{T}}+\mathrm{I}}=\frac{\mathsf{R}^{-1}-\mathrm{I}}{\mathsf{R}^{-1}+\mathrm{I}}=\frac{\mathrm{I}-\mathsf{R}}{\mathrm{I}+\mathsf{R}}=-\mathsf{C}. \tag{14.2}$$

We now consider specific properties of the transformations in three-dimensional case. After expansion of R in (14.1) in a Taylor series of powers of C according to

$$\mathsf{R}=\mathrm{I}+2\left(\mathsf{C}+\mathsf{C}^2+\mathsf{C}^3+\cdots\right),\quad \mathsf{C}=-\mathrm{I}+2\left(\mathsf{R}-\mathsf{R}^2+\mathsf{R}^3-\cdots\right),\quad \left(\mathsf{C}^{\mathrm{T}}\right)^2=\mathsf{C}^2, \tag{14.3}$$

and reduction of powers of C higher than or equal 3 by the Hamilton-Cayley identity (6) (third equation) to powers lower than 3 taking into account the antisymmetry of C with the consequence $\mathsf{C}^3=-[\mathsf{C}]\mathsf{C}$, we find the following reduced relations between R and C

$$\mathsf{R}=\mathrm{I}+2\frac{\mathsf{C}+\mathsf{C}^2}{1+[\mathsf{C}]},\quad \mathsf{R}^{-1}=\mathsf{R}^{\mathrm{T}}=\mathrm{I}-2\frac{\mathsf{C}-\mathsf{C}^2}{1+[\mathsf{C}]},\quad \mathsf{C}=\frac{\mathsf{R}-\mathsf{R}^{\mathrm{T}}}{1+\langle\mathsf{R}\rangle}=-\mathsf{C}^{\mathrm{T}}, \tag{14.4}$$

with the invariants

$$\langle\mathsf{R}\rangle=[\mathsf{R}]=\frac{3-[\mathsf{C}]}{1+[\mathsf{C}]},\quad |\mathsf{R}|=1,\quad \langle\mathsf{C}\rangle=0,\quad [\mathsf{C}]=-\frac{1}{2}\langle\mathsf{C}^2\rangle=\frac{3-\langle\mathsf{R}\rangle}{1+\langle\mathsf{R}\rangle},\quad |\mathsf{C}|=0. \tag{14.5}$$

They do not contain powers of R and C higher than quadratic ones. Furthermore using (13.1) and (14.1) we have

[6]It is not possible to represent operators of improper orthogonal transformations R' with $|\mathsf{R}'|=-1=\mathrm{e}^{\mathrm{i}\pi}$ in this way since from $\mathsf{R}'\equiv\frac{\mathrm{I}+\mathsf{C}'}{\mathrm{I}-\mathsf{C}'}$ and antisymmetry $\mathsf{C}'=-\mathsf{C}'^{\mathrm{T}}$ automatically follows $|\mathsf{R}'|=+1$, for example, from $|\mathsf{R}'|=\exp\left(2\langle\mathrm{Arth}(\mathsf{C}')\rangle\right)=\mathrm{e}^0=1$ according to $|\mathrm{e}^{\mathsf{A}}|=\mathrm{e}^{\langle\mathsf{A}\rangle}$ or from $|\mathsf{R}'|=\frac{|\mathrm{I}+\mathsf{C}'|}{|\mathrm{I}-\mathsf{C}'|}=1$ due to $|\mathrm{I}\pm\mathsf{C}'|=1+[\mathsf{C}']$ for antisymmetric matrices C'. However, with modifications not considered here a coordinate-invariant treatment of the case $|\mathsf{R}'|=-1$ is possible. We mention that the Cayley representation is often used in the alternative form $\mathsf{U}=\frac{\mathrm{I}+\mathrm{i}\mathsf{H}}{\mathrm{I}-\mathrm{i}\mathsf{H}}$ or $\mathrm{i}\mathsf{H}=\frac{\mathsf{U}-\mathrm{I}}{\mathsf{U}+\mathrm{I}}$ as one of the possible correspondences between unitary operators U and Hermitean operators H [7].

$$\mathsf{C} = \mathrm{th}\left(\frac{\Phi}{2}\right) = \frac{\Phi}{\sqrt{[\Phi]}}\mathrm{tg}\left(\frac{\sqrt{[\Phi]}}{2}\right), \quad \Phi = 2\,\mathrm{Arth}(\mathsf{C}) = 2\frac{\mathrm{arctg}\left(\sqrt{[\mathsf{C}]}\right)}{\sqrt{[\mathsf{C}]}}\mathsf{C}, \tag{14.6}$$

with the invariants

$$[\Phi] = \left(2\mathrm{arctg}\left(\sqrt{[\mathsf{C}]}\right)\right)^2 = |\boldsymbol{\varphi}|^2, \quad [\mathsf{C}] = \mathrm{tg}^2\left(\frac{\sqrt{[\Phi]}}{2}\right) = \mathrm{tg}^2\left(\frac{|\boldsymbol{\varphi}|}{2}\right). \tag{14.7}$$

Independently on the choice of the chosen sign of $\sqrt{[\Phi]}$ and $\sqrt{[\mathsf{C}]}$ the relations between C and Φ are true since they are involved only in the combinations $2\frac{\mathrm{tg}\left(\frac{\sqrt{[\Phi]}}{2}\right)}{\sqrt{[\Phi]}}$ and $\frac{\mathrm{arctg}\left(\sqrt{[\mathsf{C}]}\right)}{\sqrt{[\mathsf{C}]}}$, respectively.

A further specifics of the three-dimensional case of the Cayley representation is that a real antisymmetric tensor C (or operator in Euclidean space) with its 3 independent components can be mapped $\mathsf{C} \leftrightarrow \boldsymbol{c}$ onto a three-dimensional real vector $\boldsymbol{c}$ which in representation by vector indices in analogy to (13.1) takes on the form (see also ((10) and (13.1)); $C_{ik} \equiv \varepsilon_{ijk} c_j \leftrightarrow c_j \equiv \frac{1}{2}\varepsilon_{ijk} C_{ik}$)

$$\mathsf{C} \leftrightarrow C_{ik} \equiv [\boldsymbol{c}]_{ik} = \varepsilon_{ijk} c_j, \quad \boldsymbol{c} \leftrightarrow c_j = \frac{1}{2}\varepsilon_{ijk} C_{ik}, \quad \Rightarrow \quad \mathsf{C} \equiv [\boldsymbol{c}] \leftrightarrow \boldsymbol{c}, \tag{14.8}$$

from which follows, for example

$$\begin{aligned} &\mathsf{C}\boldsymbol{x} = [\boldsymbol{c}]\boldsymbol{x} = [\boldsymbol{c},\boldsymbol{x}], \quad \mathsf{C}[\boldsymbol{x},\boldsymbol{y}] = [\boldsymbol{c}][\boldsymbol{x},\boldsymbol{y}] = [\boldsymbol{c},[\boldsymbol{x},\boldsymbol{y}]] = (\boldsymbol{c}\boldsymbol{y})\boldsymbol{x} - (\boldsymbol{c}\boldsymbol{x})\boldsymbol{y}, \\ &\boldsymbol{y}\mathsf{C} = \boldsymbol{y}[\boldsymbol{c}] = [\boldsymbol{y},\boldsymbol{c}], \quad [\boldsymbol{x},\boldsymbol{y}]\mathsf{C} = [\boldsymbol{x},\boldsymbol{y}][\boldsymbol{c}] = [[\boldsymbol{x},\boldsymbol{y}],\boldsymbol{c}] = (\boldsymbol{x}\boldsymbol{c})\boldsymbol{y} - (\boldsymbol{y}\boldsymbol{c})\boldsymbol{x}. \end{aligned} \tag{14.9}$$

For the squared operator $\mathsf{C} \equiv [\boldsymbol{c}]$ we find (C^2 is a symmetric operator)

$$\mathsf{C}^2 = [\boldsymbol{c}]^2 = \boldsymbol{c}\cdot\boldsymbol{c} - |\boldsymbol{c}|^2 \mathrm{I}, \quad \Rightarrow \quad \langle \mathsf{C}^2 \rangle = -2|\boldsymbol{c}|^2, \quad [\mathsf{C}] = -\frac{1}{2}\langle \mathsf{C}^2 \rangle = |\boldsymbol{c}|^2. \tag{14.10}$$

From this using (14.6) one finds the following relations between the vector parameters $\boldsymbol{c}$ and $\boldsymbol{\varphi}$

$$\boldsymbol{c} = \frac{\boldsymbol{\varphi}}{|\boldsymbol{\varphi}|}\mathrm{tg}\left(\frac{|\boldsymbol{\varphi}|}{2}\right), \quad \boldsymbol{\varphi} = \frac{\boldsymbol{c}}{|\boldsymbol{c}|}2\mathrm{arctg}(|\boldsymbol{c}|), \quad 0 \le |\boldsymbol{\varphi}| \le \pi, \quad 0 \le |\boldsymbol{c}| \le +\infty, \tag{14.11}$$

with the consequence

$$\begin{aligned} &|\boldsymbol{c}| = \mathrm{tg}\left(\frac{|\boldsymbol{\varphi}|}{2}\right), \quad \frac{\boldsymbol{c}}{|\boldsymbol{c}|} = \frac{\boldsymbol{\varphi}}{|\boldsymbol{\varphi}|}, \quad \Rightarrow \quad \boldsymbol{c} = \frac{\boldsymbol{c}}{|\boldsymbol{c}|}|\boldsymbol{c}| = \frac{\boldsymbol{\varphi}}{|\boldsymbol{\varphi}|}\mathrm{tg}\left(\frac{|\boldsymbol{\varphi}|}{2}\right), \\ &\cos\left(\frac{|\boldsymbol{\varphi}|}{2}\right) = \pm\frac{1}{\sqrt{1+|\boldsymbol{c}|^2}}, \quad \sin\left(\frac{|\boldsymbol{\varphi}|}{2}\right) = \frac{|\boldsymbol{c}|}{\sqrt{1+|\boldsymbol{c}|^2}}, \quad 0 \le |\boldsymbol{\varphi}| \le \pi. \end{aligned} \tag{14.12}$$

The direct relation to the trigonometric functions in the rotation matrix R_{kl} given in (12.5) by the vector parameter $\boldsymbol{\varphi}$ is

$$\cos(|\boldsymbol{\varphi}|) = \frac{1-|\boldsymbol{c}|^2}{1+|\boldsymbol{c}|^2}, \quad \sin(|\boldsymbol{\varphi}|) = \pm\frac{2|\boldsymbol{c}|}{1+|\boldsymbol{c}|^2}, \quad 0 \le |\boldsymbol{\varphi}| \le \pi, \tag{14.13}$$

and this rotation matrix R_{kl} is therefore uniquely represented by vector parameter $\boldsymbol{c}$ as follows

$$R_{kl}=\frac{c_k c_l}{|\boldsymbol{c}|^2}+\frac{1-|\boldsymbol{c}|^2}{1+|\boldsymbol{c}|^2}\left(\delta_{kl}-\frac{c_k c_l}{|\boldsymbol{c}|^2}\right)+\frac{2}{1+|\boldsymbol{c}|^2}\varepsilon_{jkl}c_j. \tag{14.14}$$

The modulus $|\boldsymbol{c}|$ of the vector parameter $\boldsymbol{c}$ with real components (c_1,c_2,c_3) is stretched in comparison to the modulus $|\boldsymbol{\varphi}|$ of the vector parameter $\boldsymbol{\varphi}$ with real components $(\varphi_1,\varphi_2,\varphi_3)$.

All rotations of $SO(3,\mathbb{R})$ about an angle $|\boldsymbol{\varphi}|=\pi$ correspond to the parameter $|\boldsymbol{c}|\to\infty$ and in the whole region $0\le|\boldsymbol{\varphi}|\le\pi$ we have the correspondences of $\frac{\boldsymbol{c}}{|\boldsymbol{c}|}$ to $\frac{\boldsymbol{\varphi}}{|\boldsymbol{\varphi}|}$ as the rotation axis. The region to angles $\pi\le|\boldsymbol{\varphi}|\le2\pi$ can be reduced by transition $|\boldsymbol{\varphi}|\to2\pi-|\boldsymbol{\varphi}|$ to angles in the region $0\le\pi-|\boldsymbol{\varphi}|\le\pi$ with simultaneous transition to the opposite rotation axis $\frac{\boldsymbol{\varphi}}{|\boldsymbol{\varphi}|}\to-\frac{\boldsymbol{\varphi}}{|\boldsymbol{\varphi}|}$. This is not possible for the fundamental representation of $SU(2)$ and we cannot find equivalent angles within the region $0\le|\boldsymbol{\varphi}|\le2\pi$ which are equivalent by changing the direction $\frac{\boldsymbol{\varphi}}{|\boldsymbol{\varphi}|}$ of the rotation axis.

The relation (14.11) between $\boldsymbol{\varphi}$ and $\boldsymbol{c}$ maps all vectors of the three-dimensional ball of vectors $\boldsymbol{\varphi}$ with $|\boldsymbol{\varphi}|\le\pi$ onto the three-dimensional Euclidean space of vectors $\boldsymbol{c}$ where opposite points $\pm\boldsymbol{\varphi}$ of the boundary $|\boldsymbol{\varphi}|=\pi$ correspond to single points of $\boldsymbol{c}$. For $SO(3,\mathbb{R})$ the boundary corresponds to $|\boldsymbol{\varphi}|=\pi$ where opposite directions have to be identified. For $SU(2)$ the parameter $\boldsymbol{c}$ does not uniquely determine a transformation S of the form (5.13).

The two-dimensional matrix S of the fundamental representation of $SU(2)$ in (5.13) takes on the following very simple form in representation by the vector parameter $\boldsymbol{c}$ in components $\boldsymbol{c}\equiv(c_1,c_2,c_3)$

$$\mathsf{S}=\pm\frac{1}{\sqrt{1+|\boldsymbol{c}|^2}}\begin{pmatrix}1+\mathrm{i}c_3 & \mathrm{i}c_- \\ \mathrm{i}c_+ & 1-\mathrm{i}c_3\end{pmatrix},\quad c_\pm\equiv c_1\pm\mathrm{i}c_2=c_\mp^*, \tag{14.15}$$

The minus sign in front of $\frac{1}{\sqrt{1+|\boldsymbol{c}|^2}}$ appears because $\cos\left(\frac{|\boldsymbol{\varphi}|}{2}\right)=\pm\frac{1}{\sqrt{1+|\boldsymbol{c}|^2}}$ changes its sign in the region $0\le|\boldsymbol{\varphi}|\le2\pi$. The special case $c_3=0$ corresponds to the parametrization of the special element $g_{\boldsymbol{n}}$ of $SU(2)$ by Perelomov [32] (Section 4.1) where the complex parameter ζ there corresponds to our $\mathrm{i}c_-$. Furthermore, by comparison of (5.13) and (14.15) with (11.5) we find that the Cayley-Gibbs-Fyodorov parametrization possesses the following relation to the quaternion representation of $SU(2)$ matrices

$$\mathsf{S}\leftrightarrow(r_0,\boldsymbol{r})=\left(\cos\left(\frac{|\boldsymbol{\varphi}|}{2}\right),-\frac{\boldsymbol{\varphi}}{|\boldsymbol{\varphi}|}\sin\left(\frac{|\boldsymbol{\varphi}|}{2}\right)\right)=\left(\pm\frac{1}{\sqrt{1+|\boldsymbol{c}|^2}},-\frac{\boldsymbol{c}}{\sqrt{1+|\boldsymbol{c}|^2}}\right),\quad \boldsymbol{c}=-\frac{\boldsymbol{r}}{r_0}. \tag{14.16}$$

This means that the Cayley-Gibbs-Fyodorov parametrization and the quaternion representation of $SU(2)$ are connected by a simple relation but the parametrization of the group $SU(2)$ by the parameter $\boldsymbol{c}$ is not unique.

The parametrization of elements of $SU(2)$ and of the three-dimensional rotation group $SO(3,\mathbb{R})$ by the three-dimensional vector parameter $\boldsymbol{c}$ which is dual to the antisymmetric operator C of the Cayley representation was introduced by Gibbs (according to [6]) and was used by Fyodorov as the fundament of his approach to the representation theory of this group in coordinate-invariant representation and, furthermore, was extended by him to the Lorentz group [6]. This parametrization is advantageous, in particular, for the coordinate-invariant representation of the composition law of two, in general, non-commuting operators in these groups. We consider this in next Section.

15. Composition Law for the Vector Parameters $\boldsymbol{\varphi}$ and $\boldsymbol{c}$ Corresponding to Products of Rotations of $SO(3,\mathbb{R})$

As in the case of $SU(2)$ (Section 9) we derive now the composition law of the two vector parameters $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$ to a new vector parameter $\boldsymbol{\varphi}$ for the product of two rotations R_1 and R_2 according to

$$\mathsf{R} = \mathsf{R}_1\mathsf{R}_2 \leftrightarrow \boldsymbol{\varphi}, \quad \mathsf{R}_1 \leftrightarrow \boldsymbol{\varphi}_1, \quad \mathsf{R}_2 \leftrightarrow \boldsymbol{\varphi}_2. \tag{15.1}$$

This composition law can be obtained from the representation (13.11) of the rotation operator R and possesses the explicit form

$$\begin{aligned}
\cos(|\boldsymbol{\varphi}|) &= \cos(|\boldsymbol{\varphi}_1|)\cos(|\boldsymbol{\varphi}_2|) - \frac{\boldsymbol{\varphi}_1\boldsymbol{\varphi}_2}{|\boldsymbol{\varphi}_1||\boldsymbol{\varphi}_2|}\sin(|\boldsymbol{\varphi}_1|)\sin(|\boldsymbol{\varphi}_2|),\\
\frac{\boldsymbol{\varphi}}{|\boldsymbol{\varphi}|}\sin(|\boldsymbol{\varphi}|) &= \frac{1}{2}\left\{\frac{\boldsymbol{\varphi}_1}{|\boldsymbol{\varphi}_1|}\left(\sin(|\boldsymbol{\varphi}_1|)(1+\cos(|\boldsymbol{\varphi}_2|)) - \frac{\boldsymbol{\varphi}_1\boldsymbol{\varphi}_2}{|\boldsymbol{\varphi}_1||\boldsymbol{\varphi}_2|}(1-\cos(|\boldsymbol{\varphi}_1|))\sin(|\boldsymbol{\varphi}_2|)\right)\right.\\
&\quad + \frac{\boldsymbol{\varphi}_2}{|\boldsymbol{\varphi}_2|}\left((1+\cos(|\boldsymbol{\varphi}_1|))\sin(|\boldsymbol{\varphi}_2|) - \frac{\boldsymbol{\varphi}_1\boldsymbol{\varphi}_2}{|\boldsymbol{\varphi}_1||\boldsymbol{\varphi}_2|}\sin(|\boldsymbol{\varphi}_1|)(1-\cos(|\boldsymbol{\varphi}_2|))\right)\\
&\quad \left. - \frac{[\boldsymbol{\varphi}_1,\boldsymbol{\varphi}_2]}{|\boldsymbol{\varphi}_1||\boldsymbol{\varphi}_2|}\left(\sin(|\boldsymbol{\varphi}_1|)\sin(|\boldsymbol{\varphi}_2|) - \frac{\boldsymbol{\varphi}_1\boldsymbol{\varphi}_2}{|\boldsymbol{\varphi}_1||\boldsymbol{\varphi}_2|}(1-\cos(|\boldsymbol{\varphi}_1|))(1-\cos(|\boldsymbol{\varphi}_2|))\right)\right\}.
\end{aligned} \tag{15.2}$$

It seems to be also possible to find it from the composition law for $SU(2)$ using the formulae (9.2) and (9.3).

We calculate now the antisymmetric operator C for the product of two three-dimensional rotations $\mathsf{R} = \mathsf{R}_1\mathsf{R}_2$ (*i.e.*, $\mathsf{R}_1^{-1} = \mathsf{R}_1^{\mathrm{T}}$, $\mathsf{R}_2^{-1} = \mathsf{R}_2^{\mathrm{T}}$). If we substitute $\mathsf{R} \to \mathsf{R}_1\mathsf{R}_2$ in the formula for C in last of Equations (14.4) and express R_1 and R_2 by the antisymmetric operators C_1 and C_2 we find

$$\begin{aligned}
\mathsf{C} &= \frac{\mathsf{R}_1\mathsf{R}_2 - (\mathsf{R}_1\mathsf{R}_2)^{-1}}{1+\langle \mathsf{R}_1\mathsf{R}_2\rangle}\\
&= \frac{\left(\mathrm{I} + 2\frac{\mathsf{C}_1+\mathsf{C}_1^2}{1+[\mathsf{C}_1]}\right)\left(\mathrm{I} + 2\frac{\mathsf{C}_2+\mathsf{C}_2^2}{1+[\mathsf{C}_1]}\right) - \left(\mathrm{I} - 2\frac{\mathsf{C}_2-\mathsf{C}_2^2}{1+[\mathsf{C}_2]}\right)\left(\mathrm{I} - 2\frac{\mathsf{C}_1-\mathsf{C}_1^2}{1+[\mathsf{C}_1]}\right)}{1+\langle \mathsf{R}_1\mathsf{R}_2\rangle}
\end{aligned}$$

$$
\begin{aligned}
&= 4\frac{\left(1+[C_2]\right)C_1+\left(1+[C_1]\right)C_2+[C_1,C_2]+C_1^2C_2+C_2C_1^2+C_1C_2^2+C_2^2C_1+\left[C_1^2,C_2^2\right]}{\left(1+\langle R_1R_2\rangle\right)\left(1+[C_1]\right)\left(1+[C_2]\right)} \\
&= 4\frac{\left(1+\frac{1}{2}\langle C_1C_2\rangle\right)\left(C_1+C_2+[C_1,C_2]\right)}{\left(1+\langle R_1R_2\rangle\right)\left(1+[C_1]\right)\left(1+[C_2]\right)}, \quad \left(\left\langle C_1^2\right\rangle=-2[C_1],\left\langle C_2^2\right\rangle=-2[C_2]\right),
\end{aligned}
\tag{15.3}
$$

where we used the specific three-dimensional identities for general antisymmetric operators C_1 and C_2

$$
C_1^2C_2+C_2C_1^2=\frac{1}{2}\left(\langle C_1C_2\rangle C_1+\left\langle C_1^2\right\rangle C_2\right), \quad \left[C_1^2,C_2^2\right]=\frac{1}{2}\langle C_1C_2\rangle[C_1,C_2], \tag{15.4}
$$

which can be directly checked. From the first of these identities and after multiplication of it with C_2 and then forming the traces we find scalar identities for general antisymmetric operators C_1 and C_2 as follows

$$
\left\langle C_1^2C_2\right\rangle=0, \quad \left\langle C_1^2C_2^2\right\rangle=\frac{1}{4}\left(\langle C_1C_2\rangle^2+\left\langle C_1^2\right\rangle\left\langle C_2^2\right\rangle\right), \quad \left\langle (C_1C_2)^2\right\rangle=\frac{1}{2}\langle C_1C_2\rangle^2. \tag{15.5}
$$

Furthermore, starting from first of Equations (14.4) with $R=R_1R_2$ and using the identities (15.5) we find

$$
1+\langle R_1R_2\rangle=\frac{4\left(1+\frac{1}{2}\langle C_1C_2\rangle\right)^2}{\left(1+[C_1]\right)\left(1+[C_2]\right)}. \tag{15.6}
$$

Inserting the relation (15.6) into (15.3) this provides the composition law for two rotations in the following final operator form

$$
R=R_1R_2, \quad \Leftrightarrow \quad C=\frac{C_1+C_2+[C_1,C_2]}{1+\frac{1}{2}\langle C_1C_2\rangle}=-C^T, \tag{15.7}
$$

with the only non-vanishing invariant $[C]=-\frac{1}{2}\left\langle C^2\right\rangle$ of the antisymmetric operator C

$$
\begin{aligned}
&\left\langle C^2\right\rangle=\frac{\left\langle C_1^2\right\rangle+\left\langle C_2^2\right\rangle+2\langle C_1C_2\rangle+\frac{1}{2}\left(\langle C_1C_2\rangle^2-\left\langle C_1^2\right\rangle\left\langle C_2^2\right\rangle\right)}{\left(1+\frac{1}{2}\langle C_1C_2\rangle\right)^2}=-2[C], \\
&\Rightarrow 1+[C]=1-\frac{1}{2}\left\langle C^2\right\rangle=\frac{\left(1-\frac{1}{2}\left\langle C_1^2\right\rangle\right)\left(1-\frac{1}{2}\left\langle C_2^2\right\rangle\right)}{\left(1+\frac{1}{2}\langle C_1C_2\rangle\right)^2}=\frac{\left(1+[C_1]\right)\left(1+[C_2]\right)}{\left(1+\frac{1}{2}\langle C_1C_2\rangle\right)^2}.
\end{aligned}
\tag{15.8}
$$

From (15.7) using the definition (14.8) of the vector parameter $\boldsymbol{c}$ by the antisymmetric operator C follows the composition law expressed by the vector parameter $\boldsymbol{c}$ in agreement with [6] [37][7]

[7]Fyodorov introduces in formula (2.4) in [6] c_j by $C_{kl}\equiv c_{kl}^{\times}=\varepsilon_{kjl}c_j$ but its inversion has to be then $c_j=\frac{1}{2}\varepsilon_{kjl}C_{kl}$ that by the sign is not identical with our (14.8) and is only a mistake since it is correct in earlier [5]. Pars [37] (Equation (7.9.2), without citation) has a minus sign in front of the vector product but this substitution changes only the direction of the rotation.

$$\boldsymbol{c} \equiv \langle \boldsymbol{c}_1, \boldsymbol{c}_2 \rangle = \frac{\boldsymbol{c}_1 + \boldsymbol{c}_2 + [\boldsymbol{c}_1, \boldsymbol{c}_2]}{1 - \boldsymbol{c}_1 \boldsymbol{c}_2}, \tag{15.9}$$

and is near to a formula of Rodrigues cited, e.g., in [33] and [35] (p. 17). The appearance of the vector product $[\boldsymbol{c}_1, \boldsymbol{c}_2]$ in this relation is the consequence that the order of two rotations $\mathsf{R}_1\mathsf{R}_2$ does not commute. A linearization is obtained by stretching $R\boldsymbol{c} \to \boldsymbol{x}$ (**Appendix C**). From (15.9) follows for the modulus $|\boldsymbol{c}|$

$$|\boldsymbol{c}| \equiv \sqrt{\boldsymbol{c}^2} = \pm \frac{\sqrt{(\boldsymbol{c}_1 + \boldsymbol{c}_2)^2 + [\boldsymbol{c}_1, \boldsymbol{c}_2]^2}}{1 - \boldsymbol{c}_1 \boldsymbol{c}_2}, \quad \Rightarrow \quad 1 + \boldsymbol{c}^2 = \frac{(1 + \boldsymbol{c}_1^2)(1 + \boldsymbol{c}_2^2)}{(1 - \boldsymbol{c}_1 \boldsymbol{c}_2)^2}. \tag{15.10}$$

Our derivation of the composition law (15.9) of two vector parameters $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ distinguishes from the more directly obtained of Fyodorov by using specific three-dimensional identities for antisymmetric operators. The composition of two vector parameters $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ to a new vector parameter $\boldsymbol{c}$ for the product of two rotations $\mathsf{R}_1\mathsf{R}_2 = \mathsf{R}$ can be obtained from (12.5) but this is also possible using (9.2)-(9.4). The composition law (9.3) for the vector parameter $\boldsymbol{\varphi}$ is more complicated than that for the vector parameter $\boldsymbol{c}$ and can be also obtained from (15.9) using (14.8) and (14.6). This is an advantage of using the vector parameter $\boldsymbol{c}$ which, however, is nonlinear in $\boldsymbol{\varphi}$.

Following Fyodorov ([6]) we introduced in (9.5) and used in (15.9) the convenient symbol $\langle \boldsymbol{c}_1, \boldsymbol{c}_2 \rangle$ for the composition of two vector parameters $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ corresponding to the product $\mathsf{R}_1\mathsf{R}_2$ or $\mathsf{S}_1\mathsf{S}_2$. It allows to formulate some composition rules in a simple way. In general, due to non-commutativity but associativity of the group multiplication one has

$$\langle \boldsymbol{c}_1, \boldsymbol{c}_2 \rangle \neq \langle \boldsymbol{c}_2, \boldsymbol{c}_1 \rangle, \quad \langle \langle \boldsymbol{c}_1, \boldsymbol{c}_2 \rangle, \boldsymbol{c}_3 \rangle = \langle \boldsymbol{c}_1, \langle \boldsymbol{c}_2, \boldsymbol{c}_3 \rangle \rangle \equiv \langle \boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3 \rangle, \tag{15.11}$$

meaning that one may omit the inner brackets in the composition of three transformations. Explicitly, we calculate in considered case ($[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]$ denotes the volume product of three vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$)

$$\langle \boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3 \rangle = \frac{(1 - \boldsymbol{c}_2\boldsymbol{c}_3)\boldsymbol{c}_1 + (1 + \boldsymbol{c}_1\boldsymbol{c}_3)\boldsymbol{c}_2 + (1 - \boldsymbol{c}_1\boldsymbol{c}_2)\boldsymbol{c}_3 + [\boldsymbol{c}_1, \boldsymbol{c}_2] + [\boldsymbol{c}_1, \boldsymbol{c}_3] + [\boldsymbol{c}_2, \boldsymbol{c}_3]}{1 - \boldsymbol{c}_1\boldsymbol{c}_2 - \boldsymbol{c}_1\boldsymbol{c}_3 - \boldsymbol{c}_2\boldsymbol{c}_3 - [\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3]}. \tag{15.12}$$

An important special case of (15.12) corresponds to $\mathsf{R}' = \mathsf{R}_0\mathsf{R}\mathsf{R}_0^{-1}$ of $SO(3, \mathbb{R})$ where R and R_0 which provide all conjugate elements R'. Since in both cases the transition to the inverse element means the transition $\boldsymbol{\varphi} \leftrightarrow -\boldsymbol{\varphi}$ or $\boldsymbol{c} \leftrightarrow -\boldsymbol{c}$, respectively, we have to calculate $\boldsymbol{c}' = \langle \boldsymbol{c}_0, \boldsymbol{c}, -\boldsymbol{c}_0 \rangle$ for which we find from (15.12)

$$\begin{aligned} \boldsymbol{c}' = \langle \boldsymbol{c}_0, \boldsymbol{c}, -\boldsymbol{c}_0 \rangle &\equiv \frac{(1 - |\boldsymbol{c}_0|^2)\boldsymbol{c} + 2(\boldsymbol{c}_0\boldsymbol{c})\boldsymbol{c}_0 - 2[\boldsymbol{c}_0, \boldsymbol{c}]}{1 + |\boldsymbol{c}_0|^2}, \\ \Rightarrow \quad |\boldsymbol{c}'|^2 = |\boldsymbol{c}|^2, \quad \boldsymbol{c}'\boldsymbol{c}_0 &= \boldsymbol{c}\boldsymbol{c}_0, \end{aligned} \tag{15.13}$$

This means that the vector parameters $\boldsymbol{c}$ to conjugate elements of the groups $SO(3, \mathbb{R})$ possess the same modulus (length) $|\boldsymbol{c}|$ and therefore correspond to rotations about the same angle but, in general, to different rotation axes $\boldsymbol{n} \equiv \frac{\boldsymbol{c}}{|\boldsymbol{c}|} = \frac{\boldsymbol{\varphi}}{|\boldsymbol{\varphi}|}$ and $\boldsymbol{n}' \equiv \frac{\boldsymbol{c}'}{|\boldsymbol{c}'|} = \frac{\boldsymbol{\varphi}'}{|\boldsymbol{\varphi}'|}$. If the rotation axes of $\boldsymbol{c}$ and of $\boldsymbol{c}_0$ are parallel that means $[\boldsymbol{c}_0, \boldsymbol{c}] = \boldsymbol{0}$ then one finds

$$[\boldsymbol{c}_0, \boldsymbol{c}] = \boldsymbol{0}, \quad \Rightarrow \quad \boldsymbol{c}' = \langle \boldsymbol{c}_0, \boldsymbol{c}, -\boldsymbol{c}_0 \rangle \equiv \frac{\left(1-|\boldsymbol{c}_0|^2\right)\boldsymbol{c} + 2\left(|\boldsymbol{c}_0|^2\right)\boldsymbol{c}}{1+|\boldsymbol{c}_0|^2} = \boldsymbol{c}. \tag{15.14}$$

The formulae (15.13) and (15.14) show that rotations around arbitrary axes but with the same rotation angle belong to the same class of conjugate elements and that with different rotation angles to different classes of conjugate elements.

16. Invariant Integration over Group $SO(3,\mathbb{R})$ and $SU(2)$

The s-dimensional irreducible representations $D_k^i(g)$ of a finite group G with N elements $g \in G$ possess for any fixed $h \in G$ the following orthogonality relations in index notation for operators, e.g., [20] [26]

$$\frac{1}{N}\sum_{g\in G} D_k^i(g) D_l^j\left(g^{-1}\right) = \frac{1}{N}\sum_{g\in G} D_k^i(gh) D_l^j\left((gh)^{-1}\right) = \frac{1}{s}\delta_k^j \delta_l^i. \tag{16.1}$$

In transition to a Lie group one has to substitute $g \in G$ by the chosen parameter, in our case of $SO(3,\mathbb{R})$, by the vector parameter $\boldsymbol{c}$ or by the vector parameter $\boldsymbol{\varphi}$ and the summation over the discrete elements g by integration over the chosen parameter, where g^{-1} must be substituted by the corresponding parameter for the inverse element, in our case by $-\boldsymbol{c}$ that means in our first considered case and h by a fixed parameter $\boldsymbol{c}_0$

$$\begin{aligned} &\frac{1}{V_G}\int_G \mathrm{d}^3 c\, \mu(\boldsymbol{c}) D_k^i(\boldsymbol{c}) D_l^j(-\boldsymbol{c}) \\ &= \frac{1}{V_G}\int_G \mathrm{d}^3 c\, \mu(\boldsymbol{c}) D_k^i\left(\langle \boldsymbol{c}, \boldsymbol{c}_0 \rangle\right) D_l^j\left(-\langle \boldsymbol{c}, \boldsymbol{c}_0 \rangle\right) = \frac{1}{s}\delta_k^j \delta_l^i, \end{aligned} \tag{16.2}$$

with V_G the group volume in chosen parametrization defined by[8]

$$V_G \equiv \int_G \mathrm{d}^3 c\, \mu(\boldsymbol{c}) = \int_G \mathrm{d}c_1 \wedge \mathrm{d}c_2 \wedge \mathrm{d}c_3\, \mu(\boldsymbol{c}). \tag{16.3}$$

The function $\frac{\mu(\boldsymbol{c})}{V_G}$ corresponding to $\frac{1}{N}$ in (16.1) is an "invariant" measure which gives the weight of every group element to parameter $\boldsymbol{c}$ in the chosen parametrization and depends on it.

In this Section we derive the invariance of a certain measure of the group $SO(3,\mathbb{R})$ and by parameter transformation of a measure of $SU(2)$. For this purpose we make the substitutions $\boldsymbol{c} \to \boldsymbol{c}', \boldsymbol{c}_1 \to \boldsymbol{c}, \boldsymbol{c}_2 \to d\boldsymbol{c}'$ in the composition formula (15.9), leading to

$$\boldsymbol{c}' = \frac{\boldsymbol{c} + \boldsymbol{c}_0 + |\boldsymbol{c}, \boldsymbol{c}_0|}{1 - \boldsymbol{c}\boldsymbol{c}_0}, \quad \Rightarrow \quad 1 + \boldsymbol{c}'^2 = \frac{\left(1+\boldsymbol{c}^2\right)\left(1+\boldsymbol{c}_0^2\right)}{\left(1-\boldsymbol{c}\boldsymbol{c}_0\right)^2}, \tag{16.4}$$

[8]In our case of vector parameter $\boldsymbol{c}$ the weight $\mu(\boldsymbol{c})$ depends only on the modulus $|\boldsymbol{c}|$.

where the fixed $\boldsymbol{c}_0$ transforms an arbitrary vector $\boldsymbol{c}$ into a certain vector $\boldsymbol{c}'$. The general composition law $\boldsymbol{c} \to \boldsymbol{c}'$ is nonlinear. We calculate now the differential change of $\boldsymbol{c} + \mathrm{d}\boldsymbol{c}'$ for a differential change of $\boldsymbol{c} + \mathrm{d}\boldsymbol{c}$ with fixed $\boldsymbol{x}_0$

$$\mathrm{d}c_i' = \frac{\partial c_i'}{\partial c_l}\mathrm{d}c_l = \left\{ \frac{\delta_{il} - \varepsilon_{ikl}c_{0,k}}{1 - \boldsymbol{c}\boldsymbol{c}_0} + \frac{\left(c_i + c_{0,i} + \varepsilon_{ijk}c_j c_{0,k}\right)c_{0,l}}{\left(1 - \boldsymbol{c}\boldsymbol{c}_0\right)^2} \right\}\mathrm{d}c_l. \tag{16.5}$$

or in coordinate invariant notation

$$\mathrm{d}\boldsymbol{c}' = \frac{\left(1 - \boldsymbol{c}\boldsymbol{c}_0\right)\left(\mathrm{I} - \left[\boldsymbol{c}_0\right]\right) + \left(\boldsymbol{c} + \boldsymbol{c}_0 + \left[\boldsymbol{c}, \boldsymbol{c}_0\right]\right) \cdot \boldsymbol{c}_0}{\left(1 - \boldsymbol{c}\boldsymbol{c}_0\right)^2}\mathrm{d}\boldsymbol{c}. \tag{16.6}$$

With substitutions $\alpha = \gamma \to 1 - \boldsymbol{c}\boldsymbol{c}_0, \beta \to 1$ and $\boldsymbol{b} \to \boldsymbol{c} + \boldsymbol{c}_0 + \left[\boldsymbol{c}, \boldsymbol{c}_0\right]$, $\tilde{\boldsymbol{b}} = \boldsymbol{c} \to \boldsymbol{c}_0$ we find from (D.2) in **Appendix D** for the determinant in numerator of (16.6) the astonishingly simple result

$$\left|\left(1 - \boldsymbol{c}\boldsymbol{c}_0\right)\left(\mathrm{I} - \left[\boldsymbol{c}_0\right]\right) + \left(\boldsymbol{c} + \boldsymbol{c}_0 + \left[\boldsymbol{c}, \boldsymbol{c}_0\right]\right) \cdot \boldsymbol{c}_0\right| = \left(1 - \boldsymbol{c}\boldsymbol{c}_0\right)^2\left(1 + \boldsymbol{c}_0^2\right)^2. \tag{16.7}$$

Now we calculate from (16.5) using the determinant (16.7) and the identity in (16.4) the changes of the volumes

$$\mathrm{d}c_1' \wedge \mathrm{d}c_2' \wedge \mathrm{d}c_3' = \frac{\left(1 + \boldsymbol{c}_0^2\right)^2}{\left(1 - \boldsymbol{c}\boldsymbol{c}_0\right)^4}\mathrm{d}c_1 \wedge \mathrm{d}c_2 \wedge \mathrm{d}c_3 = \frac{\left(1 + \boldsymbol{c}'^2\right)^2}{\left(1 + \boldsymbol{c}^2\right)^2}\mathrm{d}c_1 \wedge \mathrm{d}c_2 \wedge \mathrm{d}c_3 \tag{16.8}$$

This is not an invariant volume element for the group but if we divide both sides by $\left(1 + \boldsymbol{c}'^2\right)^2$ we find

$$\mathrm{d}V_G = \frac{\mathrm{d}c_1' \wedge \mathrm{d}c_2' \wedge \mathrm{d}c_3'}{\left(1 + \boldsymbol{c}'^2\right)^2} = \frac{\mathrm{d}c_1 \wedge \mathrm{d}c_2 \wedge \mathrm{d}c_3}{\left(1 + \boldsymbol{c}^2\right)^2}. \tag{16.9}$$

However, this is invariant after the transformation $\boldsymbol{c} \to \boldsymbol{c}'$ and does not depend on the chosen fixed parameter $\boldsymbol{c}_0$. For the group volume we calculate

$$V_G = \int_{-\infty}^{+\infty}\mathrm{d}c_1\int_{-\infty}^{+\infty}\mathrm{d}c_2\int_{-\infty}^{+\infty}\mathrm{d}c_3\frac{1}{\left(1 + \left(c_1^2 + c_2^2 + c_3^2\right)\right)^2} = \int_0^{\infty}\frac{4\pi\mathrm{d}|\boldsymbol{c}||\boldsymbol{c}|^2}{\left(1 + |\boldsymbol{c}|^2\right)^2} = \pi^2. \tag{16.10}$$

Thus the invariant measure of $SO(3, \mathbb{R})$ with integration of $\boldsymbol{c}$ over the whole space $\boldsymbol{c} \in \mathbb{R}^3$ is

$$\frac{1}{\pi^2}\frac{\mathrm{d}c_1 \wedge \mathrm{d}c_2 \wedge \mathrm{d}c_3}{\left(1 + \boldsymbol{c}^2\right)^2}, \quad \mu(\boldsymbol{c}) \equiv \frac{1}{\left(1 + \boldsymbol{c}^2\right)^2}, \tag{16.11}$$

for certain translation-invariant functions (e.g., representations of the group, in particular, the identity representation). This result agrees with Fyodorov [6] (§. 6) who give a modified derivation, likely, the only known such result.

From relation (14.11) between the vector parameters $\boldsymbol{c}$ and $\boldsymbol{\varphi}$ follows

$$\mathrm{d}c_i = \frac{\partial c_i}{\partial \varphi_k}\mathrm{d}\varphi_k = \left\{ \frac{\mathrm{tg}\left(\frac{|\boldsymbol{\varphi}|}{2}\right)}{|\boldsymbol{\varphi}|}\left(\delta_{ik} - \frac{\varphi_i\varphi_k}{|\boldsymbol{\varphi}|^2}\right) + \frac{1}{2\cos^2\left(\frac{|\boldsymbol{\varphi}|}{2}\right)}\frac{\varphi_i\varphi_k}{|\boldsymbol{\varphi}|^2} \right\}\mathrm{d}\varphi_k, \tag{16.12}$$

and using the determinant of the transformation matrix between $\mathrm{d}c_i$ and $\mathrm{d}\varphi_k$ we find

$$\mathrm{d}^3c = \frac{tg^2\left(\frac{|\boldsymbol{\varphi}|}{2}\right)}{2|\boldsymbol{\varphi}|^2\cos^2\left(\frac{|\boldsymbol{\varphi}|}{2}\right)}\mathrm{d}^3\varphi, \quad \mathrm{d}^3c \equiv \mathrm{d}c_1 \wedge \mathrm{d}c_2 \wedge \mathrm{d}c_3, \quad \mathrm{d}^3\varphi \equiv \mathrm{d}\varphi_1 \wedge \mathrm{d}\varphi_2 \wedge \mathrm{d}\varphi_3. \tag{16.13}$$

Due to (14.11) and taking into account $\frac{1}{1+|\boldsymbol{c}|^2} = \cos^2\left(\frac{|\boldsymbol{\varphi}|}{2}\right)$ (see (14.8)), we may choose the volume element $\mathrm{d}V_G$ of the invariant integration over the group $SO(3,\mathbb{R})$ as follows

$$\mathrm{d}V_G = \frac{1}{8}\left(\frac{\sin\left(\frac{|\boldsymbol{\varphi}|}{2}\right)}{\frac{|\boldsymbol{\varphi}|}{2}}\right)^2 \mathrm{d}^3\varphi = \frac{1}{2}\sin^2\left(\frac{|\boldsymbol{\varphi}|}{2}\right)\mathrm{d}|\boldsymbol{\varphi}| \wedge \mathrm{d}^2n, \quad \boldsymbol{n} \equiv \frac{\boldsymbol{\varphi}}{|\boldsymbol{\varphi}|}, \tag{16.14}$$

where d^2n denotes the area element of the surface $\mathbb{S}^2$ of unit vectors $|\boldsymbol{n}| = 1$ of a three-dimensional ball $\mathbb{B}$ and the integration of $|\boldsymbol{\varphi}|$ goes from $|\boldsymbol{\varphi}| = 0$ up to $|\boldsymbol{\varphi}| = \pi$ for $SO(3,\mathbb{R})$ and from $|\boldsymbol{\varphi}| = 0$ up to $|\boldsymbol{\varphi}| = 2\pi$ for $SU(2)$ with weight $\mathrm{d}|\boldsymbol{\varphi}||\boldsymbol{\varphi}|^2 \to \mathrm{d}|\boldsymbol{\varphi}|\frac{1}{2}\sin^2\left(\frac{|\boldsymbol{\varphi}|}{2}\right)$

$$\begin{aligned} V_G &= 4\pi\int_0^{\pi}\mathrm{d}|\boldsymbol{\varphi}|\frac{1}{2}\sin^2\left(\frac{|\boldsymbol{\varphi}|}{2}\right) = \pi^2, \quad \left(SO(3,\mathbb{R})\right) \\ V_G &= 4\pi\int_0^{2\pi}\mathrm{d}|\boldsymbol{\varphi}|\frac{1}{2}\sin^2\left(\frac{|\boldsymbol{\varphi}|}{2}\right) = 2\pi^2, \quad \left(SU(2)\right). \end{aligned} \tag{16.15}$$

This suggests that the invariant measure of $SO(3,\mathbb{R})$ using the vector parameter $\boldsymbol{\varphi}$ is

$$\frac{1}{8\pi^2}\left(\frac{\sin\left(\frac{|\boldsymbol{\varphi}|}{2}\right)}{\frac{|\boldsymbol{\varphi}|}{2}}\right)^2 \mathrm{d}^3\varphi, \quad \mu(\boldsymbol{\varphi}) \equiv \frac{1}{8}\left(\frac{\sin\left(\frac{|\boldsymbol{\varphi}|}{2}\right)}{\frac{|\boldsymbol{\varphi}|}{2}}\right)^2, \quad \left(SO(3,\mathbb{R})\right), \tag{16.16}$$

with integration of appropriate functions over a ball of radius $|\boldsymbol{\varphi}| = \pi$ and the invariant measure of $SU(2)$ is

$$\frac{1}{16\pi^2}\left(\frac{\sin\left(\frac{|\boldsymbol{\varphi}|}{2}\right)}{\frac{|\boldsymbol{\varphi}|}{2}}\right)^2 \mathrm{d}^3\varphi, \quad \mu(\boldsymbol{\varphi}) \equiv \frac{1}{8}\left(\frac{\sin\left(\frac{|\boldsymbol{\varphi}|}{2}\right)}{\frac{|\boldsymbol{\varphi}|}{2}}\right)^2, \quad \left(SU(2)\right), \tag{16.17}$$

with integration of appropriate functions over a ball of radius $|\boldsymbol{\varphi}| = 2\pi$. Result (16.16) is similar to that for mean value and invariant measure in the parametrization by Euler angles (Lyubarskij [20], §. 16).

17. Stereographic Projection of Unit Sphere from North Pole to an Extended Complex Equator Plane and Its Connection to 3D Rotations

We now deal with a relation between rotations of the surface $\mathbb{S}^2$ of a unit ball $\mathbb{B}^3$ in 3-dimensional space (Riemann sphere) and fractional linear transformation (Möbius transformations) of an extended complex plane $\overline{\mathbb{C}} \equiv \mathbb{C} \cup \infty$ by stereographic projection of the unit sphere onto the complex plane.

We consider a plane $\mathbb{R}^2 \to \mathbb{C}$ with the Cartesian coordinates (u,v) embedded as Equatorial plane $z=0$ into a 3-dimensional Euclidean space $\mathbb{R}^3$ with the Cartesian coordinates (x,y,z). In the 3-dimensional space $\mathbb{R}^3$ we consider the surface of a sphere $\mathbb{S}^2$ with unit radius $R=1$ described by

$$x^2+y^2+z^2=1, \tag{17.1}$$

and make the stereographic projection of this spherical surface from the North Pole $(x,y,z)=(0,0,1)$ onto the plane with the coordinates $(x=u, y=v)$. The relation of the coordinates of the unit sphere (x,y,z) with their projection points (u,v) onto the Equatorial plane is given by the nonlinear mapping (see Figure 6)

$$\begin{aligned} &(x,y,z) \to (u,v) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right), \quad \left(z = \pm\sqrt{1-x^2-y^2} \equiv \pm\sqrt{1-r^2}\right), \\ &s \equiv \sqrt{u^2+v^2} = \frac{\sqrt{x^2+y^2}}{1-z} \equiv \frac{r}{1-z}, \quad \left(\frac{s}{R} = \frac{s-r}{z}, \quad r \equiv \sqrt{x^2+y^2}, \quad R=1\right), \end{aligned} \tag{17.2}$$

with the unique inversion ($0 \le r \le 1, 0 \le s \le \infty$)

$$\begin{aligned} &(u,v) \to (x,y,z) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right), \quad \Rightarrow \quad x^2+y^2+z^2=1, \\ &s = \frac{r}{1-z} = \frac{r}{1\mp\sqrt{1-r^2}} = \frac{1\pm\sqrt{1-r^2}}{r}, \quad \Rightarrow \quad 0 \le (1-rs)^2 = 1-r^2 \le 1. \end{aligned} \tag{17.3}$$

The Northern part of the spherical surface $\mathbb{S}^2$ is mapped onto the outer part $u^2+v^2>1$ of the unit disc ($\mathbb{D}: u^2+v^2 \le 1$) and the Southern part of the sphere to the inner part $u^2+v^2<1$ of this unit disc $\mathbb{D}$. The Equator E of the sphere is mapped onto the border circle $u^2+v^2=1$ of the unit disc $\mathbb{D}$, the South pole to the center $(u,v)=(0,0)$ and the North Pole to the infinity point of the extended complex plane $\overline{\mathbb{C}}$.

Instead of the real coordinates (u,v) and (x,y) we introduce now pairs of complex conjugate coordinates (w,w^*) and (z_+,z_-) in the following way

$$\begin{aligned} &w \equiv u+\mathrm{i}v, \quad w^* \equiv u-\mathrm{i}v, \quad u \equiv \frac{w+w^*}{2}, \quad v \equiv -\mathrm{i}\frac{w-w^*}{2}, \\ &z_+ \equiv x+\mathrm{i}y, \quad z_- \equiv x-\mathrm{i}y, \quad x \equiv \frac{z_++z_-}{2}, \quad y \equiv -\mathrm{i}\frac{z_+-z_-}{2}. \end{aligned} \tag{17.4}$$

The Equation (17.1) for the unit sphere is then

$$z_+z_-+z^2=1. \tag{17.5}$$

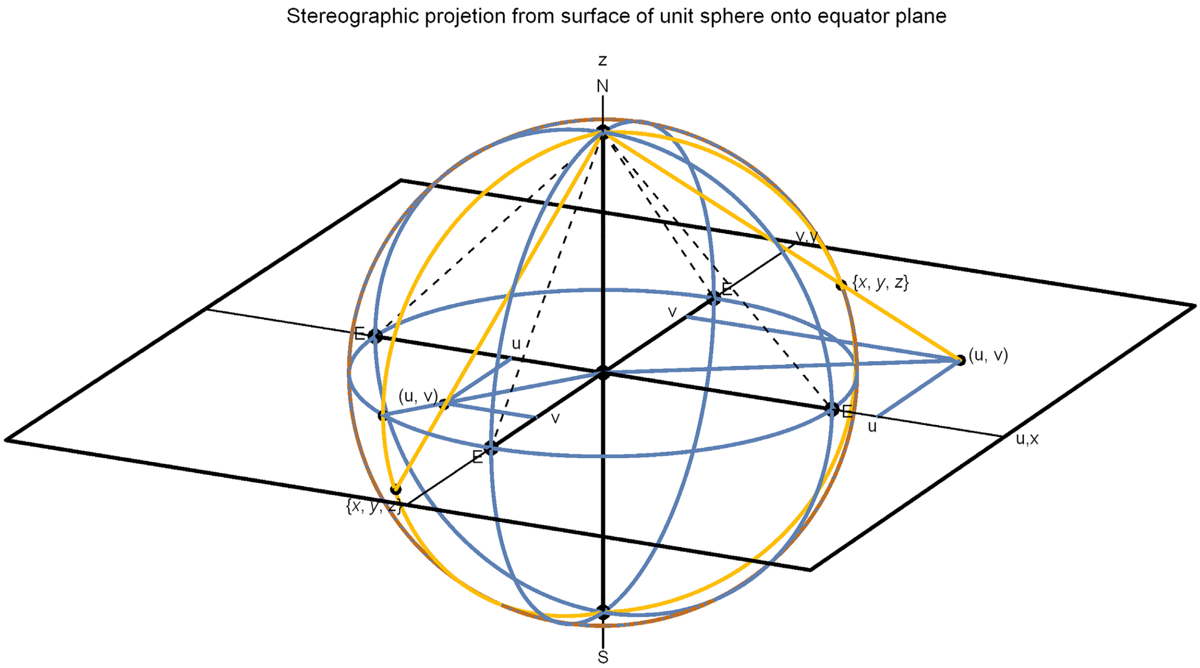

Figure 6. Stereographic projection of the surface of the unit sphere from North Pole onto the Equator plane. There are shown the projections of two points from the surface of the sphere with coordinates (x,y,z) onto the Equator plane with coordinates (u,v), one with $z>0$ outside to the unit disc and the second with $z<0$ inside to the unit disc in the Equator plane. A 3D-rotation of the surface of the sphere makes a unique fractional (or Möbius) transformation of the Equator plane described by formulae in the text.

The mapping (17.2) takes on the form

$$(z_+, z_-, z) \to (w, w^*) = \left(\frac{z_+}{1-z}, \frac{z_-}{1-z}\right), \quad z = \pm\sqrt{1-z_+z_-}, \tag{17.6}$$

with the unique inversion

$$(w, w^*) \to (z_+, z_-, z) = \left(\frac{2w}{ww^*+1}, \frac{2w^*}{ww^*+1}, \frac{ww^*-1}{ww^*+1}\right). \tag{17.7}$$

Since in the following we use multiplication of complex numbers we can identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ which we extend by a single point ∞ to the extended complex plane $\overline{\mathbb{C}}$. The North Pole $(z_+, z_-, z) = (0,0,1)$ is now mapped by the stereographic projection onto this unique new element ∞ of $\overline{\mathbb{C}}$.

We now consider a fractional linear transformation $(w, w^*) \to (w', w'^*)$ of the extended complex plane $\overline{\mathbb{C}}$ as follows

$$w' = \frac{\kappa w + \mu}{\lambda w + \nu}, \quad w'^* = \frac{\kappa^* w^* + \mu^*}{\lambda^* w^* + \nu^*}, \quad \left|\pm\begin{pmatrix} \kappa & \lambda \\ \mu & \nu \end{pmatrix}\right| = \kappa\nu - \lambda\mu = 1, \tag{17.8}$$

where $(\kappa, \lambda, \mu, \nu)$ are 3 independent, in general, complex numbers. The determinant of the coefficient matrix is equal to +1 that means unimodularity. The fractional transformations of this kind as is well known and can easily be checked form a group which as group of matrices $\begin{pmatrix} \kappa, & \lambda \\ \mu, & \nu \end{pmatrix}$ is the special linear group

$SL(2,\mathbb{C})$. For the transformed coordinates (z'_+, z'_-, z') using the connection (17.7) for the transformed (primed) coordinates and the transformation (17.8) we find

$$\begin{aligned} z'_+ &= \frac{2w'}{w'w'^*+1} = \frac{2(\kappa w+\mu)(\lambda^* w^* + \nu^*)}{(\kappa w+\mu)(\kappa^* w^* + \mu^*)+(\lambda w+\nu)(\lambda^* w^* + \nu^*)}, \quad z'_- = z'^*_+, \\ z' &= \frac{w'w'^*-1}{w'w'^*+1} = \frac{(\kappa w+\mu)(\kappa^* w^* + \mu^*)-(\lambda w+\nu)(\lambda^* w^* + \nu^*)}{(\kappa w+\mu)(\kappa^* w^* + \mu^*)+(\lambda w+\nu)(\lambda^* w^* + \nu^*)}. \end{aligned} \tag{17.9}$$

If we now specialize the fractional linear transformation supposing

$$\nu = \kappa^*, \quad \mu = -\lambda^*, \quad \Leftrightarrow \quad \kappa = \nu^*, \quad \lambda = -\mu^*, \quad \Rightarrow \quad \begin{pmatrix} \kappa & \lambda \\ \mu & \nu \end{pmatrix} = \begin{pmatrix} \kappa & \lambda \\ -\lambda^* & \kappa^* \end{pmatrix}, \tag{17.10}$$

which specializes the transformation from a general $SL(2,\mathbb{C})$ to a $SU(2)$ matrix $\begin{pmatrix} \kappa & \lambda \\ \mu & \nu \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$ the relation (17.9) can be expressed

$$\begin{aligned} z'_+ &= \frac{2\left(-\kappa\mu(ww^*-1)+\kappa^2 w-\mu^2 w^*\right)}{\underbrace{(\kappa\nu-\lambda\mu)}_{=1}(ww^*+1)} = \kappa^2 z_+ - \mu^2 z_- - 2\kappa\mu z, \quad z'_- = z'^*_+, \\ z' &= \frac{(\kappa\nu+\lambda\mu)(ww^*-1)-2\kappa\lambda w+2\mu\nu w^*}{\underbrace{(\kappa\nu-\lambda\mu)}_{=1}(ww^*+1)} = -\kappa\lambda z_+ + \mu\nu z_- + (\kappa\nu+\lambda\mu)z. \end{aligned} \tag{17.11}$$

and using (17.7) as the following linear transformation $(z_+, z, z_-) \leftrightarrow (z'_+, z', z'_-)$

$$(z'_+, z', z'_-) = (z_+, z, z_-)\mathsf{R}, \quad \Leftrightarrow \quad \begin{pmatrix} z'_+ \\ z' \\ z'_- \end{pmatrix} = \mathsf{R}^{\mathrm{T}} \begin{pmatrix} z_+ \\ z \\ z_- \end{pmatrix}, \tag{17.12}$$

with the (complex) rotation matrix in representation with respect to the (complex) basis (z_+, z, z_-)

$$\begin{aligned} \mathsf{R} &\equiv \begin{pmatrix} \kappa^2 & -\kappa\lambda & -\lambda^2 \\ -2\kappa\mu & \kappa\nu+\lambda\mu & 2\lambda\nu \\ -\mu^2 & \mu\nu & \nu^2 \end{pmatrix} = \begin{pmatrix} \kappa^2 & -\kappa\lambda & -\lambda^2 \\ 2\kappa\lambda^* & \kappa\kappa^*-\lambda\lambda^* & 2\kappa^*\lambda \\ -\lambda^{*2} & -\kappa^*\lambda^* & \kappa^{*2} \end{pmatrix}, \\ |\mathsf{R}| &= (\kappa\nu-\lambda\mu)^3 = 1, \quad \langle \mathsf{R} \rangle = [\mathsf{R}] = (\kappa+\nu)^2 - 1. \end{aligned} \tag{17.13}$$

with the two possible identifications distinguished by sign

$$\begin{pmatrix} \kappa & \lambda \\ \mu & \nu \end{pmatrix} = \pm \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \equiv \pm \mathsf{S}, \quad \nu = \kappa^*, \quad \mu = -\lambda^*, \tag{17.14}$$

this leads to the same rotation matrix R in (12.1) explicitly given in (12.2). In representation with respect to the real basis (x, y, z) we find[9]

$$(x', y', z') = (x, y, z)\mathsf{R}, \quad \Leftrightarrow \quad \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \mathsf{R}^{\mathrm{T}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{17.15}$$

[9]We consider R as operator and do not distinguish the notation for R in representations with respect to different basis vectors; see also footnote before (12.3).

with the explicit form of the real matrix R and the same invariant as in (17.13)

$$\mathsf{R}=\begin{pmatrix}\frac{1}{2}\left(\kappa^{2}-\lambda^{2}-\mu^{2}+\nu^{2}\right) & -\frac{\mathrm{i}}{2}\left(\kappa^{2}+\lambda^{2}-\mu^{2}-\nu^{2}\right) & -\kappa\lambda+\mu\nu\\ \frac{\mathrm{i}}{2}\left(\kappa^{2}-\lambda^{2}+\mu^{2}-\nu^{2}\right) & \frac{1}{2}\left(\kappa^{2}+\lambda^{2}+\mu^{2}+\nu^{2}\right) & -\mathrm{i}\left(\kappa\lambda+\mu\nu\right)\\ -\kappa\mu+\lambda\nu & \mathrm{i}\left(\kappa\mu+\lambda\nu\right) & \kappa\nu+\lambda\mu\end{pmatrix}, \tag{17.16}$$

$$|\mathsf{R}|=\left(\kappa\nu-\lambda\mu\right)^{3}=1,\quad \langle\mathsf{R}\rangle=[\mathsf{R}]=\left(\kappa+\nu\right)^{2}-1.$$

Thus a 3D rotation of the surface of a unit sphere $\mathbb{S}^{3}$ corresponds in unique way to a fractional linear (Möbius) transformations of the extended complex plane $\bar{\mathbb{C}}$ of the form (17.8) equivalent to two different (by sign) unitary unimodular transformations of $SU(2)$. Another special linear fractional transformation is mentioned by Gürsey [21] (around and after Equation (4.27) there).

The mappings in the above considerations are made according to the following scheme

$$\begin{array}{ccc}
(x,y,z),\quad \left(x^{2}+y^{2}+z^{2}=1\right) & \leftrightarrow & (u,v),\quad \left(w,w^{*}\right)\equiv\left(u+\mathrm{i}v,u-\mathrm{i}v\right)\\
\uparrow & \text{stereogr. projection} & \uparrow\\
\text{3D rotation (17.15) with (17.13)} & & \text{fract. linear transf. (17.8) with (17.10) (17.17)}\\
\downarrow & \text{stereogr. projection} & \downarrow\\
(x',y',z'),\quad \left(x'^{2}+y'^{2}+z'^{2}=1\right) & \leftrightarrow & (u',v'),\quad \left(w',w'^{*}\right)\equiv\left(u'+\mathrm{i}v',u'-\mathrm{i}v'\right).
\end{array}$$

The general Möbius transformations studied in the complex analysis are much richer in content than touched here (e.g., Needham [38] with a great number of wonderful illustrations).

18. Conclusion

This article collects some basic mathematical aspects of the Lie group $SU(2)$ embedded in higher Lie groups such as $Sp(4,\mathbb{R})$ and inhomogeneous Lie group $I.SU(2)$ and starts from the middle of the nineteens with collecting material. It was after working in applications of the group $SU(1,1)$ in quantum optics and with the aim to treat such problems as light polarization of beams in reflection and refraction and to discuss the forms of polarized and unpolarized light. It works mainly with coordinate-invariant methods using with some advantage, the vector parameter $\boldsymbol{\varphi}$ instead of Euler angles (α,β,γ) and with advantage, the vector parameter $\boldsymbol{c}$ for $SO(3,\mathbb{R})$ of F.I. Fyodorov from Minsk. In this way some difficulties were met, for example, of the root diagrams for inhomogeneous groups or in the result for the invariant measures of $SO(3,\mathbb{R})$ which I tried to overcome or to represent as problem (**Appendix C**). A continuation of the article with more aspects of physical applications to polarization of light beams is possible and it is hoped to become realized.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendix A. Relation between Vector Parameter $\boldsymbol{\varphi}$ and Euler Angles (α,β,γ) for Group $SU(2)$

Using the vector parameter $\boldsymbol{\varphi}$, we obtained in (5.13) the matrix of the fundamental representation of $SU(2)$ in the form

$$\mathsf{S}=\begin{pmatrix}\cos\left(\frac{|\boldsymbol{\varphi}|}{2}\right)+\mathrm{i}\frac{\varphi_3}{|\boldsymbol{\varphi}|}\sin\left(\frac{|\boldsymbol{\varphi}|}{2}\right) & \mathrm{i}\frac{\varphi_-}{|\boldsymbol{\varphi}|}\sin\left(\frac{|\boldsymbol{\varphi}|}{2}\right)\\ \mathrm{i}\frac{\varphi_+}{|\boldsymbol{\varphi}|}\sin\left(\frac{|\boldsymbol{\varphi}|}{2}\right) & \cos\left(\frac{|\boldsymbol{\varphi}|}{2}\right)-\mathrm{i}\frac{\varphi_3}{|\boldsymbol{\varphi}|}\sin\left(\frac{|\boldsymbol{\varphi}|}{2}\right)\end{pmatrix}$$
$$\equiv\begin{pmatrix}\sqrt{T}\,\mathrm{e}^{\mathrm{i}\vartheta} & \sqrt{R}\,\mathrm{e}^{-\mathrm{i}\chi}\\ -\sqrt{R}\,\mathrm{e}^{\mathrm{i}\chi} & \sqrt{T}\,\mathrm{e}^{-\mathrm{i}\vartheta}\end{pmatrix},\quad |\boldsymbol{\varphi}|\equiv\sqrt{\varphi_1^2+\varphi_2^2+\varphi_3^2}=\sqrt{\varphi_-\varphi_++\varphi_3^2}, \tag{A.1}$$

and with abbreviations (T for transmission and R for reflection in application to reflection problems)

$$T\equiv\cos^2\left(\frac{|\boldsymbol{\varphi}|}{2}\right)+\frac{\varphi_3^2}{|\boldsymbol{\varphi}|^2}\sin^2\left(\frac{|\boldsymbol{\varphi}|}{2}\right)=1-\frac{\varphi_-\varphi_+}{|\boldsymbol{\varphi}|^2}\sin^2\left(\frac{|\boldsymbol{\varphi}|}{2}\right),$$
$$R\equiv\frac{\varphi_-\varphi_+}{|\boldsymbol{\varphi}|^2}\sin^2\left(\frac{|\boldsymbol{\varphi}|}{2}\right)=\left(1-\frac{\varphi_3^2}{|\boldsymbol{\varphi}|^2}\right)\sin^2\left(\frac{|\boldsymbol{\varphi}|}{2}\right),\quad T+R=1. \tag{A.2}$$

On the other side, using the Euler angles (α,β,γ), one obtains representations of the form [3] [26]

$$\mathsf{S}=\begin{pmatrix}\exp\left(\mathrm{i}\frac{\alpha+\gamma}{2}\right)\cos\left(\frac{\beta}{2}\right) & \mathrm{e}^{\mathrm{i}\delta_0}\exp\left(-\mathrm{i}\frac{\alpha-\gamma}{2}\right)\sin\left(\frac{\beta}{2}\right)\\ -\mathrm{e}^{-\mathrm{i}\delta_0}\exp\left(\mathrm{i}\frac{\alpha-\gamma}{2}\right)\sin\left(\frac{\beta}{2}\right) & \exp\left(-\mathrm{i}\frac{\alpha+\gamma}{2}\right)\cos\left(\frac{\beta}{2}\right)\end{pmatrix}$$
$$=\begin{pmatrix}\exp\left(\mathrm{i}\frac{\gamma}{2}\right) & 0\\ 0 & \exp\left(-\mathrm{i}\frac{\gamma}{2}\right)\end{pmatrix}\begin{pmatrix}\cos\left(\frac{\beta}{2}\right) & \mathrm{e}^{\mathrm{i}\delta_0}\sin\left(\frac{\beta}{2}\right)\\ -\mathrm{e}^{-\mathrm{i}\delta_0}\sin\left(\frac{\beta}{2}\right) & \cos\left(\frac{\beta}{2}\right)\end{pmatrix}. \tag{A.3}$$

The fixed phase δ_0 which can be chosen arbitrarily is introduced here to bring in correspondence representations by different authors, for example, $\delta_0=0$ by Biedenharn and Louck [18] (here $(\alpha,\beta,\gamma)\to(-\alpha,-\beta,-\gamma)$), by Landau and Lifshits [3] (Equation (58.6)), by Hamermesh [26] (Equation (9.65)) and $\delta_0=-\frac{\pi}{2}$ by Lyubarski [20] (Equation (53.3)).

With $\varphi_\pm\equiv\varphi_1\pm\mathrm{i}\varphi_2$, we find from (A.1) the inversion

$$\varphi_1=-\mathrm{i}\left(S_{12}+S_{21}\right)\frac{\arcsin(\theta)}{\theta},\quad \varphi_2=\left(S_{12}-S_{21}\right)\frac{\arcsin(\theta)}{\theta},$$
$$\varphi_3=-\mathrm{i}\left(S_{11}-S_{22}\right)\frac{\arcsin(\theta)}{\theta},\quad \theta\equiv\sqrt{1-\left(\frac{S_{11}+S_{22}}{2}\right)^2}\equiv\sin\left(\frac{|\boldsymbol{\varphi}|}{2}\right). \tag{A.4}$$

Inserting the matrix elements according to (A.1), we get the following relation of $\boldsymbol{\varphi}=(\varphi_1,\varphi_2,\varphi_3)$ to the Euler angles (α,β,γ)

$$\begin{aligned}
\varphi_1 &= -2\sin\left(\frac{\alpha-\gamma-\delta_0}{2}\right)\sin\left(\frac{\beta}{2}\right)\frac{\arcsin(\theta)}{\theta},\\
\varphi_2 &= -2\cos\left(\frac{\alpha-\gamma-\delta_0}{2}\right)\sin\left(\frac{\beta}{2}\right)\frac{\arcsin(\theta)}{\theta},\\
\varphi_3 &= 2\sin\left(\frac{\alpha+\gamma}{2}\right)\cos\left(\frac{\beta}{2}\right)\frac{\arcsin(\theta)}{\theta},\\
|\boldsymbol{\varphi}| &= 2\left|\arccos\left\{\cos\left(\frac{\alpha+\gamma}{2}\right)\cos\left(\frac{\beta}{2}\right)\right\}\right|,
\end{aligned} \tag{A.5}$$

with θ expressed by the Euler angles

$$\theta \equiv \sqrt{1-\cos^2\left(\frac{\alpha+\gamma}{2}\right)\cos^2\left(\frac{\beta}{2}\right)}. \tag{A.6}$$

For the inversion of these relations, we first make the inversion of (A.1) that provides

$$\exp(\mathrm{i}\alpha) = \sqrt{-\mathrm{e}^{\mathrm{i}2\delta_0}\frac{S_{11}S_{21}}{S_{22}S_{12}}}, \quad \sin\left(\frac{\beta}{2}\right) = \sqrt{-S_{12}S_{21}}, \quad \exp(\mathrm{i}\gamma) = \sqrt{-\mathrm{e}^{-\mathrm{i}2\delta_0}\frac{S_{11}S_{12}}{S_{22}S_{21}}}. \tag{A.7}$$

Inserting here the matrix elements (A.1), we find the relations

$$\begin{aligned}
\alpha &= \operatorname{arctg}\left(\frac{\varphi_3}{|\boldsymbol{\varphi}|}\operatorname{tg}\left(\frac{|\boldsymbol{\varphi}|}{2}\right)\right) + \operatorname{arctg}\left(\frac{\varphi_2}{\varphi_1}\right) + \delta_0 \mp \frac{\pi}{2},\\
\beta &= 2\arcsin\left(\frac{\sqrt{\varphi_1^2+\varphi_2^2}}{|\boldsymbol{\varphi}|}\sin\left(\frac{|\boldsymbol{\varphi}|}{2}\right)\right),\\
\gamma &= \operatorname{arctg}\left(\frac{\varphi_3}{|\boldsymbol{\varphi}|}\operatorname{tg}\left(\frac{|\boldsymbol{\varphi}|}{2}\right)\right) - \operatorname{arctg}\left(\frac{\varphi_2}{\varphi_1}\right) - \delta_0 \pm \frac{\pi}{2}.
\end{aligned} \tag{A.8}$$

This shows that the relations between the vector parameter $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \varphi_3)$ and the Euler angles (α, β, γ) are not very simple and by no means evident and well suited for the calculation of the invariant measure in the vector parameter $\boldsymbol{\varphi}$.

The main disadvantage of the Euler angles (α, β, γ) in comparison to the vector parameter $(\varphi_1, \varphi_2, \varphi_2)$ is that they are not well appropriate for coordinate-invariant considerations and, therefore, for emphasizing invariance properties of the system. An advantage of the Euler angles in comparison to the discussed vector parameter is that the transformation matrices are easily factorized in these variables, whereas the disentanglement of the representation matrices with regard to different components of the vector parameter $(\varphi_1, \varphi_2, \varphi_2)$ leads to more complicated but important relations (see Section 10).

Appendix B. A Curious Problem with Divergence of Traces of Commutators in Lie Algebras

In the Lie algebra the commutator of two operators $[X, Y] \equiv XY - YX$ takes on the role of the product of operators. The trace of an arbitrary operator Z we denote by $\langle Z \rangle$. Then we have

$$\left[X,Y\right]\equiv XY-YX,\quad \Rightarrow\quad \left\langle\left[X,Y\right]\right\rangle=\left\langle XY\right\rangle-\left\langle YX\right\rangle=0. \tag{B.1}$$

However, in the special case of a pair of annihilation and creation operator $X=a,Y=a^{\dagger}$ we find a non-vanishing, even infinite trace

$$\left[a,a^{\dagger}\right]=I,\quad \Rightarrow\quad \left\langle\left[a,a^{\dagger}\right]\right\rangle=\left\langle I\right\rangle\to+\infty. \tag{B.2}$$

One may think that a possible reason for this is that $\left\langle AB\right\rangle=\left\langle BA\right\rangle$ is no more correct for the case of infinite matrices or operators due to problems of divergences.

The identity operator I can be resolved to the infinite sum over number-state operator $|n\rangle\langle n|$ with

$$I=\sum_{n=0}^{\infty}|n\rangle\langle n|,\quad \langle m|n\rangle=\delta_{m,n},\quad a|n\rangle=\sqrt{n}|n-1\rangle,\quad a^{\dagger}|n\rangle=\sqrt{n+1}|n+1\rangle. \tag{B.3}$$

Using this we find for the trace of $\left[a,a^{\dagger}\right]$

$$\left\langle\left[a,a^{\dagger}\right]\right\rangle=\left\langle I\right\rangle=\sum_{n=0}^{\infty}\langle n|I|n\rangle=\sum_{n=0}^{\infty}1\to+\infty. \tag{B.4}$$

On the other side forming the trace over $aa^{\dagger}$ and $a^{\dagger}a$ separately

$$\begin{aligned}\left\langle aa^{\dagger}\right\rangle&=\sum_{n=0}^{\infty}\langle n|aa^{\dagger}|n\rangle=\sum_{n=0}^{\infty}(n+1)\langle n+1|n+1\rangle=\sum_{n=0}^{\infty}(n+1)\to+\infty,\\ \left\langle a^{\dagger}a\right\rangle&=\sum_{n=0}^{\infty}\langle n|a^{\dagger}a|n\rangle=\sum_{n=0}^{\infty}n\langle n-1|n-1\rangle=\sum_{n=1}^{\infty}n=\sum_{n=0}^{\infty}(n+1)\to+\infty,\end{aligned} \tag{B.5}$$

Thus we obtained in this calculation formally the same trace over $aa^{\dagger}$ and $a^{\dagger}a$

$$\left\langle aa^{\dagger}\right\rangle=\left\langle a^{\dagger}a\right\rangle,\quad \Rightarrow\quad \left\langle\left[a,a^{\dagger}\right]\right\rangle=0. \tag{B.6}$$

This is the result which we would expect since for finite operators A and B we have in every case $\left\langle AB\right\rangle=\left\langle BA\right\rangle$.

It is clear that the contradiction results from calculations with divergent sums and by reordering the positive and negative sum terms in different way we may find different results. There remains the problem which of the calculations for expectation values of commutators is the correct one if needed and why. Due to $\left\langle\left[A,B\right]\right\rangle=0$ in "general case" we tend to $\left\langle\left[a,a^{\dagger}\right]\right\rangle=0$ in "most" cases.

Appendix C. Transformation of Parameter c for $SO(3,\mathbb{R})$ and Limiting Case

If we make in (16.4) the substitutions

$$\boldsymbol{c}\equiv\frac{\boldsymbol{x}}{R},\quad \boldsymbol{c}_0\equiv\frac{\boldsymbol{x}_0}{R},\quad \boldsymbol{c}'\equiv\frac{\boldsymbol{x}'}{R}, \tag{C.1}$$

then we find

$$\frac{\boldsymbol{x}'}{R}=\boldsymbol{c}'=\frac{\boldsymbol{c}+\boldsymbol{c}_0+\left[\boldsymbol{c},\boldsymbol{c}_0\right]}{1-\boldsymbol{c}\boldsymbol{c}_0}=\frac{\frac{\boldsymbol{x}}{R}+\frac{\boldsymbol{x}_0}{R}+\frac{\left[\boldsymbol{x},\boldsymbol{x}_0\right]}{R^2}}{1-\frac{\boldsymbol{x}\boldsymbol{x}_0}{R^2}},\quad \Rightarrow\quad \boldsymbol{x}'=\frac{\boldsymbol{x}+\boldsymbol{x}_0+\frac{\left[\boldsymbol{x},\boldsymbol{x}_0\right]}{R}}{1-\frac{\boldsymbol{x}\boldsymbol{x}_0}{R^2}}. \tag{C.2}$$

In the limit $R \to \infty$

$$R \to \infty: \quad \boldsymbol{x}' \to \boldsymbol{x} + \boldsymbol{x}_0, \tag{C.3}$$

it makes the transition to a translation of an arbitrary vector $\boldsymbol{x}$ to a new vector $\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{x}_0$ with fixed $\boldsymbol{x}_0$.

Appendix D. Invariants and Complementary Operator to a Special Three-Dimensional Operator $\mathsf{A} = \alpha \mathrm{I} + \beta \boldsymbol{b} \cdot \tilde{\boldsymbol{b}} + \gamma [\boldsymbol{c}]$

We consider the following special three-dimensional operator A in three-dimensional Euclidean space ($g_{ij} = \delta_{ij}$) which is a little more generally as needed in Section 16 but suited also for many other applications

$$\mathsf{A} = \alpha \mathrm{I} + \beta \boldsymbol{b} \cdot \tilde{\boldsymbol{b}} + \gamma [\boldsymbol{c}], \tag{D.1}$$

where $[\boldsymbol{c}]$ is an antisymmetric operator to vector $\boldsymbol{c}$ (see 10) and calculate its invariants and the complementary operator. According to (7) we have first to calculate its powers A^2 and A^3 and then to form their traces that for length we do not write down. Then with (7) we can find for their invariants

$$\begin{aligned}
\langle \mathsf{A} \rangle &= 3\alpha + \beta \tilde{\boldsymbol{b}} \boldsymbol{b}, \\
[\mathsf{A}] &= 3\alpha^2 + 2\alpha\beta \tilde{\boldsymbol{b}} \boldsymbol{b} + \beta\gamma \left[\tilde{\boldsymbol{b}}, \boldsymbol{b}, \boldsymbol{c} \right] + \gamma^2 \boldsymbol{c}^2, \\
|\mathsf{A}| &= \alpha^2 \left(\alpha + \beta \left(\tilde{\boldsymbol{b}} \boldsymbol{b} \right) \right) + \alpha\beta\gamma \left[\tilde{\boldsymbol{b}}, \boldsymbol{b}, \boldsymbol{c} \right] + \alpha\gamma^2 \boldsymbol{c}^2 + \beta\gamma^2 \left(\tilde{\boldsymbol{b}} \boldsymbol{c} \right) \left(\boldsymbol{c} \boldsymbol{b} \right) \\
&= \left(\alpha + \beta \tilde{\boldsymbol{b}} \boldsymbol{b} \right) \left(\alpha^2 + \gamma^2 \boldsymbol{c}^2 \right) + \alpha\beta\gamma \left[\tilde{\boldsymbol{b}}, \boldsymbol{b}, \boldsymbol{c} \right] + \beta\gamma^2 \left[\tilde{\boldsymbol{b}}, \boldsymbol{c} \right] [\boldsymbol{c}, \boldsymbol{b}].
\end{aligned} \tag{D.2}$$

The complementary operator $\overline{\mathsf{A}} \equiv [\mathsf{A}] \mathrm{I} - \langle \mathsf{A} \rangle \mathsf{A} + \mathsf{A}^2$ to A possesses the following representation

$$\begin{aligned}
\overline{\mathsf{A}} &= \left\{ \alpha \left(\alpha + \beta \tilde{\boldsymbol{b}} \boldsymbol{b} \right) + \beta\gamma \left[\tilde{\boldsymbol{b}}, \boldsymbol{b}, \boldsymbol{c} \right] \right\} \mathrm{I} - \alpha\beta \boldsymbol{b} \cdot \tilde{\boldsymbol{b}} + \gamma^2 \boldsymbol{c} \cdot \boldsymbol{c} - \left(\alpha + \beta \tilde{\boldsymbol{b}} \boldsymbol{b} \right) \gamma [\boldsymbol{c}] \\
&\quad + \beta\gamma \left(\boldsymbol{b} \cdot \left[\tilde{\boldsymbol{b}}, \boldsymbol{c} \right] + [\boldsymbol{c}, \boldsymbol{b}] \cdot \tilde{\boldsymbol{b}} \right), \quad \langle \overline{\mathsf{A}} \rangle = [\mathsf{A}], \quad \mathsf{A}^{-1} = \frac{\overline{\mathsf{A}}}{|\mathsf{A}|}.
\end{aligned} \tag{D.3}$$

One may consider many special cases of these formulae.