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Single Charged Particle Motion in a Flat Surface with Static Electromagnetic Field and Quantum Hall Effect

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Abstract

Taking into account the non separable solution for the quantum problem of the motion of a charged particle in a flat surface of lengths L_x and L_y with transversal static magnetic field $\boldsymbol{B}$ and longitudinal static electric field $\boldsymbol{E}$, the quantum current, the transverse (Hall) and longitudinal resistivities are calculated for the state $n=0$ and $j=0$. We found that the transverse resistivity is proportional to an integer number, due to the quantization of the magnetic flux, and longitudinal resistivity can be zero for times $t \gg L_x B/cE$. In addition, using a modified periodicity of the solution, a modified quantization of the magnetic flux is found which allows to have IQHE and FQHE of any filling factor of the form $\nu = k/l$, with $k, l \in \mathcal{Z}$.

Keywords

Landau's Gauge, Quantum Hall Effect, Degeneration

1. Introduction

There are a lot of literature dealing with the phenomenon of Quantum Hall Effect [1]-[8], and most of them use the Landau's solution of the eigenvalue problem associated to the charged particle motion in a flat surface with static transversal magnetic field to the surface. This brings about the known Landau's levels for the energies and a separable variable solution for the eigenfunctions [9]. However, it has been shown that a non separable of variables solution exists for this problem with the same Landau's levels [10] [11], and these levels are numerable degenerated [12], determining the operators which causes this degeneration. In addition, the quantization of the magnetic flux appears naturally [10],

$$\frac{m\omega_c}{\hbar}A = 2\pi l, \quad l \in \mathcal{Z}, \quad \omega_c = \frac{qB}{mc}, \tag{1}$$

where m is the mass of the charge q, c is the speed of light, ω_c is the so called cyclotron frequency, B is the magnitude of the static magnetic field, $A = L_x L_y$ is the area of the sample, and $2\pi\hbar = h$ is the Planck's constant. As we mentioned before, Landau's separable solution is normally used to try to explain the so called Integer Quantum and Fractional Quantum Hall Effects (IQHE and FQHE) [4] [5] [6] [7], which were first discovered experimentally [1] [2] [3]. The IQHE is normally explained as a single particle phenomenon; meanwhile, the FQHE is explained as a many particle event [4] [5] [6]. Experimentally, both of them occur in highly impure samples, where these impurities have the effect of extending the range of magnetic field intensity where the resistivity is quantized [2] [3] [7]. The main characteristic of the IQHE or FQHE is the resistivity (or voltage) which appears on the transverse motion of the charges, so called Hall's resistivity ρ_H. This Hall's resistivity acquires a constant value on certain regions of the magnetic field, and within these regions, the longitudinal resistivity is zero. The values of these constant ρ_H turn out to be inverse to an integer number (IQHE) or proportional to an integer number (FQHE) multiplied by the constant h/q^2, called von Klitzing constant [2] [3] ($h/q^2 \approx 25812.80745\,\Omega$). In this paper, we calculate the quantum current and the expected value of the transverse and longitudinal resistivities for a single charged particle motion on a flat surface using the non separable solution in the lowest Landau level ($n = 0$) and using the first wave function ($j = 0$).

2. Quantum Current

The Hamiltonian associated to the motion of a charge particle q with mass m on a flat surface of lengths L_x and L_y with transverse magnetic field $\boldsymbol{B} = (0,0,B)$ and longitudinal electric field $\boldsymbol{E} = (0,E,0)$ is given by

$$\hat{H} = \frac{1}{2m}\left(\boldsymbol{p} - \frac{q}{c}\boldsymbol{A}\right)^2 + qV, \tag{2}$$

where $\boldsymbol{A}$ is the vector potential, $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, and V is the scalar potential, $\boldsymbol{E} = -\nabla V$. The Schrödinger's equation,

$$i\hbar\frac{\partial \Psi}{\partial t} = \hat{H}\Psi, \tag{3}$$

can be written, using the operator $\boldsymbol{p} = -i\hbar\nabla$, as

$$i\hbar\frac{\partial \Psi}{\partial t} = \frac{1}{2m}\left[-\hbar^2\nabla^2 + i\frac{\hbar q}{c}\left(\nabla\cdot\boldsymbol{A} + \boldsymbol{A}\cdot\nabla\right) + \frac{q^2A^2}{c^2}\right]\Psi + qV\Psi. \tag{4}$$

Taking the usual complex conjugated to this expression, a similar equation is gotten for the function Ψ^*. Multiplying this one by Ψ, (4) by Ψ^* and subtracting both, the following continuity equation is obtained

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{J} = 0, \tag{5}$$

where ρ and $\boldsymbol{J}$ are defined as

$$\rho = \Psi \cdot \Psi^* \tag{6}$$

and

$$\boldsymbol{J} = \frac{i\hbar}{2m}\left(\Psi \nabla \Psi^* - \Psi^* \nabla \Psi\right) - \frac{q}{mc}\rho \boldsymbol{A}. \tag{7}$$

Since Ψ is a scalar complex function, it can be written as $\Psi = |\Psi| e^{i\theta}$, where $|\Psi|$ and θ are real functions, and θ is the argument of the function. Then, the current is given by

$$\boldsymbol{J} = \left(\frac{\hbar}{m}\nabla\theta - \frac{q}{mc}\boldsymbol{A}\right)|\Psi|^2. \tag{8}$$

For the general solution of (3), the function θ can be very complicated expression of all variables. However, for a particular state solution of the system, say

$$\psi_n(\boldsymbol{x}, t) = e^{i\phi_n(x,t)} f_n(\boldsymbol{x}), \tag{9}$$

the argument is just $\theta = \phi_n(\boldsymbol{x}, t)$, and the current associated to this state of the system is given by

$$\boldsymbol{J}_n = \left(\frac{\hbar}{m}\nabla\phi_n - \frac{q}{mc}\boldsymbol{A}\right)|f_n|^2. \tag{10}$$

3. Single Charged Particle Current

The non separable solution of (3) using the Landau's gauge $\boldsymbol{A} = B(-y, 0, 0)$ and the longitudinal constant electric field $\boldsymbol{E} = (0, E, 0)$ was given as

$$f_n^0 = \frac{1}{\sqrt{2^n n! L_y}}\left(\frac{m\omega_c}{\pi\hbar}\right)^{1/4} e^{i\phi_n} e^{-\frac{m\omega_c}{2\hbar}(x - c\mathcal{E}t/B)^2} H_n\left(\sqrt{\frac{m\omega_c}{\hbar}}(x - c\mathcal{E}t/B)\right), \tag{11a}$$

where $\mathcal{E} = qE$, ω_c is the cyclotron frequency (1), and ϕ_n is given by

$$\phi_n = -\left[\hbar\omega_c\left(n + \frac{1}{2}\right) - \frac{mc^2\mathcal{E}}{2B^2}\right]\frac{t}{\hbar} - \frac{m\omega_c}{\hbar}\left(x - \frac{c\mathcal{E}t}{B}\right)\left(y - \frac{mc^2\mathcal{E}}{qB^2}\right). \tag{11b}$$

These functions are degenerated in the sense that for each Landau's level ($\hbar\omega_c(n + 1/2)$), one has a numerable solutions $f_n^j = (\hat{p}_x)^j f_n^0$, $j \in Z$. Thus, the expressions (11a) define the state of the system. Using this function ϕ_n in (10) and for the index of degeneration $j = 0$, we have

$$\boldsymbol{J}_n = \left[\frac{cE}{B}\hat{\boldsymbol{i}} - \omega_c\left(x - \frac{c\mathcal{E}t}{B}\right)\hat{\boldsymbol{j}}\right]|f_n^0|^2. \tag{12}$$

In particular, for the ground state of Landau's energy, it follows that the components of the current are

$$J_0^x = \frac{c\mathcal{E}}{B}\left|f_0^0\right|^2, \tag{13}$$

and

$$J_0^y = -\omega_c\left(x - \frac{c\mathcal{E}t}{B}\right)\left|f_0^0\right|^2. \tag{14}$$

The electric conductivity along the x-axis is called Hall's conductivity and is given by

$$\sigma_H = \frac{q}{\mathcal{E}}J_0^x = \frac{qc}{B}\left|f_0^0\right|^2. \tag{15}$$

Thus, the Hall's resistivity is $\rho_H = 1/\sigma_H$, and the expected value of the resistivity in the state f_0^0 is

$$\left\langle f_0^0 \middle| \rho_H \middle| f_0^0 \right\rangle = \int_0^{L_x}\int_0^{L_y}\frac{\left|f_0^0\right|^2}{\sigma_H}\mathrm{d}x\mathrm{d}y = \frac{BA}{qc}. \tag{16}$$

Now, multiplying and dividing this quantity by $m\omega_c/\hbar$ and making some rearrangements, one gets

$$\left\langle f_0^0 \middle| \rho_H \middle| f_0^0 \right\rangle = \frac{\hbar}{q^2}\left(\frac{m\omega_c}{\hbar}A\right), \tag{17}$$

and taking into consideration the magnetic field flux quantization (1), it follows that

$$\left\langle f_0^0 \middle| \rho_H \middle| f_0^0 \right\rangle = \frac{h}{q^2}l, \quad l \in \mathcal{Z}. \tag{18}$$

The expected value in the state f_0^0 of the longitudinal resistivity ρ_y is

$$\left\langle f_0^0 \middle| \rho_y \middle| f_0^0 \right\rangle = \int_0^{L_x}\int_0^{L_y}\frac{\left|f_0^0\right|^2 \mathrm{d}x\mathrm{d}y}{\sigma_y} = \frac{\mathcal{E}}{q}\int_0^{L_x}\int_0^{L_y}\frac{\left|f_0^0\right|^2 \mathrm{d}x\mathrm{d}y}{J_0^y} \tag{19}$$

$$= -\frac{\mathcal{E}}{q\omega_c}\int_0^{L_x}\int_0^{L_y}\frac{\mathrm{d}x\mathrm{d}y}{x - \frac{cEt}{B}} = -\frac{\mathcal{E}L_y}{q\omega_c}\ln\left(1 - \frac{L_x B}{c\mathcal{E}t}\right) \approx 0 \tag{20}$$

since one has normally in the experiments that $L_x B/c\mathcal{E}t \ll 1$, that is, the time in the experiments are such that

$$t \gg \frac{L_x B}{c\mathcal{E}}. \tag{21}$$

For example, on the reference [2] and with respect the voltage gate V_g, one has that $BL_x/c\mathcal{E} = BA/cV_g \sim 4.5\times10^{-8}\,\mathrm{sec}$. So, the condition (21) is well satisfied in this experiment.

Note that the expression (18) implies a filling factor $\nu = 1/l$, which correspond to the IQHE phenomenon for $l = 1$ and to the FQHE phenomenon for $l > 1$. However, this result is valid for an analysis of a single charged particle, and both QHE phenomena appear due to the quantization of the magnetic flux (1). In addition, one must note that this analysis is still valid for any $n > 0$ and $j = 0$.

4. Full IQHE and FQHE

The quantization of the magnetic flux (1) arises from the periodicity of the solutions of the Hamiltonian [10], which can be expressed using (11a) for $\mathcal{E}=0$ as

$$f_n^0\left(L_x, y+L_y, t\right)=f_n^0\left(L_x, y, t\right). \tag{22}$$

However (and also for $\mathcal{E}=0$), let us assume that $L_y=Nl_y$ where $l_y \ll L_y$ and $N\in\mathcal{Z}^+$, that is, the total area L_xL_y is covered with slices of area L_xl_y, with horizontal length L_x and width l_y. Let us impose the periodicity condition of the form

$$f_n^0\left(L_x, y+kl_y, t\right)=f_n^0\left(L_x, y, t\right), \quad k\in\mathcal{Z}, \tag{23}$$

such that with the phase (11b), one gets

$$\frac{m\omega_c}{\hbar}L_xkl_y=2\pi l, \quad l\in\mathcal{Z} \tag{24}$$

which brings about the relation

$$\frac{m\omega_c}{\hbar}a=2\pi\frac{l}{k}, \quad \text{with } a=L_xl_y. \tag{25}$$

Using (1) and making some rearrangements, the magnetic field can be given by

$$B=\alpha\frac{l}{k}, \quad \text{with } \alpha=\frac{hc}{qa} \tag{26}$$

and using (25) in (17), the expected value of the Hall resistivity would be

$$\left\langle f_0^0\left|\rho_H\right|f_0^0\right\rangle=\frac{h}{q^2}\frac{l}{k}, \quad k,l\in\mathcal{Z}, \tag{27}$$

implying now a filling factor of $\nu=k/l$, which represents the full IQHE (for $l=1$) and FQHE (for $l>1$). To determine the magnetic values B where these phenomena occur, one looks for the value B_0 where the first IQHE ($l=k=1$) appears, which intersect the normal linear dependence behavior straight line, and this defines $\alpha=B_0$. Then, one uses the resulting expression

$$B=B_0\frac{l}{k} \tag{28}$$

to find the other quantized magnetic fields which correspond to IQHE or FQHE. For example, on the experimental data shown on the reference [3], one sees that $B_0\approx 5\ \text{T}$ for $l=k=1$ (corresponding to an area $a\approx 8.27\times10^{-4}\ \mu\text{m}^2$), and the other FQHE are matched quite well for $l=3$ and $k=1$, that is $B\approx 15\ \text{T}$. Another example is shown on the reference [8] page 886, one sees that $B_0\approx 9.8\ \text{T}$ for $l=k=1$ (corresponding to an area $a\approx 4.22\times10^{-4}\ \mu\text{m}^2$), and the other IQHE and FQHE magnetic fields are matched quite well for $l>1$ and $k>1$. In addition, on reference [13] page 207, one sees that $B_0\approx 4.2\ \text{T}$ for $l=k=1$ (corresponding to an area $a=9.85\times10^{-4}\ \mu\text{m}^2$), and the other IQHE and FQHE magnetic fields are matched quite well for $l>1$ and $k>1$. Finally, on reference [14] page 156801-2, one sees that $B_0\approx 5.3\ \text{T}$ for $l=k=1$ (corresponding to an area $a = 7.8\times10^{-4}\ \mu m^2$), and for the filling factor $\nu = 3/4$ one gets $B = 4B_0/3 = 7.06\ \mathrm{T}$, which is approximately the experimental value reported.

5. Conclusion

Using the known non-separable solution for the quantum motion of a charged particle in a flat surface with static fields, in the state $n = 0$ and $j = 0$, the Hall and the longitudinal resistivities were calculated. For the quantization of the magnetic flux, which can appear from the simple periodicity on the y-direction, the results bring about the IQHE and FQHE phenomena since from the expression (18) it appears a filling factor of $1/l$ for a single charged particle due to the quantization of the magnetic flux. If $l = 1$, one gets the IQHE phenomenon, and if $l > 1$, one gets the FQHE phenomenon. However, it is not possible to say anything about filling factors of the form $\nu = k/l$. For a more extended quantization of the magnetic flux (25), which appears of the extended periodicity (23), one gets also IQHE and FQHE but with a filling factor of $\nu = k/l$.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Wakabayashi, J.I. and Kawaji, S. (1978) *Journal of the Physical Society of Japan*, **44**, 1839-1849. https://doi.org/10.1143/JPSJ.44.1839

[2] Klitzing, K.V., Dorda, G. and Pepper, M. (1980) *Physical Review Letters*, **45**, 494-497. https://doi.org/10.1103/PhysRevLett.45.494

[3] Tsui, D.C., Stormer, H.L. and Gossard, A.C. (1982) *Physical Review Letters*, **48**, 1559-1562. https://doi.org/10.1103/PhysRevLett.48.1559

[4] Laughlin, R.B. (1981) *Physical Review B*, **23**, 5632-5633. https://doi.org/10.1103/PhysRevB.23.5632

[5] Laughlin, R.B. (1983) *Physical Review B*, **27**, 3383-3389. https://doi.org/10.1103/PhysRevB.27.3383

[6] Laughlin, R. B. (1983) *Physical Review Letters*, **50**, 1395-1398. https://doi.org/10.1103/PhysRevLett.50.1395

[7] Prange, R.E. (1981) *Physical Review B*, **23**, 4802-4805. https://doi.org/10.1103/PhysRevB.23.4802

[8] Stormer, H.L. (1999) *Reviews of Modern Physics*, **71**, 875-889. https://doi.org/10.1103/RevModPhys.71.875

[9] Landau, L.D. and Lifshitz, E.M. (2013) Quantum Mechanics: Non-Relativistic Theory. Volume 3, Elsevier, Amsterdam.

[10] López, G.V. and Lizarraga, J.A. (2020) *Journal of Modern Physics*, **11**, 1731-1742. https://doi.org/10.4236/jmp.2020.1110106

[11] López, G.V., Lizarraga, J.A. and Bravo, O.J.P. (2021) *Journal of Modern Physics*, **12**, 1464-1474. https://doi.org/10.4236/jmp.2021.1210088

[12] Lizarraga, J.A. and López, G.V. (2022) *Journal of Modern Physics*, **13**, 122-126.

https://doi.org/10.4236/jmp.2022.132010

[13] Stern, A. (2007) *Annals of Physics*, **232**, 204-249. https://doi.org/10.1016/j.aop.2007.10.008

[14] Wang, C., *et al*. (2022) *Physical Review Letters*, **129**, Article ID: 156801. https://doi.org/10.1103/PhysRevLett.129.156801