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# An Outline of the Grand Unified Theory of Gauge Fields

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## Abstract

This paper attempts to propose a grand unified guiding principle of gauge fields from the mathematical and physical picture of fiber bundles: it is believed that our universe may have more fundamental interactions than the four fundamental interactions, and the gauge fields of these fundamental interactions are just a unified gauge potential on the fiber bundle manifold or the components connected to the bottom manifold, that is, our universe; these components can meet the transformation of gauge potential, and even can be transformed from a fundamental interaction gauge potential to another fundamental interaction gauge potential, and can be summarized into a unified equation, namely the expression of the generalized gauge equation, corresponding to the gauge transformation invariance; so gauge transformation invariance is a necessary condition to unify field theory, but quantization of field is not a necessary condition; the four (or more) fundamental interaction fields of the universe are unified into a universal gauge field defined by the connection of the principal fiber bundle on the cosmic base manifold.

## Keywords

Gauge Field, Principal Fiber Bundle, Gauge Transformation Invariance, Grand Unified Theory of Physics

## 1. Introduction

In 1954, Yang Zhenning and Mills extended the $U(1)$ group local gauge symmetry to the $SU(2)$ group, which represents the local gauge symmetry of isotope spin in the non-Abelian group, and established the Yang-Mills gauge field theory [1]. Following the idea of gauge field theory, there is a gauge symmetry group $U(1)\times SU(2)$, which has achieved great success in unifying electromagnetic interaction and weak interaction. Such a theory has been proved to be consistent with renormalization in the sense of spontaneous symmetry breaking Higgs mechanism [2] [3] [4] [5]. Moreover, the idea of gauge field theory provides a successful unified description of the dynamics (QCD) of gauge symmetry group $SU(3)$ [6] [7] of electric weak interaction and strong interaction characterized by quantum color, so as to construct a quantum gauge field theory dominated by symmetry, that is, gauge group $U(1)\times SU(2)\times SU(3)$ with spontaneous symmetry breaking, the standard model. This is a milestone achievement of gauge field theory in unifying electromagnetic, weak and strong fundamental interactions.

This led many scholars after Einstein to try to expand the gauge "quantum" field theory to the category of gravity, hoping to establish a grand unified theory of four fundamental interactions, including gravity and electromagnetic forces [8] [9] [10]. But until now, we may believe that the gravitational field is a gauge field, but the quantization theory of the gravitational field has always been inconsistent with the microscopic quantum field theory [11] [12] [13], this has also become an exciting point for the creation of the famous superstring theory [14] [15], which still inspires us to think about a question today: can we say for sure that gravity can be quantized?

The second question is whether there are more than four fundamental interactions in nature? It seems that no principle can limit the fundamental interactions in nature to four kinds, namely, gravitational, electromagnetic, and weak and strong interactions. Dark matter and dark energy put forward interpretations of this question from the perspective of astrophysics or cosmic scale [16] [17]. Are dark matter and dark energy the real existence or the representation of some unknown fundamental interactions? In fact, so far, many normative unified field theories have attempted to construct some direct product forms of very specific structural groups [18] [19] [20] [21]. These direct product forms often correspond to some specific basic particle fields, thus explaining the existing basic particles and astrophysical theories and experimental phenomena, which is like establishing the world's unity on some specific basic particles. If they also can be decomposed, or can it decay? Or are there not only four fundamental interactions in the world (for example, dark energy does not exist, but a cosmic expansion caused by fundamental interactions on a cosmic scale)?

And what is the "over-distance" effect of quantum entanglement? The so-called force field interaction between two objects that we know is just the exchange of virtual particles between two objects, which seems to be invalid here, and the interaction seems to be a whole that occurs on two entangled objects at the same time [22] [23].

On the other hand, as the mathematical basis of the gauge field theory, the theory of principal fiber bundles has been greatly developed [24]. In a sense, the gauge theory of physics is just the principal fiber bundle (principal bundle) theory in mathematics [25]. The question is how to deeply understand or develop the relevant concepts or meanings of physics implied by the principal bundle theory? Thanks Professor Liang Canbin for teaching and building bridges in this area over the years [26]. It is based on these "plays" and "bridges" that the author can consider the above-mentioned issues about the grand unification of physics in this paper, and try to propose an outline program for the grand unification of gauge fields from the mathematical physics picture of principal fiber bundles.

## 2. Basic Points of the Outline

Why is it so difficult to unify gravity and electromagnetic force among the four fundamental interactions? Why must the gauge transformation remain unchanged? Is there a unified outline program of cosmic gauge field? How to understand the "teleportation" between quantum entangled states? What is the interaction between entanglements? After a long time of meditation, the author of this paper proposes a "unified outline program for the gauge field of space and time in the universe", the main points of which are as follows:

1) The whole universe structure can be simply described as the structure of a region ordinary principal bundle and associated bundle, which can be called "the principal associated bundles picture of the universe": the universe is the bottom manifold *M*; the structure group of the principal bundle is defined as a Lie group $G$ (also a manifold), which reflects the laws or rules of the universe; then the fiber bundle of principal bundle, bottom manifold and group $G$ form a direct product of manifold, which can be expressed as $P = G \times M$. It is recorded as $P(M,G)$. The fiber bundle associated to principal bundle (Associated bundle) is composed of manifold $Q(M,F)$ and fiber bundles determined by the coordination of *P*, *M*, *G*, *F* and related maps. The structure of the principal bundle and associated bundle can be shown by the following "principal associated bundles picture of the universe" (**Figure 1**).

Here, the precise definitions and mathematical and physical expressions of the mappings $\tau$, $\hat{\tau}$, $T_U$, $\hat{T}_U$, $\pi$, $\hat{\pi}$, etc., will be given one after another in the following sections.

2) The cross section of principal bundle $\sigma : M \to P$ represents a choice of gauge (embodied as an internal frame of gauge field); the transformation between two different cross sections of the principal bundle gives a gauge transformation, and vice versa. The cross section of associated bundle $\hat{\sigma} : M \to Q$ represents a physical gauge field (or elementary particle field) on the base manifold.

$$
\begin{tikzcd}
 & & G \times F \arrow[dl, "\tau"'] \arrow[dr, "\hat{\tau}"] & & \\
P \times G & P \arrow[l, "T_U"'] \arrow[dr, "\pi"'] & & Q \arrow[r, "\hat{T}_U"] \arrow[dl, "\hat{\pi}"] & P \times F \\
 & & M & &
\end{tikzcd}
$$

**Figure 1.** The picture of the principal associated bundles of the universe. *M* represents our universe, and above *M* stands the high-dimensional space-time (heaven) of the laws or rules governing the universe.

3) The gauge potential corresponds to the connection of the principal bundle, and the gauge field strength corresponds to the curvature on the principal bundle. So it can be said that the connection is the gauge potential and the curvature is the gauge field strength.

4) The gauge potential on the bottom manifold $M$ is a 1-form field function of the Lie algebra taking $G$ in a space-time region. Different gauge potentials satisfy the formula of gauge potential transformation, but from $P$, these different gauge potentials are just the gauge potentials of the principal bundle (connection field $\tilde{\boldsymbol{\omega}}$) projection component on the base manifold under the cross section transformation, total gauge potential $\tilde{\boldsymbol{\omega}}$ is invariant, and the transformation between cross sections is the gauge transformation; this general gauge potential $\tilde{\boldsymbol{\omega}}$ corresponds to a cosmic space-time gauge field, and the four fundamental interactions in reality only correspond to the projection components of this cosmic space-time gauge field under different gauge choices (internal frame choices); this is the meaning of gauge transformation invariance of gauge field.

5) There may be many components of connection $\tilde{\boldsymbol{\omega}}$, so there are not only four fundamental interactions in the universe, but also five, six, and so on. These corresponding fundamental interactions are spatio-temporal regional field functions of different gauge choices (internal frame choices), and may also be the superposition or combination of four (or other fundamental) interactions.

6) The so-called gauge transformation may be a transformation from one fundamental interaction (such as electromagnetic force) to another fundamental interaction (such as gravity) in the picture of the principal associated bundles of the universe; the so-called gauge invariance means that the gauge field of space and time in the universe is invariant under the gauge transformation, and gravity, electromagnetic force, or strong or weak force are all the expressions of its components. The gauge transformation is only the transformation between these components, and the gauge invariance is that the total space-time gauge field does not change with the gauge transformation (showing the Lagrangian invariance of the physical field).

7) Quantization is not necessarily a necessary condition for the unification of the gauge field, but the invariance of gauge transformation is the necessary condition for the unification of the gauge field!

8) The four fundamental fields of the universe (gravity, electromagnetic force, weak force, and strong force) are unified in one cosmic space-time gauge potential $\tilde{\boldsymbol{\omega}}$, which corresponds to a cosmic space-time gauge field, and the mutual transformation between the four fundamental gauge fields can be described by a generalized gauge potential Equation (referred to as GG equation):

$$\sigma_V^* \tilde{\boldsymbol{\omega}}(Y) = \mathcal{A}d_{g_{UV}(x)^{-1}} \sigma_U^* \tilde{\boldsymbol{\omega}}(Y) + L^{-1}_{g_{UV}(x)*} g_{UV*}(Y), \forall x \in U \cap V, Y \in T_x M \quad \text{(GGE)}$$

here, $\sigma_V^*$ and $\sigma_U^*$ is the pull back mapping of the principal bundle cross section to the bottom manifold region $V$ or $U$ respectively, showing that all the different gauge potentials are just the components of the cosmic space-time unified potential in different regions of the bottom manifold; the physical and mathematical meanings of other symbols will be introduced later.

Let's use mathematical physics to construct a strict framework for the above arguments; the main references are [25] (Chapter 6, Appendix 3), [26] (Appendix G and I in Volume II and III) however, the author of this paper has made some generalization in the process of derivation and proof.

## 3. Relevant Mathematical Physics Framework

### 3.1. Concepts of Principal Fiber Bundle

The principal fiber bundle $P(M,G)$ consists of the bundle manifold *P*, the base manifold *M*, and the Lie group $G$ of the structure group, and satisfies the following three conditions:

1) $G$ has a free right action $R$ on *P*, $R: P\times G\to P$; here free means: if $g\neq e$, then there is $pg\neq p, \forall g\in G, p\in P$. Since the dimension of $P$ is greater than *G*, so this is an embedding map. It makes fibers also Lie groups (both manifolds and groups), and is isomorphic to *G*.

2) There is a $C^{\infty}$ projection on to mapping $\pi: P\to M$, and satisfies: $\pi^{-1}[\pi(p)]=\{pg|g\in G\}$, $\forall p\in P$. In order to ensure the existence of $\pi^{-1}$, $[\,]$ represents a singleton subset.

3) For each $x\in M$, the existence of open neighborhood $U\subset M$ and differential homeomorphism (local trivialization mapping) $T_U:\pi^{-1}[U]\to U\times G$ can be expressed as: $T_U(p)=(\pi(p),S_U(p))$, $\forall p=\pi^{-1}[U]$, where mapping $S_U$ is defined as $\pi^{-1}[U]\to G$, and meets (*i.e.*, the "core" requirement of local trivialization $T_U$: corresponding to $p\to pg$ on fiber or orbit): $S_U(pg)=S_U(p)g$, $\forall g\in G$. Here $S_U:\pi^{-1}[U]\to G$ is not a differential homeomorphism, because $\pi^{-1}[U]$ manifold is larger than *G*, but $S_U:\pi^{-1}[x]\to G$ is a differential homeomorphism. If *p* is a special point, it is marked as $\check{p}_U(x)$, $S_U(\check{p}_U)=e$, and $R_{\check{p}_U}: G\to\pi^{-1}[x]$ is also a differential homeomorphism, so $R_p=S_U^{-1}$.

Here $\pi^{-1}[x], x\in U\subset M$ can be called the fiber above the point $x\in M$; $\pi^{-1}[\pi(p)]=\{pg|g\in G\}$ is equal to the group $G$ right acting $R$ to the orbit passing through point *p*, that is, the fiber is equal to the orbit, $\{pg|g\in G\}\subset P$, where the fiber bundle forms a bundle manifold; the principal fiber bundle $P(M,G)$ can be abbreviated as *P*.

### 3.2. Concept of Associated Bundle

Let $P(M,G)$ be the principal bundle. If manifold $F$ is selected, make $G$ have a left action $\chi$ on $F$ (no free requirement), $\chi: G\times F\to F$, *i.e.* $\chi_g(f)=gf, \forall g\in G, f\in F$. Then the free right action $R$ of group $G$ on $P$ and the left action $\chi$ of group $G$ on $F$ jointly induce the free right action $\xi$ of $G$ on $P\times F$, $\xi:(P\times F)\times G\to P\times F$, which is defined as $\xi_g(p,f):=(pg,g^{-1}f)\in P\times F$, here $\forall g\in G, p\in P, f\in F$. If $\tau: P\times F\to P$ represents the natural projection mapping, that is, $\tau(p,f):=p, \forall p\in P, f\in F$, each track of $\xi$ on $P\times F$ is regarded as an element, the set of all elements is defined as *Q*, any element $q\in Q$ represents a orbit on $P\times F$, and $Q$ is a manifold, which we call the associated bundle. Here $q$ can be defined as $q\equiv\left\{pg,g^{-1}f\middle|g\in G\right\}\equiv p\cdot f$, which is called the orbit passing $(p,f)$ point and is formed by $\xi_g(p,f)$. Note that $p\cdot f=pg\cdot g^{-1}f=p\cdot gg^{-1}f=p'\cdot f'$.

### 3.3. Relations of Natural Projection Mapping of Principal Associated Bundles

1) $\hat{\tau}:P\times F\to Q$, is defined as $\hat{\tau}(p,f):=p\cdot f\in Q$, *i.e.* $\forall p\in P$, $\hat{\tau}_p:F\to Q$; from this, we can define a topology for *Q*, let $\phi\subset Q$ is open if and only if $\hat{\tau}^{-1}[\phi]\subset P\times F$ is open, so $Q$ is topological space, $\hat{\tau}$ is a continuous mapping. Not only that, but also $Q$ can be proved to be a manifold.

2) $\hat{\pi}:Q\to M$, defined as $\hat{\pi}(q):=\pi(p)\in M$, $\forall q=p\cdot f\in Q$. Hence more precisely, $\hat{\tau}_p:F\to\hat{\pi}^{-1}[x]$, $x\equiv\pi(p)$ while $R_p:G\to\pi^{-1}[x]$, $x\equiv\pi(p)$, and $\hat{\tau}_p,R_p$ is a differential homeomorphic mapping. In other words, $\hat{\tau}_p$ or $R_p$ brings the manifold structure of $F$ or $G$ respectively to the fiber $\hat{\pi}^{-1}[x]$ of the companion bundle $Q$ or the fiber $\pi^{-1}[x]$ of the principal bundle *P*.

3) $\tau:P\times F\to P,\tau(p,f):=p,\forall p\in P,\ f\in F$.

4) $\pi:P\to M$, and meet: $\pi^{-1}[\pi(p)]=\left\{pg\middle|g\in G\right\}$, $\forall p\in P$.

Here, the relevant definitions in (3), (4) have been given by 3.1 and 3.2.

If for every $x\in M$ there exists an open neighborhood $U\subset M$, its inverse image $\pi^{-1}[U]$ is diffeomorphic to the product manifold $U\times G$, *i.e.* $\pi^{-1}[U]=U\times G$, then the corresponding $T_U$ is said to be locally trivial, and the corresponding principal bundle is locally trivial; if $U=M$, that is, $\pi^{-1}[U]=P=M\times G$, is said to be integrally trivial. It can be said that any principal bundle is locally trivial. Therefore, the local trivialization can be extended to the principal associated bundles picture of the universe (see **Figure 2**).

Where, note that $T_U:\pi^{-1}[U]\to U\times G$, $T_U(p):=(\pi(p),S_U(p))$, therefore, local trivial $\hat{T}_U:\pi^{-1}[U]\to U\times F$, $\hat{T}_U(p):=(\hat{\pi}(q),\breve{f}_U)$; here, $S_U(\breve{p}_U)=e$, $q=\breve{p}_U\cdot f=\breve{p}_U\cdot\breve{f}_U$. So $Q$ is an associated bundle of the principal bundle *P*, and $F$ is called the typical fiber of *Q*. If there is $U\cap V\neq\varnothing$, then one has $\breve{p}_V=\breve{p}_U g_{UV}(x)\Rightarrow g_{UV}(x)\breve{f}_U=\breve{f}_V$. In addition, the local cross section of the associated bundle can be defined as $\hat{\sigma}:U\to Q$, namely satisfying $\pi(\hat{\sigma}(x))=x$.

$$
\begin{array}{ccc}
 & \pi^{-1}[U]\times F & \\
 & \tau\downarrow \qquad\qquad \downarrow\hat{\tau} & \\
U\times G \overset{T_U}{\leftarrow} \pi^{-1}[U] & & \hat{\pi}^{-1}[U] \overset{\hat{T}_U}{\rightarrow} U\times F \\
 & \pi\downarrow \qquad\qquad \downarrow\hat{\pi} & \\
 & U \quad = \quad U &
\end{array}
$$

**Figure 2.** The more specific structure of the principal associated bundles of the universe, $\pi^{-1}[U]$ represents the principal fiber bundle on *U*; *U* refers to the overall trivial or locally trivial region of *M*.

### 3.4. Frame Bundle

$M$ is a set as $n$-dimensional manifold, $P \equiv \left\{ x, \left\{ e_\mu \right\} \middle| x \in M \right\}$, $\left\{ e_\mu \right\}$ is a $T_x M$ base, abbreviated as $e_\mu T_x$ stands the tangent space for $x \in M$. Then $P$ can be proved to be a $n + n^2$ dimensional manifold. Now select $GL(n)$ as the structure group $G$, then a frame bundle can be constructed by the following three steps:

1) Define the right action $R$ of matrix group $GL(n)$ on $P$, $R : P \times GL(n) \to P$ is: $R_g\left(x, e_\nu g_\mu^\nu\right)$, here $g_\mu^\nu$ represents a matrix element of $g$.

2) Define projection mapping $\pi : P \to M$, namely $\pi\left(x, e_\mu\right) := x, \forall \left(x, e_\mu\right) \in P$.

3) Define local trivial $T_U : \pi^{-1}[U] \to U \times G$, namely $T_U\left(x, e_\mu\right) := (x, h)$, here $h \equiv S_U\left(x, e_\mu\right) \in G$, $\left.\frac{\partial}{\partial x^\nu}\right|_x h_\mu^\nu = e_\mu$, and $S_U(pg) = S_U(p) g, \forall g \in G$. So $T_U$ is differential homeomorphic.

The principal bundle $P(M, GL(n))$ constructed by the above three steps is called the frame bundle, which is recorded as $FM$.

### 3.5. Tangent Bundle

On the basis of $FM$, take the manifold $F = \mathbb{R}^n$, then $F$ is a vector space, and $f \in F$ can be expressed as a column matrix composed of $n$ real numbers, that is, $\left(f^1, \cdots, f^n\right)$; so we can define the left action $\chi : G \times F \to F$ as $\left(\chi_g(f)\right)^\mu := g_\nu^\mu f^\nu, \forall g \in GL(n), f \in F$; then a $\xi$ is determined by the right and left actions, $\xi : (P \times F) \times G \to P \times F$, $\xi_g : P \times F \to P \times F$, specifically $\xi_g(p, f) = \left(pg, g^{-1} f\right) \Rightarrow \xi_g\left(x, e_\mu; f^\rho\right) = \left(x, e_\nu g_\mu^\nu; \left(g^{-1}\right)_\sigma^\rho f^\sigma\right)$. Here $\left(x, e_\mu; f^\rho\right) \in P \times F$ can produce $v \equiv e_\mu f^\mu \in T_x M$, and on the same orbit $v = e_\mu f^\mu = v' = e'_\mu f'^\mu$; that is to say, each $q \in \hat{\pi}^{-1}[x]$ point (represents an orbit) 1-1 corresponds to a vector $v$ in $T_x M$, and all different $v$ in $T_x M$ correspond to different $q$ above it to form a tangent bundle $\hat{\pi}^{-1}[x]$, that is, $\hat{\pi}^{-1}[x] \xleftrightarrow{\text{1-1}} T_x M$; so the tangent bundle $Q = P \times F$ is the associated bundle of $FM$. Furthermore, $Q$ can be regarded as the tangent bundle $TM$ of $M$, so any cross section $\hat{\sigma} : U \to Q$ (because 1-1 corresponds to the vector of the tangent space on $U$) is a vector field on $U \subset M$. Since it is a vector field, at least it is preliminarily explained that the section $\hat{\sigma} : U \to Q$ is related to coordinates, different section corresponds to different coordinate, and there is a relationship of coordinate transformation between sections.

On the basis of $FM$ if the manifold $F = \left(\mathbb{R}^n\right)^* = \mathcal{T}_{\mathbb{R}^n}(0,1)$, $f = \left(f_1, \cdots, f_n\right) \in F$, $\left(\chi_g(f)\right)_\mu := \left(g^{-1}\right)_\mu^\nu f_\nu$, then given any point $\left(x, e_\mu; f_\rho\right) \in P \times F$, one can produce: $\beta \equiv e^\mu f_\mu \in T_x^* M$ (the dual space of $T_x M$), and there is $\beta = \beta'$ on the same orbital; all the different $\beta$ in $T_x^* M$ correspond to the different $q$ above it to form a cotangent bundle $\hat{\pi}^{-1}[x]$, that is, $\hat{\pi}^{-1}[x] \xleftrightarrow{\text{1-1}} T_x^* M$; so the cotangent bundle $Q = P \times F$ is also an associated bundle of $FM$. Any of its cross sections $\hat{\sigma} : U \to Q$ is a covector field on $U \subset M$. Note that in general, $\chi : G \times F \to F$ is a Lie transformation group: $\hat{G} \equiv \left\{ \chi_g : F \to F \middle| g \in G \right\}$.

In addition, one of the simplest structures of the principal associated bundle of the universe constructed by us may be the frame bundle plus the associated tangent bundle. However, in order to make this structure of the principal associated bundles of the universe able to accommodate the most universal gauge field of the universe and the corresponding unknown fundamental interactions, we still use the general abstract structure of the principal associated bundles to replace the concrete model.

## 4. Principal Bundle and Gauge Field

### 4.1. Gauge Selection and Cross Section

Definition: Let $P(M,G)$ be the principal bundle, $U$ be the open subset of $M$, $C^\infty$ mapping $\sigma: U \to P$ is called a local cross section, if $\pi(\sigma(x)) = x, \forall x \in U$. Here if $U = M$, then $\sigma: M \to P$, which is called the overall section. Next, in the case of local cross section, we further explore the physical meaning of the local cross section: let $\mathbb{R}^4$ be an open subset of the bottom manifold $M$, and $G$ is the structural group to construct a non-trivial principal bundle $\mathbb{R}^4 \times G$, where the free right-hand action of $G$ on $P$ is $R: (\mathbb{R}^4 \times G) \times G \to \mathbb{R}^4 \times G$, namely $\forall g_1 \in G$, define $R_{g_1}: \mathbb{R}^4 \times G \to \mathbb{R}^4 \times G$ as: $R_{g_1}(x, g_2) := (x, g_2 g_1)$, $\forall (x, g_1) \in \mathbb{R}^4 \times G$. Let $\sigma: \mathbb{R}^4 \to P$ and $\sigma': \mathbb{R}^4 \to P$ be the local sections of $P$ respectively, then $\forall x \in \mathbb{R}^4$ has a unique group element field $g: \mathbb{R}^4 \to G$ such that: $\forall g(x) \in G, x \in \mathbb{R}^4, \sigma'(x) = \sigma(x) g(x)^{-1}$. Therefore, there exists a representation group element such that $U(x) \equiv \rho(g(x)) \in \hat{G}$, $\rho: G \to \hat{G}$, thus creating a local gauge transformation: $\phi'(x) = U(x)\phi(x) \equiv \rho(g(x))\phi(x), \forall \phi(x) \in V$, where $V$ is the representation space of $\hat{G}$, and $\hat{G}$ is a representation of $G$. At this time, $\phi(x)$ is actually a column matrix, and $\rho(g(x))$ is a square matrix. Then define the associated bundle: choose $F = V$, define the left action $\chi: G \times F \to F$ as: $\forall g_1 \in G$, $\chi_{g_1}: F \to F$, $\chi_{g_1}(f_1) := \rho(g_1)(f_1)$, $\forall f_1 \in F$, then there is an associated bundle $\phi(x) \equiv q = p \cdot f = \sigma(x) \cdot f(x) \in \hat{\pi}^{-1}[x] \subset Q$, where $f: M \to F = V$, $\forall f(x) \in F = V$. So $\phi(x)$ is determined by the cross sections $\sigma$ and $f$. In addition, $g(x)$ can generate: 1) $\sigma'(x) = \sigma(x) g(x)^{-1}$, 2) $f'(x) = \chi_{g(x)} f(x) = \rho(g(x)) f(x) = g(x) f(x)$ (*i.e.* gauge transformation), which is equivalent to

$$\begin{aligned}\Phi'(x) &= \sigma'(x) \cdot f'(x) = \sigma(x) g(x)^{-1} \cdot g(x) f(x) \\ &= \sigma(x) \cdot g(x)^{-1} g(x) f(x) = \sigma(x) \cdot f(x) = \Phi(x) \in \hat{\pi}^{-1}[x]\end{aligned}.$$

Furthermore, we can generally say that the so-called local (global) gauge transformation for the principal bundle is actually the transformation cross section $\sigma(x) \to \sigma'(x)$. This is equivalent to the transformation of the frame and the transformation of the components (of a physical field) under the internal frame field: $f(x) = \phi(x) \to f'(x) = \phi'(x)$, but the overall physical field (internal vector $\Phi(x)$) is unchanged, *i.e.* $\Phi'(x) = \Phi(x)$. The so-called gauge selection is to select different cross sections, while a cross section on the associated bundle $\hat{\sigma}$ is just a physical field $\Phi(x)$! In a word, the change of the cross section on the principal bundle is the change of the internal frame. If the internal frame is changed, it is equivalent to the transformation of a gauge. Therefore, to select a cross section of the principal bundle is to select a gauge. Note that the construction and related proofs here are relatively general. Even if $\mathbb{R}^4$ is changed to a submanifold of general $M$, the conclusion is still correct, and there is no restriction on the type of gauge potential or the type of fundamental interaction.

### 4.2. Invariance of Global Gauge Transformation of Physical Field

Assume that the physical fields $\phi_1$ and $\phi_2$ are two independent particle fields in Minkowski space-time $\left(\mathbb{R}^4, \eta_{ab}\right)$, have the same mass $m$, and spin 0, then according to the spirit of quantum field theory, they must respectively obey Klein-Gordon equation:

$$\partial^a \partial_a \phi_j - m^2 \phi_j = 0,\ j = 1,2 \tag{1}$$

The total Lagrangian density of both is:

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 = -\frac{1}{2}\left[\left(\partial^a \phi_1\right)\partial_a \phi_1 + m^2 \phi_1^2 + \left(\partial^a \phi_2\right)\partial_a \phi_2 + m^2 \phi_2^2\right] \tag{2}$$

Introducing a complex scalar field

$$\begin{cases} \phi = \dfrac{1}{\sqrt{2}}\left(\phi_1 + i\phi_2\right) \\ \overline{\phi} = \dfrac{1}{\sqrt{2}}\left(\phi_1 - i\phi_2\right) \end{cases} \tag{3}$$

Then the Klein-Gordon Equation (1) becomes:

$$\begin{cases} \partial^a \partial_a \phi - m^2 \phi = 0 \\ \partial^a \partial_a \overline{\phi} - m^2 \overline{\phi} = 0 \end{cases} \tag{4}$$

Then the total Lagrangian density becomes:

$$\mathcal{L} = -\left[\left(\partial^a \overline{\phi}\right)\partial_a \phi + m^2 \phi \overline{\phi}\right] \tag{5}$$

Then an overall gauge transformation can be introduced as follows:

$$\begin{cases} \phi' = \mathrm{e}^{-iq\theta}\phi \\ \overline{\phi}' = \mathrm{e}^{iq\theta}\overline{\phi} \end{cases} \tag{6}$$

where $\theta$ is a real number, $q$ is an integer, it is actually a group $U(1) = \left\{\mathrm{e}^{-iq\theta}\Big|_{\theta\in\mathbb{R}}\right\}$, more generally: $U(1) \to \hat{G}$, $\mathrm{e}^{-iq\theta} \to diag\left(\mathrm{e}^{-iq_1\theta}, \cdots, \mathrm{e}^{-iq_N\theta}\right)$; hence one has

$$\begin{cases} \partial_a \phi' = \mathrm{e}^{-iq\theta}\partial_a \phi \\ \partial_a \overline{\phi}' = \mathrm{e}^{iq\theta}\partial_a \overline{\phi} \end{cases} \tag{7}$$

therefore, the total Lagrangian density (5) remains unchanged, namely

$$\mathcal{L} = -\left[\left(\partial^a \overline{\phi}'\right)\partial_a \phi' + m^2 \phi' \overline{\phi}'\right] = -\left[\left(\partial^a \overline{\phi}\right)\partial_a \phi + m^2 \phi \overline{\phi}\right] \tag{8}$$

It can be seen that the total Lagrangian density is invariant under the field transformation of the gauge transformation (6), which reflects the symmetry under the field transformation and is different from the space-time symmetry. Therefore, we call this invariance the internal symmetry of the field, which, like the space-time symmetry, can lead to the conservation law and can be expressed by the Noether theorem [27].

### 4.3. Local Gauge Invariance for Non-Abelian Fields

In gauge transformation (6) $\theta$ does not change with time and space points, which is called the overall gauge transformation. The converse is called local gauge transformation. If the multiplication of the corresponding Lie group $G$ is commutative, it is called an Abelian field, otherwise it is called a non-Abelian field. This non-Abelian field is also called Yang-Mills field, which was proposed by Yang-Mills in 1954 by extending $G$ from $U(1)$ to $SU(2)$ [1]. From then on, it was the first time to open a river that the gauge field theory unified the fundamental physical interaction. Following the basic spirit of Yang-Mills field, we discuss the local gauge invariance of a generalized non-Abelian field:

First, Equation (8) is generalized to the local gauge transformation:

$$\begin{cases} \phi'(x) = U(\boldsymbol{\theta}(x))\phi(x) = \mathrm{e}^{-i\boldsymbol{L}\cdot\boldsymbol{\theta}(x)}\phi(x) \\ \bar{\phi}'(x) = \bar{\phi}(x)U(\boldsymbol{\theta}(x))^{-1} = \bar{\phi}(x)\mathrm{e}^{i\boldsymbol{L}\cdot\boldsymbol{\theta}(x)} \end{cases} \tag{9}$$

Then, we make the following assumptions:

1) The structure group and its representation group of the system are denoted as $G$ and $\rho: G \to \hat{G}$, $\dim G \equiv R$, the physical field concerned is a multi-component complex particle field $\phi(x) \in V$ (representing space), which can be called the generalized Yang-Mills field, can be accompanied by $R$ potentials (gauge potentials). These gauge potentials (also known as Yang-Mills potentials) are $R$ complementary potential fields $\left\{ A_a^r \middle| r = 1, \cdots, R \right\}$ that can be generalized in general manifolds (including space-time manifolds of general relativity). Their coordinate components are marked as $A_\mu^r(x)$.

2) The total Lagrangian density of the system can be expressed as:

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_{\mathrm{YM}} \tag{10}$$

Here $\mathcal{L}_{\mathrm{YM}}$ is the Lagrangian density of the generalized Yang-Mills field, $\mathcal{L}_1$ can be expressed as:

$$\mathcal{L}_1 = \mathcal{L}_0\left(\phi(x), D_\mu\phi(x); \bar{\phi}(x), D_\mu\bar{\phi}(x)\right) \tag{11}$$

Here $D_\mu\phi(x)$ and $D_\mu\bar{\phi}(x)$ are respectively defined as:

$$\begin{cases} D_\mu\phi(x) = \partial_\mu\phi(x) - ik\boldsymbol{L}\cdot\boldsymbol{A}_\mu(x)\phi(x) \\ D_\mu\bar{\phi}(x) = \partial_\mu\bar{\phi}(x) + ik\bar{\phi}(x)\boldsymbol{L}\cdot\boldsymbol{A}_\mu(x) \end{cases} \tag{12}$$

which can also be defined as a derivative covariant operator, where $k$ is defined as a coupling constant (and if $G = U(1)$, then $k = e$ $\leftrightarrow$ Electromagnetic gauge field; if $G = SO(N)$), then $k = -1$ $\leftrightarrow$ Gravitational gauge field); and $\boldsymbol{L}\cdot\boldsymbol{A}_\mu(x)$ is defined as:

$$\boldsymbol{L}\cdot\boldsymbol{A}_\mu(x) = L_1A_\mu^1(x) + \cdots + L_RA_\mu^R(x) = L_rA_\mu^r(x) \tag{13}$$

Then under the local gauge transformation (9), formula (8) can be expressed as:

$$\begin{cases} D'_\mu \phi'(x) = U(\boldsymbol{\theta}(x)) D_\mu \phi(x) \\ D'_\mu \bar{\phi}'(x) = \left[D_\mu \bar{\phi}(x)\right] U(\boldsymbol{\theta}(x))^{-1} \end{cases} \tag{14}$$

and make $\mathcal{L}_1$ unchanged. To ensure these, we require the transformation form of $A^r_\mu(x)$ is as follows:

$$\text{(a)} \quad \begin{aligned} D'_\mu \phi'(x) &= \partial_\mu \phi'(x) - ik\boldsymbol{L} \cdot \boldsymbol{A}'_\mu(x)\phi'(x) \\ &= U(\boldsymbol{\theta}(x))\partial_\mu \phi(x) + \left[\partial_\mu U(\boldsymbol{\theta}(x))\right]\phi(x) - ik\boldsymbol{L} \cdot \boldsymbol{A}'_\mu(x) U(\boldsymbol{\theta}(x))\phi(x) \end{aligned} \tag{15}$$

$$\text{(b)} \quad \begin{aligned} U(\boldsymbol{\theta}(x)) D_\mu \phi(x) &= U(\boldsymbol{\theta}(x))\left[\partial_\mu \phi(x) - ik\boldsymbol{L} \cdot \boldsymbol{A}_\mu(x)\phi(x)\right] \\ &= U(\boldsymbol{\theta}(x))\partial_\mu \phi(x) - ikU(\boldsymbol{\theta}(x))\boldsymbol{L} \cdot \boldsymbol{A}_\mu(x)\phi(x) \end{aligned} \tag{16}$$

Obviously, considering formula (14), and comparing formula (16) with formula (15), to make (a) = (b), we only need to have:

$$\partial_\mu U(\boldsymbol{\theta}(x)) - ik\boldsymbol{L} \cdot \boldsymbol{A}'_\mu(x) U(\boldsymbol{\theta}(x)) = -ikU(\boldsymbol{\theta}(x))\boldsymbol{L} \cdot \boldsymbol{A}_\mu(x)\phi(x), \tag{17}$$

after sorting out, it is found that for group $U(1)$, there is:

$$A'^1_\mu(x) = A^1_\mu(x) - e^{-1}\partial_\mu \theta(x) \tag{18}$$

For general case, defining $-iL_r \equiv \rho_* e_r \in \hat{\mathcal{G}}$, here $\hat{\mathcal{G}}$ is the representation of a Lie algebra of $G$, or Lie algebra of $\hat{G}$; $e_r$ is a basis of Lie algebra of $\hat{\mathcal{G}}$; $\rho_*$ is a forward mapping of $\rho$, then $\hat{A}_\mu(x) \equiv -i\boldsymbol{L} \cdot \boldsymbol{A}_\mu(x) = -iL_r A^r_\mu(x) \in \hat{\mathcal{G}}$ can be defined. Therefore, Equation (17) becomes the following equation, which can be also proved that its right side belongs to $\hat{\mathcal{G}}$.

$$\hat{A}'_\mu(x) = U(\boldsymbol{\theta}(x))\hat{A}_\mu(x)U(\boldsymbol{\theta}(x))^{-1} - k^{-1}\partial_\mu U(\boldsymbol{\theta}(x))U(\boldsymbol{\theta}(x))^{-1} \tag{19}$$

what needs to be emphasized here is that formula (19) is essentially a form of "generalized gauge transformation", that is, the general transformation formula between two connections of the bottom manifold in the intersection domain: the generalized gauge equation, if we also look at the gauge potential as it is a connection. We shall return to this issue later.

Now let's introduce the generalized Yang-Mills field strength: because $R$ gauge potentials $A^r_\mu(x)$ have been introduced, so there should be $R$ gauge field strengths $F^r_{\mu\nu}(r = 1,\cdots,R)$, which can be expressed as:

$$F^r_{\mu\nu}(x) = \partial_\mu A^r_\nu - \partial_\nu A^r_\mu + k\sum\nolimits_{s,t=1}^{R} C^r_{st} A^s_\mu(x) A^t_\nu(x), (r = 1,\cdots,R) \tag{20}$$

where $C^r_{st}$ represents the structural constant of the Lie algebra $\hat{\mathcal{G}}$ of $G$ under the basis $\{e_r\}$. So it can be defined:

$$\mathcal{L}_{\text{YM}} := -\frac{1}{16\pi}\sum\nolimits_{r=1}^{R} F^r_{\mu\nu} F^{r\mu\nu} \tag{21}$$

here, the metric $g^{\mu\alpha}$, $g^{\nu\beta}$ can be used to lift subscripts of $F^r_{\alpha\beta}$:

The following task is to prove that $\mathcal{L}_{\text{YM}}$ is also invariant under transformation (19). For this, we first introduce the simplified notation: $\hat{F}_{\mu\nu}(x) \equiv -iL_r F^r_{\mu\nu}(x) \in \hat{\mathcal{G}}$, similar Equation (20) can be changed to

$$\hat{F}_{\mu\nu}(x)=\partial_\mu\hat{A}_\nu(x)-\partial_\nu\hat{A}_\mu(x)+k\left[\hat{A}_\mu(x),\hat{A}_\nu(x)\right] \tag{22}$$

where $\left[\hat{A}_\mu(x),\hat{A}_\nu(x)\right]$ is the Lie brackets of the Lie algebra elements $\hat{A}_\mu(x)$ and $\hat{A}_\nu(x)$. From this, it can be proved that $\mathcal{L}_{\text{YM}}$ is invariant under the transformation (19), that is, $\mathcal{L}'_{\text{YM}}(x)=\mathcal{L}_{\text{YM}}(x)$. The proof is as follows:

Firstly we prove that $16\pi\lambda_\rho\mathcal{L}_{\text{YM}}=tr\left(F_{\mu\nu}F^{\mu\nu}\right)$. Let $\{e_r\}$ be the orthonormal basis, and let $F_{\mu\nu}(x)=e_rF^r_{\mu\nu}(x)\in\hat{\mathcal{G}}$, then we can introduce $\rho$, so that $\rho_*F_{\mu\nu}(x)=\hat{F}_{\mu\nu}(x)\in\hat{\mathcal{G}}$, then we have

$$\begin{aligned}tr\left(F_{\mu\nu}F^{\mu\nu}\right)&=tr\left[\left(\rho_*F_{\mu\nu}\right)\left(\rho_*F^{\mu\nu}\right)\right]=tr\left[\left(\rho_*\sum\nolimits_{r=1}^{R}F^r_{\mu\nu}e_r\right)\left(\rho_*\sum\nolimits_{s=1}^{R}F^{s\mu\nu}e_s\right)\right]\\&=\sum\nolimits_{r,s=1}^{R}F^r_{\mu\nu}F^{s\mu\nu}tr\left[\left(\rho_*e_r\right)\left(\rho_*e_s\right)\right]=\sum\nolimits_{r,s=1}^{R}F^r_{\mu\nu}F^{s\mu\nu}\tilde{K}\left(e_r,e_s\right)\\&=\lambda_\rho\sum\nolimits_{r,s=1}^{R}F^r_{\mu\nu}F^{s\mu\nu}K\left(e_r,e_s\right)=-\lambda_\rho\sum\nolimits_{r,s=1}^{R}F^r_{\mu\nu}F^{s\mu\nu}\delta_{rs}=16\pi\lambda_\rho\mathcal{L}_{\text{YM}}\end{aligned} \tag{23}$$

where $\tilde{K}$ is the generalized Cartan metric satisfying $\tilde{K}=\lambda_\rho K$, and $K\left(e_r,e_s\right)=-\delta_{rs}$. The last step uses formula (21).

Secondly, we can prove that $tr\left(\hat{F}'_{\mu\nu}\hat{F}'^{\mu\nu}\right)=tr\left(\hat{F}_{\mu\nu}\hat{F}^{\mu\nu}\right)$, namely

$$tr\left(\hat{F}'_{\mu\nu}\hat{F}'^{\mu\nu}\right)=tr\left(U\hat{F}_{\mu\nu}U^{-1}U\hat{F}^{\mu\nu}U^{-1}\right)=tr\left(U^{-1}U\hat{F}_{\mu\nu}\hat{F}^{\mu\nu}\right)=tr\left(\hat{F}_{\mu\nu}\hat{F}^{\mu\nu}\right) \tag{24}$$

where the last step used the rotation of the matrix under the *tr* operator. Therefore, from formula (24), $\mathcal{L}'_{\text{YM}}(x)=\mathcal{L}_{\text{YM}}(x)$.

q.e.d.

### 4.4. Gauge Potential and Connection

The above discussion (including the generalization of Yang-Mills potential) shows that in a very general principal associated bundles structure, *i.e.* in the equation $\phi'(x)=U\left(\boldsymbol{\theta}(x)\right)\phi(x)$, one can choose a definition, $U\left(\boldsymbol{\theta}(x)\right)\equiv\rho\left(g(x)\right)\in\hat{G}$, to construct a local gauge transformation: $\phi'(x)=U(x)\phi(x)\equiv\rho\left(g(x)\right)\phi(x)$, $\forall\phi(x)\in V$. Here $V$ is the representation space of $\hat{G}$. So here $\phi(x)$ is a column matrix, $\rho\left(g(x)\right)$ is a square matrix. Reselect $F=V$, $\forall f(x)\in F=V$, define $\Phi(x)\equiv\sigma(x)\cdot f(x)\in\hat{\pi}^{-1}[x]\subset Q$, where $f(x)\in F$ (a typical fiber), one can introduce $\Phi'(x)\equiv\sigma'(x)\cdot f'(x)\in\hat{\pi}^{-1}[x]=\Phi(x)$. In addition, for the principal bundle *FM* and associated bundle *TM*, there is $q=\sigma(x)\cdot f^\mu=\left(x,e_\mu\right)f^\mu=e_\mu f^\mu=e'_\mu f'^\mu\equiv V$ which is called space-time vector, while $\Phi(x)$ can be called internal vector; $\sigma(x)$ is called the internal frame, and $f(x)$ is called the component of the internal vector expanded with the internal frame. But if $\sigma(x)g(x)^{-1}=\sigma'(x)$, $f(x)=f'(x)$, then: $\Phi(x)=\Phi'(x)$. In the above discussion, we also see that in order to ensure the invariance of the total Lagrangian density $\mathcal{L}$ under the local gauge transformation, that is, equivalent invariance for the cross section transformation to the principal bundle *P*, the gauge potential $A_\mu(x)$ must be introduced, and make it under gauge transform (*i.e.* the section transform $\sigma'(x)=\sigma(x)g(x)^{-1}$, where $g(x)$ is a group element field) to change to $A'_\mu(x)$, then $A_\mu(x)\to A'_\mu(x)$ (that is, the transformation of gauge potential satisfying Equation (19)) also corresponds to an absolute invariance, that is, the invariance of the connection on the principal bundle $\tilde{\boldsymbol{\omega}}$. In other words, the connection on the principal bundle is the gauge potential, which is invariant under the transformation of the gauge potential. For this reason, we firstly give three definitions of connection on the principal bundle:

1) A connection on the principal bundle $P(M,G)$ is to specify a horizontal subspace $H_p \subset T_pP$ for each point $p \in P$, satisfying:

a) $T_pP = V_p \oplus H_p$, $\forall p \in P$,

b) $R_{g^*}\left[H_p\right] = H_{pg}$, $\forall p \in P, g \in G$,

c) $H_p$ gives a $C^\infty$ $n$-dimensional distribution on $P$.

Here $V_p := \left\{X \in T_pP \middle| \pi_*(X) = 0\right\}$ is the vertical subspace.

2) A connection on the principal bundle $P(M,G)$ is a first-order differential form (abbreviation: 1 form) field $\tilde{\boldsymbol{\omega}}$ of a Lie algebra $\mathcal{G}$ value of $C^\infty$ on $P$, and satisfies:

a) $\tilde{\boldsymbol{\omega}}_p\left(A_p^*\right) = A, \forall A \in \mathcal{G}, p \in P$,

b) $\tilde{\boldsymbol{\omega}}_{pg}\left(R_{g^*}X\right) = \mathcal{A}d_{g^{-1}}\tilde{\boldsymbol{\omega}}_p(X), \forall p \in P, g \in G, X \in T_pP$,

Here $A_p^*$ is a vertical vector field generated by $A$ on $p \in P$; while $\mathcal{A}d_{g^{-1}}$ represents an automorphism called adjoint isomorphism constructed by element $g^{-1}$ through an isomorphic mapping $I_{g^{-1}} := g^{-1}hg, \forall h \in G$ to induce a push forward mapping (tangent mapping) as $I_{g^{-1}*} := \mathcal{A}d_{g^{-1}}$ at $I_{g^{-1}}(e) = e$ point, that is, $\mathcal{A}d_{g^{-1}} := \mathcal{G} \to \mathcal{G}$ is a linear transformation on Lie algebra $\mathcal{G}$.

3) A connection on the principal bundle $P(M,G)$ is to the local trivial $T_U : \pi^{-1}[U] \to U \times G$ specifies a 1-form field $\boldsymbol{\omega}_U$ of the $\mathcal{G}$ value of $C^\infty$ on $U$, that is a connection on $U \subset M$. At this time, if $T_V : \pi^{-1}[V] \to V \times G$ is another local triviality, that is, $U \cap V \neq \varnothing$, and the conversion function from $T_U$ to $T_V$ is $g_{UV}$, then there is

$$\boldsymbol{\omega}_V(Y) = \mathcal{A}d_{g_{UV}(x)^{-1}}\boldsymbol{\omega}_U(Y) + L^{-1}_{g_{UV}(x)*}g_{UV*}(Y) = B + A, \forall x \in U \cap V, Y \in T_xM \quad (25)$$

where $L^{-1}_{g_{UV}(x)}$ is the inverse map of the left translation $L_{g_{UV}(x)}$ generated by $g_{UV}(x) \in G$, $L^{-1}_{g_{UV}(x)*} \equiv \left(L^{-1}_{g_{UV}(x)}\right)_*$.

Furthermore, it can be shown that the three definitions are equivalent to each other. But the emphases are different: definition (1) means that the connection is the horizontal subspace of the point-tangent space on the fiber of the principal bundle; definition (2) means that the connection is the 1-form field with Lie algebra value on the fiber point of the principal bundle; definition (3) means that the connection is the 1-form field of a point on the fiber-bottom manifold of the principal bundle, that is, $\boldsymbol{\omega}_U \equiv \sigma_U^*\tilde{\boldsymbol{\omega}} \in \Lambda_U(1,\mathcal{G})$, where the lower * indicates the push forward mapping, and the upper * indicates the pull back mapping. The connection coordination formula (25) in the third definition of connection is a form of generalized gauge potential transformation, where the local trivial $T_U$ corresponds to the local cross section $\sigma_U := U \to P$, and there is a pull back mapping

$$\begin{cases} \boldsymbol{\omega}_U := \sigma_U^* \tilde{\boldsymbol{\omega}} \\ \boldsymbol{\omega}_V := \sigma_V^* \tilde{\boldsymbol{\omega}} \end{cases} \tag{26}$$

Proof of the Equation (25): Firstly

$$\boldsymbol{\omega}_V\big|_x(Y) = \left(\sigma_V^* \tilde{\boldsymbol{\omega}}\big|_{\sigma_V^*(x)}\right)(Y) = \tilde{\boldsymbol{\omega}}_{\sigma_V(x)}(\sigma_{V*}Y) \tag{27}$$

but (the last step, Leibniz's law, can be proved in the work [25])

$$\begin{aligned} \sigma_{V*}Y &= \sigma_{V*}\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 \eta(t) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 \sigma_V(\eta(t)) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 \left[\sigma_U(\eta(t))g_{UV}(\eta(t))\right] \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 \left[\sigma_U(\eta(t))g_{UV}(x)\right] + \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 \left[\sigma_U(x)g_{UV}(\eta(t))\right] \end{aligned} \tag{28}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 \left[\sigma_U(\eta(t))g_{UV}(x)\right] = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 R_{g_{UV}(x)}\sigma_U(\eta(t)) = R_{g_{UV}(x)*}\sigma_{U*}(Y) \tag{29}$$

$$\begin{aligned} \sigma_U(x)g_{UV}(\eta(t)) &= \sigma_U(x)g_{UV}(\eta(t)) = \sigma_V(x)g_{UV}(x)^{-1}g_{UV}(\eta(t)) \\ &= \sigma_V(x)\left[L_{g_{UV}(x)^{-1}}g_{UV}(\eta(t))\right] = R_{\sigma_V(x)}\left[L_{g_{UV}(x)^{-1}}g_{UV}(\eta(t))\right] \end{aligned} \tag{30}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 \left[\sigma_U(x)g_{UV}(\eta(t))\right] = R_{\sigma_V(x)*}\left[L^{-1}_{g_{UV}(x)*}g_{UV*}(Y)\right] = R_{\sigma_V(x)*}A = A^*_{\sigma_V(x)} \tag{31}$$

Therefore

$$\sigma_{V*}Y = R_{g_{UV}(x)*}\sigma_{U*}(Y) + A^*_{\sigma_V(x)} \tag{32}$$

$$\begin{aligned} \boldsymbol{\omega}_V\big|_x(Y) &= \tilde{\boldsymbol{\omega}}\big|_{\sigma_V(x)}(\sigma_{V*}Y) = \tilde{\boldsymbol{\omega}}\big|_{\sigma_V(x)}\left(R_{g_{UV}(x)*}\sigma_{U*}(Y)\right) + \tilde{\boldsymbol{\omega}}\big|_{\sigma_V(x)}\left(A^*_{\sigma_V(x)}\right) \\ &= \tilde{\boldsymbol{\omega}}\big|_{\sigma_V(x)}\left(R_{g_{UV}(x)*}\sigma_{U*}(Y)\right) + A = \tilde{\boldsymbol{\omega}}_{pg}\left(R_{g*}X\right) + A \\ &= \mathcal{A}d_{g^{-1}}\tilde{\boldsymbol{\omega}}_p(X) + A = B + A \end{aligned} \tag{33}$$

where

$$\begin{aligned} \mathcal{A}d_{g^{-1}}\tilde{\boldsymbol{\omega}}_p(X) &= \mathcal{A}d_{g_{UV}(x)^{-1}}\,\tilde{\boldsymbol{\omega}}\big|_{\sigma_U(x)}(\sigma_{U*}Y) = \mathcal{A}d_{g_{UV}(x)^{-1}}\left(\sigma_U^*\,\tilde{\boldsymbol{\omega}}\big|_{\sigma_U(x)}\right)(Y) \\ &= \mathcal{A}d_{g_{UV}(x)^{-1}}\,\boldsymbol{\omega}_U\big|_x(Y) = B \end{aligned} \tag{34}$$

q.e.d.

If the structure group $G$ is a $N \times N$ matrix group, $\mathrm{V} \equiv \{N \times N \text{ matrix}\}$, then $\mathrm{V}$ is a vector space, and $G \subset \mathrm{V}$, $\mathcal{G} \subset \mathrm{V}$, then Equation (25) can be expressed in a simplified form:

$$\boldsymbol{\omega}_V = g_{UV}^{-1}\boldsymbol{\omega}_U g_{UV} + g_{UV}^{-1}dg_{UV} \tag{35}$$

For example, if we take the general gauge potential on the bottom manifold (that is, not limited to the electromagnetic gauge potential, it may also include the gravitational gauge potential, etc.) as: $A_\mu^r(x) \to A_\mu'^r(x)$ (1 form field of real or complex value), then there is: $e_r A_a^r \in \Lambda_U(1,\mathcal{G})$, where $e_r$ is the basis in Lie algebra $\mathcal{G}$. In addition, $\boldsymbol{\omega} = \sigma^*\tilde{\boldsymbol{\omega}} \to \boldsymbol{\omega}' = \sigma'^*\tilde{\boldsymbol{\omega}}$, $\forall \Lambda_U(1,\mathcal{G})$, where $\Lambda_U(1,\mathcal{G})$ is a set of 1 form fields of Lie algebra $\mathcal{G}$ on $U$. So we can define: $\omega_a \equiv k e_r A_a^r \in \Lambda_M(1,\mathcal{G})$, or $\boldsymbol{\omega} \equiv k e_r \boldsymbol{A}^r \equiv k\boldsymbol{A} \in \Lambda_M(1,\mathcal{G})$, note that here $M$ is a suitable general bottom manifold (that is, our "universe" manifold satisfying the local trivial condition, which can be much more complicated than 4 dimensional real or complex manifolds and is the same as $M$ in the structure diagram of the principal associated bundles of the universe). Now we want to prove that the $\boldsymbol{\omega}$ defined in this way satisfies the transformation relation (25) of the connection definition (3) above, $\boldsymbol{\omega}_V(Y) \to \boldsymbol{\omega}_U(Y)$.

Proof: Let $\omega_\mu \equiv ke_r A^r_\mu \in \Lambda_M(0, \mathcal{G})$, then $\omega_\mu(x) \equiv ke_r A^r_\mu(x) \in \mathcal{G}$, $\boldsymbol{\omega} = \omega_\mu dx^\mu$, now let again $\sigma_V(x) = \sigma_U(x) g_{UV}(x) = \sigma'(x) = \sigma(x) g^{-1}(x)$, then Equation (25) becomes

$$\boldsymbol{\omega}'(Y) = \mathcal{A}d_{g(x)}\boldsymbol{\omega}(Y) + L_{g(x)*}g_*^{-1}(Y), \forall x \in M, Y \in T_x M \tag{36}$$

If one taking $Y \equiv \left.\frac{\partial}{\partial x^\mu}\right|_{x_0}$, then one gets $\boldsymbol{\omega}(Y) = \omega_\nu(x_0) dx^\nu\left(\left.\frac{\partial}{\partial x^\mu}\right|_{x_0}\right) = \omega_\mu(x_0)$, so proving Equation (36) only needs to prove

$$\omega'_\mu(x_0) = \mathcal{A}d_{g(x_0)}\omega_\mu(x_0) + L_{g(x_0)*}g_*^{-1}\left(\left.\frac{\partial}{\partial x^\mu}\right|_{x_0}\right) \tag{37}$$

Then, by considering $\omega_a \equiv ke_r A^r_a \equiv kA_a$, and using the formula:

$$\begin{aligned}\hat{\omega}_\mu(x) &\equiv k\hat{A}_\mu(x) \equiv k\left(-iL_r A^r_\mu(x)\right) = k\left(\rho_*(e_r) A^r_\mu(x)\right) \\ &= \rho_*\left(k\left(e_r A^r_\mu(x)\right)\right) = \rho_*\left(\omega_\mu(x)\right) \in \hat{\mathcal{G}},\end{aligned}$$

one can change Equation (19) as

$$\begin{aligned}\hat{\omega}'_\mu(x) &= \rho(g(x))\hat{\omega}_\mu(x)\rho(g(x))^{-1} - \left[\partial_\mu\rho(g(x))\right]\rho(g(x))^{-1} \\ &= \rho(g(x))\hat{\omega}_\mu(x)\rho(g(x))^{-1} + \rho(g(x))\left[\partial_\mu\rho(g(x))^{-1}\right] \\ &= \rho(g(x))\hat{\omega}_\mu(x)\rho(g(x))^{-1} + \rho(g(x))\left[\left.\frac{\partial}{\partial x^\mu}\right|_x \rho(g(x))^{-1}\right]\end{aligned} \tag{38}$$

Then, by defining $\left.\frac{\partial}{\partial x^\mu}\right|_x \rho(g(x))^{-1} = \left.\frac{\partial}{\partial t}\right|_0 \rho(g(\eta(t)))^{-1}$, the second item of the above formula can be proved to be ([26], Volume III p. 308)

$$\begin{aligned}&\rho(g(x))\left[\left.\frac{\partial}{\partial x^\mu}\right|_x \rho(g(x))^{-1}\right] \\ &= \left.\frac{\partial}{\partial t}\right|_0 \rho(g(x))\rho(g(\eta(t)))^{-1} = \left.\frac{\partial}{\partial t}\right|_0 L_{\rho(g(x))}\rho(g(\eta(t)))^{-1} \\ &= \left.\frac{\partial}{\partial t}\right|_0 L_{\rho(g(x))}\rho\left(g^{-1}(\eta(t))\right) = L_{\rho(g(x))*}\rho_* g_*^{-1}\left.\frac{\partial}{\partial t}\right|_0 \eta(t) \\ &= L_{\rho(g(x))*}\rho_* g_*^{-1}\left.\frac{\partial}{\partial x^\mu}\right|_x = \rho_*\left[L_{g(x)*}g_*^{-1}\left.\frac{\partial}{\partial x^\mu}\right|_x\right]\end{aligned} \tag{39}$$

Then the first item of Equation (38) is:

$$\begin{aligned}\rho(g(x))\hat{\omega}_\mu(x)\rho(g(x))^{-1} &= g_1 B g_1^{-1} = \mathcal{A}d_{g_1}B = \mathcal{A}d_{\rho(g(x))}\hat{\omega}_\mu(x) \\ &= \mathcal{A}d_{\rho(g(x))}\rho_*\left(\omega_\mu(x)\right) = \rho_*\left[\mathcal{A}d_{\rho(g(x))}\omega_\mu(x)\right]\end{aligned} \tag{40}$$

therefore

$$\hat{\omega}'_\mu(x) = \rho_*\left[ \mathcal{A}d_{\rho(g(x))}\omega_\mu(x) + L_{g(x)*}g_*^{-1}\frac{\partial}{\partial x^\mu}\bigg|_x \right] \tag{41}$$

So from $\hat{\omega}'_\mu(x) = \rho_*(\omega'_\mu(x))$, Equation (36) can be obtained.

q.e.d.

The above proof shows that the connection on the principal bundle is invariable, but the connection on the base manifold is variable and varies with the selection of cross sections. The transformation relationship is the same as that of the gauge potential. What is invariable is the connection on the principal bundle $\tilde{\boldsymbol{\omega}}$, therefore $\tilde{\boldsymbol{\omega}}$ is the (quite generalized) gauge potential! The author should emphasize that Equation (25) is actually a generalized gauge potential transformation, and $\boldsymbol{\omega}'(Y)$ or $\boldsymbol{\omega}(Y)$ is just components of $\tilde{\boldsymbol{\omega}}$, which include not only the electromagnetic gauge potential and other three basic interaction gauge potentials, but also the gravitational gauge potential; the transformation relationship between these components is exactly the gauge potential transformation relationship (25), but no matter how the gauge transformation, $\tilde{\boldsymbol{\omega}}$ is the same. This has implicitly pointed out that there is a unified universal gauge field corresponding to the invariable gauge potential in the universe!

Not only that, but it can be seen from formula (20) that the gauge potential can define the gauge field strength. At this time, if there are physically: $A_\mu^r \to A_\mu'^r, F_{\mu\nu}^r \to F_{\mu\nu}'^r$, so what transformation does it correspond to in mathematics? What is the constant quantity?

## 5. Gauge Field and Space-Time Curvature

### 5.1. Gauge Field Strength and Curvature

The answer to the above question is the curvature transformation on the bottom manifold, that is, $\boldsymbol{\Omega} \to \boldsymbol{\Omega}'$, and the constant "original quantity" is the curvature on the principal bundle, denoted as $\hat{\boldsymbol{\Omega}}$. Now first define the curvature on the principal bundle:

Let $K$ be a manifold and $\mathcal{G}$ be a Lie algebra, then $\Lambda_K(i,\mathcal{G})$ is the set of smooth $i$-form fields of all Lie algebras $\mathcal{G}$ valued on $K$, namely vector space. Set up $\boldsymbol{\varphi} \in \Lambda_K(i,\mathcal{G})$, $\boldsymbol{\psi} \in \Lambda_K(j,\mathcal{G})$, $[\boldsymbol{\varphi},\boldsymbol{\psi}] \in \Lambda_K(i+j,\mathcal{G})$, then there is

$$[\boldsymbol{\varphi},\boldsymbol{\psi}](X_1,\cdots,X_{i+j}) := \frac{1}{i!j!}\sum_\pi \delta_\pi\left[\boldsymbol{\varphi}\left(X_{\pi(1)},\cdots,X_{\pi(i)}\right),\boldsymbol{\psi}\left(X_{\pi(i+1)},\cdots,X_{\pi(i+j)}\right)\right] \tag{42}$$

Here $\pi$ represents the arrangement, $\delta_\pi$ is defined as the even permutation is 1, and the odd permutation is −1, then $[\boldsymbol{\varphi},\boldsymbol{\psi}](X_1,\cdots,X_{i+j}) \in \Lambda_K(o,\mathcal{G})$ becomes a Lie bracket when $(X_1,\cdots,X_{i+j})$ is fixed. If $i=2, j=1$, then

$$\begin{aligned}
&[\boldsymbol{\varphi},\boldsymbol{\psi}](X_1,X_2,X_3) \\
&= \frac{1}{2}\Big(\big[\boldsymbol{\varphi}(X_1,X_2),\boldsymbol{\psi}(X_3)\big] + \big[\boldsymbol{\varphi}(X_3,X_1),\boldsymbol{\psi}(X_2)\big] + \big[\boldsymbol{\varphi}(X_2,X_3),\boldsymbol{\psi}(X_1)\big] \\
&\quad - \big[\boldsymbol{\varphi}(X_2,X_1),\boldsymbol{\psi}(X_3)\big] - \big[\boldsymbol{\varphi}(X_1,X_3),\boldsymbol{\psi}(X_2)\big] - \big[\boldsymbol{\varphi}(X_3,X_2),\boldsymbol{\psi}(X_1)\big]\Big)
\end{aligned} \tag{43}$$

In addition, if $\boldsymbol{\omega} \in \Lambda_K(1,\mathcal{G})$, then one has

$$[\boldsymbol{\omega},\boldsymbol{\omega}](X_1,X_2) = [\boldsymbol{\omega}(X_1),\boldsymbol{\omega}(X_2)] - [\boldsymbol{\omega}(X_2),\boldsymbol{\omega}(X_1)] = 2[\boldsymbol{\omega}(X_1),\boldsymbol{\omega}(X_2)] \quad (44)$$

Usually, $\Lambda_K(i+j,\mathcal{G})$ is called the graded Lie algebra, if the Lie bracket operations are also defined in $\Lambda_K(i+j,\mathcal{G})$.

The covariant exterior differential is defined as follows: let the principal bundle with connection $(P,\tilde{\boldsymbol{\omega}})$ have $\boldsymbol{\varphi} \in \Lambda_P(i,\mathcal{G})$, define:

1) $\boldsymbol{\varphi}^H \in \Lambda_P(i,\mathcal{G})$ is the horizontal component of $\boldsymbol{\varphi}$,

$$\boldsymbol{\varphi}^H(X_1,\cdots,X_i) := \boldsymbol{\varphi}(X_1^H,\cdots,X_i^H) \quad (45)$$

where $X_1,\cdots,X_i$ is an arbitrary vector field on *P*, respectively.

2) The covariant external differential is defined as:

$$D\boldsymbol{\varphi} := (d\boldsymbol{\varphi})^H \in \Lambda_P(i+1,\mathcal{G}) \quad (46)$$

3) The curvature $\tilde{\boldsymbol{\Omega}}$ of a connection $\tilde{\boldsymbol{\omega}} \in \Lambda_P(1,\mathcal{G})$ is defined as (note that connections on the principal bundle can determine the curvature.)

$$\tilde{\boldsymbol{\Omega}} := D\tilde{\boldsymbol{\omega}} \equiv (d\tilde{\boldsymbol{\omega}})^H \in \Lambda_P(2,\mathcal{G}) \quad (47)$$

Then it can be proved that:

$$\tilde{\boldsymbol{\Omega}} = d\tilde{\boldsymbol{\omega}} + \frac{1}{2}[\tilde{\boldsymbol{\omega}},\tilde{\boldsymbol{\omega}}] \quad (48)$$

Proof:

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$$\begin{aligned}\tilde{\boldsymbol{\Omega}}_p(X,Y) = D\tilde{\boldsymbol{\omega}}_p(X,Y) = (d\tilde{\boldsymbol{\omega}})^H\Big|_p(X,Y) = d\tilde{\boldsymbol{\omega}}\big|_p(X^H,Y^H) \\ = d\tilde{\boldsymbol{\omega}}_p(X,Y) + \frac{1}{2}[\tilde{\boldsymbol{\omega}},\tilde{\boldsymbol{\omega}}]_p(X,Y) \\ = d\tilde{\boldsymbol{\omega}}_p(X,Y) + [\tilde{\boldsymbol{\omega}}_p(X),\tilde{\boldsymbol{\omega}}_p(Y)]\end{aligned} \quad (49)$$

therefore

$$d\tilde{\boldsymbol{\omega}}\big|_p(X^H,Y^H) = d\tilde{\boldsymbol{\omega}}_p(X,Y) + [\tilde{\boldsymbol{\omega}}_p(X),\tilde{\boldsymbol{\omega}}_p(Y)] \quad (50)$$

Then there are three cases:

a) $X,Y \in H_p \Rightarrow \tilde{\boldsymbol{\omega}}_p(X) = 0, \tilde{\boldsymbol{\omega}}_p(Y) = 0$, Equation (49) holds;

b) $X,Y \in V_p \Rightarrow X = A_p^*, Y = B_p^*, 0 = d\tilde{\boldsymbol{\omega}}_p(X,Y) + [\tilde{\boldsymbol{\omega}}_p(X),\tilde{\boldsymbol{\omega}}_p(Y)]$,

hence

$$d\tilde{\boldsymbol{\omega}}\big|_p(A_p^*,B_p^*) = e_r\, d\tilde{\boldsymbol{\omega}}^r\Big|_p(A_p^*,B_p^*) = -[\tilde{\boldsymbol{\omega}}_p(A_p^*),\tilde{\boldsymbol{\omega}}_p(B_p^*)] \quad (51)$$

The formula is used here

$$\begin{aligned}d\tilde{\boldsymbol{\omega}}(A^*,B^*) &= e_r d\tilde{\boldsymbol{\omega}}^r(A^*,B^*) \\ &= e_r A^*(\tilde{\boldsymbol{\omega}}^r(B^*)) - e_r B^*(\tilde{\boldsymbol{\omega}}^r(A^*)) - e_r\tilde{\boldsymbol{\omega}}^r([A^*,B^*]) \\ &= e_r A^*(\tilde{\boldsymbol{\omega}}^r(B^*)) - e_r B^*(\tilde{\boldsymbol{\omega}}^r(A^*)) - \tilde{\boldsymbol{\omega}}([A^*,B^*]) \\ &= A^*(e_r\tilde{\boldsymbol{\omega}}^r(B^*)) - B^*(e_r\tilde{\boldsymbol{\omega}}^r(A^*)) - \tilde{\boldsymbol{\omega}}([A^*,B^*])\end{aligned}$$

$$
\begin{aligned}
&= A^*\left(\tilde{\boldsymbol{\omega}}\left(B^*\right)\right) - B^*\left(\tilde{\boldsymbol{\omega}}\left(A^*\right)\right) - \tilde{\boldsymbol{\omega}}\left(\left[A^*, B^*\right]\right) \\
&= d\tilde{\boldsymbol{\omega}}\left(B^*\right)\left(A^*\right) - d\tilde{\boldsymbol{\omega}}\left(A^*\right)\left(B^*\right) - \tilde{\boldsymbol{\omega}}\left(\left[A^*, B^*\right]\right) \\
&= -\tilde{\boldsymbol{\omega}}\left(\left[A^*, B^*\right]\right) = -\tilde{\boldsymbol{\omega}}\left(\left[A, B\right]^*\right) = -\left[A, B\right] \\
&= -\left[\tilde{\boldsymbol{\omega}}_p\left(A_p^*\right), \tilde{\boldsymbol{\omega}}_p\left(B_p^*\right)\right]
\end{aligned}
\tag{52}
$$

and used definitions and equations:

$$
\begin{aligned}
e_r A^*\left(\tilde{\boldsymbol{\omega}}^r\left(B^*\right)\right) &:= e_r d\tilde{\boldsymbol{\omega}}^r\left(B^*\right)\left(A^*\right) = de_r\tilde{\boldsymbol{\omega}}^r\left(B^*\right)\left(A^*\right) \\
&= d\tilde{\boldsymbol{\omega}}\left(B^*\right)\left(A^*\right) = A^*\left(\tilde{\boldsymbol{\omega}}\left(B^*\right)\right)
\end{aligned}
\tag{53}
$$

Also $\tilde{\boldsymbol{\omega}}\left(B^*\right)$, $\tilde{\boldsymbol{\omega}}\left(A^*\right)$ is a constant scalar field with a value of $\mathcal{G}$, $\therefore d\tilde{\boldsymbol{\omega}}\left(B^*\right) = 0$, $d\tilde{\boldsymbol{\omega}}\left(A^*\right) = 0$.

c) $X \in V_p, Y \in H_p \Rightarrow X = A_p^*, Y = \tilde{\boldsymbol{Z}}_p$ ($\tilde{\boldsymbol{Z}}_p$ that is, the value of a horizontal lift field at *p*), then one gets

$$
\begin{aligned}
d\tilde{\boldsymbol{\omega}}\big|_p(0,\cdot) = 0 &= d\tilde{\boldsymbol{\omega}}\big|_p\left(A_p^*, \tilde{\boldsymbol{Z}}_p\right) + \left[\tilde{\boldsymbol{\omega}}_p\left(A_p^*\right), \tilde{\boldsymbol{\omega}}_p\left(\tilde{\boldsymbol{Z}}_p\right)\right] \\
&= d\tilde{\boldsymbol{\omega}}\big|_p\left(A_p^*, \tilde{\boldsymbol{Z}}_p\right) + \left[A, 0\right] = d\tilde{\boldsymbol{\omega}}\big|_p\left(A_p^*, \tilde{\boldsymbol{Z}}_p\right) \\
&= A_p^*\left(\tilde{\boldsymbol{\omega}}_p\left(\tilde{\boldsymbol{Z}}_p\right)\right) - \tilde{\boldsymbol{Z}}_p\left(\tilde{\boldsymbol{\omega}}_p\left(A_p^*\right)\right) - 0 \\
&= 0 - \tilde{\boldsymbol{Z}}_p\left(\tilde{\boldsymbol{\omega}}_p\left(A_p^*\right)\right) - 0 = 0
\end{aligned}
\tag{54}
$$

A theorem has been used here: $\left[A^*, \tilde{\boldsymbol{Z}}\right] = 0$, and a vector field acting on a constant scalar field is 0, $\tilde{\boldsymbol{Z}}(A) = 0$, so there is

$$
\begin{aligned}
d\tilde{\boldsymbol{\omega}}\left(A^*, \tilde{\boldsymbol{Z}}\right) &= A^*\left(\tilde{\boldsymbol{\omega}}\left(\tilde{\boldsymbol{Z}}\right)\right) - \tilde{\boldsymbol{Z}}\left(\tilde{\boldsymbol{\omega}}\left(A^*\right)\right) - \tilde{\boldsymbol{\omega}}\left(\left[A^*, \tilde{\boldsymbol{Z}}\right]\right) \\
&= A^*\left(\tilde{\boldsymbol{\omega}}\left(\tilde{\boldsymbol{Z}}\right)\right) - \tilde{\boldsymbol{Z}}\left(\tilde{\boldsymbol{\omega}}\left(A^*\right)\right) \\
&= 0 - \tilde{\boldsymbol{Z}}\left(\tilde{\boldsymbol{\omega}}\left(A^*\right)\right) = -\tilde{\boldsymbol{Z}}(A) = 0
\end{aligned}
\tag{55}
$$

q.e.d.

Furthermore, it can be proved that:

$$D\tilde{\boldsymbol{\Omega}} = 0 \tag{56}$$

In fact, by definition (47), one gets,

$$D\tilde{\boldsymbol{\Omega}} = \left(d\tilde{\boldsymbol{\Omega}}\right)^H \tag{57}$$

that is

$$
\begin{aligned}
&D\tilde{\boldsymbol{\Omega}}(X,Y,Z) = \left(d\tilde{\boldsymbol{\Omega}}\right)^H(X,Y,Z) \\
&= \left(d\tilde{\boldsymbol{\Omega}}\right)\left(X^H, Y^H, Z^H\right) = \frac{1}{2}d\left[\tilde{\boldsymbol{\omega}}, \tilde{\boldsymbol{\omega}}\right]\left(X^H, Y^H, Z^H\right) \\
&= \frac{1}{2}\left\{\left[d\tilde{\boldsymbol{\omega}}, \tilde{\boldsymbol{\omega}}\right] - \left[\tilde{\boldsymbol{\omega}}, d\tilde{\boldsymbol{\omega}}\right]\right\}\left(X^H, Y^H, Z^H\right) = \left[d\tilde{\boldsymbol{\omega}}, \tilde{\boldsymbol{\omega}}\right]\left(X^H, Y^H, Z^H\right) \\
&= \frac{1}{2}\left\{\left[d\tilde{\boldsymbol{\omega}}\left(X^H, Y^H\right), \tilde{\boldsymbol{\omega}}\left(Z^H\right)\right] + \left[d\tilde{\boldsymbol{\omega}}\left(Z^H, X^H\right), \tilde{\boldsymbol{\omega}}\left(Y^H\right)\right]\right. \\
&\quad \left. + \left[d\tilde{\boldsymbol{\omega}}\left(Y^H, Z^H\right), \tilde{\boldsymbol{\omega}}\left(X^H\right)\right] - \left[d\tilde{\boldsymbol{\omega}}\left(X^H, Z^H\right), \tilde{\boldsymbol{\omega}}\left(Y^H\right)\right]\right.
\end{aligned}
$$

$$
\begin{aligned}
&-\left[d\tilde{\omega}\left(Z^H,Y^H\right),\tilde{\omega}\left(X^H\right)\right]-\left[d\tilde{\omega}\left(Y^H,X^H\right),\tilde{\omega}\left(Z^H\right)\right]\Big\} \\
&=\frac{1}{2}\Big\{\left[d\tilde{\omega}\left(X^H,Y^H\right),0\right]+\left[d\tilde{\omega}\left(Z^H,X^H\right),0\right]+\left[d\tilde{\omega}\left(Y^H,Z^H\right),0\right] \\
&-\left[d\tilde{\omega}\left(X^H,Z^H\right),0\right]-\left[d\tilde{\omega}\left(Z^H,Y^H\right),0\right]-\left[d\tilde{\omega}\left(Y^H,X^H\right),0\right]\Big\} \\
&=0
\end{aligned}
\tag{58}
$$

where

$$d\tilde{\mathbf{\Omega}}=0+\frac{1}{2}d\left[\tilde{\omega},\tilde{\omega}\right]=\frac{1}{2}\left(\left[d\tilde{\omega},\tilde{\omega}\right]-\left[\tilde{\omega},d\tilde{\omega}\right]\right)=\left[d\tilde{\omega},\tilde{\omega}\right] \tag{59}$$

q.e.d.

And there is also a theorem that holds:

$$d\tilde{\mathbf{\Omega}}=\left[\tilde{\mathbf{\Omega}},\tilde{\omega}\right] \tag{60}$$

Proof:

$$\left[\tilde{\mathbf{\Omega}},\tilde{\omega}\right]=\left[d\tilde{\omega},\tilde{\omega}\right]+\frac{1}{2}\left[\left[\tilde{\omega},\tilde{\omega}\right],\tilde{\omega}\right]=\left[d\tilde{\omega},\tilde{\omega}\right]+0=d\tilde{\mathbf{\Omega}} \tag{61}$$

Again by the previous definition:

$$\boldsymbol{\omega}_U := \sigma_U^*\tilde{\omega}, \boldsymbol{\omega}\equiv ke_rA_\mu^r dx^\mu = k\boldsymbol{A}, \boldsymbol{A}\equiv e_rA_\mu^r dx^\mu$$

one can introduce:

$$\mathbf{\Omega}_U := \sigma_U^*\tilde{\mathbf{\Omega}}, \boldsymbol{F}\equiv\frac{1}{2}e_rF_{\mu\nu}^r dx^\mu\wedge dx^\nu \tag{62}$$

From this we can prove the theorem: Let $\boldsymbol{F}\equiv\frac{1}{2}e_rF_{\mu\nu}^r dx^\mu\wedge dx^\nu$, then $\mathbf{\Omega}_U=k\boldsymbol{F}$, where $k$ is a scale coefficient, and $\boldsymbol{F}$ can be defined as the gauge field strength.

Proof:

$$
\begin{aligned}
\mathbf{\Omega}&=d\boldsymbol{\omega}+\frac{1}{2}\left[\boldsymbol{\omega},\boldsymbol{\omega}\right]=d\left(ke_rA_\mu^r dx^\mu\right)+\frac{1}{2}\left[ke_sA_\mu^s dx^\mu, ke_tA_\nu^t dx^\nu\right] \\
&=ke_r dA_\mu^r\wedge dx^\mu+\frac{1}{2}k^2\left[e_s,e_t\right]A_\mu^sA_\nu^t dx^\mu\wedge dx^\nu \\
&=\frac{1}{2}ke_r\left(\partial_\mu A_\nu^r-\partial_\nu A_\mu^r\right)dx^\mu\wedge dx^\nu+\frac{1}{2}k^2C_{st}^r e_rA_\mu^sA_\nu^t dx^\mu\wedge dx^\nu \\
&=\frac{1}{2}ke_r\left(\partial_\mu A_\nu^r-\partial_\nu A_\mu^r+kC_{st}^rA_\mu^sA_\nu^t\right)dx^\mu\wedge dx^\nu \\
&=\frac{1}{2}ke_rF_{\mu\nu}^r dx^\mu\wedge dx^\nu=k\boldsymbol{F}
\end{aligned}
\tag{63}
$$

here, $\left[e_s,e_t\right]=C_{st}^r e_r$, as well

$$
\begin{aligned}
dA_\mu^r\wedge dx^\mu&=\frac{\partial A_\mu^r}{\partial x^\nu}dx^\nu\wedge dx^\mu=\left(\partial_\nu A_\mu^r\right)dx^\nu\wedge dx^\mu \\
&=\left(\partial_{[\mu}A_{\nu]}^r\right)dx^\mu\wedge dx^\nu=\frac{1}{2}\left(\partial_\mu A_\nu^r-\partial_\nu A_\mu^r\right)dx^\mu\wedge dx^\nu
\end{aligned}
\tag{64}
$$

The above proof shows that the defined $\tilde{\mathbf{\Omega}}$ is the gauge field strength, and the defined $\tilde{\omega}$ is the gauge potential, which is constant. However, $\mathbf{\Omega}$ and $\boldsymbol{\omega}$ change with the cross section, and the corresponding transformation also corresponds to the transformation of the gauge potential. These transformations are caused by the transformation of the cross section (that is, the gauge transformation), but the "grand quantities" connection $\tilde{\boldsymbol{\omega}}$ and curvature $\tilde{\boldsymbol{\Omega}}$ in $(P,\tilde{\boldsymbol{\omega}})$ and $(P,\tilde{\boldsymbol{\Omega}})$ are constant, and the change are only their components $\boldsymbol{\omega} \to \boldsymbol{\omega}', \boldsymbol{\Omega} \to \boldsymbol{\Omega}'$, respectively.

## 5.2. Transformation Relation of Curvature

Under the cross section transformation $\sigma$, the transformation relationship of $\boldsymbol{\omega} \to \boldsymbol{\omega}'$ on the bottom manifold (that is, our world) is (25); what is the transformation relationship of $\boldsymbol{\Omega} \to \boldsymbol{\Omega}'$? To do this, we need to prove the following theorems:

$$\text{(Theorem A)}\quad \boldsymbol{\Omega}_U = d\boldsymbol{\omega}_U + \frac{1}{2}[\boldsymbol{\omega}_U, \boldsymbol{\omega}_U] \tag{65}$$

Proof:

$$\begin{aligned} \boldsymbol{\Omega}_U &= \sigma_U^*\tilde{\boldsymbol{\Omega}} = \sigma_U^*\left(d\tilde{\boldsymbol{\omega}} + \frac{1}{2}[\tilde{\boldsymbol{\omega}},\tilde{\boldsymbol{\omega}}]\right) \\ &= d\left(\sigma_U^*\tilde{\boldsymbol{\omega}}\right) + \frac{1}{2}\left[\sigma_U^*\tilde{\boldsymbol{\omega}}, \sigma_U^*\tilde{\boldsymbol{\omega}}\right] \\ &= d\boldsymbol{\omega}_U + \frac{1}{2}[\boldsymbol{\omega}_U, \boldsymbol{\omega}_U] \end{aligned} \tag{66}$$

(Theorem B) Equation (63) corresponds to Equation (20), namely

$$\begin{cases} \boldsymbol{\Omega} = d\boldsymbol{\omega} + \frac{1}{2}[\boldsymbol{\omega},\boldsymbol{\omega}] \\ \Updownarrow \\ F_{\mu\nu}^r(x) = \partial_\mu A_\nu^r - \partial_\nu A_\mu^r + k\sum_{s,t=1}^{R} C_{st}^r A_\mu^s(x) A_\nu^t(x), (r = 1,\cdots,R) \end{cases} \tag{67}$$

Proof: the same as Equation (63),

$$\begin{aligned} \boldsymbol{\Omega} &= ke_r d\left(A_\mu^r dx^\mu\right) + \frac{1}{2}[e_s, e_t] k^2 A_\mu^s A_\nu^t dx^\mu \wedge dx^\nu \\ &= \frac{1}{2} ke_r \left(\partial_\mu A_\nu^r - \partial_\nu A_\mu^r + C_{st}^r e_r k^2 A_\mu^s A_\nu^t\right) dx^\mu \wedge dx^\nu \\ &= \frac{1}{2} ke_r F_{\mu\nu}^r dx^\mu \wedge dx^\nu \\ &\therefore F_{\mu\nu}^r = \partial_\mu A_\nu^r - \partial_\nu A_\mu^r + C_{st}^r e_r k^2 A_\mu^s A_\nu^t \end{aligned}$$

Here Equations (62) and (63) are considered.

(Theorem C) Let $g_{UV}: U \cap V \to G$ be the local trivial transformation function from $T_U$ to $T_V$, then on $U \cap V$ of the bottom manifold one gets

$$\boldsymbol{\Omega}_V = \mathcal{A}d_{g_{UV}^{-1}} \boldsymbol{\Omega}_U \tag{68}$$

Proof:

$\forall x \in U \cap V, X, Y \in T_x M$, one has

$$\boldsymbol{\Omega}_V(X,Y) = \left(\sigma_V^*\tilde{\boldsymbol{\Omega}}\right)(X,Y) = \tilde{\boldsymbol{\Omega}}\left(\sigma_V^* X, \sigma_V^* Y\right) \tag{69}$$

Again considering Equations (31) and (32), one gets

$$\sigma_V^* X = R_{g_{UV}(x)*}\sigma_{U*}X + \left[L_{g_{UV}(x)*}^{-1} g_{UV}(X)\right]_{\sigma_V(x)}^* = X_1 + X_2 \tag{70}$$

$$\sigma_V^* Y = R_{g_{UV}(x)*}\sigma_{U*}Y + \left[L_{g_{UV}(x)*}^{-1} g_{UV}(Y)\right]_{\sigma_V(x)}^* = Y_1 + Y_2 \tag{71}$$

Hence Equation (69) changes

$$\begin{aligned}\mathbf{\Omega}_V(X,Y) &= \tilde{\mathbf{\Omega}}(X_1+X_2, Y_1+Y_2) \\ &= \tilde{\mathbf{\Omega}}(X_1,Y_1) + \tilde{\mathbf{\Omega}}(X_1,Y_2) + \tilde{\mathbf{\Omega}}(X_2,Y_1) + \tilde{\mathbf{\Omega}}(X_2,Y_2)\end{aligned} \tag{72}$$

The action objects of the last three items on the right side of the above formula involve the vertical vector $X_2$ or $Y_2$, so they are all 0, so it can be obtained

$$\begin{aligned}\mathbf{\Omega}_V(X,Y) &= \tilde{\mathbf{\Omega}}(X_1,Y_1) = \tilde{\mathbf{\Omega}}\left(R_{g_{UV}(x)*}\sigma_{U*}X, R_{g_{UV}(x)*}\sigma_{U*}Y\right) \\ &= \left(R_{g_{UV}(x)}^*\tilde{\mathbf{\Omega}}\right)(\sigma_{U*}X, \sigma_{U*}Y) = \left(\mathcal{A}d_{g_{UV}^{-1}}\tilde{\mathbf{\Omega}}\right)(\sigma_{U*}X, \sigma_{U*}Y) \\ &= \mathcal{A}d_{g_{UV}^{-1}}\left(\sigma_U^*\tilde{\mathbf{\Omega}}\right)(X,Y) = \mathcal{A}d_{g_{UV}^{-1}(x)}\mathbf{\Omega}_U(X,Y)\end{aligned} \tag{73}$$

here, the theorem is used: $R_g^*\tilde{\mathbf{\Omega}} = \mathcal{A}d_{g^{-1}}\tilde{\mathbf{\Omega}}, \forall g \in G$.

q.e.d.

(Theorem D) If the structure group is a matrix group, formula (68) can be further expressed as:

$$\mathbf{\Omega}_V = g_{UV}^{-1}\mathbf{\Omega}_U g_{UV} \tag{74}$$

Therefore, under the cross section transformation $\sigma$, the transformation relationship of $\boldsymbol{\omega} \to \boldsymbol{\omega}'$ on the bottom manifold (that is, the world we assume) is the Equation (25), and the corresponding transformation relationship of $\mathbf{\Omega} \to \mathbf{\Omega}'$ is the Equation (68)!

### 5.3. Gravitational Gauge Field

Physics has roughly unified the three fundamental interactions of electromagnetism, weak and strong, but the fundamental interaction of gravitation has not been unified, which makes us pay special attention to the problem of the gravitational gauge field. 1) Is gravity a gauge field? The answer is: suppose it is. Many authors have proposed gravitational gauge theory in this regard, such as Duan Yishi's general theory of relativity and gravitational gauge theory and Yue-Liang Wu's unified theory of gauge field [25] [28]; 2) the general gauge equation proved above (25) and the corresponding gauge potential in Equation (68) also include the gravitational gauge potential? The answer is yes! In fact, in Chapter 6 and Appendix 3 of Duan Yishi's book, the author has proved relatively well that in the principal fiber bundle where the $SO(n)$ group is the structural group, the connection is the gravitational gauge potential, and the curvature is the gravitational gauge field strength. And there is a very classic saying, "The gauge theory of physics is just the principal fiber bundle (principal bundle) theory in mathematics" [25]. Considering these circumstantial evidences, the author of this paper confirms **that Equations (25) and (68) are some forms of GGE**

**(generalized gauge equations) that are generally valid for the gauge potential of fundamental interaction; the connection of space-time is the gauge potential, and the curvature of space-time is the gauge field strength; no matter for gravitation or for electromagnetic force and weak and strong interaction forces, Equations (25) and (68) are true for the four fundamental interactions.**

So under the condition that the structure group is large enough, the generalized gauge Equation (GGE) can express the transformation of the gauge potential on the bottom manifold (our universe) from one fundamental interaction gauge potential to another fundamental interaction gauge potential, such as the transformation from electromagnetic gauge potential to gravitational gauge potential, while the universal unified gauge potential on the principal bundle (similar to the "heaven" of high-dimensional space-time) is unchanged, these local gauge potentials on the base manifold are just the projection components of the choice of the unified gauge potential under different gauge (or the cross section of principal bundle). However, this transformation is not only the transformation within the same gauge field as understood in the traditional view, but may be the transformation between different gauge fields, or even the transformation between gauge potentials with cross fundamental interactions. We can call this transformation from gauge potentials with one fundamental interaction to gauge potentials with another fundamental interaction as "cross fundamental gauge potential transformation". This is exactly the meaning of the great unity of physics revealed by Equation (25) and the corresponding Equation (68); it is also the outline significance of the grand unification of the cosmic gauge field revealed by the structural picture of the principal associated bundles of the universe.

### 5.4. Connection of Entangled States

The above "grand unified outline" can first be used to explain the "paradox" of quantum entangled states. What is quantum entanglement first? The rough answer of quantum theory is that if the wave functions of two particles cannot be written as the scalar product of two wave functions, then the two particles are entangled. Fan Hongyi believed in the book [29] that the entangled state is the common eigenstate of the commutator $P_1 \pm P_2$ and $Q_1 \mp Q_2$, *i.e.* $[P_1 \pm P_2, Q_1 \mp Q_2]$, here the momentum operator of two particle is respectively $P_1$ or, $P_2$ and the position operator of two particle is respectively as $Q_1$ or $Q_2$. Hence for $[P_1 + P_2, Q_1 - Q_2]$, it can be expressed as

$$\begin{cases} (Q_1 - Q_2)|\eta\rangle = \sqrt{2}\eta_1 |\eta\rangle \\ (P_1 + P_2)|\eta\rangle = \sqrt{2}\eta_2 |\eta\rangle \end{cases} \tag{75}$$

Here, the form of the entangled state $|\eta\rangle$ is

$$|\eta\rangle = \exp\left(-\frac{1}{2}|\eta|^2 + \eta a_1^\dagger - \eta^* a_2^\dagger + a_1^\dagger a_2^\dagger\right)|00\rangle \tag{76}$$

Among them, $\eta = \eta_1 + \eta_2$, $Q_j = \frac{a_j + a_j^\dagger}{\sqrt{2}}$, $P_j = \frac{a_j - a_j^\dagger}{\sqrt{2}i}$; and $|\eta\rangle$ satisfies the completeness relation and is also orthogonal.

One of the strangeness of the entangled state is hidden in the Schmidt decomposition in its coordinate representation:

$$|\eta\rangle = \mathrm{e}^{-i\eta_1\eta_2} \int_{-\infty}^{+\infty} \mathrm{d}q \, |q\rangle_1 \otimes \left| q - \sqrt{2}\eta_1 \right\rangle_2 \mathrm{e}^{i\eta_2 q} \tag{77}$$

The above formula shows that when particle 1 is in its coordinate eigenstate $|q\rangle_1$, particle 2 is simultaneously in its coordinate eigenstate $\left| q - \sqrt{2}\eta_1 \right\rangle$, and these two particles have a distance, and the distance may even be great. Why does the state change of particle 1 cause the state of particle 2 to change correspondingly "simultaneously"? From the general gauge Equation (25), if all the interactions in the universe are regarded as the interaction of the gauge field and a kind of space-time regional force, then the various strange performances of the quantum entanglement state are easy to understand, so our answer is as follows:

1) The so-called quantum entanglement means that the space-time of two particles establishes a connection, thus a gauge potential appears, and a gauge field acts on the two particles.

2) This connection is a special connection, the regional gauge potential, which makes the sum of the two particles' momentum and the difference of their coordinates limit the common eigenstates of the two particles in this region, and as a whole, it is affected by the regional gauge field at the same time. This is different from the common sense of connection or gauge potential. The difference lies in the "simultaneous regional integration".

3) Any operation to change the state of one of the particles is equivalent to changing the structure of the gauge potential in this area at the same time, which must affect the role of all gauge fields in this area at the same time, and must also affect the corresponding state of another particle.

In a word, quantum entanglement is to establish the gauge field of the whole region between two particles, so that the whole of the two particles are affected by the gauge field at the same time. Any change in the state of one particle must change the state of the other particle at the same time. Here, the gauge effect is simultaneous for the whole region, without the meaning of time and transmission speed. In fact, the quantum entangled state may be the common eigenstate of the operator of the relative distance between the two quanta and the total momentum operator of the two quanta, which indicates that the relative distance between the two quanta can change in the space time region where the entanglement occurs, and their total momentum can also change, but the region of the space time regional force that causes the entanglement will also change, still covering the entire region where the entangled state is located. Therefore, any "operation or measurement" behavior that changes a single quantum state will "simultaneously" cause the gauge field force of the whole space-time region enveloped to act on another entangled quantum to change its state. There is no superluminal force transmission, which is the concept of assumption that the role of this special global gauge field force is still considered as a local part.

## 6. Conclusions and Outlook

A) In this paper, a general unified program of physics is proposed, and the unified formula is the generalized gauge equation GGE or Equation (25). All interactions in the world are unified in the gauge potential (gauge field) in the universe picture of the principal associated bundles, and the fundamental interaction of our universe on the bottom manifold is just the representation of the component of the gauge potential of this principal bundle, which follows the transformation of the generalized gauge equation from one component to another, or even the transformation of one fundamental interaction into another.

B) The invariance of the gauge transformation or the satisfaction of the generalized gauge Equation (EEG) is a necessary condition for the universal unified field theory, but the quantization of the field is not a necessary condition for the universal unified field theory.

C) Outlook: 1) The fundamental interactions may not be limited to four types, and the discovery of new fundamental interactions is possible. 2) The fundamental interaction may be transformed into each other, and the basic equation of transformation is just GGE. 3) It is found that the structure group can express more gauge field components; so it is an important task for physics in the future to simplify and solve EEG so that one can concretely express the transformation relationship between any two gauge field components, especially the transformation relationship between electromagnetic force and gravity (which is extremely important for solving human aerospace dynamics).

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## Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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