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# Optic Axes and Elliptic Cone Equation in Coordinate-Invariant Treatment

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## Abstract

We derive for crystal optics in coordinate-invariant way the cone approximation of refraction vectors in the neighborhood of optic axes and determine its invariants and eigenvectors. It proved to describe an elliptic cone. The second invariant of the operator of the wave equation with respect to similarity transformations determines the special cases of degeneration including the optic axes where the polarization of the waves due to self-intersection of the dispersion surface is not uniquely determined. This second invariant is included in all investigations and it is taken into account in the illustrations. It is biquadratic in the refraction vectors and the corresponding forth-order surface in three-dimensional space splits in two separate shells and a non-rational product decomposition describing this is found. We give also a more general classification of all possible solutions of an equation with an arbitrary three-dimensional operator.

## Keywords

Permittivity Tensor, Principal Permittivities, Three-Dimensional Operator of Wave Equation, Operator Invariants, Refraction Vector, Ray Vector, Cone Approximation in Neighborhood of Optic Axis, Conical Refraction

## A Remark to Notations

Three-dimensional vectors: bold letters, e.g., $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\cdots$, $\boldsymbol{ab}$ scalar products, $[\boldsymbol{a},\boldsymbol{b}]$ vector products, $[\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}]$ volume products, $\boldsymbol{a}\cdot\boldsymbol{b}$ dyadic products, Three-dimensional operators: serif-less letters $\mathsf{A},\mathsf{B},\cdots$, $\mathsf{AB}$ operator products, $\mathsf{A}\boldsymbol{a},\tilde{\boldsymbol{a}}\mathsf{A}$ products of operators with vectors, $\tilde{\boldsymbol{a}}\mathsf{A}\boldsymbol{a}$ bilinear (and quadratic $\tilde{\boldsymbol{a}}=\boldsymbol{a}$) forms, $\langle\mathsf{A}\rangle,[\mathsf{A}],|\mathsf{A}|$ trace, second invariant and determinant of $\mathsf{A}$.

In three-dimensional Euclidean spaces with a symmetric metric tensor $g^{ij}$ with $g^{ij}g_{jk}=\delta^i_k$ the dual anti-symmetric pseudo-tensor $b_{jl}\equiv\epsilon_{jkl}b^k=-b_{lj}$ to a vector $b^k$ ($\epsilon_{jkl}$ Levi-Cicita symbol) can be also seen as antisymmetric operator $b_l^i \equiv g^{ij}\epsilon_{jkl}b^k$ and in coordinate-invariant form we can write this antisymmetric operator as $[\boldsymbol{b}]$ with the advantage that vector and also volume products can be written only by displacement of the squared brackets, e.g., $\boldsymbol{a}[\boldsymbol{b}] \equiv [\boldsymbol{a},\boldsymbol{b}]$, $[\boldsymbol{b}]\boldsymbol{c} \equiv [\boldsymbol{b},\boldsymbol{c}]$, $\boldsymbol{a}[\boldsymbol{b}]\boldsymbol{c} \equiv [\boldsymbol{a},\boldsymbol{b}]\boldsymbol{c} \equiv \boldsymbol{a}[\boldsymbol{b},\boldsymbol{c}] \equiv [\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}]$.

In physical texts it makes sometimes difficulties to write vectors by small bold letters and operators by serif-less Capital letters. In case of Greek letters, 'Latex' (and also printing) does not provide serif-less letters. In these cases I write as a compromise operators by bold letters, e.g., $\boldsymbol{\varepsilon}$ such as for vectors to distinguish them, in particular, from scalars. This means that in present physical text one must know which kind of quantities one has: scalars, vectors or operators.

## 1. Introduction

The presence of optic axes in the general case of biaxial crystal causes very interesting effects for the propagation of light in the neighborhood of such directions and belongs to the exceptional cases which are more difficult to describe in comparison to the propagation into other directions. The reason is that the polarization and the group velocity in direction of optic axes is not uniquely determined and depends very much on small deviations of the refraction vectors from the optic axis. William Rowan Hamilton forecasted in 1828 the effect of conical refraction in direction of an optic axis or binormal. The conical approximation of the dispersion surface (surface of refraction vectors) in the neighborhood of an optic axis is sufficiently well understood concerning the plane spanned by the optic axis and the cone axis which in general case of biaxial crystals form an angle but not fully concerning the plane perpendicular to it containing the optic axis or the cone axis. In particular, some loose remarks in treatments make the impression that the mentioned cone approximation is evidently considered as approximation by a circular cone that, however, is not true. In reality it is an elliptic cone. Although in many cases if the main (or principal) permittivities in direction of the tensor axes are not very different the differences of the elliptic cone to a circular cone are small, but this is a principal question. Interesting treatments are to find in the comprehensive encyclopedic article of Szivessy [1] and in vol. 8 of the course of theoretical physics of Landau, Lifshits (complemented by Pitayevski) [2] although the last authors announce their own treatment as highly schematically. Further interesting representations including optic axes are to find, e.g., in books of Sommerfeld [3], of Born and Wolf [4], of Ditchburn [5] and of Sivukhin [6].

New in present article is that we derive the full three-dimensional elliptic cone equation for the approximation of refraction vectors near the optic axis in coordinate-invariant way and that we make general remarks how the wave-equation operator is connected in case of optic axes with the vanishing of the second invariant of this operator and with vanishing of its complementary operator. The vanishing of the second invariant is a necessary but not a sufficient requirement for the presence of an optic axis. In particular, we illustrate in most of the figures the surface of vanishing of the second invariant in connection with the necessary vanishing of the determinant of the wave-equation operator in three-dimensional space of refraction vectors for the cases of solutions near the optic axes. Furthermore, we calculate principal formulae for the polarization of the electric field near to propagation in direction of optic axes. Coordinate-invariant methods from the starting equations up to the results without using specialized coordinate systems which are not directly connected with vectors and operators with physical meaning for the problem were first described and applied in optics of anisotropic media by F.I. Fyodorov [7] [8] and later applied for the description of the Lorentz group in [9] where they are also very useful and the most of our articles (e.g., [10] [11]) in the seventieths work also with such methods. Depart from the introduction of the notion of axes of a second-rank tensor I did not find in [7] [8] and in other renowned sources a detailed treatment of the elliptic cone equation for optic axes. For this purpose I searched in Internet for it where though I found rich experimental material and also some known theoretical representations but not the mentioned equation and the relation to the second invariant of the wave-equation operator.

We begin in Section 2 with a general consideration and classification to three-dimensional operator equations together with its possible degenerative cases and specializes this in Sections 3 - 5 to the wave equation for anisotropic media. In Section 6 we derive the cone approximation for refraction vectors in the neighborhood of an optic axis and diagonalize the cone operator in Section 7. In Section 8 we describe the intersection of the plane containing the optic axis and the direction perpendicular to it. Section 9 provides a general formula for polarization of the electric field in the neighborhood of the optic axis. In Section 10 we discuss shortly the transition to uniaxial media. Section 11 considers possible degenerations to optic axis in the complex domain and discusses but cannot fully solve whether cases with vanishing of determinant and second invariant of the wave-equation operator are possible which are not connected with optic axes. In Section 12 we make short remarks to a duality between optics with refraction operators and ray optics in connection with the cone approximation in the neighborhood of an optic axis. Some more mathematical problems and an ethical problem are discussed in the Appendices.

## 2. Vanishing of the Second Invariant $[\mathsf{L}]$ of the Operator $\mathsf{L}$ of the Wave Equation and the Case of Optic Axes

First, we consider the general case of the algebra of three-dimensional operators $\mathsf{L}$ for which right-hand solutions $\boldsymbol{e}$ and left-hand solutions $\tilde{\boldsymbol{e}}$ of the equations

$$\mathbf{0} = \mathsf{L}\boldsymbol{e}, \quad \mathbf{0} = \tilde{\boldsymbol{e}}\mathsf{L}, \quad \left(\text{or } L^i_j e^j = 0, \tilde{e}_i L^i_j = 0\right), \tag{2.1}$$

should be calculated. This may be also expressed in the way that $\boldsymbol{e}$ and $\tilde{\boldsymbol{e}}$ are eigen-solutions (eigenvectors) of the operator $\mathsf{L}$ to eigenvalue $\lambda = 0$. Usually, the operator $\mathsf{L}$ depends on parameters and this means that the parameters must be chosen in such way that $\mathsf{L}$ possesses at least one eigenvalue $\lambda = 0$ to get a solution of $\mathsf{L}\boldsymbol{e} = \mathbf{0}$.

An arbitrary three-dimensional operator $\mathsf{L}$ satisfies the three-dimensional Hamilton-Cayley identity (e.g., Gantmakher [12]) with $\mathsf{I} \equiv \mathsf{L}^0$ the three-dimensional identity operator

$$0 = \mathsf{L}^3 - \langle \mathsf{L} \rangle \mathsf{L}^2 + [\mathsf{L}]\mathsf{L} - |\mathsf{L}|\mathsf{I}, \tag{2.2}$$

where the invariants with respect to similarity transformation of the operator $\mathsf{L}$ (trace, second invariant, determinant) are defined by[1]

$$\langle \mathsf{L} \rangle \equiv L_i^i, \quad [\mathsf{L}] \equiv \frac{1}{2}\left(\langle \mathsf{L} \rangle^2 - \langle \mathsf{L}^2 \rangle\right), \quad |\mathsf{L}| \equiv \frac{1}{6}\left(\langle \mathsf{L} \rangle^3 - 3\langle \mathsf{L} \rangle \langle \mathsf{L}^2 \rangle + 2\langle \mathsf{L}^3 \rangle\right). \tag{2.3}$$

The complementary operator $\overline{\mathsf{L}}$ to operator $\mathsf{L}$ is defined by

$$\overline{\mathsf{L}} \equiv \mathsf{L}^2 - \langle \mathsf{L} \rangle \mathsf{L} + [\mathsf{L}]\mathsf{I}, \quad \Rightarrow \quad \overline{\mathsf{L}}\mathsf{L} = \mathsf{L}\overline{\mathsf{L}} = |\mathsf{L}|\mathsf{I}, \quad \langle \overline{\mathsf{L}} \rangle = [\mathsf{L}]. \tag{2.4}$$

The eigenvalues $\lambda$ of the operator $\mathsf{L}$ to right-hand and left-hand eigenvectors $\boldsymbol{e}$ and $\tilde{\boldsymbol{e}}$ satisfy in accordance with the Hamilton-Cayley identity (2) the equation

$$0 = \lambda^3 - \langle \mathsf{L} \rangle \lambda^2 + [\mathsf{L}]\lambda - |\mathsf{L}|. \tag{2.5}$$

Necessary condition for solutions of (2.1) or the presence of an eigenvalue $\lambda = 0$ of $\mathsf{L}$ is the vanishing of the determinant of $\mathsf{L}$ that means $|\mathsf{L}| = 0$ leading from (2.5) to the from

$$0 = \lambda\left(\lambda^2 - \langle \mathsf{L} \rangle \lambda + [\mathsf{L}]\right), \tag{2.6}$$

with, at least, one simple eigenvalue $\lambda = 0$

$$\lambda = \lambda_0 = 0. \tag{2.7}$$

The other two eigenvalues $\lambda$ of $\mathsf{L}$ satisfy then the equation

$$0 = \lambda^2 - \langle \mathsf{L} \rangle \lambda + [\mathsf{L}], \tag{2.8}$$

with the solutions

$$\lambda = \lambda_\pm = \frac{1}{2}\left\{\langle \mathsf{L} \rangle \pm \sqrt{\langle \mathsf{L} \rangle^2 - 4[\mathsf{L}]}\right\} = \frac{1}{2}\left\{\langle \mathsf{L} \rangle \pm \sqrt{2\langle \mathsf{L}^2 \rangle - \langle \mathsf{L} \rangle^2}\right\}. \tag{2.9}$$

In case of $|\mathsf{L}| = 0$ the Hamilton-Cayley identity (2.2) using definition (2.4) takes on the form

$$0 = L^3 - \langle \mathsf{L} \rangle \mathsf{L}^2 + [\mathsf{L}]\mathsf{L} = \mathsf{L}\overline{\mathsf{L}} = \overline{\mathsf{L}}\mathsf{L}. \tag{2.10}$$

By successive application of the Hamilton-Cayley identity follows the general relation

[1]Our notation of the invariants possesses the advantage that they are easily to see and to recognize as such in formulae and text. Furthermore, since letter $\mathsf{L}$ in $\mathsf{L}^0 = \mathsf{I}$ is not specific for a certain $\mathsf{L}$ we prefer the special notation $\mathsf{I}$ for the identity operator in (1). However, there are also formulae which are not specific for a certain dimension where it is then appropriate to write the identity operator, e.g., $\mathsf{L}^0$ if $\mathsf{L}$ determines the dimension of $\mathsf{I}$.

$$\overline{\mathrm{L}}^2 = [\mathrm{L}]\overline{\mathrm{L}} - |\mathrm{L}|\left(\langle \mathrm{L}\rangle \mathrm{I} - \mathrm{L}\right), \tag{2.11}$$

which in case of vanishing determinant $|\mathrm{L}| = 0$ can be written in the form

$$\Pi \equiv \frac{\overline{\mathrm{L}}}{\langle \overline{\mathrm{L}}\rangle} = \frac{\overline{\mathrm{L}}}{[\mathrm{L}]}, \quad \Pi^2 = \Pi, \quad \langle \Pi \rangle = 1. \tag{2.12}$$

Thus $\Pi \boldsymbol{a} \neq \boldsymbol{0}$ and $\tilde{\boldsymbol{a}}\Pi \neq \boldsymbol{0}$ for arbitrary vectors $\boldsymbol{a}$ and $\tilde{\boldsymbol{a}}$ are right-hand or left-hand eigen-solutions of $\mathrm{L}$ to eigenvalue $\lambda = 0$, respectively, and $\Pi$ is projection operator to obtain such solutions. For $\langle \overline{\mathrm{L}}\rangle = [\mathrm{L}] = 0$ these projection operators are no more defined and these are degenerate cases.

Since in present article we mainly investigate degenerate cases such as optic axes we continue our further investigations at once with the case that apart from the determinant $|\mathrm{L}|$ also the second invariant $[\mathrm{L}]$ of the operator $\mathrm{L}$ vanishes that means

$$|\mathrm{L}| = 0, \quad [\mathrm{L}] = 0. \tag{2.13}$$

In this case the Hamilton-Cayley identity (2.2) takes on the following form

$$0 = \mathrm{L}^3 - \langle \mathrm{L}\rangle \mathrm{L}^2 = \mathrm{L}\left(\mathrm{L}^2 - \langle \mathrm{L}\rangle \mathrm{L}\right) = \left(\mathrm{L}^2 - \langle \mathrm{L}\rangle \mathrm{L}\right)\mathrm{L}, \tag{2.14}$$

with the invariants with respect to similarity transformations defined in (2.3) and the complementary operator $\overline{\mathrm{L}}$ in (2.4) which specializes here to

$$\overline{\mathrm{L}} \equiv \mathrm{L}^2 - \langle \mathrm{L}\rangle \mathrm{L}, \quad \Rightarrow \quad 0 = \mathrm{L}\overline{\mathrm{L}} = \overline{\mathrm{L}}\mathrm{L}, \quad \langle \overline{\mathrm{L}}\rangle = [\mathrm{L}] = 0. \tag{2.15}$$

Due to vanishing of $[\mathrm{L}]$ in the denominator one-dimensional projection operators $\Pi$ to eigenvalue $\lambda = 0$ of $\mathrm{L}$ are no more existing. We have to distinguish two subcases

$$\begin{aligned} &|\mathrm{L}| = 0, [\mathrm{L}] = 0, \quad \Rightarrow \quad \mathrm{L}^3 - \langle \mathrm{L}\rangle \mathrm{L}^2 = 0, \quad \overline{\mathrm{L}} \equiv \mathrm{L}^2 - \langle \mathrm{L}\rangle \mathrm{L}, \quad \overline{\mathrm{L}}^2 = 0: \\ &1. \quad \overline{\mathrm{L}} \neq 0, \quad \Rightarrow \quad \overline{\mathrm{L}}^2 = 0, \quad \mathrm{L}\overline{\mathrm{L}} = \overline{\mathrm{L}}\mathrm{L} = 0, \\ &2. \quad \overline{\mathrm{L}} = 0, \quad \Rightarrow \quad \mathrm{L}^2 = \langle \mathrm{L}\rangle \mathrm{L}, \quad \Rightarrow \quad \left(\frac{\mathrm{L}}{\langle \mathrm{L}\rangle}\right)^2 = \frac{\mathrm{L}}{\langle \mathrm{L}\rangle}. \end{aligned} \tag{2.16}$$

In first subcase of $\overline{\mathrm{L}} \neq 0$ it is proportional to the dyadic product of $\boldsymbol{e}\cdot\tilde{\boldsymbol{e}}$ and can be normalized as following

$$\overline{\mathrm{L}} = \boldsymbol{e}\cdot\tilde{\boldsymbol{e}}, \quad \Rightarrow \quad [\mathrm{L}] = \langle \overline{\mathrm{L}}\rangle = \tilde{\boldsymbol{e}}\boldsymbol{e} = 0, \quad \overline{\mathrm{L}}^2 = (\tilde{\boldsymbol{e}}\boldsymbol{e})\boldsymbol{e}\cdot\tilde{\boldsymbol{e}} = 0. \tag{2.17}$$

In this case we find right-hand and left-hand solutions proportional to $\boldsymbol{e}$ and $\tilde{\boldsymbol{e}}$ which are orthogonal to each other. Their scalar product $\tilde{\boldsymbol{e}}\boldsymbol{e}$ cannot be normalized. This belongs to a Jordan normal form (e.g., [12]) of L to a twofold degenerate eigenvalue $\lambda = 0$.

In the second subcase $\overline{\mathrm{L}} = 0$ we find according to the definition (2.15) under the assumption $\langle \mathrm{L}\rangle \neq 0$

$$0 = \mathrm{L}\left(\mathrm{L} - \langle \mathrm{L}\rangle \mathrm{I}\right), \quad \Rightarrow \quad \left(\mathrm{L} - \langle \mathrm{L}\rangle \mathrm{I}\right)^2 = \underbrace{\mathrm{L}^2 - \langle \mathrm{L}\rangle \mathrm{L}}_{\overline{\mathrm{L}}=0} - \langle \mathrm{L}\rangle \underbrace{\left(\mathrm{L} - \langle \mathrm{L}\rangle \mathrm{I}\right)}_{\neq 0}, \tag{2.18}$$

and we may define projection operators $\Pi'$ to twofold degenerate eigenvalue $\lambda = 0$ according to

$$\Pi' \equiv \frac{\langle \mathsf{L} \rangle \mathsf{I} - \mathsf{L}}{\langle \mathsf{L} \rangle}, \quad \Pi'^2 = \Pi', \quad \langle \Pi' \rangle = 2, \quad \langle \langle \mathsf{L} \rangle \mathsf{I} - \mathsf{L} \rangle = 2 \langle \mathsf{L} \rangle \neq 0, \tag{2.19}$$

and any vector $\Pi' \boldsymbol{a} \neq \boldsymbol{0}$ is right-hand vector and any vector $\tilde{\boldsymbol{a}} \Pi' \neq \boldsymbol{0}$ left-hand eigenvector of $\mathsf{L}$ to degenerate eigenvalue $\lambda = 0$ according to

$$\mathsf{L} \Pi' \boldsymbol{a} = -\frac{\overline{\mathsf{L}}}{\langle \mathsf{L} \rangle} \boldsymbol{a} = \boldsymbol{0}, \quad \tilde{\boldsymbol{a}} \mathsf{L} \Pi' = -\tilde{\boldsymbol{a}} \frac{\overline{\mathsf{L}}}{\langle \mathsf{L} \rangle} = \boldsymbol{0}, \tag{2.20}$$

This is the case of optic axes of $\mathsf{L}$ with $\Pi'$ the two-dimensional projection operator to determine to twofold degenerate eigenvalue $\lambda = 0$ possible eigenvectors as solutions of Equation (2.1).

In case of $|\mathsf{L}| = 0$ and $[\mathsf{L}] = 0$ according to (2.8) and (2.9) the third, in general, non-degenerate eigenvalue of $\mathsf{L}$ is $\lambda = \langle \mathsf{L} \rangle \neq 0$ and the projection operator $\Pi_\lambda$ to this eigenvalue is

$$\Pi_\lambda \equiv \frac{\mathsf{L}}{\langle \mathsf{L} \rangle}, \quad \Pi_\lambda^2 = \frac{\mathsf{L}^2}{\langle \mathsf{L} \rangle^2} = \frac{\mathsf{L}}{\langle \mathsf{L} \rangle} = \Pi_\lambda, \quad \langle \Pi_\lambda \rangle = 1, \quad \left( \lambda = \langle \mathsf{L} \rangle \right). \tag{2.21}$$

Any vector $\Pi_\lambda \boldsymbol{b} \neq \boldsymbol{0}$ is right-hand eigenvector and any vector $\tilde{\boldsymbol{b}} \Pi_\lambda \neq \boldsymbol{0}$ is left-hand eigenvector of $\mathsf{L}$ to the eigenvalue $\lambda = \langle \mathsf{L} \rangle$ that is also clear without the formalism. The vector $\tilde{\boldsymbol{b}} \Pi_\lambda \neq \boldsymbol{0}$ is perpendicular to all two-fold degenerate right-hand eigenvectors of $\mathsf{L}$ to eigenvalue zero and the analogous property is true for the left-hand eigenvalues. Since $\Pi_\lambda$ is projection operator with trace equal to 1 it is proportional to a dyadic product $\boldsymbol{d} \cdot \tilde{\boldsymbol{d}}$ with scalar product $\tilde{\boldsymbol{d}} \boldsymbol{d} = 1$ that means

$$\overline{\mathsf{L}} = 0, \quad \Rightarrow \quad \frac{\mathsf{L}}{\langle \mathsf{L} \rangle} = \boldsymbol{d} \cdot \tilde{\boldsymbol{d}}, \quad \Rightarrow \quad \mathsf{L} = \langle \mathsf{L} \rangle \boldsymbol{d} \cdot \tilde{\boldsymbol{d}}, \quad \tilde{\boldsymbol{d}} \boldsymbol{d} = 1. \tag{2.22}$$

The operator $\mathsf{L}$ is in case of $\overline{\mathsf{L}} = 0$ proportional to a dyadic product of two mutually normalized vectors $\boldsymbol{d}$ and $\tilde{\boldsymbol{d}}$. All (twofold degenerate) solutions $\boldsymbol{e}$ and $\tilde{\boldsymbol{e}}$ of $\mathsf{L} \boldsymbol{e} = \boldsymbol{0}$ and $\tilde{\boldsymbol{e}} \mathsf{L} = \boldsymbol{0}$ according to

$$\overline{\mathsf{L}} = 0, \quad \Rightarrow \quad \tilde{\boldsymbol{d}} \boldsymbol{e} = \tilde{\boldsymbol{e}} \boldsymbol{d} = 0, \quad \Pi_\lambda \Pi' = \Pi' \Pi_\lambda = 0, \quad \Pi' + \Pi_\lambda = \mathsf{I}. \tag{2.23}$$

are perpendicular to $\boldsymbol{d}$ or $\tilde{\boldsymbol{d}}$, respectively.

For completeness we will classify the last possible degenerate case, the vanishing of all three invariants of operator $\mathsf{L}$ with three subcases although it does not possess essential importance for optic axes

$$\begin{aligned} & |\mathsf{L}| = 0, \quad [\mathsf{L}] = 0, \quad \langle \mathsf{L} \rangle = 0, \quad \Rightarrow \quad \mathsf{L}^3 = 0: \\ & 1. \quad \mathsf{L} \neq 0, \quad \mathsf{L}^2 \neq 0, \quad \mathsf{L}^3 = 0, \quad \Rightarrow \quad \overline{\mathsf{L}} = \mathsf{L}^2, \\ & 2. \quad \mathsf{L} \neq 0, \quad \mathsf{L}^2 = 0, \quad \Rightarrow \quad \overline{\mathsf{L}} = \mathsf{L}, \\ & 3. \quad \mathsf{L} = 0. \end{aligned} \tag{2.24}$$

The first two subcases belong to two different Jordan normal forms. For example, in first subcase one may find a right-hand vector $\boldsymbol{a}_0$ with $\mathsf{L} \boldsymbol{a}_0 = \boldsymbol{a}_1 \neq \boldsymbol{0}$, $\mathsf{L} \boldsymbol{a}_1 = \boldsymbol{a}_2 \neq \boldsymbol{0}$, $\mathsf{L} \boldsymbol{a}_2 = \boldsymbol{0}$ and analogously for left-hand vectors. Orthogonality relations between these vectors are also easily to obtain. The third subcase is a rare case where for certain very special cases $\mathsf{L}$ vanishes and they possess importance as special case embedded into the neighborhood of more general cases (following Sections).

We try now to collect the results to eigenvalue $\lambda = 0$ in a scheme. Our notations are the invariants of three-dimensional operators plus the complementary operators and projection operators as follows

$$|\mathsf{L}|, \quad [\mathsf{L}], \quad \langle\mathsf{L}\rangle, \quad \overline{\mathsf{L}} \equiv [\mathsf{L}]\mathsf{I} - \langle\mathsf{L}\rangle\mathsf{L} + \mathsf{L}^2, \quad \overline{\mathsf{L}}' \equiv \langle\mathsf{L}\rangle\mathsf{I} - \mathsf{L}, \quad \overline{\mathsf{L}}'' \equiv \mathsf{I}. \tag{2.25}$$

This leads to the schematic tree:

$$\begin{array}{llllll}
|\mathsf{L}| = 0, \ [\mathsf{L}] \neq 0, & & & & & \Pi = \dfrac{\overline{\mathsf{L}}}{[\mathsf{L}]}, \quad \Pi^2 = \Pi, \quad \langle\Pi\rangle = 1, \\
& [\mathsf{L}] = 0, \ \overline{\mathsf{L}} \neq 0, & & & \overline{\mathsf{L}}^2 = 0, & \\
& & \overline{\mathsf{L}} = 0, \ \langle\mathsf{L}\rangle \neq 0, & & & \Pi' = \dfrac{\overline{\mathsf{L}}'}{\langle\mathsf{L}\rangle}, \quad \Pi'^2 = \Pi', \quad \langle\Pi'\rangle = 2, \\
& & & \langle\mathsf{L}\rangle = 0 & \mathsf{L} \neq 0, \ \mathsf{L}^2 \neq 0 \quad \mathsf{L}^3 = 0, & \\
& & & & \mathsf{L} \neq 0, \ \mathsf{L}^2 = 0, & \\
& & & & \mathsf{L} = 0, & \Pi'' = \mathsf{I}, \quad \Pi''^2 = \Pi'', \quad \langle\Pi''\rangle = 3.
\end{array} \tag{2.26}$$

It is possible that for a given operator $\mathsf{L}$ not all cases are realized, for example, the case $|\mathsf{L}| = 0$, $[\mathsf{L}] = 0$ but $\overline{\mathsf{L}} \neq 0$. From next Section on we consider a special operator $\mathsf{L}$ which concerns the optics of homogeneous anisotropic media.

## 3. Preparations for Treatment of Wave Propagation in Direction of Optic Axes

The energy and momentum propagation of quasiplane and quasimonochromatic waves becomes complicated for refraction vectors $\boldsymbol{n} \equiv \frac{c}{\omega}\boldsymbol{k}$ in direction of optic axes due to non-uniqueness of the group velocity $\boldsymbol{v}$ or ray vectors $\boldsymbol{s}$. For reflection and refraction problems the refraction vectors $\boldsymbol{n}$ or the wave vectors $\boldsymbol{k}$ and (circular) frequency $\omega$ are the most fundamental notions in comparison to the ray vectors or group velocities and we start with this concept. In this Section we begin to consider by coordinate-invariant methods exceptional and degenerate cases to which, in particular, belong the singular cases of optic axes. The presented approach is of principal importance since it can be applied also to other operator equations with operators different from our considered specialized operator $\mathsf{L}$. Thus we continue the investigations of Section 2 to degenerate cases for the following special operator (1) of the wave optics of anisotropic media[2]

[2]We neglect in this article widely frequency dispersion and therefore we do not write the frequency $\omega$ as additional variable in $\mathsf{L}(\boldsymbol{n})$. Taking it into account it is more appropriate to use the "three-dimensional" operator $\mathsf{L}(\boldsymbol{k},\omega) \equiv \frac{c^2}{\omega^2}(\boldsymbol{k}\cdot\boldsymbol{k} - \boldsymbol{k}^2\mathsf{I}) + \boldsymbol{\varepsilon}(\boldsymbol{k},\omega)$ with $\boldsymbol{k}$ the wave vector. In last case $|\mathsf{L}| = 0$ and $[\mathsf{L}] = 0$ become three-dimensional (hyper-) surfaces in the four-dimensional $(\boldsymbol{k},\omega)$ -space.

$$\mathsf{L}(\boldsymbol{n}) = \boldsymbol{n}\cdot\boldsymbol{n} - \boldsymbol{n}^2\mathsf{I} + \boldsymbol{\varepsilon}, \tag{3.1}$$

which by solution of the equation $\mathsf{L}\boldsymbol{E}_0 = \boldsymbol{0}$ provides solution for the amplitude $\boldsymbol{E}_0 \equiv \boldsymbol{E}_0(\boldsymbol{n})$ of the electric field ($\boldsymbol{n} \equiv \frac{c}{\omega}\boldsymbol{k}$ are refraction vectors). A certain symmetry of the frequency-dependent permittivity tensor $\boldsymbol{\varepsilon}$ we do not assume in this Section. The invariants of $\mathsf{L}$ are

$$\begin{aligned}
&\langle\mathsf{L}\rangle = -2\boldsymbol{n}^2 + \langle\boldsymbol{\varepsilon}\rangle,\\
&[\mathsf{L}] = (\boldsymbol{n}^2)^2 - \langle\boldsymbol{\varepsilon}\rangle\boldsymbol{n}^2 - \boldsymbol{n}\boldsymbol{\varepsilon}\boldsymbol{n} + [\boldsymbol{\varepsilon}],\\
&|\mathsf{L}| = (\boldsymbol{n}^2)\boldsymbol{n}\boldsymbol{\varepsilon}\boldsymbol{n} - \langle\boldsymbol{\varepsilon}\rangle\boldsymbol{n}\boldsymbol{\varepsilon}\boldsymbol{n} + \boldsymbol{n}\boldsymbol{\varepsilon}^2\boldsymbol{n} + |\boldsymbol{\varepsilon}|,
\end{aligned} \tag{3.2}$$

and the complementary operator $\overline{\mathsf{L}}$ to $\mathsf{L}$ defined in (4)

$$\overline{\mathsf{L}} = \left(\boldsymbol{n}^2 - \langle\boldsymbol{\varepsilon}\rangle\right)\boldsymbol{n}\cdot\boldsymbol{n} + \boldsymbol{\varepsilon}\boldsymbol{n}\cdot\boldsymbol{n} + \boldsymbol{n}\cdot\boldsymbol{n}\boldsymbol{\varepsilon} - (\boldsymbol{n}\boldsymbol{\varepsilon}\boldsymbol{n})\mathsf{I} + \overline{\boldsymbol{\varepsilon}}, \quad \langle\overline{\mathsf{L}}\rangle = [\mathsf{L}]. \tag{3.3}$$

In case $[\mathsf{L}] = 0$, the projection operator $\Pi$ defined in (2.12) with explicitly given invariant $[\mathsf{L}]$ in (3.2) and complementary operator $\overline{\mathsf{L}}$ in (3.3) does not exist. We can distinguish then two partial cases. The first case is

$$[\mathsf{L}] = \langle\overline{\mathsf{L}}\rangle = 0, \quad \overline{\mathsf{L}} \equiv \mathsf{L}^2 - \langle\mathsf{L}\rangle\mathsf{L} \neq 0, \quad \Rightarrow \quad \overline{\mathsf{L}}^2 = 0, \tag{3.4}$$

where $\overline{\mathsf{L}}^2 = 0$ follows from (2.11) in connection with the requirement $|\mathsf{L}| = 0$ and the supposition $[\mathsf{L}] = 0$ from the general relation for arbitrary three-dimensional operators

$$\overline{\mathsf{L}}^2 = [\mathsf{L}]\overline{\mathsf{L}} + |\mathsf{L}|\left(\mathsf{L} - \langle\mathsf{L}\rangle\mathsf{I}\right). \tag{3.5}$$

However, from $\overline{\mathsf{L}}^2 = 0$ in (3.4) not automatically follows also $\overline{\mathsf{L}} = 0$. In the case (3.4) due to $\overline{\mathsf{L}} \neq 0$ there exist vectors $\boldsymbol{a}$ and $\tilde{\boldsymbol{a}}$ such that $\overline{\mathsf{L}}\boldsymbol{a} \neq \boldsymbol{0}$ and $\tilde{\boldsymbol{a}}\overline{\mathsf{L}} \neq \boldsymbol{0}$ and they are the only linearly independent right-hand and left-hand eigenvectors of $\mathsf{L}$ to eigenvalue 0, respectively, and thus their polarization is non-degenerate but their scalar product is vanishing and, therefore, they are orthogonal to each other

$$\left(\tilde{\boldsymbol{a}}\overline{\mathsf{L}}\right)\left(\overline{\mathsf{L}}\boldsymbol{a}\right) = \tilde{\boldsymbol{a}}\overline{\mathsf{L}}^2\boldsymbol{a} = 0, \quad \Rightarrow \quad \tilde{\boldsymbol{a}}\overline{\mathsf{L}}^2\boldsymbol{E}_0 = 0. \tag{3.6}$$

This case is not possible in lossless media ($\boldsymbol{\varepsilon}^{\mathrm{T}} = \boldsymbol{\varepsilon}^*$, $\boldsymbol{\varepsilon}$ is Hermitean operator) with real refraction vectors $\boldsymbol{n}$ since then the left-hand eigenvectors $\tilde{\boldsymbol{a}}\overline{\mathsf{L}}$ of $\mathsf{L}$ are proportional to $\boldsymbol{E}_0^*$ and the right-hand eigenvectors $\overline{\mathsf{L}}\boldsymbol{a}$ to $\boldsymbol{E}_0$ and $\boldsymbol{E}_0^*\boldsymbol{E}_0$ is non-vanishing in every case. However, this case may happen for evanescent waves with complex refraction vectors or in lossy media not dealt with in present paper.

We consider the first case, the vanishing of $|\mathsf{L}|$ and of $[\mathsf{L}]$ but $\overline{\mathsf{L}}$ must not necessarily be vanishing

$$\begin{aligned}
&|\mathsf{L}| = (\boldsymbol{n}^2)\boldsymbol{n}\boldsymbol{\varepsilon}\boldsymbol{n} - \langle\boldsymbol{\varepsilon}\rangle\boldsymbol{n}\boldsymbol{\varepsilon}\boldsymbol{n} + \boldsymbol{n}\boldsymbol{\varepsilon}^2\boldsymbol{n} + |\boldsymbol{\varepsilon}| = 0,\\
&[\mathsf{L}] = (\boldsymbol{n}^2)^2 - \langle\boldsymbol{\varepsilon}\rangle\boldsymbol{n}^2 - \boldsymbol{n}\boldsymbol{\varepsilon}\boldsymbol{n} + [\boldsymbol{\varepsilon}] = 0.
\end{aligned} \tag{3.7}$$

By multiplication of the first equation with $\boldsymbol{n}^2$ and the second equation with

$\boldsymbol{n\varepsilon n}$ one finds an equivalent equation for one of these equations

$$\left(\boldsymbol{n}^2\right)\left(\boldsymbol{n}\boldsymbol{\varepsilon}^2\boldsymbol{n}\right)+\left(\boldsymbol{n\varepsilon n}\right)^2+\left|\boldsymbol{\varepsilon}\right|\boldsymbol{n}^2-\left[\boldsymbol{\varepsilon}\right]\boldsymbol{n\varepsilon n}=0. \tag{3.8}$$

Two of the three Equations (3.7) and (3.8) are independent and must be solved to get the relations between the components of the refraction vectors $\boldsymbol{n}$ in case of $\left|\mathsf{L}\right|=0$ and $\left[\mathsf{L}\right]=0$ but $\overline{\mathsf{L}}\neq 0$ that is fairly complicated.

We consider now the second case which is the important case of twofold degeneration of the eigenvalue $\lambda=0$ of the operator $\mathsf{L}$ for a given refraction vector $\boldsymbol{n}$ and thus the case of optic axes or binormals. In this case not only $\overline{\mathsf{L}}^2$ according to (4) is vanishing but $\overline{\mathsf{L}}$ itself and we have

$$\overline{\mathsf{L}}\equiv\mathsf{L}^2-\left\langle\mathsf{L}\right\rangle\mathsf{L}+\left[\mathsf{L}\right]\mathsf{I}=0,\quad\Rightarrow\quad\left\langle\overline{\mathsf{L}}\right\rangle=\left[\mathsf{L}\right]=0,\quad\Rightarrow\quad\mathsf{L}^2-\left\langle\mathsf{L}\right\rangle\mathsf{L}=0. \tag{3.9}$$

From (3.9) follows that in this case the, in general, non-vanishing and non-degenerate eigenvalue of $\mathsf{L}$ is $\lambda=\left\langle\mathsf{L}\right\rangle$ and from $\mathsf{L}^2-\left\langle\mathsf{L}\right\rangle\mathsf{L}=0$ that vectors $\mathsf{L}\boldsymbol{a}\neq\boldsymbol{0}$ and $\tilde{\boldsymbol{a}}\mathsf{L}\neq\boldsymbol{0}$ for arbitrary appropriate $\boldsymbol{a}$ and $\tilde{\boldsymbol{a}}$ are right-hand and left-hand eigenvectors of $\mathsf{L}$ to this eigenvalue $\left\langle\mathsf{L}\right\rangle$, respectively

$$\mathsf{L}\mathsf{L}=\left\langle\mathsf{L}\right\rangle\mathsf{L},\quad\Rightarrow\quad\mathsf{L}\left(\mathsf{L}\boldsymbol{a}\right)=\left\langle\mathsf{L}\right\rangle\mathsf{L}\boldsymbol{a},\quad\left(\tilde{\boldsymbol{a}}\mathsf{L}\right)\mathsf{L}=\left\langle\mathsf{L}\right\rangle\tilde{\boldsymbol{a}}\mathsf{L}. \tag{3.10}$$

From $\mathsf{L}^2=\left\langle\mathsf{L}\right\rangle\mathsf{L}$ follow for the special operator (3.1) by multiplication from right and from left with the refraction vector $\boldsymbol{n}$ the identities

$$\begin{aligned}\left(\boldsymbol{n}\cdot\boldsymbol{n}-\left(\boldsymbol{n}^2\right)\mathsf{I}+\boldsymbol{\varepsilon}\right)^2\boldsymbol{n}&=\left(\boldsymbol{n}\cdot\boldsymbol{n}-\left(\boldsymbol{n}^2\right)\mathsf{I}+\boldsymbol{\varepsilon}\right)\boldsymbol{\varepsilon n}=\left(-2\boldsymbol{n}^2+\left\langle\boldsymbol{\varepsilon}\right\rangle\right)\boldsymbol{\varepsilon n},\\ \boldsymbol{n}\left(\boldsymbol{n}\cdot\boldsymbol{n}-\left(\boldsymbol{n}^2\right)\mathsf{I}+\boldsymbol{\varepsilon}\right)^2&=\boldsymbol{n\varepsilon}\left(\boldsymbol{n}\cdot\boldsymbol{n}-\left(\boldsymbol{n}^2\right)\mathsf{I}+\boldsymbol{\varepsilon}\right)=\boldsymbol{n\varepsilon}\left(-2\boldsymbol{n}^2+\left\langle\boldsymbol{\varepsilon}\right\rangle\right),\end{aligned} \tag{3.11}$$

and many other more complicated identities can be derived in similar way. By scalar multiplication of the vectorial Equation (3.11) with the vector $\boldsymbol{n}$ we get the scalar requirement

$$\left(2\boldsymbol{n}^2-\left\langle\boldsymbol{\varepsilon}\right\rangle\right)\boldsymbol{n\varepsilon n}+\boldsymbol{n}\boldsymbol{\varepsilon}^2\boldsymbol{n}=0. \tag{3.12}$$

Using the dispersion equation $\left|\mathsf{L}\right|=0$ with $\left|\mathsf{L}\right|$ explicitly given in (3.2) we see that from (3.12) also follows

$$\boldsymbol{n}^2\left(\boldsymbol{n\varepsilon n}\right)-\left|\boldsymbol{\varepsilon}\right|=0, \tag{3.13}$$

as a condition for optic axes which has to be satisfied plus in addition with a further independent one of the above derived equations from $\left|\mathsf{L}\right|=0$ and $\left[\mathsf{L}\right]=0$, for example

$$\left\langle\boldsymbol{\varepsilon}\right\rangle\boldsymbol{n\varepsilon n}-\boldsymbol{n}\boldsymbol{\varepsilon}^2\boldsymbol{n}-2\left|\boldsymbol{\varepsilon}\right|=0. \tag{3.14}$$

This equation possesses the advantage that it is only quadratically in the components of the refraction vector $\boldsymbol{n}$ and, therefore, with only two solutions for one component in dependence on the other components. That the first of the considered two cases is more general as the second can be seen from the fact that it is obviously impossible to derive a second-degree equation for the components of the refraction vectors.

According to (2.19) and (2.21) the operators $\Pi'$ and $\Pi_\lambda$ for $\lambda=\left\langle\mathsf{L}\right\rangle$ are projection operators in case of $\overline{\mathsf{L}} = 0$ and together with (3.11) we find in our case more explicitly

$$\Pi' \equiv \mathsf{I} - \frac{\mathsf{L}}{\langle \mathsf{L} \rangle} = \frac{\left(\boldsymbol{n}^2 - \langle \boldsymbol{\varepsilon} \rangle\right)\mathsf{I} + \boldsymbol{n}\cdot\boldsymbol{n} + \boldsymbol{\varepsilon}}{2\boldsymbol{n}^2 - \langle \boldsymbol{\varepsilon} \rangle} = \mathsf{I} - \frac{\boldsymbol{\varepsilon}\boldsymbol{n}\cdot\boldsymbol{n}\boldsymbol{\varepsilon}}{\boldsymbol{n}\boldsymbol{\varepsilon}^2\boldsymbol{n}}, \quad \Pi'^2 = \Pi', \quad \langle \Pi' \rangle = 2,$$
$$\Pi_\lambda \equiv \frac{\mathsf{L}}{\langle \mathsf{L} \rangle} = \frac{\boldsymbol{n}\cdot\boldsymbol{n} - \left(\boldsymbol{n}^2\right)\mathsf{I} + \boldsymbol{\varepsilon}}{-2\boldsymbol{n}^2 + \langle \boldsymbol{\varepsilon} \rangle} = \frac{\boldsymbol{\varepsilon}\boldsymbol{n}\cdot\boldsymbol{n}\boldsymbol{\varepsilon}}{\boldsymbol{n}\boldsymbol{\varepsilon}^2\boldsymbol{n}}, \quad \Pi_\lambda^2 = \Pi_\lambda, \quad \langle \Pi_\lambda \rangle = 1, \quad \left(\lambda = \langle \mathsf{L} \rangle\right). \tag{3.15}$$

Due to $\boldsymbol{n}\boldsymbol{\varepsilon}\Pi'\boldsymbol{e} = 0$ right-hand eigenvectors $\boldsymbol{e}$ of $\mathsf{L}$ are polarized perpendicular to the plane determined by the normal vector $\boldsymbol{n}\boldsymbol{\varepsilon}$.

The necessary (but not sufficient) condition $[\mathsf{L}] = 0$ for optic axes takes on the following form

$$\left(\boldsymbol{n}^2\right)^2 - \langle \boldsymbol{\varepsilon} \rangle \boldsymbol{n}^2 - \boldsymbol{n}\boldsymbol{\varepsilon}\boldsymbol{n} + [\boldsymbol{\varepsilon}] = 0. \tag{3.16}$$

By multiplication of (3.16) with $\boldsymbol{n}^2$ and elimination of $\left(\boldsymbol{n}^2\right)\boldsymbol{n}\boldsymbol{\varepsilon}\boldsymbol{n}$ by means of (3.13) we find that the condition

$$\left(\boldsymbol{n}^2\right)^3 - \langle \boldsymbol{\varepsilon} \rangle \left(\boldsymbol{n}^2\right)^2 + [\boldsymbol{\varepsilon}]\boldsymbol{n}^2 - |\boldsymbol{\varepsilon}| = 0. \tag{3.17}$$

is satisfied for the case of an optic axes. This is the eigenvalue equation for the tensor $\boldsymbol{\varepsilon}$ with the eigenvalues $\boldsymbol{n}^2$. Since the eigenvalues of $\boldsymbol{\varepsilon}$ are the values $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ to eigenvectors $\{\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3\}$ for an optic axis at least one of the equations $\boldsymbol{n}^2 = \varepsilon_i, (i = 1,2,3)$ has to be satisfied. In next Section we see that this is the eigenvalue $\varepsilon_2$ in the ordered sequence (4.4) of eigenvalues. It is clear that not all of the derived scalar identities for the presence of optic axes are independent from each other and there are many possibilities to derive them and also further ones.

Due to (3.11) which can be written

$$\boldsymbol{\varepsilon}^2\boldsymbol{n} + \left(\boldsymbol{n}^2 - \langle \boldsymbol{\varepsilon} \rangle\right)\boldsymbol{\varepsilon}\boldsymbol{n} + \left(\boldsymbol{n}\boldsymbol{\varepsilon}\boldsymbol{n}\right)\boldsymbol{n} = \boldsymbol{0}, \tag{3.18}$$

the three vectors $\{\boldsymbol{n}, \boldsymbol{\varepsilon}\boldsymbol{n}, \boldsymbol{\varepsilon}^2\boldsymbol{n}\}$ are linear dependent. This can be also expressed by the vanishing of the volume product

$$\left[\boldsymbol{n}, \boldsymbol{\varepsilon}\boldsymbol{n}, \boldsymbol{\varepsilon}^2\boldsymbol{n}\right] = 0. \tag{3.19}$$

Therefore the three vectors $\{\boldsymbol{n}, \boldsymbol{\varepsilon}\boldsymbol{n}, \boldsymbol{\varepsilon}^2\boldsymbol{n}\}$ lie in the same plane spanned by two of the three vectors.

## 4. Coordinate-Invariant Treatment in Case of Electrically Anisotropic Media with Real Symmetric Permittivity Tensor

We apply now the general discussions of last Sections to the classical crystal optics with the special operator (3.1) of the wave equation for the amplitude of the electric field $\boldsymbol{E}_0$ for refraction vectors $\boldsymbol{n}$ that means

$$\mathsf{L}(\boldsymbol{n}) = \boldsymbol{n}\cdot\boldsymbol{n} - \boldsymbol{n}^2\mathsf{I} + \boldsymbol{\varepsilon} \equiv \mathsf{L}, \quad \boldsymbol{\varepsilon} \equiv \boldsymbol{\varepsilon}(\omega), \quad \left(\mathsf{L}\boldsymbol{E}_0 = \boldsymbol{0}\right), \tag{4.1}$$

and extend it by consideration of a symmetrical permittivity tensor $\boldsymbol{\varepsilon}$ (or with indices $\varepsilon_{ij} = \varepsilon_{ji} \to \boldsymbol{\varepsilon}$) in principal axes form with real unit vectors $\boldsymbol{c}_i$ (axes notations by letter $\boldsymbol{c}$ according to Fyodorov [7])

$$\boldsymbol{\varepsilon} \equiv \varepsilon_1 \boldsymbol{c}_1 \cdot \boldsymbol{c}_1 + \varepsilon_2 \boldsymbol{c}_2 \cdot \boldsymbol{c}_2 + \varepsilon_3 \boldsymbol{c}_3 \cdot \boldsymbol{c}_3 = \boldsymbol{\varepsilon}^{\mathrm{T}}, \quad \boldsymbol{c}_i \boldsymbol{c}_j = \delta_{ij}. \tag{4.2}$$

For the $n$-th powers of the tensor $\boldsymbol{\varepsilon}$ we find from (4.2)

$$\boldsymbol{\varepsilon}^n = \varepsilon_1^n \boldsymbol{c}_1 \cdot \boldsymbol{c}_1 + \varepsilon_2^n \boldsymbol{c}_2 \cdot \boldsymbol{c}_2 + \varepsilon_3^n \boldsymbol{c}_3 \cdot \boldsymbol{c}_3. \tag{4.3}$$

We suppose that the real-valued and non-negative principal (or main) values $\varepsilon_i$ are ordered according to[3]

$$0 \le \varepsilon_1 \le \varepsilon_2 \le \varepsilon_3, \quad \varepsilon_i = \varepsilon_i^*, \quad (i = 1,2,3), \quad \varepsilon_i \equiv \varepsilon_i(\omega). \tag{4.4}$$

The directions of the vectors $\boldsymbol{c}_i$ are called the principal axes of the tensor $\boldsymbol{\varepsilon}$. Depending on the symmetry of the medium the normalized vectors $\boldsymbol{c}_i$ in direction of the principal axes may or may not depend on the frequency. The permittivity tensor $\boldsymbol{\varepsilon}$ is now by this proposition not only symmetric but becomes also a Hermitean one, *i.e.* $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{\mathrm{T}} = \boldsymbol{\varepsilon}^{\dagger}$. Our further treatment here distinguishes from the very interesting considerations of Fyodorov [7] [8] insofar that we from beginning on involve more the operator $\mathsf{L}$, its invariants and the complementary operator.

The invariants of $\mathsf{L}$ were already given in (3.2) but have now to be specialized according to (4.2) and are

$$\begin{aligned} &\langle \mathsf{L} \rangle = -2\boldsymbol{n}^2 + \langle \boldsymbol{\varepsilon} \rangle, \quad \langle \boldsymbol{\varepsilon} \rangle = \varepsilon_1 + \varepsilon_2 + \varepsilon_3, \\ &[\mathsf{L}] = (\boldsymbol{n}^2)^2 - (\langle \boldsymbol{\varepsilon} \rangle \boldsymbol{n}^2 + \boldsymbol{n\varepsilon n}) + [\boldsymbol{\varepsilon}], \quad [\boldsymbol{\varepsilon}] = \varepsilon_1\varepsilon_2 + \varepsilon_1\varepsilon_3 + \varepsilon_2\varepsilon_3, \\ &|\mathsf{L}| = (\boldsymbol{n}^2)\boldsymbol{n\varepsilon n} - (\langle \boldsymbol{\varepsilon} \rangle \boldsymbol{n\varepsilon n} - \boldsymbol{n\varepsilon}^2\boldsymbol{n}) + |\boldsymbol{\varepsilon}|, \quad |\boldsymbol{\varepsilon}| = \varepsilon_1\varepsilon_2\varepsilon_3, \end{aligned} \tag{4.5}$$

and the complementary operator $\overline{\mathsf{L}}$ to operator $\mathsf{L}$ with the identity $\langle \overline{\mathsf{L}} \rangle = [\mathsf{L}]$ is

$$\begin{aligned} &\overline{\mathsf{L}} \equiv \mathsf{L}^2 - \langle \mathsf{L} \rangle \mathsf{L} + [\mathsf{L}]\mathsf{I} = (\boldsymbol{n}^2 - \langle \boldsymbol{\varepsilon} \rangle)\boldsymbol{n} \cdot \boldsymbol{n} + \boldsymbol{\varepsilon n} \cdot \boldsymbol{n} + \boldsymbol{n} \cdot \boldsymbol{n\varepsilon} - (\boldsymbol{n\varepsilon n})\mathsf{I} + \overline{\boldsymbol{\varepsilon}}, \\ &\overline{\boldsymbol{\varepsilon}} \equiv \boldsymbol{\varepsilon}^2 - \langle \boldsymbol{\varepsilon} \rangle \boldsymbol{\varepsilon} + [\boldsymbol{\varepsilon}]\mathsf{I} = \varepsilon_2\varepsilon_3 \boldsymbol{c}_1 \cdot \boldsymbol{c}_1 + \varepsilon_1\varepsilon_3 \boldsymbol{c}_2 \cdot \boldsymbol{c}_2 + \varepsilon_1\varepsilon_2 \boldsymbol{c}_3 \cdot \boldsymbol{c}_3. \end{aligned} \tag{4.6}$$

The general formulae for the invariants of a complementary operator $\overline{\mathsf{L}}$ to a three-dimensional operator $\mathsf{L}$ are

$$\langle \overline{\mathsf{L}} \rangle = [\mathsf{L}], \quad [\overline{\mathsf{L}}] = |\mathsf{L}|\langle \mathsf{L} \rangle, \quad |\overline{\mathsf{L}}| = |\mathsf{L}|^2. \tag{4.7}$$

If in addition to $|\mathsf{L}| = 0$ which in our case is the dispersion equation for the refraction vectors also $[\mathsf{L}] = 0$ then the complementary operator $\overline{\mathsf{L}}$ to $\mathsf{L}$ can be vanishing $\overline{\mathsf{L}} = 0$ or non-vanishing but $\overline{\mathsf{L}}^2$ according to (3.5) is in every case vanishing. The first case $\overline{\mathsf{L}} = 0$ is the case of optic axes.

We introduce the abbreviations $n_i \equiv \boldsymbol{n c}_i$. The determinant $|\mathsf{L}|$ written in principal axes components takes on the form

[3]This means that we neglect absorption or, more generally, dissipation and also do not take into consideration cases of negative principal values of the permittivities.

$$
\begin{aligned}
|\mathsf{L}| &= \left(n_1^2+n_2^2+n_3^2\right)\left(\varepsilon_1 n_1^2+\varepsilon_2 n_2^2+\varepsilon_3 n_3^2\right) \\
&\quad -\left(\left(\varepsilon_1+\varepsilon_2+\varepsilon_3\right)\left(\varepsilon_1 n^2+\varepsilon_2 n_2^2+\varepsilon_3 n_3^2\right)-\left(\varepsilon_1^2 n^2+\varepsilon_2^2 n_2^2+\varepsilon_3^2 n_3^2\right)\right)+\varepsilon_1\varepsilon_2\varepsilon_3 \\
&= \left(n_1^2+n_2^2+n_3^2\right)\left(\varepsilon_1 n_1^2+\varepsilon_2 n_2^2+\varepsilon_3 n_3^2\right) \\
&\quad -\left(\varepsilon_1\left(\varepsilon_2 n_2^2+\varepsilon_3 n_3^2\right)+\varepsilon_2\left(\varepsilon_1 n_1^2+\varepsilon_3 n_3^2\right)+\varepsilon_3\left(\varepsilon_1 n_1^2+\varepsilon_2 n_2^2\right)\right)+\varepsilon_1\varepsilon_2\varepsilon_3 \\
&= \left(n_1^2+n_2^2+n_3^2-\varepsilon_2\right)\left(\varepsilon_1 n_1^2+\varepsilon_2 n_2^2+\varepsilon_3 n_3^2-\varepsilon_1\varepsilon_3\right)-\left(\varepsilon_3-\varepsilon_2\right)\left(\varepsilon_2-\varepsilon_1\right)n_2^2,
\end{aligned}
\tag{4.8}
$$

plus further forms by permutation of the indices. In the planes perpendicular to the principal optic axes $n_i = 0$ from the dispersion equation $|\mathsf{L}| = 0$ in the form (4.8) we find the well-known relations

$$
\begin{aligned}
n_3 &\equiv \boldsymbol{n}\boldsymbol{c}_3 = 0, \quad \Rightarrow \quad \left(n_1^2+n_2^2-\varepsilon_3\right)\left(\varepsilon_1 n_1^2+\varepsilon_2 n_2^2-\varepsilon_1\varepsilon_2\right)=0, \\
n_2 &\equiv \boldsymbol{n}\boldsymbol{c}_2 = 0, \quad \Rightarrow \quad \left(n_1^2+n_3^2-\varepsilon_2\right)\left(\varepsilon_1 n_1^2+\varepsilon_3 n_3^2-\varepsilon_1\varepsilon_3\right)=0, \\
n_1 &\equiv \boldsymbol{n}\boldsymbol{c}_1 = 0, \quad \Rightarrow \quad \left(n_2^2+n_3^2-\varepsilon_1\right)\left(\varepsilon_2 n_2^2+\varepsilon_3 n_3^2-\varepsilon_2\varepsilon_3\right)=0,
\end{aligned}
\tag{4.9}
$$

that means the dispersion surface decomposes in every case in the plane perpendicular to a principal axis into a circle and an ellipse (but not in general). Due to ordering (4.4) only in the case $n_2 = 0$ from the three principal planes (4.9) we have intersections of the circle and the ellipse and therefore real optic axes (**Figure 1**; similar pictures are, for example, also in [2] (§99, pp. 409, 410) and in [4] [7]. In cases of $\varepsilon_1 = \varepsilon_2 \neq \varepsilon_3$ and $\varepsilon_1 \neq \varepsilon_2 = \varepsilon_3$ the three cases (4.9) reduce to two different cases and the medium becomes optically uniaxial.

The invariant $[\mathsf{L}]$ in principal axes form takes on the following form

$$
\begin{aligned}
[\mathsf{L}] &= \left(\left(n_1^2+n_2^2+n_3^2\right)-\left(\varepsilon_1+\varepsilon_2+\varepsilon_3\right)\right)\left(n_1^2+n_2^2+n_3^2\right) \\
&\quad -\left(\varepsilon_1 n_1^2+\varepsilon_2 n_2^2+\varepsilon_3 n_3^2\right)+\varepsilon_1\varepsilon_2+\varepsilon_1\varepsilon_3+\varepsilon_2\varepsilon_3,
\end{aligned}
\tag{4.10}
$$

that can be reduced to

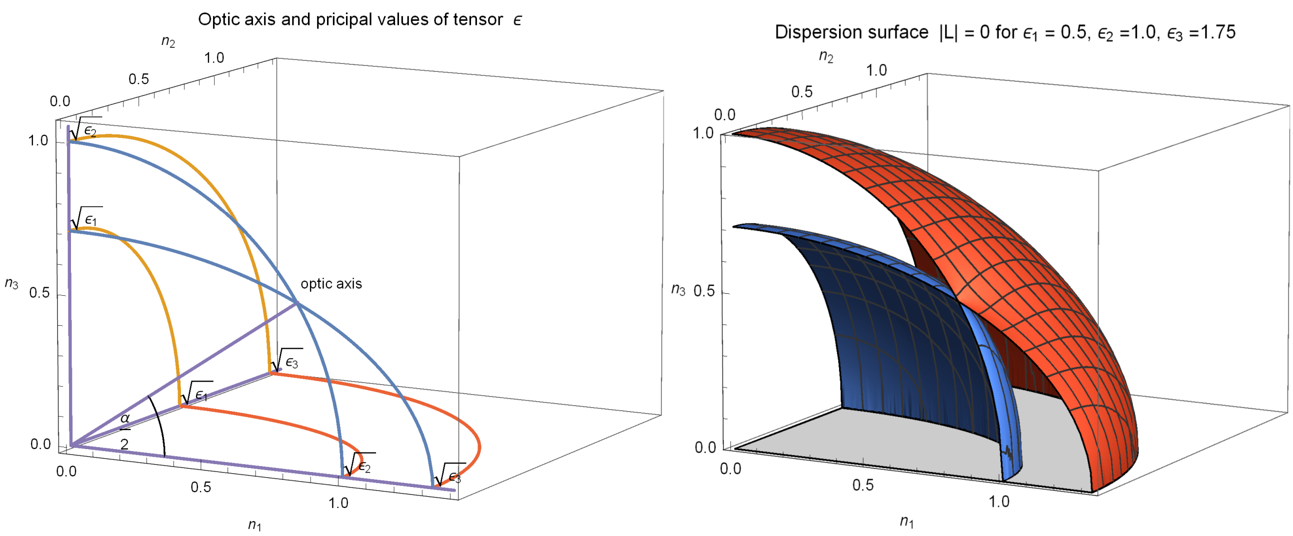


**Figure 1.** T he geometry of the dispersion surface $|\mathsf{L}| = 0$ with an optic axis in connection with the principal values $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ of the permittivity tensor $\boldsymbol{\varepsilon}$. In particular, the left-hand figure is a well-known picture for illustration of the dispersion surface (e.g, [1] [2]) which in next Sections is complemented by more detailed figures.

$$\begin{aligned}
[\mathsf{L}] &= \left(n_1^2+n_2^2+n_3^2\right)^2 - \left(\varepsilon_1\left(2n_1^2+n_2^2+n_3^2\right)+\varepsilon_2\left(n_1^2+2n_2^2+n_3^2\right)\right. \\
&\quad \left. +\varepsilon_3\left(n_1^2+n_2^2+2n_3^2\right)\right)+\varepsilon_1\varepsilon_2+\varepsilon_1\varepsilon_3+\varepsilon_2\varepsilon_3 \\
&= \left(n_1^2+n_2^2+n_3^2\right)^2 - \left(\left(2\varepsilon_1+\varepsilon_2+\varepsilon_3\right)n_1^2+\left(\varepsilon_1+2\varepsilon_2+\varepsilon_3\right)n_2^2\right. \\
&\quad \left. +\left(\varepsilon_1+\varepsilon_2+2\varepsilon_3\right)n_3^2\right)+\varepsilon_1\varepsilon_2+\varepsilon_1\varepsilon_3+\varepsilon_2\varepsilon_3.
\end{aligned} \tag{4.11}$$

In analogy to (9) the second invariant $[\mathsf{L}]$ can be factorized in the special cases $n_i=0$. We write here explicitly down only the special case $n_2=0$

$$\begin{aligned}
& n_2=0, \quad \Rightarrow \\
& \left[\mathsf{L}\left(n_1,0,n_3\right)\right] = \left(n_1^2+n_3^2\right)^2 - \left(\left(2\varepsilon_1+\varepsilon_2+\varepsilon_3\right)n_1^2+\left(\varepsilon_1+\varepsilon_2+2\varepsilon_3\right)n_3^2\right) \\
& \qquad +\varepsilon_1\varepsilon_2+\varepsilon_1\varepsilon_3+\varepsilon_2\varepsilon_3 \\
& = \left(n_1^2+n_3^2-\frac{2\varepsilon_1+\varepsilon_2+\varepsilon_3}{2}\right)^2 - \left(\left(\varepsilon_3-\varepsilon_1\right)n_3^2+\varepsilon_1^2+\left(\frac{\varepsilon_3-\varepsilon_2}{2}\right)^2\right) \\
& = \left(n_1^2+n_3^2-\frac{2\varepsilon_1+\varepsilon_2+\varepsilon_3}{2}+\sqrt{\left(\varepsilon_3-\varepsilon_1\right)n_3^2+\varepsilon_1^2+\left(\frac{\varepsilon_3-\varepsilon_2}{2}\right)^2}\right) \\
& \quad \cdot\left(n_1^2+n_3^2-\frac{2\varepsilon_1+\varepsilon_2+\varepsilon_3}{2}-\sqrt{\left(\varepsilon_3-\varepsilon_1\right)n_3^2+\varepsilon_1^2+\left(\frac{\varepsilon_3-\varepsilon_2}{2}\right)^2}\right).
\end{aligned} \tag{4.12}$$

This is a non-rational factorization which is only unique with the additional requirement that the expression within the square root is positive (only positive sum terms) under the supposition (4.4) of ordering the permittivities but not in other case.

The general form of this non-rational factorization is

$$\begin{aligned}
[\mathsf{L}(\boldsymbol{n})] &= \left(n_1^2+n_2^2+n_3^2-\frac{2\varepsilon_1+\varepsilon_2+\varepsilon_3}{2}\right)^2 - \left(\left(\varepsilon_2-\varepsilon_1\right)n_2^2+\left(\varepsilon_3-\varepsilon_1\right)n_3^2+\varepsilon_1^2+\left(\frac{\varepsilon_3-\varepsilon_2}{2}\right)^2\right) \\
&= \left(n_1^2+n_1^2+n_3^2-\frac{2\varepsilon_1+\varepsilon_2+\varepsilon_3}{2}+\sqrt{\left(\varepsilon_2-\varepsilon_1\right)n_2^2+\left(\varepsilon_3-\varepsilon_1\right)n_3^2+\varepsilon_1^2+\left(\frac{\varepsilon_3-\varepsilon_2}{2}\right)^2}\right) \\
&\quad \cdot\left(n_1^2+n_1^2+n_3^2-\frac{2\varepsilon_1+\varepsilon_2+\varepsilon_3}{2}-\sqrt{\left(\varepsilon_2-\varepsilon_1\right)n_2^2+\left(\varepsilon_3-\varepsilon_1\right)n_3^2+\varepsilon_1^2+\left(\frac{\varepsilon_3-\varepsilon_2}{2}\right)^2}\right),
\end{aligned} \tag{4.13}$$

which easily can be specialized to $n_3=0$ and $n_1=0$. It shows that in the real domain of this three-dimensional space the considered surface consists of two shells with no self-intersections but in the complex domain it possesses such points for the vanishing of the root.

The complementary operator $\overline{\mathsf{L}}$ to $\mathsf{L}$ expressed in principal axes form is

$$\begin{aligned}
\overline{\mathsf{L}} &= \left(n_1^2+n_2^2+n_3^2\right)\left(n_1\boldsymbol{c}_1+n_2\boldsymbol{c}_2+n_3\boldsymbol{c}_3\right)\cdot\left(n_1\boldsymbol{c}_1+n_2\boldsymbol{c}_2+n_3\boldsymbol{c}_3\right) \\
&\quad -\varepsilon_1\left(n_1^2\left(\boldsymbol{c}_2\cdot\boldsymbol{c}_2+\boldsymbol{c}_3\cdot\boldsymbol{c}_3\right)+\left(n_2\boldsymbol{c}_2+n_3\boldsymbol{c}_3\right)\cdot\left(n_2\boldsymbol{c}_2+n_3\boldsymbol{c}_3\right)\right) \\
&\quad -\varepsilon_2\left(n_2^2\left(\boldsymbol{c}_1\cdot\boldsymbol{c}_1+\boldsymbol{c}_3\cdot\boldsymbol{c}_3\right)+\left(n_1\boldsymbol{c}_1+n_3\boldsymbol{c}_3\right)\cdot\left(n_1\boldsymbol{c}_1+n_3\boldsymbol{c}_3\right)\right) \\
&\quad -\varepsilon_3\left(n_3^2\left(\boldsymbol{c}_1\cdot\boldsymbol{c}_1+\boldsymbol{c}_2\cdot\boldsymbol{c}_2\right)+\left(n_1\boldsymbol{c}_1+n_2\boldsymbol{c}_2\right)\cdot\left(n_1\boldsymbol{c}_1+n_2\boldsymbol{c}_2\right)\right) \\
&\quad +\varepsilon_2\varepsilon_3\boldsymbol{c}_1\cdot\boldsymbol{c}_1+\varepsilon_1\varepsilon_3\boldsymbol{c}_2\cdot\boldsymbol{c}_2+\varepsilon_1\varepsilon_2\boldsymbol{c}_3\cdot\boldsymbol{c}_3.
\end{aligned} \tag{4.14}$$

It is easy to check $\langle\overline{\mathsf{L}}\rangle = [\mathsf{L}]$ explicitly given in (4.10).

If $[\mathsf{L}] \neq 0$, we can determine according to (2.12) a "one-dimensional" projection operator $\Pi$ for the determination of polarization vectors proportional to solutions $\boldsymbol{E}_0$ in the following way

$$\Pi \equiv \frac{\overline{\mathsf{L}}}{\langle\overline{\mathsf{L}}\rangle} = \frac{\overline{\mathsf{L}}}{[\mathsf{L}]}, \quad ([\mathsf{L}] \neq 0), \quad \Rightarrow \quad \Pi^2 = \Pi, \quad \langle\Pi\rangle = 1, \tag{4.15}$$

and we obtain explicitly specialized by (4.1)

$$\Pi = \frac{(\boldsymbol{n}^2)\boldsymbol{n}\cdot\boldsymbol{n} - \langle\boldsymbol{\varepsilon}\rangle\boldsymbol{n}\cdot\boldsymbol{n} + \boldsymbol{\varepsilon}\boldsymbol{n}\cdot\boldsymbol{n} + \boldsymbol{n}\cdot\boldsymbol{n}\boldsymbol{\varepsilon} - (\boldsymbol{n}\boldsymbol{\varepsilon}\boldsymbol{n})\mathsf{I} + \overline{\boldsymbol{\varepsilon}}}{(\boldsymbol{n}^2)^2 - \langle\boldsymbol{\varepsilon}\rangle\boldsymbol{n}^2 - \boldsymbol{n}\boldsymbol{\varepsilon}\boldsymbol{n} + [\boldsymbol{\varepsilon}]}, \quad \Pi^2 = \Pi, \quad \langle\Pi\rangle = 1. \tag{4.16}$$

An arbitrary vector $\overline{\mathsf{L}}\boldsymbol{a} \neq \boldsymbol{0}$ and an arbitrary vector $\tilde{\boldsymbol{a}}\overline{\mathsf{L}} \neq \boldsymbol{0}$ is a right-hand or left-hand eigenvector of $\mathsf{L}$ to eigenvalue zero, respectively, but usually non-normalized. Due to orthonormality of vectors $\boldsymbol{c}_i$ it is not difficult to calculate from representation (4.14) the products of the operator $\overline{\mathsf{L}}$ with vectors $\boldsymbol{c}_i$, for example

$$\begin{aligned} \overline{\mathsf{L}}\boldsymbol{c}_2 &= \left(n_2^2\left(n_1^2+n_2^2+n_3^2\right) - \varepsilon_1\left(n_1^2+n_2^2\right) - \varepsilon_3\left(n_2^2+n_3^2\right) + \varepsilon_1\varepsilon_3\right)\boldsymbol{c}_2 \\ &\quad + n_2\left(\left(n_1^2+n_2^2+n_3^2\right)\left(n_1\boldsymbol{c}_1 + n_3\boldsymbol{c}_3\right) - \left(\varepsilon_3 n_1\boldsymbol{c}_1 + \varepsilon_1 n_3\boldsymbol{c}_3\right)\right). \end{aligned} \tag{4.17}$$

This is the sum of a vector proportional to vector $\boldsymbol{c}_2$ and a linear combination of vectors $\boldsymbol{c}_1$ and $\boldsymbol{c}_3$ in the plane perpendicular to $\boldsymbol{c}_2$. Divided by $[\mathsf{L}]$ according to (4.10) the operator $\overline{\mathsf{L}}$ is projection operators to the operator $\mathsf{L}$ in case of $|\mathsf{L}| = 0$ but $[\mathsf{L}] \neq 0$. Different choice of vectors for the determination of eigenvectors lead to more or less favorable representation of the eigenvectors. Many special cases could be discussed.

## 5. The Case of Optic axes $|\mathsf{L}| = 0$ and $\overline{\mathsf{L}} = \boldsymbol{0}$ with Real-Valued Refraction Vectors

We continue now the considerations to optic axes. Novel is that we integrate the vanishing $[\mathsf{L}] = 0$ of the second invariant of the operator $\mathsf{L}$ and take it into account also in the illustrations. Both surfaces together are represented in **Figure 2**. The surface $[\mathsf{L}] = 0$ is a two-shell surface of forth degree with no intersection in the real region and no rational factorization but the inner shell touches the dispersion surface $|\mathsf{L}| = 0$ at the points of self-intersections that means at the optic axes. This is better to see in pictures of intersection of both these surface in three-dimensional space of refraction vectors $\boldsymbol{n}$ with the plane $n_2 = 0$ and no intersections in the planes $n_3 = 0$ and $n_1 = 0$ in **Figure 3**.

Starting from Equation (4.9) we consider the special case $n_2 = 0$. This case leads to two real solutions for a circle and an ellipse with self-intersection and by multiplication of the separated equations with factors to the following two equations

$$\begin{aligned} n_2 = 0: \quad \Rightarrow \quad 0 &= \varepsilon_1\varepsilon_3\left(n_1^2 + n_3^2 - \varepsilon_2\right), \\ 0 &= \varepsilon_2\left(\varepsilon_1 n_1^2 + \varepsilon_3 n_3^2 - \varepsilon_1\varepsilon_3\right). \end{aligned} \tag{5.1}$$

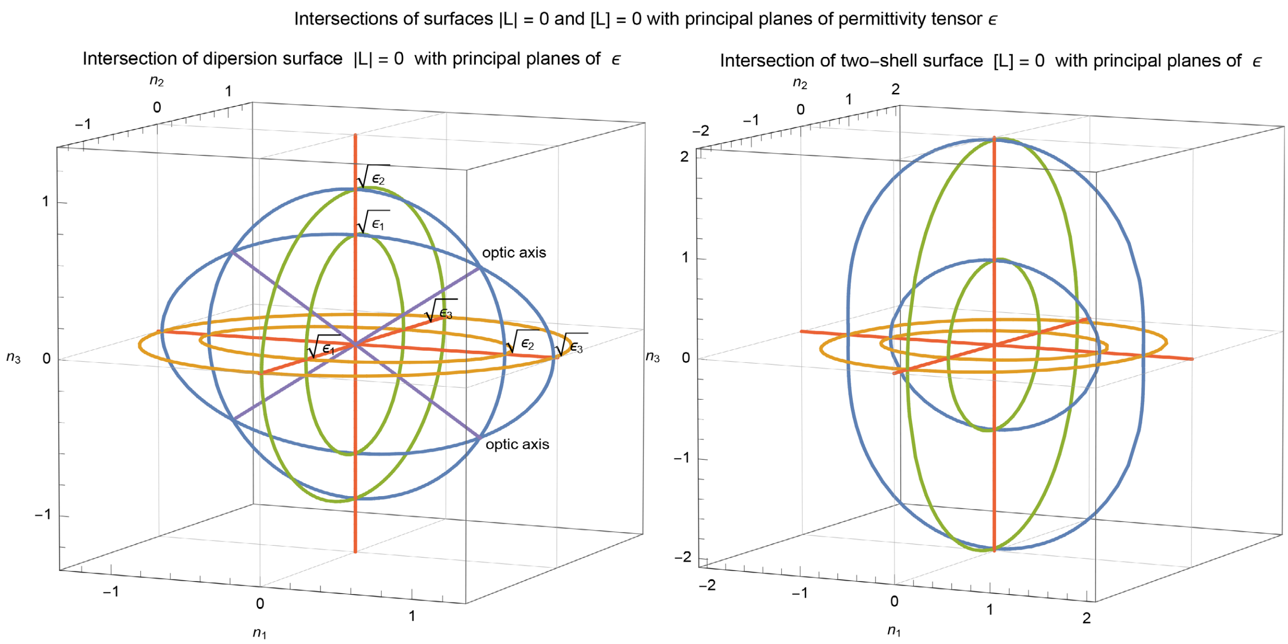


**Figure 2.** Intersections of dispersion surface $|\mathrm{L}|=0$ (blue, green, yellow) and the surface $[\mathrm{L}]=0$ with the principal planes of the permittivity tensor $\boldsymbol{\varepsilon}$. The optic axes are in the plane $n_2=0$. The surface $[\mathrm{L}]=0$ on right-hand picture possesses two shells without intersection but only the inner shell intersects the dispersion surface $|\mathrm{L}|=0$ at the optic axes. The chosen values for the Figures are $\varepsilon_1=0.5$, $\varepsilon_2=1.0$, $\varepsilon_3=1.75$, the same as in **Figure 1**. Pay attention that the scales in both partial pictures are different!

By forming the difference of these equations we find (agrees, e.g., with [2] (§99, Equation (99.5))

$$\varepsilon_1(\varepsilon_3-\varepsilon_2)n_1^2=\varepsilon_3(\varepsilon_2-\varepsilon_1)n_3^2. \tag{5.2}$$

Thus the components $n_1\equiv \boldsymbol{n}\boldsymbol{c}_1$ and $n_3\equiv \boldsymbol{n}\boldsymbol{c}_3$ are related to each other with 4 possible solutions. One can check by (4.9) that besides $|\mathrm{L}|=0$ for the points of intersection also the second invariant vanishes, *i.e.*, $[\mathrm{L}]=0$. In general, $n_3$ can take on complex values and then for $n_1$ result also complex values. The case (5.2) leads to optic axes which among others (see Section 11) are present for real refraction (and wave) vectors. Since this case is connected with a self-intersection of the dispersion surface in its neighborhood one has different non-degenerate polarizations of the two branches and the polarization for the optic axes becomes twofold degenerate. We investigate now the special case that $n_3$ and therefore also $n_1$ take on real values.

We denote the angle between the two possible solutions of $n_3$ for $n_1\geq 0$ of the corresponding refraction vectors $\boldsymbol{n}$ by $\alpha$ and have then the relations (easily calculable from $\mathrm{tg}\left(\frac{\alpha}{2}\right)$ but for convenience many are given; see **Figure 1** and **Figure 3**)

$$\mathrm{tg}\left(\frac{\alpha}{2}\right)\equiv\frac{n_3}{n_1}=\sqrt{\frac{\varepsilon_1(\varepsilon_3-\varepsilon_2)}{\varepsilon_3(\varepsilon_2-\varepsilon_1)}},\quad \mathrm{tg}\left(\frac{\pi-\alpha}{2}\right)=\frac{\cos\left(\frac{\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}=\sqrt{\frac{\varepsilon_3(\varepsilon_2-\varepsilon_1)}{\varepsilon_1(\varepsilon_3-\varepsilon_2)}}, \tag{5.3}$$

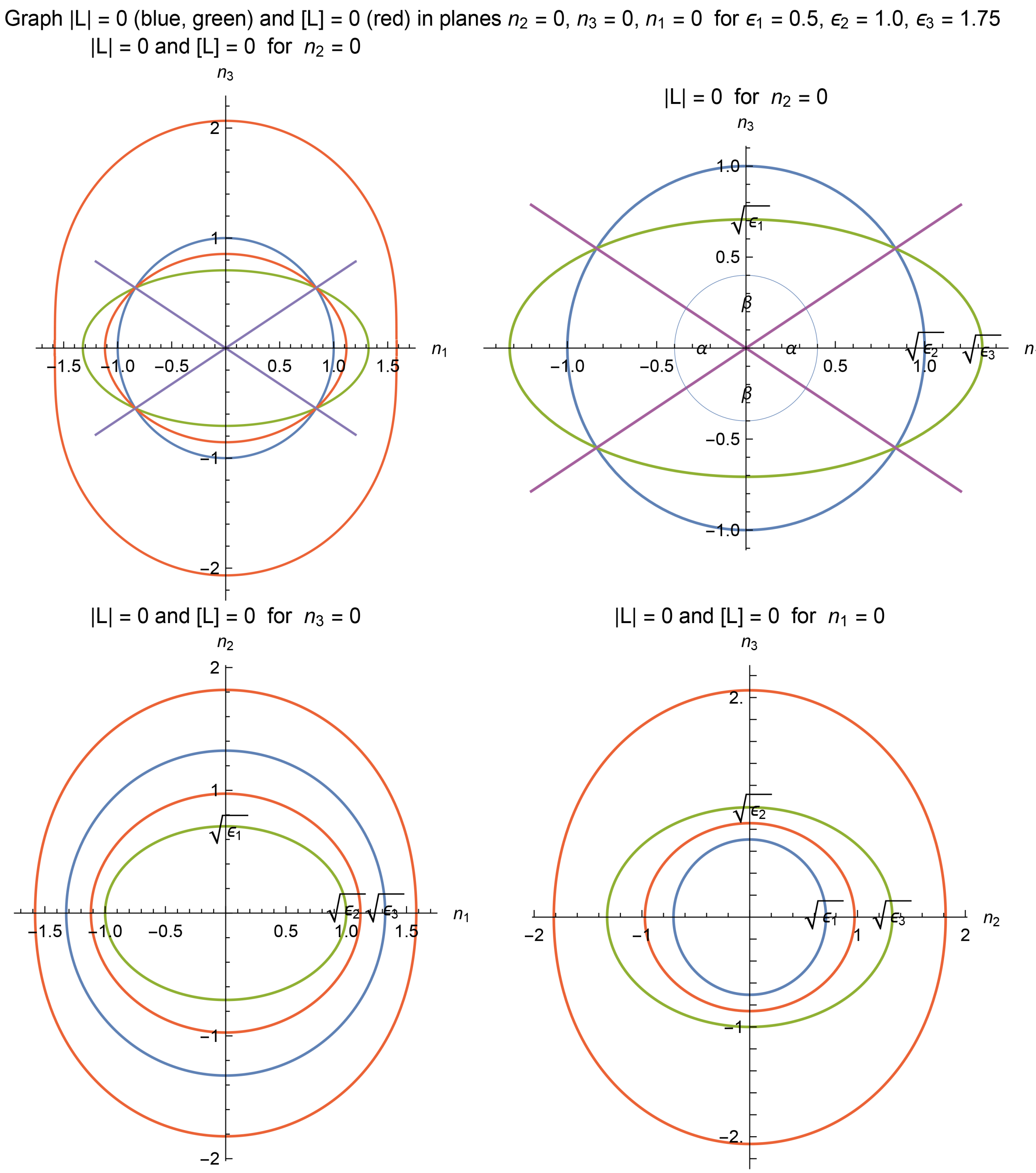


**Figure 3.** Intersections of dispersion surface $|\mathrm{L}|=0$ (blue, green) and the surface $[\mathrm{L}]=0$ (red) with the planes perpendicular to vectors $\boldsymbol{c}_2$ (upper curves) and $\boldsymbol{c}_3$ and $\boldsymbol{c}_1$ (lower curves). The dispersion curves consist in every case of these planes (but not in general) of a circle (blue) and an ellipse (green) whereas the curves $[\mathrm{L}]=0$ are two-sheet forth-order curves (red) without self-intersections as two-dimensional surfaces in the three-dimensional space for real wave vectors. In case of optic axes one sheet of the two-sheet curves intersects the dispersion curves for the optic axes exactly in the point of intersection of the dispersion curves. We have chosen $\varepsilon_1=0.5$, $\varepsilon_2=1.0$, $\varepsilon_3=1.75$ from which results $\alpha=66.4218^{\circ}$, $\beta=113.5782^{\circ}$.

or the Cosines and Sines, correspondingly

$$\begin{aligned}
&\cos\left(\frac{\alpha}{2}\right)=\frac{1}{\sqrt{1+\mathrm{tg}^2\left(\frac{\alpha}{2}\right)}}=\sqrt{\frac{\varepsilon_3\left(\varepsilon_2-\varepsilon_1\right)}{\varepsilon_2\left(\varepsilon_3-\varepsilon_1\right)}},\\
&\cos(\alpha)=\frac{\varepsilon_2\left(\varepsilon_1+\varepsilon_3\right)-2\varepsilon_1\varepsilon_3}{\varepsilon_2\left(\varepsilon_3-\varepsilon_1\right)}=-\cos(\beta),\\
&\sin\left(\frac{\alpha}{2}\right)=\frac{\mathrm{tg}\left(\frac{\alpha}{2}\right)}{\sqrt{1+\mathrm{tg}^2\left(\frac{\alpha}{2}\right)}}=\sqrt{\frac{\varepsilon_1\left(\varepsilon_3-\varepsilon_2\right)}{\varepsilon_2\left(\varepsilon_3-\varepsilon_1\right)}},\\
&\sin(\alpha)=\frac{2\sqrt{\varepsilon_1\varepsilon_3\left(\varepsilon_3-\varepsilon_2\right)\left(\varepsilon_2-\varepsilon_1\right)}}{\varepsilon_2\left(\varepsilon_3-\varepsilon_1\right)}=\sin(\beta),\\
&\beta\equiv\pi-\alpha,\quad \cos(\alpha)+\cos(\beta)=0,\\
&\sin(\alpha)-\sin(\beta)=0,\quad \cos(\alpha+\beta)=-1.
\end{aligned}\tag{5.4}$$

For the illustration in many figures in next Sections we choose the values $\varepsilon_1=0.5$, $\varepsilon_2=1.0$, $\varepsilon_3=1.75$ which are sufficiently different to show characteristic features but do not correspond to the majority of experimental values with smaller differences. For these values we obtain an angle $\alpha=66.4218^\circ$.

From the derived relations (e.g., (5.3)) together with $n_2=0$ in (4.9) follows for the optic axes

$$\begin{aligned}
&n_1\equiv \boldsymbol{n}\boldsymbol{c}_1=\sqrt{\frac{\varepsilon_3\left(\varepsilon_2-\varepsilon_1\right)}{\varepsilon_3-\varepsilon_1}},\quad n_3\equiv \boldsymbol{n}\boldsymbol{c}_3=\sqrt{\frac{\varepsilon_1\left(\varepsilon_3-\varepsilon_2\right)}{\varepsilon_3-\varepsilon_1}},\\
&n^2=\left(\boldsymbol{n}\boldsymbol{c}_1\right)^2+\left(\boldsymbol{n}\boldsymbol{c}_3\right)^2=\varepsilon_2,\quad \varepsilon_1\left(\boldsymbol{n}\boldsymbol{c}_1\right)^2+\varepsilon_3\left(\boldsymbol{n}\boldsymbol{c}_3\right)^2=\varepsilon_1\varepsilon_3.
\end{aligned}\tag{5.5}$$

By reason which become soon clear we make now a sidestep of our considerations by introduction of two unit vectors $\boldsymbol{c}_\pm$ in direction of the optic axes as follows

$$\boldsymbol{c}_\pm\equiv\sqrt{\frac{\frac{1}{\varepsilon_1}-\frac{1}{\varepsilon_2}}{\frac{1}{\varepsilon_1}-\frac{1}{\varepsilon_3}}}\boldsymbol{c}_1\pm\sqrt{\frac{\frac{1}{\varepsilon_2}-\frac{1}{\varepsilon_3}}{\frac{1}{\varepsilon_1}-\frac{1}{\varepsilon_3}}}\boldsymbol{c}_3=\sqrt{\frac{\varepsilon_3\left(\varepsilon_2-\varepsilon_1\right)}{\varepsilon_2\left(\varepsilon_3-\varepsilon_1\right)}}\boldsymbol{c}_1\pm\sqrt{\frac{\varepsilon_1\left(\varepsilon_3-\varepsilon_2\right)}{\varepsilon_2\left(\varepsilon_3-\varepsilon_1\right)}}\boldsymbol{c}_3,\tag{5.6}$$

with the inversion

$$\begin{aligned}
&\boldsymbol{c}_1=\sqrt{\frac{\frac{1}{\varepsilon_1}-\frac{1}{\varepsilon_3}}{\frac{1}{\varepsilon_1}-\frac{1}{\varepsilon_2}}}\frac{\boldsymbol{c}_++\boldsymbol{c}_-}{2}=\sqrt{\frac{\varepsilon_2\left(\varepsilon_3-\varepsilon_1\right)}{\varepsilon_3\left(\varepsilon_2-\varepsilon_1\right)}}\frac{\boldsymbol{c}_++\boldsymbol{c}_-}{2},\\
&\boldsymbol{c}_3=\sqrt{\frac{\frac{1}{\varepsilon_1}-\frac{1}{\varepsilon_3}}{\frac{1}{\varepsilon_2}-\frac{1}{\varepsilon_3}}}\frac{\boldsymbol{c}_+-\boldsymbol{c}_-}{2}=\sqrt{\frac{\varepsilon_2\left(\varepsilon_3-\varepsilon_1\right)}{\varepsilon_1\left(\varepsilon_3-\varepsilon_2\right)}}\frac{\boldsymbol{c}_+-\boldsymbol{c}_-}{2}.
\end{aligned}\tag{5.7}$$

Their normalization and scalar products are

$$\boldsymbol{c}_{\pm}^{2}=1,\quad \boldsymbol{c}_{+}\boldsymbol{c}_{-}=\frac{\varepsilon_{2}\left(\varepsilon_{1}+\varepsilon_{3}\right)-2\varepsilon_{1}\varepsilon_{3}}{\varepsilon_{2}\left(\varepsilon_{3}-\varepsilon_{1}\right)}=\cos(\alpha),\quad \boldsymbol{c}_{2}\boldsymbol{c}_{\pm}=0. \tag{5.8}$$

The representation of the symmetrical permittivity tensor $\boldsymbol{\varepsilon}$ in (4.2) and of its inverse $\boldsymbol{\varepsilon}^{-1}$ by the vectors $\boldsymbol{c}_{\pm}$ are

$$\begin{aligned}
\boldsymbol{\varepsilon}&=\varepsilon_{2}\mathsf{I}-\left(\varepsilon_{2}-\varepsilon_{1}\right)\boldsymbol{c}_{1}\cdot\boldsymbol{c}_{1}+\left(\varepsilon_{3}-\varepsilon_{2}\right)\boldsymbol{c}_{3}\cdot\boldsymbol{c}_{3}\\
&=\varepsilon_{2}\mathsf{I}-\frac{\varepsilon_{2}\left(\varepsilon_{3}-\varepsilon_{1}\right)}{4\varepsilon_{1}\varepsilon_{3}}\left\{\left(\varepsilon_{3}+\varepsilon_{1}\right)\left(\boldsymbol{c}_{+}\cdot\boldsymbol{c}_{-}+\boldsymbol{c}_{-}\cdot\boldsymbol{c}_{+}\right)-\left(\varepsilon_{3}-\varepsilon_{1}\right)\left(\boldsymbol{c}_{+}\cdot\boldsymbol{c}_{+}+\boldsymbol{c}_{-}\cdot\boldsymbol{c}_{-}\right)\right\},\\
\boldsymbol{\varepsilon}^{-1}&=\frac{1}{\varepsilon_{2}}\mathsf{I}+\left(\frac{1}{\varepsilon_{1}}-\frac{1}{\varepsilon_{2}}\right)\boldsymbol{c}_{1}\cdot\boldsymbol{c}_{1}-\left(\frac{1}{\varepsilon_{2}}-\frac{1}{\varepsilon_{3}}\right)\boldsymbol{c}_{3}\cdot\boldsymbol{c}_{3}\\
&=\frac{1}{\varepsilon_{2}}\mathsf{I}+\frac{1}{2}\left(\frac{1}{\varepsilon_{1}}-\frac{1}{\varepsilon_{3}}\right)\left(\boldsymbol{c}_{+}\cdot\boldsymbol{c}_{-}+\boldsymbol{c}_{-}\cdot\boldsymbol{c}_{+}\right).
\end{aligned} \tag{5.9}$$

We see that in these representations only the tensor $\boldsymbol{\varepsilon}^{-1}$ takes on a relatively simple form. This is a slightly varied variant of the method of Fyodorov [7] [8] where directly the optic axes of the refraction vectors are involved. The original method of Fyodorov of introduction of axes of a symmetric tensor we represent in Appendix B.

The entirety of possible refraction vectors $(\pm)\boldsymbol{n}_{\pm}$ in (or in opposite) direction of the optic axes is

$$\boldsymbol{n}_{\pm}^{(\pm)}=(\pm)\sqrt{\varepsilon_{2}}\boldsymbol{c}_{\pm}=(\pm)\frac{\sqrt{\varepsilon_{3}\left(\varepsilon_{2}-\varepsilon_{1}\right)}\boldsymbol{c}_{1}\pm\sqrt{\varepsilon_{1}\left(\varepsilon_{3}-\varepsilon_{2}\right)}\boldsymbol{c}_{3}}{\sqrt{\varepsilon_{3}-\varepsilon_{1}}}. \tag{5.10}$$

We omit now the trivial signs (±) and may calculate from (5.10) immediately

$$\begin{aligned}
\boldsymbol{n}_{\pm}&=\sqrt{\frac{\varepsilon_{3}\left(\varepsilon_{2}-\varepsilon_{1}\right)}{\varepsilon_{3}-\varepsilon_{1}}}\boldsymbol{c}_{1}\pm\sqrt{\frac{\varepsilon_{1}\left(\varepsilon_{3}-\varepsilon_{2}\right)}{\varepsilon_{3}-\varepsilon_{1}}}\boldsymbol{c}_{3},\quad \boldsymbol{n}_{\pm}\boldsymbol{n}_{\pm}=\varepsilon_{2},\\
\boldsymbol{\varepsilon}\boldsymbol{n}_{\pm}&=\sqrt{\frac{\varepsilon_{3}\left(\varepsilon_{2}-\varepsilon_{1}\right)}{\varepsilon_{3}-\varepsilon_{1}}}\varepsilon_{1}\boldsymbol{c}_{1}\pm\sqrt{\frac{\varepsilon_{1}\left(\varepsilon_{3}-\varepsilon_{2}\right)}{\varepsilon_{3}-\varepsilon_{1}}}\varepsilon_{3}\boldsymbol{c}_{3},\quad \boldsymbol{n}_{\pm}\boldsymbol{\varepsilon}\boldsymbol{n}_{\pm}=\varepsilon_{1}\varepsilon_{3},\\
\boldsymbol{\varepsilon}^{2}\boldsymbol{n}_{\pm}&=\sqrt{\frac{\varepsilon_{3}\left(\varepsilon_{2}-\varepsilon_{1}\right)}{\varepsilon_{3}^{2}-\varepsilon_{1}}}\varepsilon_{1}^{2}\boldsymbol{c}_{1}\pm\sqrt{\frac{\varepsilon_{1}\left(\varepsilon_{3}-\varepsilon_{2}\right)}{\varepsilon_{3}-\varepsilon_{1}}}\varepsilon_{3}^{2}\boldsymbol{c}_{3}\\
&=-\varepsilon_{1}\varepsilon_{3}\boldsymbol{n}_{\pm}+\left(\varepsilon_{1}+\varepsilon_{3}\right)\boldsymbol{\varepsilon}\boldsymbol{n}_{\pm},\quad \boldsymbol{n}_{\pm}\boldsymbol{\varepsilon}^{2}\boldsymbol{n}_{\pm}=\varepsilon_{1}\varepsilon_{3}\left(\varepsilon_{1}+\varepsilon_{3}-\varepsilon_{2}\right),
\end{aligned} \tag{5.11}$$

and furthermore

$$\boldsymbol{n}_{\pm}\boldsymbol{n}_{\mp}=\frac{\varepsilon_{2}\left(\varepsilon_{1}+\varepsilon_{3}\right)-2\varepsilon_{1}\varepsilon_{3}}{\varepsilon_{3}-\varepsilon_{1}},\quad \frac{\boldsymbol{n}_{\pm}\boldsymbol{n}_{\mp}}{\boldsymbol{n}_{\pm}\boldsymbol{n}_{\pm}}=\frac{\varepsilon_{2}\left(\varepsilon_{1}+\varepsilon_{3}\right)-2\varepsilon_{1}\varepsilon_{3}}{\varepsilon_{2}\left(\varepsilon_{3}-\varepsilon_{1}\right)}=\cos(\alpha). \tag{5.12}$$

The last expression is the Cosine of the angle between the optic axes (see (5.3)). Since the right-hand sides of the three vectors on the left-hand side in (5.11) are linear combinations of only two vectors $\boldsymbol{c}_{1}$ and $\boldsymbol{c}_{3}$ they are linearly dependent and the relation is given. From (5.11) follows

$$\begin{aligned}
&\left(\boldsymbol{n}_{\pm}\boldsymbol{n}_{\pm}\right)\left(\boldsymbol{n}_{\pm}\boldsymbol{\varepsilon}\boldsymbol{n}_{\pm}\right)-|\boldsymbol{\varepsilon}|=0,\\
&\left(\boldsymbol{n}_{\pm}\boldsymbol{n}_{\pm}\right)\left(\boldsymbol{n}_{\pm}\boldsymbol{\varepsilon}^{2}\boldsymbol{n}_{\pm}\right)-\left(\boldsymbol{n}_{\pm}\boldsymbol{\varepsilon}\boldsymbol{n}_{\pm}\right)^{2}=\varepsilon_{1}\varepsilon_{3}\left(\varepsilon_{3}-\varepsilon_{2}\right)\left(\varepsilon_{2}-\varepsilon_{1}\right).
\end{aligned} \tag{5.13}$$

For the following vector products we find

$$
\begin{aligned}
\left[\boldsymbol{n}_{\pm},\boldsymbol{\varepsilon}\boldsymbol{n}_{\pm}\right] &= \mp\sqrt{\varepsilon_1\varepsilon_3\left(\varepsilon_3-\varepsilon_2\right)\left(\varepsilon_2-\varepsilon_1\right)}\boldsymbol{c}_2, \\
\left[\boldsymbol{n}_{\pm},\boldsymbol{\varepsilon}^2\boldsymbol{n}_{\pm}\right] &= \mp\sqrt{\varepsilon_1\varepsilon_3\left(\varepsilon_3-\varepsilon_2\right)\left(\varepsilon_2-\varepsilon_1\right)}\left(\varepsilon_3+\varepsilon_1\right)\boldsymbol{c}_2, \\
\left[\boldsymbol{\varepsilon}\boldsymbol{n}_{\pm},\boldsymbol{\varepsilon}^2\boldsymbol{n}_{\pm}\right] &= \mp\sqrt{\varepsilon_1\varepsilon_3\left(\varepsilon_3-\varepsilon_2\right)\left(\varepsilon_2-\varepsilon_1\right)}\left(\varepsilon_3^2+\varepsilon_1\varepsilon_3+\varepsilon_1^2\right)\boldsymbol{c}_2 \\
&= \mp\sqrt{\varepsilon_1\varepsilon_3\left(\varepsilon_3-\varepsilon_2\right)\left(\varepsilon_2-\varepsilon_1\right)}\underbrace{\left(\left(\varepsilon_1+\varepsilon_3\right)^2-\varepsilon_1\varepsilon_3\right)}_{\geq 3\varepsilon_1\varepsilon_3}\boldsymbol{c}_2.
\end{aligned}
\tag{5.14}
$$

All these vector products are proportional to the vector $\boldsymbol{c}_2$ perpendicular to the plane spanned by the vectors $\boldsymbol{c}_1$ and $\boldsymbol{c}_3$. Since $\boldsymbol{n}_{\pm}$ lies in last mentioned plane the volume product of three vectors $\boldsymbol{n}_{\pm},\boldsymbol{\varepsilon}\boldsymbol{n}_{\pm},\boldsymbol{\varepsilon}^2\boldsymbol{n}_{\pm}$ is vanishing

$$
\left[\boldsymbol{n}_{\pm},\boldsymbol{\varepsilon}\boldsymbol{n}_{\pm},\boldsymbol{\varepsilon}^2\boldsymbol{n}_{\pm}\right]=0. \tag{5.15}
$$

This means that the three vectors $\boldsymbol{n}_{\pm},\boldsymbol{\varepsilon}\boldsymbol{n}_{\pm},\boldsymbol{\varepsilon}^2\boldsymbol{n}_{\pm}$ are in the plane perpendicular to vector $\boldsymbol{c}_2$

## 6. The Neighborhood $n=n_0+\delta n$ of Refraction Vectors to the Refraction Vector $n_0$ for an Optic Axis in the Cone Approximation

We investigate in this Section the dispersion surface in the neighborhood of an optic axis and denote the refraction vector to an optic axis by $\boldsymbol{n}_0$ and consider its neighborhood for small deviations $\boldsymbol{n}_0+\delta\boldsymbol{n}$ from $\boldsymbol{n}_0$. The vector $\boldsymbol{n}_0$ can be any of the 4 calculated vectors

$$
\boldsymbol{n}_{\pm}^{(\pm)}=(\pm)\sqrt{\varepsilon_2}\boldsymbol{c}_{\pm}, \tag{6.1}
$$

in (5.10). The quadratic forms given in (5.11) are unspecific which special vector $\boldsymbol{n}_0=\boldsymbol{n}_{\pm}^{(\pm)}$ is meant and are

$$
\boldsymbol{n}_0^2=\varepsilon_2,\quad \boldsymbol{n}_0\boldsymbol{\varepsilon}_0\boldsymbol{n}_0=\varepsilon_1\varepsilon_3,\quad \boldsymbol{n}_0\boldsymbol{\varepsilon}_0^2\boldsymbol{n}_0=\varepsilon_1\varepsilon_3\left(\varepsilon_1+\varepsilon_3-\varepsilon_2\right). \tag{6.2}
$$

and with $\boldsymbol{n}_0$ we will denote one of the two possible vectors $\boldsymbol{n}_{\pm}$ according to

$$
\boldsymbol{n}_0\equiv\sqrt{\varepsilon_2}\boldsymbol{c}_{\pm}, \tag{6.3}
$$

but do not change it in correlated calculations. Thus we have

$$
\begin{aligned}
\boldsymbol{n}_0 &= \sqrt{\frac{\varepsilon_3\left(\varepsilon_2-\varepsilon_1\right)}{\varepsilon_3-\varepsilon_1}}\boldsymbol{c}_1\pm\sqrt{\frac{\varepsilon_1\left(\varepsilon_3-\varepsilon_2\right)}{\varepsilon_3-\varepsilon_1}}\boldsymbol{c}_3, \\
\boldsymbol{\varepsilon}\boldsymbol{n}_0 &= \sqrt{\frac{\varepsilon_3\left(\varepsilon_2-\varepsilon_1\right)}{\varepsilon_3-\varepsilon_1}}\varepsilon_1\boldsymbol{c}_1\pm\sqrt{\frac{\varepsilon_1\left(\varepsilon_3-\varepsilon_2\right)}{\varepsilon_3-\varepsilon_1}}\varepsilon_3\boldsymbol{c}_3=\boldsymbol{n}_0\boldsymbol{\varepsilon}, \\
\boldsymbol{\varepsilon}^2\boldsymbol{n}_0 &= \sqrt{\frac{\varepsilon_3\left(\varepsilon_2-\varepsilon_1\right)}{\varepsilon_3^2-\varepsilon_1}}\varepsilon_1^2\boldsymbol{c}_1\pm\sqrt{\frac{\varepsilon_1\left(\varepsilon_3-\varepsilon_2\right)}{\varepsilon_3-\varepsilon_1}}\varepsilon_3^2\boldsymbol{c}_3=\boldsymbol{n}_0\boldsymbol{\varepsilon}^2.
\end{aligned}
\tag{6.4}
$$

For relations which are specific for the signs "±" in (6.4) such as for the vector product

$$
\left[\boldsymbol{n}_0,\boldsymbol{\varepsilon}\boldsymbol{n}_0\right]=\left[\boldsymbol{n}_0,\boldsymbol{n}_0\boldsymbol{\varepsilon}\right]=\mp\sqrt{\varepsilon_1\varepsilon_3\left(\varepsilon_3-\varepsilon_2\right)\left(\varepsilon_2-\varepsilon_1\right)}\boldsymbol{c}_2, \tag{6.5}
$$

we make the agreement to use the upper sign "±" in (6.4).

From this follows that the operator $\mathsf{L}_0 \equiv \mathsf{L}(\boldsymbol{n}_0)$ degenerates in direction of the optic axes to a direct product of operators as follows represented by

$$\mathsf{L}_0 \equiv \boldsymbol{n}_0 \cdot \boldsymbol{n}_0 - \boldsymbol{n}_0^2 \mathsf{I} + \boldsymbol{\varepsilon} = \left(-2\boldsymbol{n}_0^2 + \langle \boldsymbol{\varepsilon} \rangle\right) \frac{\boldsymbol{\varepsilon}\boldsymbol{n}_0 \cdot \boldsymbol{n}_0 \boldsymbol{\varepsilon}}{\boldsymbol{n}_0 \boldsymbol{\varepsilon}^2 \boldsymbol{n}_0} = \left(\varepsilon_1 + \varepsilon_3 - \varepsilon_2\right) \frac{\boldsymbol{\varepsilon}\boldsymbol{n}_0 \cdot \boldsymbol{n}_0 \boldsymbol{\varepsilon}}{\boldsymbol{n}_0 \boldsymbol{\varepsilon}^2 \boldsymbol{n}_0},$$
$$\Rightarrow \quad \overline{\mathsf{L}_0} \equiv \left(\mathsf{L}_0\right)^2 - \langle \mathsf{L}_0 \rangle \mathsf{L} = 0, \quad \Rightarrow \quad \left[\mathsf{L}_0\right] = 0. \tag{6.6}$$

In direction of an optic axis the eigenvectors of $\mathsf{L}_0$ to the refraction $\boldsymbol{n}_0$ (for the electric field) are not uniquely determined due to $\overline{\mathsf{L}_0} = 0$ and $\left[\mathsf{L}_0\right] = 0$.

The two-dimensional projection operator $\Pi_0$ for the determination of right-hand and left-hand eigenvectors $\boldsymbol{e}_0$ of $\mathsf{L}_0$ to the twofold degenerate eigenvalue $\lambda = 0$ is

$$\Pi_0' \equiv \mathsf{I} - \frac{\mathsf{L}_0}{\langle \mathsf{L}_0 \rangle} = \mathsf{I} - \frac{\boldsymbol{\varepsilon}\boldsymbol{n}_0 \cdot \boldsymbol{n}_0 \boldsymbol{\varepsilon}}{\boldsymbol{n}_0 \boldsymbol{\varepsilon}^2 \boldsymbol{n}_0}, \quad \Rightarrow \quad \Pi_0'^2 = \Pi_0', \quad \langle \Pi_0' \rangle = 2, \tag{6.7}$$

from which follows

$$\Pi_0' \boldsymbol{\varepsilon} \boldsymbol{n}_0 = \boldsymbol{n}_0 \boldsymbol{\varepsilon} \Pi_0' = \boldsymbol{0}. \tag{6.8}$$

All vectors $\Pi_0' \boldsymbol{a} \neq \boldsymbol{0}$ are proportional to possible solutions $\boldsymbol{e}_0$ of Equation (3.1) and they span the plane perpendicular to the vector $\boldsymbol{n}_0 \boldsymbol{\varepsilon}_0$ and thus we have two-fold degeneration of the possible polarization. Both last relations were already derived in Section 3 in more general form.

Thus the transition to optic axes

$$\boldsymbol{n} \rightarrow \boldsymbol{n}_0, \quad \mathsf{L}(\boldsymbol{n}) \rightarrow \mathsf{L}(\boldsymbol{n}_0) \equiv \mathsf{L}_0, \tag{6.9}$$

is connected with the conditions

$$\left|\mathsf{L}_0\right| = 0, \quad \overline{\mathsf{L}_0} = 0. \tag{6.10}$$

We have to investigate now the dispersion surface $\mathsf{L}(\boldsymbol{n}) = 0$ and the complementary operator $\overline{\mathsf{L}}(\boldsymbol{n})$ to operator $\mathsf{L}(\boldsymbol{n})$ in a small neighborhood $\boldsymbol{n} = \boldsymbol{n}_0 + \delta\boldsymbol{n}$ of $\boldsymbol{n}_0$ and find in the first two terms of a Taylor series

$$\mathsf{L}(\boldsymbol{n}_0 + \delta\boldsymbol{n}) = \mathsf{L}_0 + \left(\frac{\partial \mathsf{L}}{\partial n_k}\right)_0 \delta n_k + \frac{1}{2}\left(\frac{\partial^2 \mathsf{L}}{\partial n_k \partial n_l}\right)_0 \delta n_k \delta n_l + \cdots. \tag{6.11}$$

Two not very simple problems wait for us to solve.

Our first problem which we now begin to investigate is to find from vanishing of the determinant $\left|\mathsf{L}(\boldsymbol{n}_0 + \delta\boldsymbol{n})\right|$ an approximate solution for the possible refraction vectors $\boldsymbol{n}_0 + \delta\boldsymbol{n}$ in the neighborhood of an optic axis $\boldsymbol{n}_0$. This will become a conical approximation for $\delta\boldsymbol{n}$. The second problem is to determine an approximation of the non-degenerate eigenvectors in the neighborhood of an optic axis that we solve in a later Section.

For the determinant of the sum of two three-dimensional operators $\mathsf{A}$ and $\mathsf{B}$ one finds the general formula

$$\left|\mathsf{A} + \mathsf{B}\right| = \left|\mathsf{A}\right| + \langle \overline{\mathsf{A}} \mathsf{B} \rangle + \langle \mathsf{A} \overline{\mathsf{B}} \rangle + \left|\mathsf{B}\right|, \quad \left(\overline{\mathsf{C}} \equiv [\mathsf{C}]\mathsf{I} - \langle \mathsf{C} \rangle \mathsf{C} + \mathsf{C}^2\right), \tag{6.12}$$

with $\overline{\mathsf{A}}$ and $\overline{\mathsf{B}}$ the complementary operators to $\mathsf{A}$ and $\mathsf{B}$, respectively. This formula is specific for three-dimensional operators and has to be generalized for higher dimension. We apply this formula for

$$\begin{aligned} \mathsf{A} &\equiv \mathsf{L}_0 = \boldsymbol{n}_0 \cdot \boldsymbol{n}_0 - \left(\boldsymbol{n}_0^2\right)\mathsf{I} + \boldsymbol{\varepsilon}, \\ \mathsf{B} &\equiv \left(\frac{\partial \mathsf{L}}{\partial n_k}\right)_0 \delta n_k = \boldsymbol{n}_0 \cdot \delta\boldsymbol{n} + \delta\boldsymbol{n} \cdot \boldsymbol{n}_0 - 2\left(\boldsymbol{n}_0 \delta\boldsymbol{n}\right)\mathsf{I}. \end{aligned} \tag{6.13}$$

Since determinant $|\mathsf{L}_0|$ and complementary operator $\overline{\mathsf{L}_0}$ of $\mathsf{L}_0$ vanish according to (6.10) we have to take into account only the last two terms $\langle \mathsf{A}\overline{\mathsf{B}}\rangle$ and $|\mathsf{B}|$ in (6.12). The term $|\mathsf{B}|$ is already in third order of $\delta\boldsymbol{n}$ and can be also neglected because we want to take into account only the first non-vanishing addition to the dispersion surface which is of second order in $\delta\boldsymbol{n}$. For the same reason one may also neglect the terms from the second and higher-order derivatives of $\mathsf{L}$ in (6.11) and find

$$0 = \left|\mathsf{L}\left(\boldsymbol{n}_0 + \delta\boldsymbol{n}\right)\right| = \left|\mathsf{L}_0 + \left(\frac{\partial \mathsf{L}}{\partial n_k}\right)_0 \delta n_k + \cdots\right| = \left|\mathsf{L}_0\right| + \left\langle \mathsf{L}_0 \overline{\left(\frac{\partial \mathsf{L}}{\partial n_k}\right)_0 \delta n_k} \right\rangle + \cdots, \tag{6.14}$$

The written term on the right-hand side is the only additional term which has to be taken into account in second-order with respect to $\delta\boldsymbol{n}$ (conical approximation). Thus we have now to calculate the complementary operator $\overline{\mathsf{B}}$ that is made for the principal structure in Appendix A. With the substitutions $\boldsymbol{x} \equiv \boldsymbol{n}_0$ and $\boldsymbol{y} \equiv \delta\boldsymbol{n}$ there one finds

$$\begin{aligned} \overline{\mathsf{B}} &= \overline{\left(\frac{\partial \mathsf{L}}{\partial n_k}\right)_0 \delta n_k} = \left(\boldsymbol{n}_0 \cdot \delta\boldsymbol{n} + \delta\boldsymbol{n} \cdot \boldsymbol{n}_0\right)^2 - \left[\boldsymbol{n}_0, \delta\boldsymbol{n}\right]^2 \mathsf{I} \\ &= \left(\boldsymbol{n}_0 \delta\boldsymbol{n}\right)\left(\boldsymbol{n}_0 \cdot \delta\boldsymbol{n} + \delta\boldsymbol{n} \cdot \boldsymbol{n}_0\right) + \left(\boldsymbol{n}_0^2\right)\delta\boldsymbol{n} \cdot \delta\boldsymbol{n} + \left(\delta\boldsymbol{n}^2\right)\boldsymbol{n}_0 \cdot \boldsymbol{n}_0 \\ &\quad - \left(\left(\boldsymbol{n}_0^2\right)\left(\delta\boldsymbol{n}^2\right) - \left(\boldsymbol{n}_0 \delta\boldsymbol{n}\right)^2\right)\mathsf{I}, \end{aligned} \tag{6.15}$$

from which follows

$$\begin{aligned} \left\langle \mathsf{L}_0 \overline{\left(\frac{\partial \mathsf{L}}{\partial n_k}\right)_0 \delta n_k} \right\rangle &= \left(\boldsymbol{n}_0^2 - \langle\boldsymbol{\varepsilon}\rangle\right)\left(\left(\boldsymbol{n}_0^2\right)\left(\delta\boldsymbol{n}^2\right) - \left(\boldsymbol{n}_0 \delta\boldsymbol{n}\right)^2\right) + \left(\boldsymbol{n}_0 \delta\boldsymbol{n}\right)\left(\delta\boldsymbol{n}\boldsymbol{\varepsilon}\boldsymbol{n}_0\right) \\ &\quad + \boldsymbol{n}_0 \boldsymbol{\varepsilon} \delta\boldsymbol{n} + \left(\boldsymbol{n}_0^2\right)\left(\delta\boldsymbol{n}\boldsymbol{\varepsilon}\delta\boldsymbol{n}\right) + \left(\boldsymbol{n}_0 \boldsymbol{\varepsilon} \boldsymbol{n}_0\right)\left(\delta\boldsymbol{n}^2\right) \\ &\equiv \delta\boldsymbol{n} \mathsf{C}_0 \delta\boldsymbol{n}, \end{aligned} \tag{6.16}$$

with the abbreviation $\mathsf{C}_0$ for the following symmetric second-rank tensor

$$\mathsf{C}_0 \equiv \left(\boldsymbol{n}_0^2 - \langle\boldsymbol{\varepsilon}\rangle\right)\left(\boldsymbol{n}_0^2 \mathsf{I} - \boldsymbol{n}_0 \cdot \boldsymbol{n}_0\right) + \boldsymbol{\varepsilon}\boldsymbol{n}_0 \cdot \boldsymbol{n}_0 + \boldsymbol{n}_0 \cdot \boldsymbol{n}_0 \boldsymbol{\varepsilon} + \left(\boldsymbol{n}_0^2\right)\boldsymbol{\varepsilon} + \left(\boldsymbol{n}_0 \boldsymbol{\varepsilon} \boldsymbol{n}_0\right)\mathsf{I}. \tag{6.17}$$

The index 0 at $\mathsf{C}_0$ means that for the vector $\boldsymbol{n}$ is inserted one of the four possible solutions $\boldsymbol{n}_\pm^{(\pm)}$ from (6.3) which can be reduced to the two given solutions in (6.3) since they are involved in $\mathsf{C}_0$ only quadratically.

The homogeneous equation with second-order terms in $\delta\boldsymbol{n}$

$$0 = \delta\boldsymbol{n} \mathsf{C}_0 \delta\boldsymbol{n} = \left(\boldsymbol{n} - \boldsymbol{n}_0\right)\mathsf{C}_0\left(\boldsymbol{n} - \boldsymbol{n}_0\right), \tag{6.18}$$

describes a cone in three-dimensional space and their solutions $\delta\boldsymbol{n}$ provide the approximation of the dispersion surface $|\mathsf{L}| = 0$ in direction to the chosen refraction vector $\boldsymbol{n}_0$ of the optic axis. This means that the whole solution for refraction vectors $\boldsymbol{n}$ in the neighborhood of a refraction vector $\boldsymbol{n}_0$ for the chosen optic axis is

$$\boldsymbol{n} = \boldsymbol{n}_0 + \delta\boldsymbol{n}, \tag{6.19}$$

where $\delta\boldsymbol{n}$ is one of the possible solutions of the cone Equation (6.3). This is represented in Figure 4.

Due to the operator and scalar identities for optic axes (see (6.6)) we have

$$\begin{aligned} \overline{\mathsf{L}_0} &= \left(\boldsymbol{n}_0^2 - \langle\boldsymbol{\varepsilon}\rangle\right)\boldsymbol{n}_0 \cdot \boldsymbol{n}_0 + \boldsymbol{\varepsilon}\boldsymbol{n}_0 \cdot \boldsymbol{n}_0 + \boldsymbol{n}_0 \cdot \boldsymbol{n}_0\boldsymbol{\varepsilon} - \left(\boldsymbol{n}_0\boldsymbol{\varepsilon}\boldsymbol{n}_0\right)\mathsf{I} + \overline{\boldsymbol{\varepsilon}} = 0, \\ \left[\mathsf{L}_0\right] &= \left(\boldsymbol{n}_0^2 - \langle\boldsymbol{\varepsilon}\rangle\right)\boldsymbol{n}_0^2 - \boldsymbol{n}_0\boldsymbol{\varepsilon}\boldsymbol{n}_0 + [\boldsymbol{\varepsilon}] = 0, \end{aligned} \tag{6.20}$$

The cone tensor $\mathsf{C}_0$ is equivalent to other forms of this tensor. A good (likely the best) possibility is to combine (6.20) to the identity

$$\left[\mathsf{L}_0\right]\mathsf{I} - \overline{\mathsf{L}_0} = \left(\boldsymbol{n}_0^2 - \langle\boldsymbol{\varepsilon}\rangle\right)\left(\boldsymbol{n}_0^2\mathsf{I} - \boldsymbol{n}_0 \cdot \boldsymbol{n}_0\right) - \boldsymbol{\varepsilon}\boldsymbol{n}_0 \cdot \boldsymbol{n}_0 - \boldsymbol{n}_0 \cdot \boldsymbol{n}_0\boldsymbol{\varepsilon} + [\boldsymbol{\varepsilon}]\mathsf{I} - \overline{\boldsymbol{\varepsilon}} = 0, \tag{6.21}$$

and to subtract it from $\mathsf{C}_0$ in (6.17). Then one obtains the equivalent tensor

$$\mathsf{C}_0 = 2\left(\boldsymbol{\varepsilon}\boldsymbol{n}_0 \cdot \boldsymbol{n}_0 + \boldsymbol{n}_0 \cdot \boldsymbol{n}_0\boldsymbol{\varepsilon}\right) + \left(\boldsymbol{n}_0^2\right)\boldsymbol{\varepsilon} + \left(\boldsymbol{n}_0\boldsymbol{\varepsilon}\boldsymbol{n}_0\right)\mathsf{I} - \langle\boldsymbol{\varepsilon}\rangle\boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}^2. \tag{6.22}$$

In comparison to (6.17) with 8 sum terms in biquadratic degree of $\boldsymbol{n}_0$ the form (6.18) seems to be simpler with only 5 sum terms in quadratic degree of $\boldsymbol{n}_0$ but this is not fully true since in calculating the powers of $\mathsf{C}_0$ from (6.17) some sum terms vanish due to $\left(\boldsymbol{n}_0^2\mathsf{I} - \boldsymbol{n}_0 \cdot \boldsymbol{n}_0\right)\boldsymbol{n}_0 = \boldsymbol{n}_0\left(\boldsymbol{n}_0^2\mathsf{I} - \boldsymbol{n}_0 \cdot \boldsymbol{n}_0\right) = \mathbf{0}$.

Our next problem is to go more into the details of the structure of the second-rank symmetric tensor $\mathsf{C}_0$ (can be also seen as operator) and to calculate its eigenvalues and eigenvectors that we make in next Section.

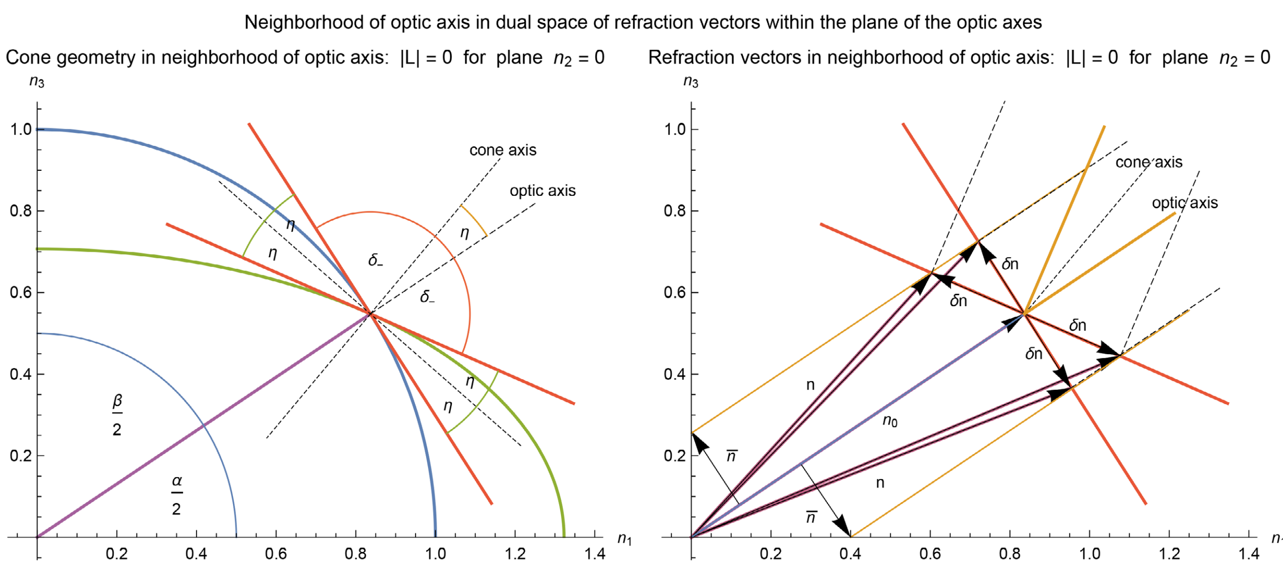


**Figure 4.** The geometry of coupled refraction vectors $\boldsymbol{n} = \boldsymbol{n}_0 + \delta\boldsymbol{n}$ in the neighborhood of optic axes in cone approximation for plane $n_2 = 0$. The tangential components $\bar{\boldsymbol{n}}$ of coupled wave vectors are the same. In the right-hand figure two cases of tangential components $\bar{\boldsymbol{n}}$ to refraction vectors $\boldsymbol{n}$ near the optic axis are drawn. In general, each possible tangential component $\bar{\boldsymbol{n}}$ goes into the equations with a certain weight. The ray vectors which are perpendicular to the cone surface, properly speaking, do not belong to this picture and it cannot be generally said at which points in real space they have their origin in applications (for example at the inside to a boundary plane).

## 7. Eigenvalues and Eigenvectors of the Cone tensor $\mathsf{C}_0$

The cone tensor $\mathsf{C}_0$ was determined in two forms (6.17) and (6.22) which are equivalent if one takes into account the identity (6.21). We calculate now first its eigenvalues and then its eigenvectors. For this purpose one has first to calculate the powers $\mathsf{C}_0^2$ and $\mathsf{C}_0^3$ that we do not write down for their many terms and then their traces. However, we made a special approach. Since the vectors $(\boldsymbol{n}_0, \boldsymbol{\varepsilon n}_0, \boldsymbol{\varepsilon n}_0)$ are in the plane spanned by the vectors $\boldsymbol{c}_1$ and $\boldsymbol{c}_3$ (see (5.11)) one eigenvalue of $\mathsf{C}_0$ and the corresponding (right- and left-hand) eigenvector proportional to $\boldsymbol{c}_2$ is easily to obtain directly from (6.22) with the result

$$\mathsf{C}_0\boldsymbol{c}_2 = \boldsymbol{c}_2\mathsf{C}_0 = -(\varepsilon_3-\varepsilon_2)(\varepsilon_2-\varepsilon_1)\boldsymbol{c}_2 \equiv \gamma_2\boldsymbol{c}_2. \tag{7.1}$$

Therefore we needed only the traces of $\mathsf{C}_0$ and $\mathsf{C}_0^2$ and could later also find the trace of $\mathsf{C}_0^3$ by $\langle \mathsf{C}_0^3\rangle = \langle \mathsf{C}_0\rangle^3 + 3\left(|\mathsf{C}_0| - \langle \mathsf{C}_0\rangle[\mathsf{C}_0]\right)$ which altogether are[4]

$$\begin{aligned}
\langle \mathsf{C}_0\rangle &= 4\varepsilon_1\varepsilon_3 - (\varepsilon_3-\varepsilon_2)(\varepsilon_2-\varepsilon_1),\\
\langle \mathsf{C}_0^2\rangle &= \left(4\varepsilon_1\varepsilon_3 + (\varepsilon_3-\varepsilon_2)(\varepsilon_2-\varepsilon_1)\right)^2,\\
\langle \mathsf{C}_0^3\rangle &= (4\varepsilon_1\varepsilon_3)^3 + 3(4\varepsilon_1\varepsilon_3)^2(\varepsilon_3-\varepsilon_2)(\varepsilon_2-\varepsilon_1) - \left((\varepsilon_3-\varepsilon_2)(\varepsilon_2-\varepsilon_1)\right)^3.
\end{aligned} \tag{7.2}$$

Using the general definitions of operator invariants (2.3) this is equivalent to

$$\begin{aligned}
\langle \mathsf{C}_0\rangle &= 4\varepsilon_1\varepsilon_3 - (\varepsilon_3-\varepsilon_2)(\varepsilon_2-\varepsilon_1),\\
[\mathsf{C}_0] &= -8\varepsilon_1\varepsilon_3(\varepsilon_3-\varepsilon_2)(\varepsilon_2-\varepsilon_1),\\
|\mathsf{C}_0| &= 4\varepsilon_1\varepsilon_3\left((\varepsilon_3-\varepsilon_2)(\varepsilon_2-\varepsilon_1)\right)^2.
\end{aligned} \tag{7.3}$$

From the equation for the eigenvalues of $\mathsf{C}_0$

$$\gamma^3 - \langle \mathsf{C}_0\rangle\gamma^2 + [\mathsf{C}_0]\gamma - |\mathsf{C}_0| = 0, \tag{7.4}$$

we obtained the following three eigenvalues

$$\begin{aligned}
\gamma_+ &= 2\left(\varepsilon_1\varepsilon_3 + \sqrt{\varepsilon_1\varepsilon_3\varepsilon_2(\varepsilon_1+\varepsilon_3-\varepsilon_2)}\right) \ge 0,\\
\gamma_- &= 2\left(\varepsilon_1\varepsilon_3 - \sqrt{\varepsilon_1\varepsilon_3\varepsilon_2(\varepsilon_1+\varepsilon_3-\varepsilon_2)}\right) \le 0,\\
\gamma_2 &= -(\varepsilon_3-\varepsilon_2)(\varepsilon_2-\varepsilon_1) = 2\varepsilon_1\varepsilon_3 - \left(\varepsilon_1\varepsilon_3 + \varepsilon_2(\varepsilon_1+\varepsilon_3-\varepsilon_2)\right) \le 0.
\end{aligned} \tag{7.5}$$

The half difference between $\gamma_-$ and $\gamma_2$ is

$$\frac{\gamma_- - \gamma_2}{2} = \frac{\varepsilon_1\varepsilon_3 + \varepsilon_2(\varepsilon_1+\varepsilon_3-\varepsilon_2)}{2} - \sqrt{\varepsilon_1\varepsilon_3\varepsilon_2(\varepsilon_1+\varepsilon_3-\varepsilon_2)} \ge 0. \tag{7.6}$$

Under the assumption (4.4) that the principal values $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ are non-negative and are ordered we find the inequalities

$$-(\varepsilon_3-\varepsilon_2)(\varepsilon_2-\varepsilon_1) = \gamma_2 \le \gamma_- \le 0 \le \gamma_+ \le 4\varepsilon_1\varepsilon_3 + (\varepsilon_3-\varepsilon_2)(\varepsilon_2-\varepsilon_1), \tag{7.7}$$

and one eigenvalue $\gamma_+$ is non-negative and two eigenvalues $(\gamma_-, \gamma_2)$ are non-positive. This follows from the inequality between arithmetic and geometric

[4]It is easy to check that the trace $\langle \mathsf{C}_0\rangle$ obtained from (6.17) or (6.22) using (6.2) agrees with the form given here.

mean in the two special forms

$$\frac{\varepsilon_1+\varepsilon_3}{2}=\frac{\varepsilon_2+(\varepsilon_1+\varepsilon_3-\varepsilon_2)}{2}\geq\sqrt{\varepsilon_2(\varepsilon_1+\varepsilon_3-\varepsilon_2)},$$
$$\frac{\varepsilon_1\varepsilon_3+\varepsilon_2(\varepsilon_1+\varepsilon_3-\varepsilon_2)}{2}\geq\sqrt{\varepsilon_1\varepsilon_3\varepsilon_2(\varepsilon_1+\varepsilon_3-\varepsilon_2)}. \tag{7.8}$$

For $0<\varepsilon_1\neq\varepsilon_2\neq\varepsilon_3$ all three eigenvalues (7.5) are genuinely different.

Knowing the eigenvalues $\gamma$ it is possible to determine the eigenvectors $\boldsymbol{c}$ directly by the operator equation, in our case from $\mathsf{C}_0\boldsymbol{c}=\gamma\boldsymbol{c}$. We give them at once in the form of projection operators $\Pi_\pm$ and $\Pi_2$ with the result (the prime at $\boldsymbol{c}'_\pm$ is written because we have already defined different vectors $\boldsymbol{c}_\pm$ in (5.6) as unit vectors in direction of the optic axes)

$$\Pi_\pm=\frac{\left(\sqrt{\varepsilon_1\varepsilon_3(\varepsilon_1+\varepsilon_3-\varepsilon_2)}\boldsymbol{n}_0\pm\sqrt{\varepsilon_2}\boldsymbol{\varepsilon}\boldsymbol{n}_0\right)\cdot\left(\sqrt{\varepsilon_1\varepsilon_3(\varepsilon_1+\varepsilon_3-\varepsilon_2)}\boldsymbol{n}_0\pm\sqrt{\varepsilon_2}\boldsymbol{n}_0\boldsymbol{\varepsilon}\right)}{2\varepsilon_1\varepsilon_3\sqrt{\varepsilon_2(\varepsilon_1+\varepsilon_3-\varepsilon_2)}\left(\sqrt{\varepsilon_2(\varepsilon_1+\varepsilon_3-\varepsilon_2)}\pm\sqrt{\varepsilon_1\varepsilon_3}\right)}$$
$$\equiv\boldsymbol{c}'_\pm\cdot\boldsymbol{c}'_\pm \tag{7.9}$$
$$\Pi_2=\frac{[\boldsymbol{n}_0,\boldsymbol{\varepsilon}\boldsymbol{n}_0]\cdot[\boldsymbol{n}_0,\boldsymbol{n}_0\boldsymbol{\varepsilon}]}{\varepsilon_1\varepsilon_3(\varepsilon_3-\varepsilon_2)(\varepsilon_2-\varepsilon_1)}=\boldsymbol{c}_2\cdot\boldsymbol{c}_2,\quad \boldsymbol{c}'_\pm\boldsymbol{c}'_\pm=\boldsymbol{c}_2^2=1,\quad \boldsymbol{c}'_+\boldsymbol{c}'_-=\boldsymbol{c}_2\boldsymbol{c}'_\pm=0,$$

with a form of the Cauchy-Schwarz-Bunyakovski inequality

$$[\boldsymbol{n}_0,\boldsymbol{n}_0\boldsymbol{\varepsilon}][\boldsymbol{n}_0,\boldsymbol{\varepsilon}\boldsymbol{n}_0]=(\boldsymbol{n}_0^2)\boldsymbol{n}_0\boldsymbol{\varepsilon}^2\boldsymbol{n}_0-(\boldsymbol{n}_0\boldsymbol{\varepsilon}\boldsymbol{n}_0)^2=\varepsilon_1\varepsilon_3(\varepsilon_3-\varepsilon_2)(\varepsilon_2-\varepsilon_1)\geq 0. \tag{7.10}$$

The operators $\Pi_\pm$ and $\Pi_2$ as projection operators to determine eigenvectors in non-degenerate case satisfy the following relations

$$\Pi_\pm^2=\Pi_\pm,\quad \Pi_2^2=\Pi_2,\quad \Pi_\pm\Pi_2=\Pi_+\Pi_-=0,$$
$$\langle\Pi_\pm\rangle=\langle\Pi_2\rangle=1,\quad \Pi_++\Pi_-+\Pi_2=\mathsf{I}. \tag{7.11}$$

The cone tensor $\mathsf{C}_0$ can now be represented by

$$\mathsf{C}_0=\gamma_+\Pi_++\gamma_-\Pi_{-1}+\gamma_2\Pi_2. \tag{7.12}$$

The combinations of permittivities in (7.5) and (7.9) can be also obtained and expressed in simple way by the quadratic forms $\boldsymbol{n}_0^2,\boldsymbol{n}_0\boldsymbol{\varepsilon}\boldsymbol{n}_0,\boldsymbol{n}_0\boldsymbol{\varepsilon}^2\boldsymbol{n}_0$ and some inequalities appear then as shown in (7.10) as the Cauchy-Schwarz-Bunyakovski inequality.

The cone approximation in combination with a part of the dispersion surface $|\mathsf{L}|=0$ at an optic axis is illustrated in **Figure 5**. In the left-hand partial picture one can see only the external part of the double cone. However, it is hardly to see that this cone is an elliptic cone since the chosen parameters ($\varepsilon_1=0.5$, $\varepsilon_2=0$, $\varepsilon_3=1.75$) are not extremely enough different. In the right-hand partial picture an intersection with the plane $n_2=0$ spanned by both optic axes is made and one may see also the internal part of the double cone and that the optic axes forms a "small" angle with the cone axis.

The complete approximate solutions for refraction vectors in the neighborhood of an optic axis is $\boldsymbol{n}=\boldsymbol{n}_0+\delta\boldsymbol{n}$. We now make for $\delta\boldsymbol{n}$ the proposition

$$\delta\boldsymbol{n}=\sigma(\delta n_+\boldsymbol{c}'_++\delta n_-\boldsymbol{c}'_-+\delta n_2\boldsymbol{c}_2). \tag{7.13}$$

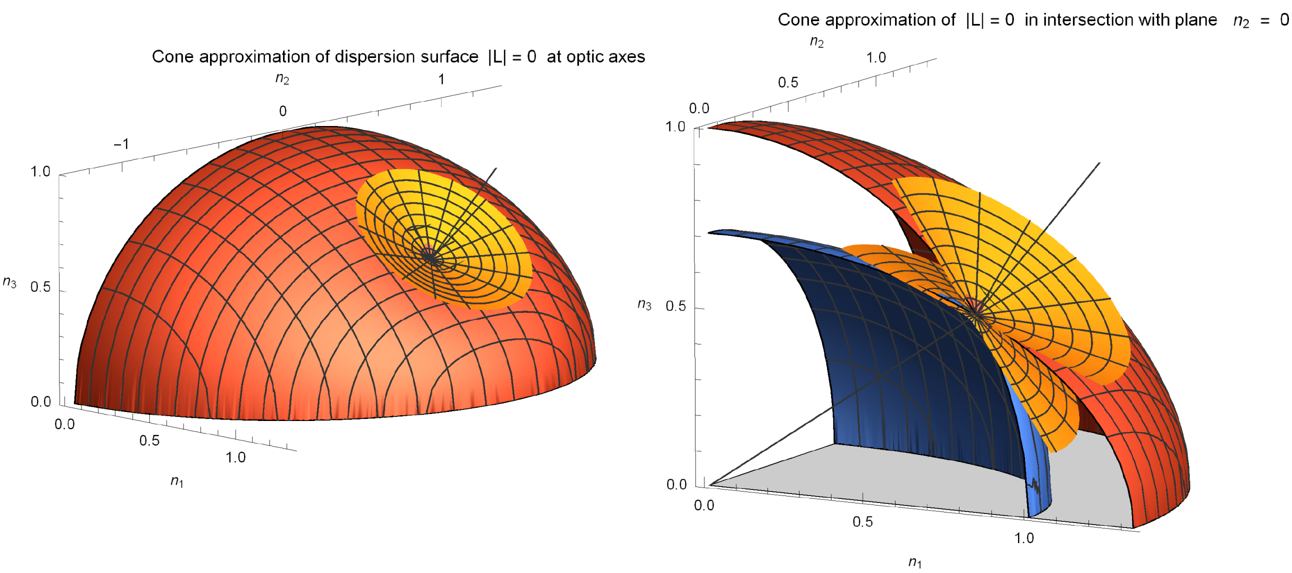


**Figure 5.** C one approximation of the dispersion surface $|\mathrm{L}| = 0$ in the neighborhood of an optic axis. The left-hand picture shows only the cone approximation of the external part of the dispersion surface in the neighborhood of an optic axis. The right-hand picture shows in addition also the approximation of the internal part of the dispersion surface by the opposite part of the double cone.

The components $(\delta n_+, \delta n_-, \delta n_2)$ are not independent from each other since they have to satisfy the cone Equation (6.18). Inserting the proposition (7.13) into the cone equation one finds due to the orthonormality (7.9) of the vectors $(\boldsymbol{c}'_+, \boldsymbol{c}'_-, \boldsymbol{c}_2)$

$$0 = \gamma_+ (\delta n_+)^2 + \gamma_- (\delta n_-)^2 + \gamma_2 (\delta n_2)^2 = \frac{(\delta n_+)^2}{\frac{1}{\gamma_+}} + \frac{(\delta n_-)^2}{\frac{1}{\gamma_-}} + \frac{(\delta n_2)^2}{\frac{1}{\gamma_2}}. \tag{7.14}$$

This is the equation for an elliptic cone with axis in direction of the vector $\boldsymbol{c}'_+$. An elliptic cone with axis in *z*-direction in coordinates $(x, y, z)$ can be written

$$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0, \quad \Rightarrow \quad z = \pm c\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2}}. \tag{7.15}$$

Thus the cone Equation (7.14) can be resolved, for example (remind that $\gamma_-$ and $\gamma_2$ are negative in comparison to $\gamma_+$)

$$\delta n_+ = \pm \frac{1}{\gamma_+} \sqrt{-\left(\gamma_- (\delta n_-)^2 + \gamma_2 (\delta n_2)^2\right)}. \tag{7.16}$$

The whole solution for $\delta \boldsymbol{n}$ is only determined up a (small) real factor $\sigma$ and therefore is

$$\delta \boldsymbol{n} = \sigma \left( \pm \frac{1}{\gamma_+} \sqrt{-\left(\gamma_- (\delta n_-)^2 + \gamma_2 (\delta n_2)^2\right)} \boldsymbol{c}'_+ + \delta n_- \boldsymbol{c}'_- + \delta n_2 \boldsymbol{c}_2 \right), \quad \left(|\sigma| \ll \varepsilon_1\right), \tag{7.17}$$

but other forms are also possible. For this and also for other purposes we calculate now the inverse eigenvalues to $(\gamma_\pm, \gamma_2)$ with the result

$$
\begin{aligned}
\frac{1}{\gamma_+} &= -\frac{\varepsilon_1\varepsilon_3 - \sqrt{\varepsilon_1\varepsilon_3\varepsilon_2(\varepsilon_1+\varepsilon_3-\varepsilon_2)}}{2\varepsilon_1\varepsilon_3(\varepsilon_3-\varepsilon_2)(\varepsilon_2-\varepsilon_1)} = -\frac{\gamma_-}{4\varepsilon_1\varepsilon_3(\varepsilon_3-\varepsilon_2)(\varepsilon_2-\varepsilon_1)} \geq 0,\\
\frac{1}{\gamma_-} &= -\frac{\varepsilon_1\varepsilon_3 + \sqrt{\varepsilon_1\varepsilon_3\varepsilon_2(\varepsilon_1+\varepsilon_3-\varepsilon_2)}}{2\varepsilon_1\varepsilon_3(\varepsilon_3-\varepsilon_2)(\varepsilon_2-\varepsilon_1)} = -\frac{\gamma_+}{4\varepsilon_1\varepsilon_3(\varepsilon_3-\varepsilon_2)(\varepsilon_2-\varepsilon_1)} \leq 0,\\
\frac{1}{\gamma_2} &= -\frac{1}{(\varepsilon_3-\varepsilon_2)(\varepsilon_2-\varepsilon_1)} = -\frac{4\varepsilon_1\varepsilon_3}{4\varepsilon_1\varepsilon_3(\varepsilon_3-\varepsilon_2)(\varepsilon_2-\varepsilon_1)} \leq 0.
\end{aligned}
\tag{7.18}
$$

The angles between vectors which we calculated in coordinate-invariant form are now to obtain in convenient way.

The half cone angle $\delta_-$ in the plane spanned by vectors $\boldsymbol{c}_1$ and $\boldsymbol{c}_3$ (plane $n_2 = 0$) is

$$
\begin{aligned}
\operatorname{tg}(\delta_-) &= \sqrt{-\frac{\gamma_+}{\gamma_-}} = \frac{\sqrt{\varepsilon_2(\varepsilon_1+\varepsilon_3-\varepsilon_2)} + \sqrt{\varepsilon_1\varepsilon_3}}{\sqrt{(\varepsilon_3-\varepsilon_2)(\varepsilon_2-\varepsilon_1)}},\\
\operatorname{tg}(2\delta_-) &= -\sqrt{\frac{(\varepsilon_3-\varepsilon_2)(\varepsilon_2-\varepsilon_1)}{\varepsilon_1\varepsilon_3}},\\
\cos(2\delta_-) &= \sqrt{\frac{\varepsilon_1\varepsilon_3}{\varepsilon_2(\varepsilon_1+\varepsilon_3-\varepsilon_2)}},\\
\sin(2\delta_-) &= -\sqrt{\frac{(\varepsilon_3-\varepsilon_2)(\varepsilon_2-\varepsilon_1)}{\varepsilon_2(\varepsilon_1+\varepsilon_3-\varepsilon_2)}}.
\end{aligned}
\tag{7.19}
$$

The angle $\eta$ between the optic axis and the cone axis of the refraction vectors is obtained from

$$
\cos(\eta) = \frac{\boldsymbol{n}_0\boldsymbol{c}_+'}{|\boldsymbol{n}_0||\boldsymbol{c}_+'|} = \frac{\sqrt{\varepsilon_2(\varepsilon_1+\varepsilon_3-\varepsilon_2)} + \sqrt{\varepsilon_1\varepsilon_3}}{\sqrt{2\left(\varepsilon_2(\varepsilon_1+\varepsilon_3-\varepsilon_2) + \sqrt{\varepsilon_1\varepsilon_3\varepsilon_2(\varepsilon_1+\varepsilon_3-\varepsilon_2)}\right)}},
\tag{7.20}
$$

with a result which equivalently can be expressed by

$$
\begin{aligned}
\cos(\eta) &= \sqrt{\frac{1}{2}\left(1+\sqrt{\frac{\varepsilon_1\varepsilon_3}{\varepsilon_2(\varepsilon_1+\varepsilon_3-\varepsilon_2)}}\right)},\\
\sin(\eta) &= \pm\sqrt{\frac{1}{2}\left(1-\sqrt{\frac{\varepsilon_1\varepsilon_3}{\varepsilon_2(\varepsilon_1+\varepsilon_3-\varepsilon_2)}}\right)},\\
\operatorname{tg}(\eta) &= \pm\sqrt{\frac{\sqrt{\varepsilon_2(\varepsilon_1+\varepsilon_3-\varepsilon_2)} - \sqrt{\varepsilon_1\varepsilon_3}}{\sqrt{\varepsilon_2(\varepsilon_1+\varepsilon_3-\varepsilon_2)} + \sqrt{\varepsilon_1\varepsilon_3}}},\\
\operatorname{tg}(2\eta) &= \sqrt{\frac{(\varepsilon_3-\varepsilon_2)(\varepsilon_2-\varepsilon_1)}{\varepsilon_1\varepsilon_3}} = \operatorname{tg}(\pi-2\delta_-).
\end{aligned}
\tag{7.21}
$$

There is the following relation between the angles $\eta$ and $\delta_-$ (see also **Figure 4**, the left-hand partial figure)

$$
\eta + \delta_- = \frac{\pi}{2}, \quad \Rightarrow \quad \cos(\eta) = \sin(\delta_-), \quad \sin(\eta) = \cos(\delta_-).
\tag{7.22}
$$

The half cone angle $\delta_2$ in the plane spanned by the unit vector $\boldsymbol{c}_+'$ of the cone axis and the unit vector $\boldsymbol{c}_2$ perpendicular to the plane of the optic axes is

$$\operatorname{tg}\left(\delta_2\right)=\sqrt{\frac{\gamma_+}{\gamma_2}}=\sqrt{\frac{2\left(\varepsilon_1\varepsilon_3+\sqrt{\varepsilon_1\varepsilon_3\varepsilon_2\left(\varepsilon_1+\varepsilon_3-\varepsilon_2\right)}\right)}{\left(\varepsilon_3-\varepsilon_2\right)\left(\varepsilon_2-\varepsilon_1\right)}},$$
$$\operatorname{tg}\left(2\delta_2\right)=\frac{2\sqrt{2\left(\varepsilon_3-\varepsilon_2\right)\left(\varepsilon_2-\varepsilon_1\right)\left(\varepsilon_1\varepsilon_3+\sqrt{\varepsilon_1\varepsilon_3\varepsilon_2\left(\varepsilon_1+\varepsilon_3-\varepsilon_2\right)}\right)}}{\left(\varepsilon_3-\varepsilon_2\right)\left(\varepsilon_2-\varepsilon_1\right)-2\left(\varepsilon_1\varepsilon_3+\sqrt{\varepsilon_1\varepsilon_3\varepsilon_2\left(\varepsilon_1+\varepsilon_3-\varepsilon_2\right)}\right)}, \tag{7.23}$$

or for the corresponding Cosines

$$\cos\left(\delta_2\right)=\sqrt{\frac{\left(\varepsilon_3-\varepsilon_2\right)\left(\varepsilon_2-\varepsilon_1\right)}{\left(\varepsilon_3-\varepsilon_2\right)\left(\varepsilon_2-\varepsilon_1\right)+2\left(\varepsilon_1\varepsilon_3+\sqrt{\varepsilon_1\varepsilon_3\varepsilon_2\left(\varepsilon_1+\varepsilon_3-\varepsilon_2\right)}\right)}},$$
$$\cos\left(2\delta_2\right)=\frac{\left(\varepsilon_3-\varepsilon_2\right)\left(\varepsilon_2-\varepsilon_1\right)-2\left(\varepsilon_1\varepsilon_3+\sqrt{\varepsilon_1\varepsilon_3\varepsilon_2\left(\varepsilon_1+\varepsilon_3-\varepsilon_2\right)}\right)}{\left(\varepsilon_3-\varepsilon_2\right)\left(\varepsilon_2-\varepsilon_1\right)+2\left(\varepsilon_1\varepsilon_3+\sqrt{\varepsilon_1\varepsilon_3\varepsilon_2\left(\varepsilon_1+\varepsilon_3-\varepsilon_2\right)}\right)}. \tag{7.24}$$

Some numerical values of the angles for the chosen permittivities $\varepsilon_1=0.5,\varepsilon_2=1.0,\varepsilon_3=1.75$ in the illustrations are[5]

$$\delta_-=73.3946^\circ,\quad \delta_2=72.6495^\circ,\quad \frac{\alpha}{2}=33.2109^\circ,\quad \frac{\beta}{2}=56.7891^\circ,$$
$$\eta=16.6055^\circ,\quad 2\eta=33.2110^\circ. \tag{7.25}$$

The numerical value $\delta_2=72.6495^\circ$ for our choice of parameters in the figures is very near to the numerical value $\delta_-=73.3946^\circ$ and results here from the small numerical difference of the arithmetic and geometric mean of $\varepsilon_1\varepsilon_3=0.875$ and $\varepsilon_2\left(\varepsilon_1+\varepsilon_3-\varepsilon_2\right)=1.25$ in (7.5) for $\gamma_2$ and $\gamma_-$ which lead to $m_a=1.0625,m_g=1.04583$ and are combined with further constants in $\gamma_2$ and $\gamma_-$ (moreover, in the formulae for the mentioned angles their square roots are relevant). It seems to be possible that in many experiments the small differences between the genuine elliptic cone and a conjectured circular cone in the conical refraction is hardly to see but this must not be the general case, in particular, since remarkable differences of the principal permittivities can be also artificially generated by external fields.

## 8. Plane Spanned by Optic Axis with Vector $\boldsymbol{n}_0$ and Vector $\boldsymbol{c}_2$ Perpendicular to Plane of Optic Axes

There is still another interesting plane of refraction vectors which is spanned by one of the two optic axes with the unit vector $\frac{1}{\sqrt{\varepsilon_2}}\boldsymbol{n}_0$ and the principal axis of the permittivity tensor with the unit vector $\boldsymbol{c}_2$ (see (4.2)) and is perpendicular to the plane of both optic axes and possesses the normal unit vector

[5]The numerical coincidence $\frac{\alpha}{2}=2\eta$ (see after (5.3)) is not a general equality and is incidentally caused by our choice of parameters with $\varepsilon_1=\varepsilon_2-\varepsilon_1=0.5$ in different formulae and was not foreseen and intended.

$\frac{1}{\sqrt{\varepsilon_2}}[\boldsymbol{n}_0,\boldsymbol{c}_2]$. A general wave vector $\boldsymbol{n}$ can be decomposed in a vector $\boldsymbol{n}'$ in this plane and a vector $\boldsymbol{n}''$ perpendicular to this plane according to

$$\boldsymbol{n}=\boldsymbol{n}'+\boldsymbol{n}'',\quad \boldsymbol{n}'\equiv\frac{\left[\boldsymbol{n},\left[\boldsymbol{n}_0,\boldsymbol{c}_2\right]\right]}{\sqrt{\varepsilon_2}},\quad \boldsymbol{n}''\equiv\frac{\left(\left[\boldsymbol{n},\boldsymbol{n}_0,\boldsymbol{c}_2\right]\right)\left[\boldsymbol{n}_0,\boldsymbol{c}_2\right]}{\varepsilon_2}. \tag{8.1}$$

Since the component $\boldsymbol{n}''$ is perpendicular to the considered plane it is omitted and the vector $\boldsymbol{n}'$ in this plane possesses two perpendicular components according to

$$\begin{aligned}\boldsymbol{n}'&\equiv\frac{1}{\sqrt{\varepsilon_2}}\left[\boldsymbol{n},\left[\boldsymbol{n}_0,\boldsymbol{c}_2\right]\right]=\left(\boldsymbol{n}\boldsymbol{c}_2\right)\frac{\boldsymbol{n}_0}{\sqrt{\varepsilon_2}}-\frac{\boldsymbol{n}\boldsymbol{n}_0}{\sqrt{\varepsilon_2}}\boldsymbol{c}_2\\&\equiv n_0\frac{\boldsymbol{n}_0}{\sqrt{\varepsilon_2}}+n_2\boldsymbol{c}_2,\quad\left|\frac{\boldsymbol{n}_0}{\sqrt{\varepsilon_2}}\right|=\left|\boldsymbol{c}_2\right|=1,\quad\boldsymbol{n}_0\boldsymbol{c}_2=0.\end{aligned} \tag{8.2}$$

Both vectors $\frac{\boldsymbol{n}_0}{\sqrt{\varepsilon_2}}$ and $\boldsymbol{c}_2$ are unit vectors and are perpendicular to each other and also to the unit vector $\frac{1}{\sqrt{\varepsilon_2}}[\boldsymbol{n}_0,\boldsymbol{c}_2]$. From (8.2) we find

$$\begin{aligned}\boldsymbol{n}'\boldsymbol{n}'&=n_0^2+n_2^2,\\\boldsymbol{n}'\boldsymbol{\varepsilon}\boldsymbol{n}'&=\frac{\varepsilon_1\varepsilon_3}{\varepsilon_2}n_0^2+\varepsilon_2n_2^2,\\\boldsymbol{n}'\boldsymbol{\varepsilon}^2\boldsymbol{n}'&=\frac{\varepsilon_1\varepsilon_3\left(\varepsilon_1+\varepsilon_3-\varepsilon_2\right)}{\varepsilon_2}n_0^2+\varepsilon_2^2n_2^2.\end{aligned} \tag{8.3}$$

The condition $\left|\mathsf{L}_0\right|=0$ makes in considered plane the transition to

$$\begin{aligned}0&=\left(n_0^2+n_2^2\right)\left(\frac{\varepsilon_1\varepsilon_3}{\varepsilon_2}n_0^2+\varepsilon_2n_2^2\right)-\left(2\varepsilon_1\varepsilon_3n_0^2+\left(\varepsilon_1+\varepsilon_3\right)\varepsilon_2n_2^2\right)+\varepsilon_1\varepsilon_2\varepsilon_3\\&=(n_0^2+n_2^2-\varepsilon_2)\left(\frac{\varepsilon_1\varepsilon_3}{\varepsilon_2}n_0^2+\varepsilon_2n_2^2-\varepsilon_1\varepsilon_3\right)-\left(\varepsilon_3-\varepsilon_2\right)\left(\varepsilon_2-\varepsilon_1\right)n_2^2,\end{aligned} \tag{8.4}$$

and the condition $\left[\mathsf{L}_0\right]=0$ the transition to

$$\begin{aligned}0&=\left(n_0^2+n_2^2\right)^2-\left(\frac{\left(\varepsilon_3+\varepsilon_2\right)\left(\varepsilon_2+\varepsilon_1\right)}{\varepsilon_2}n_0^2+\left(\varepsilon_1+2\varepsilon_2+\varepsilon_3\right)n_2^2\right)\\&\quad+\varepsilon_1\varepsilon_3+\left(\varepsilon_1+\varepsilon_3\right)\varepsilon_2\\&=\left(n_0^2+n_2^2-\varepsilon_2\right)\left(n_0^2+n_2^2-\left(\varepsilon_1+\varepsilon_2+\varepsilon_3\right)\right)+\frac{\left(\varepsilon_1\varepsilon_3-\varepsilon_2^2\right)}{\varepsilon_2}\left(n_0^2-\varepsilon_2\right).\end{aligned} \tag{8.5}$$

The curves for $\left|\mathsf{L}\right|=0$ and $\left[\mathsf{L}\right]=0$ in considered plane are shown **Figure 6**. We see that the curves from $\left[\mathsf{L}_0\right]=0$ touches the curves $\left|\mathsf{L}_0\right|=0$ in the direction of optic axes. In the special case $\varepsilon_1\varepsilon_3=\varepsilon_2^2$ these are exactly two circles with radii $\sqrt{\sqrt{\varepsilon_1\varepsilon_3}}$ and $\sqrt{\varepsilon_1+\sqrt{\varepsilon_1\varepsilon_3}+\varepsilon_3}$. In general case the curves $\left|\mathsf{L}\right|=0$ are also not two circles displaced in considered plane perpendicular to the optic axis as it is shown in **Figure 7** for two cases of extremely different principal permittivities. The graph of the dispersion surface in the intersection with considered plane is represented in **Figure 8** together with two cases of small deviations $\delta \boldsymbol{n}$ of the coupled refraction vectors to both sides of the optic axis and this is clearly

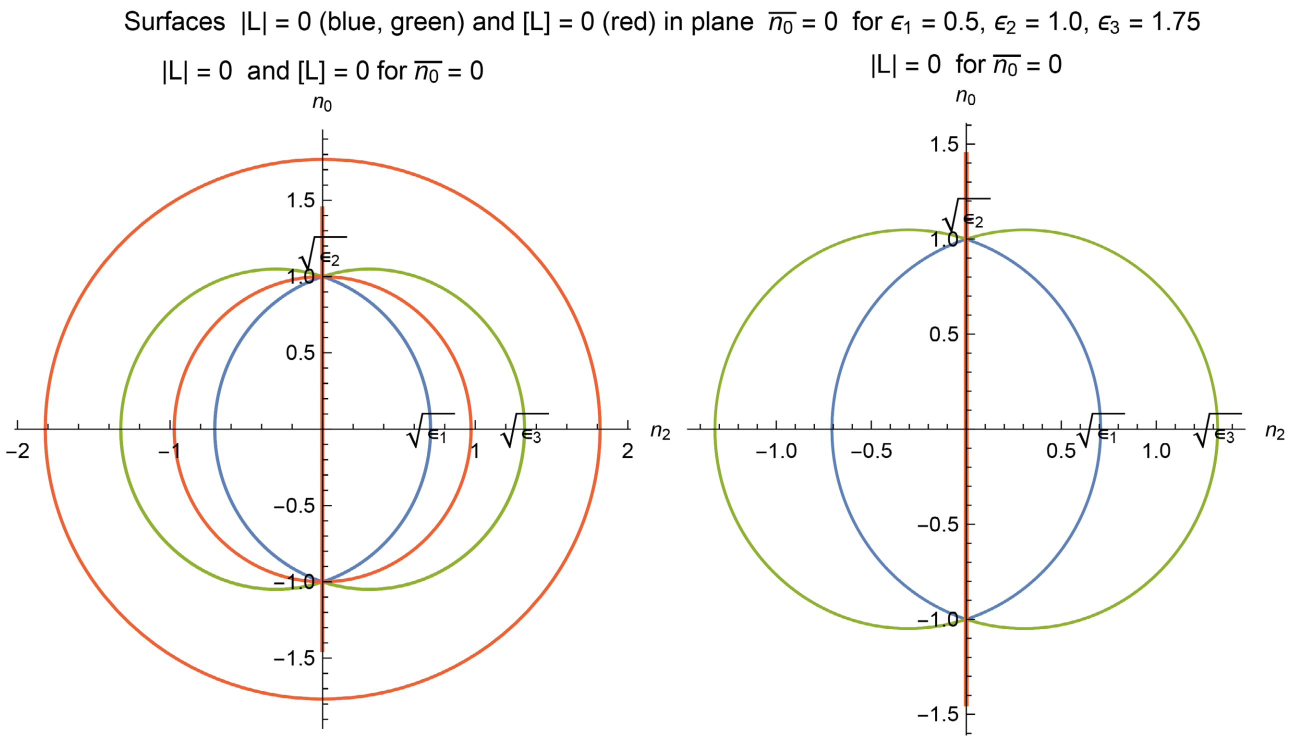


**Figure 6.** Intersections of dispersion surface $|L|=0$ (blue, green) and the surface $[L]=0$ (red) with the plane perpendicular to vector $\overline{\boldsymbol{n}_0} \equiv [\boldsymbol{n}_0, \boldsymbol{c}_2]$. The dispersion curves consist in this case of two 4th-order curves which look similar to two circles if we change on the negative side on the axis $n_2$ the colors blue and green but are not such. This can be seen if one choose essentially other parameters than in the picture which are here $\varepsilon_1 = 0.5, \varepsilon_2 = 1.0, \varepsilon_3 = 1.75$. The optic axis is here vertical. The cone axis is not contained in considered plane.

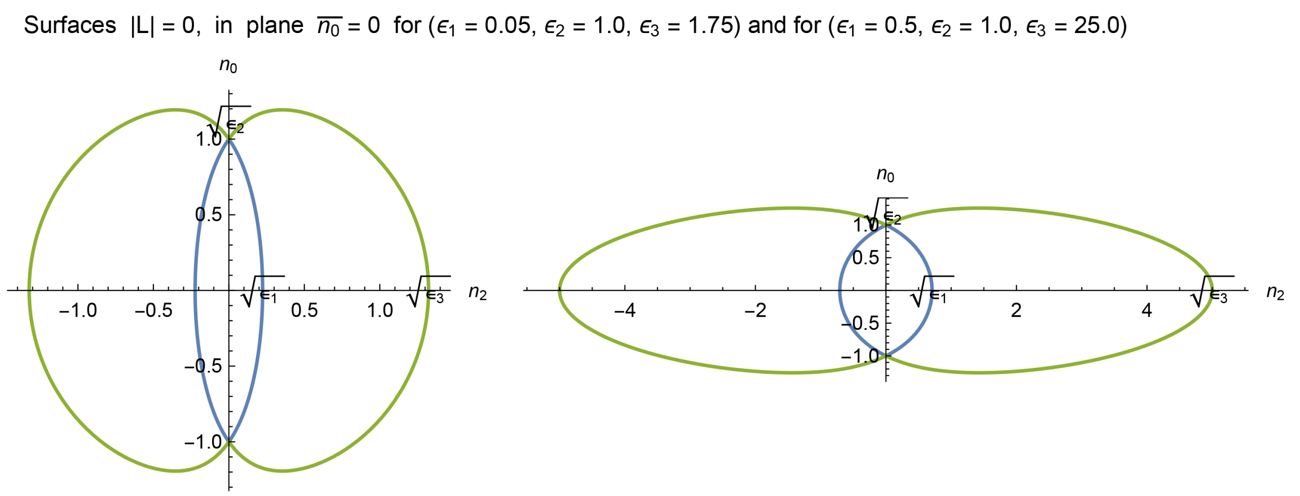


**Figure 7.** Intersections of dispersion surface $|L|=0$ (blue, green) with the plane perpendicular to vector $\overline{\boldsymbol{n}_0} \equiv [\boldsymbol{n}_0, \boldsymbol{c}_2]$ with extremely different permittivities. At the intersection points with the optic axes (vertical) the curves are not analytic. The permittivities are ($\varepsilon_1 = 0.05$, $\varepsilon_2 = 1.0$, $\varepsilon_3 = 1.75$) (left) and ($\varepsilon_1 = 0.5$, $\varepsilon_2 = 1.0$, $\varepsilon_3 = 25.0$) (right). This is more for illustration since the chosen values have little to do with really ones for media.

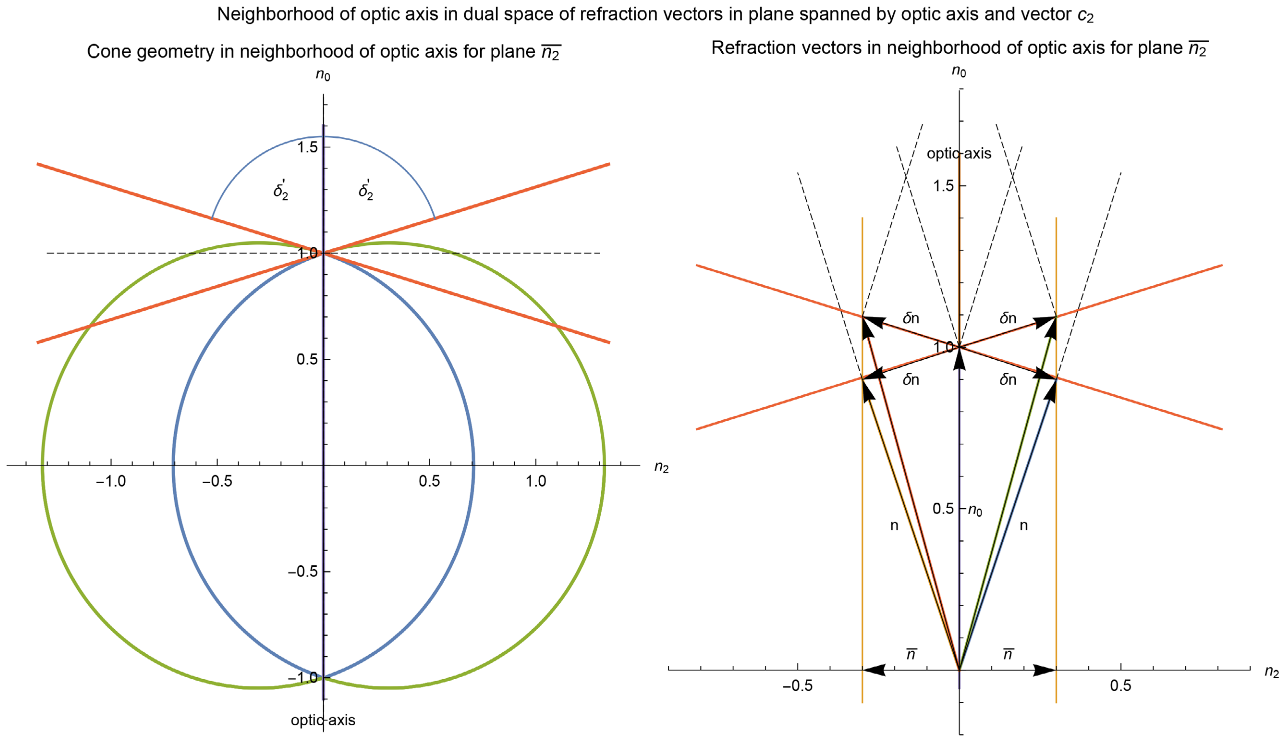


**Figure 8.** Refraction vectors in the neighborhood of the optic axis in plane with normal vector $[\boldsymbol{n}_0, \boldsymbol{c}_2]$. This plane is spanned by the vectors $\boldsymbol{n}_0$ of the optic axis and the principal vector $\boldsymbol{c}_2$ of permittivity tensor $\boldsymbol{\varepsilon}$. The cone axis does not lie in this plane and therefore also not the ray vectors and cannot really be drawn in this picture. Therefore, the angle $\delta_2'$ is (for our choice of parameters, angle between optic and cone axis $\eta \approx 16.6^\circ$, see (7.25)) small different from the angle $\delta_2$ and was not exactly calculated. The ray vectors, properly, also do not belong to the picture of refraction vectors and their origin in the real space depends on the considered device.

symmetric to optic axis.

The cone axis does not lie in considered plane spanned by the optic axis with vector $\boldsymbol{c}_+$ and the vector $\boldsymbol{c}_2$ and forms with it an angle $\eta$ and therefore cannot be shown in this Figure 8. In analogy to (7.19) the half cone angle $\delta_2$ in the plane spanned by the cone axis and the vector $\boldsymbol{c}_2$ is given by the relations

$$\operatorname{tg}(\delta_2) = \sqrt{-\frac{\gamma_+}{\gamma_2}} = \sqrt{\frac{2\left(\varepsilon_1\varepsilon_3 + \sqrt{\varepsilon_1\varepsilon_3\varepsilon_2(\varepsilon_1+\varepsilon_3-\varepsilon_2)}\right)}{(\varepsilon_3-\varepsilon_2)(\varepsilon_2-\varepsilon_1)}}, \tag{8.6}$$

Its numerical value for the used permittivities is $\delta_2 = 72.6495^\circ$ as was already given in (7.25) in comparison to the corresponding angle $\delta_-$ in the plane of the optic axes in Figure 6. Therefore, the angle $\delta_2'$ in Figure 7 is a little different from $\delta_2$ and the ray vectors cannot be (exactly) drawn in this picture.

## 9. Polarization of Electric Field in the Neighborhood of an Optic Axis

In case of $[\mathsf{L}] \to [\mathsf{L}_0] = 0$ and $\overline{\mathsf{L}_0} = 0$ the projection operator in (4.15) for the determination of polarization vectors $\boldsymbol{e}$ of the electric field becomes indeterminate and fails to act that is a consequence of the presence of an optic axis and the twofold degeneration of polarization vectors of the electric (and also magnetic) field in such directions. This twofold degeneration is removed for propagation in arbitrary small deviations from the optic axis. We deal with now these cases. Clearly, this is already made in the literature, in particular, in coordinate-invariant way in the monographs of Fyodorov [7] [8] but we are mainly interested for the neighborhood of optic axes. Our representation here distinguishes from that of Fyodorov in the way that we from beginning on involve more the operator $\mathsf{L}$ and its invariants. We make this for the classical crystal optics with only one frequency-dependent permittivity tensor $\boldsymbol{\varepsilon}(\omega)$ (or $\varepsilon_{ij}(\omega)$ written with indices) of the form (4.2) (for lossless case) and use as before the notation $\boldsymbol{n} = \boldsymbol{n}_0 + \delta\boldsymbol{n}$ for the refraction vectors in the neighborhood of the optic axis to the refraction vector $\boldsymbol{n}_0$.

We write the non-degenerate projection operators (4.15) now in the form

$$\Pi(\boldsymbol{n}_0+\delta\boldsymbol{n}) = \frac{\overline{\mathsf{L}(\boldsymbol{n}_0+\delta\boldsymbol{n})}}{\langle\overline{\mathsf{L}(\boldsymbol{n}_0+\delta\boldsymbol{n})}\rangle}, \quad \overline{\mathsf{L}_0} \equiv \overline{\mathsf{L}(\boldsymbol{n}_0)} = 0, \quad \langle\overline{\mathsf{L}_0}\rangle = [\mathsf{L}_0] = 0, \tag{9.1}$$

and make an expansion of the refraction vectors in numerator and denominator in powers of the "small" deviations from the optic axis $\delta(\boldsymbol{n})$. The general identity for the complementary operator of a sum of two operators $\mathsf{A}$ and $\mathsf{B}$ is

$$\overline{\mathsf{A}+\mathsf{B}} = \overline{\mathsf{A}} + \mathsf{AB} + \mathsf{BA} - (\langle\mathsf{B}\rangle\mathsf{A} + \langle\mathsf{A}\rangle\mathsf{B}) + (\langle\mathsf{A}\rangle\langle\mathsf{B}\rangle - \langle\mathsf{AB}\rangle)\mathsf{I} + \overline{\mathsf{B}}, \tag{9.2}$$

from which follows for its trace (can also independently be calculated from the second invariant of $\mathsf{A}+\mathsf{B}$)

$$\langle\overline{\mathsf{A}+\mathsf{B}}\rangle = [\mathsf{A}] + \langle\mathsf{A}\rangle\langle\mathsf{B}\rangle - \langle\mathsf{AB}\rangle + [\mathsf{B}] = [\mathsf{A}+\mathsf{B}]. \tag{9.3}$$

Up to linear terms in $\delta\boldsymbol{n}$ we find then from these identities or from (4.16) in the cone approximation[6]

$$\begin{aligned}\overline{\mathsf{L}(\boldsymbol{n}_0+\delta\boldsymbol{n})} &= (\boldsymbol{n}_0^2)(\boldsymbol{n}_0\cdot\delta\boldsymbol{n} + \delta\boldsymbol{n}\cdot\boldsymbol{n}_0) + 2(\boldsymbol{n}_0\delta\boldsymbol{n})\boldsymbol{n}_0\cdot\boldsymbol{n}_0 \\ &\quad - \langle\boldsymbol{\varepsilon}\rangle(\boldsymbol{n}_0\cdot\delta\boldsymbol{n} + \delta\boldsymbol{n}\cdot\boldsymbol{n}_0) + \boldsymbol{\varepsilon}(\boldsymbol{n}_0\cdot\delta\boldsymbol{n} + \delta\boldsymbol{n}\cdot\boldsymbol{n}_0) \\ &\quad + (\boldsymbol{n}_0\cdot\delta\boldsymbol{n} + \delta\boldsymbol{n}\cdot\boldsymbol{n}_0)\boldsymbol{\varepsilon} - (\boldsymbol{n}_0\boldsymbol{\varepsilon}\delta\boldsymbol{n} + \delta\boldsymbol{n}\boldsymbol{\varepsilon}\boldsymbol{n}_0)\mathsf{I},\end{aligned} \tag{9.4}$$

and for its trace

$$\begin{aligned}\langle\overline{\mathsf{L}(\boldsymbol{n}_0+\delta\boldsymbol{n})}\rangle &= \left(4(\boldsymbol{n}_0^2)(\boldsymbol{n}_0\delta\boldsymbol{n}) - 2\langle\boldsymbol{\varepsilon}\rangle(\boldsymbol{n}_0\delta\boldsymbol{n}) - (\boldsymbol{n}_0\boldsymbol{\varepsilon}\delta\boldsymbol{n} + \delta\boldsymbol{n}\boldsymbol{\varepsilon}\boldsymbol{n}_0)\right) \\ &= [\mathsf{L}(\boldsymbol{n}_0+\delta\boldsymbol{n})].\end{aligned} \tag{9.5}$$

As already said and derived in Section 7 the components of the small deviations of the refraction vectors from the optic axis are not independent from each other and are restricted by "cone" conditions of the form (7.16) or (7.17) which also have to be taken into account in (9.4). To get (non-normalized) polarization vectors of the electric field one has to apply these projection operators

[6]We write the following and wrote already some formulae before preserving a certain left-right-symmetry that means a little more general as absolutely necessary for symmetrical permittivity tensors $\boldsymbol{\varepsilon}$ that may become important in generalizations to Hermitean permittivity tensors and demonstrates more the symmetries.

$\Pi(\boldsymbol{n}_0+\delta\boldsymbol{n})$ to appropriately chosen vectors that can be made in various way. The formulae (9.4) and (9.5) are fairly complicated and it is usually a stony way to explain with them real experiments in detail. Besides conical refraction the polarization effects in direction of optic axes are diverse and difficult to calculate exactly but one may find many pictures from experiments in monographs and in Internet.

## 10. Optic Axes in Uniaxial Media

The special case of uniaxial crystal is very well known and we make only short remarks to classify them by the limiting transition from biaxial crystals to this uniaxial case. We have to distinguish two cases of the limiting transition, the case $\varepsilon_1 \to \varepsilon_2$ and the case $\varepsilon_3 \to \varepsilon_2$. In both cases the primarily two optic axes coincide to one optic axis but in different positions if we preserve the ordering relations (4.4) for the principal values of the permittivities.

As first limiting case we consider $\varepsilon_1 \to \varepsilon_2$

$$\varepsilon_1=\varepsilon_2\equiv\varepsilon_o,\quad \varepsilon_3\equiv\varepsilon_e,\quad \varepsilon_o\le\varepsilon_e. \tag{10.1}$$

From (3) follows then $\frac{\alpha}{2}=\frac{\pi}{2}$ and the cone approximation degenerates to a tangential plane to the optic axis. This becomes a uniaxial positive medium (**Figure 9**).

The invariants of the tensor $\boldsymbol{\varepsilon}$ are then

$$\begin{aligned}\langle\boldsymbol{\varepsilon}\rangle&=2\varepsilon_2+\varepsilon_3=2\varepsilon_o+\varepsilon_e,\\ [\boldsymbol{\varepsilon}]&=\varepsilon_2(\varepsilon_2+2\varepsilon_3)=\varepsilon_o(\varepsilon_o+2\varepsilon_e),\\ |\boldsymbol{\varepsilon}|&=\varepsilon_2^2\varepsilon_3=\varepsilon_o^2\varepsilon_e\end{aligned} \tag{10.2}$$

Furthermore we introduce the notations

$$n_\perp^2\equiv n_1^2+n_2^2,\quad n_\parallel^2=n_3^2, \tag{10.3}$$

The invariants of the operator $\mathsf{L}$ are

$$\begin{aligned}\langle\mathsf{L}\rangle&=-2(n_\perp^2+n_\parallel^2)+(2\varepsilon_o+\varepsilon_e),\\ [\mathsf{L}]&=(n_\perp^2+n_\parallel^2)^2-((3\varepsilon_o+\varepsilon_e)n_\perp^2+2(\varepsilon_o+\varepsilon_e)n_\parallel^2)+\varepsilon_o(\varepsilon_o+2\varepsilon_e),\\ |\mathsf{L}|&=(n_\perp^2+n_\parallel^2-\varepsilon_o)(\varepsilon_o^2n_\perp^2+\varepsilon_en_\parallel^2-\varepsilon_o\varepsilon_e).\end{aligned} \tag{10.4}$$

The invariant $[\mathsf{L}]$ can be also written in the form

$$[\mathsf{L}]=(n_\parallel^2+n_\perp^2-(\varepsilon_o+\varepsilon_e))^2+((\varepsilon_e-\varepsilon_o)n_\perp^2-\varepsilon_e^2), \tag{10.5}$$

from which results the following decomposition into a product, however, not with rational factors (two-shell surface)

$$\begin{aligned}[\mathsf{L}]=&\left(n_\perp^2+n_\parallel^2-(\varepsilon_e+\varepsilon_o)+\sqrt{\varepsilon_e^2-(\varepsilon_e-\varepsilon_o)n_\perp^2}\right)\\ &\cdot\left(n_\perp^2+n_\parallel^2-(\varepsilon_e+\varepsilon_o)-\sqrt{\varepsilon_e^2-(\varepsilon_e-\varepsilon_o)n_\perp^2}\right)\end{aligned} \tag{10.6}$$

For $n_\perp=0$ we get the special product form

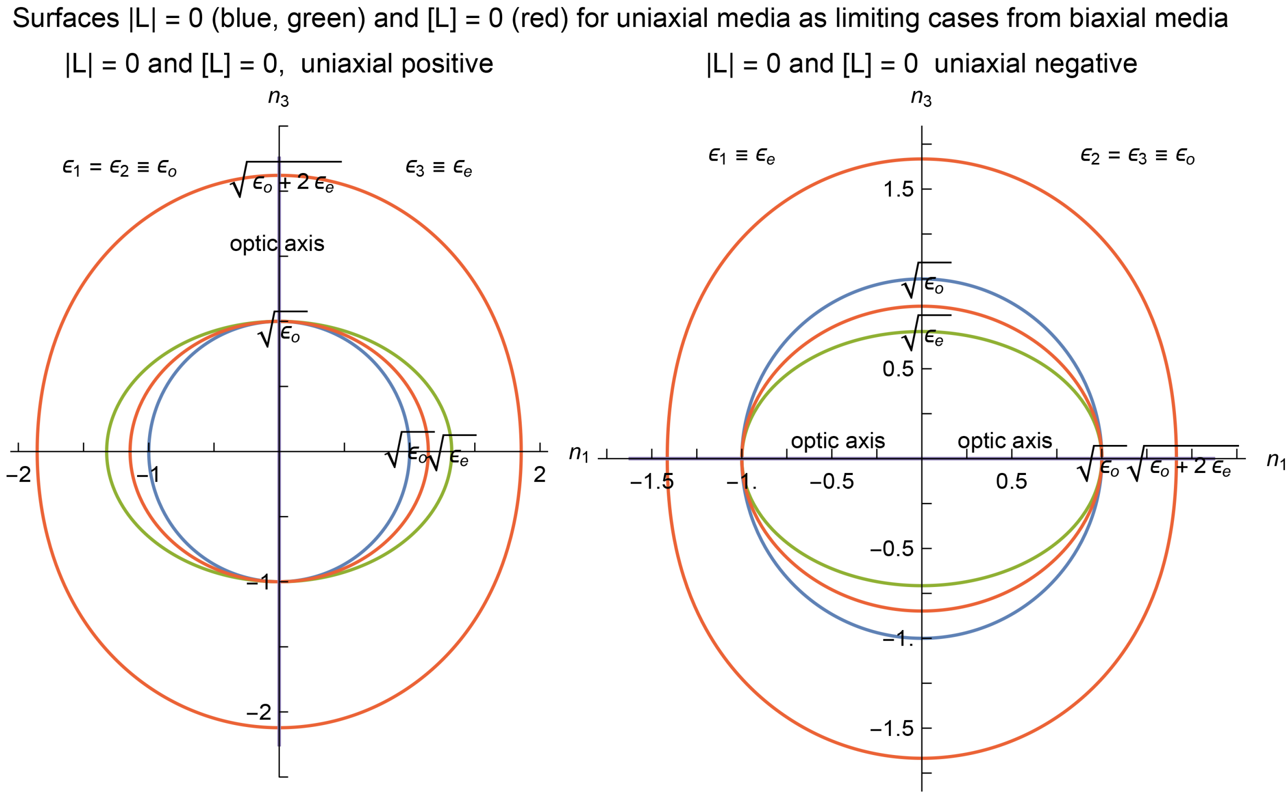


**Figure 9.** Limiting transition from optically biaxial to uniaxial media with dispersion surface $|\mathrm{L}| = 0$ (blue, green) and surface $[\mathrm{L}] = 0$ (red). The optic axes is vertical in first case and is horizontal in second case due to our limiting transition $\varepsilon_1 \to \varepsilon_2$ and $\varepsilon_3 \to \varepsilon_2$ in first and second case, respectively. The tangents of all considered surfaces at the touching points are orthogonal to the optic axes and the cones from biaxial crystals degenerate to tangential planes. From the two-shell surface $[\mathrm{L}] = 0$ only the inner shell touches the optic axes and lies between the sphere and the rotation ellipsoid of the dispersion surface $|\mathrm{L}| = 0$. In left-hand picture the chosen permittivities are $\varepsilon_o = 1.0, \varepsilon_e = 1.75$ and in right-hand picture $\varepsilon_o = 0.5, \varepsilon_e = 1.0$.

$$[\mathrm{L}] = \left(n_\perp^2 + n_\parallel^2 - \varepsilon_o\right)\left(n_\perp^2 + n_\parallel^2 - \left(2\varepsilon_e + \varepsilon_o\right)\right) \tag{10.7}$$

It touches the dispersion surfaces $|\mathrm{L}| = 0$ at the optic axes.

As second limiting case we consider now $\varepsilon_3 \to \varepsilon_2$

$$\varepsilon_1 = \varepsilon_e, \quad \varepsilon_3 = \varepsilon_2 \equiv \varepsilon_o, \quad \varepsilon_e < \varepsilon_o. \tag{10.8}$$

From (5.3) follows in this case $\frac{\alpha}{2} = 0$ and the cone approximation degenerates also to a tangential plane at the optic axis. This becomes then a uniaxial negative medium (**Figure 8**).

The invariants of the permittivity tensor $\boldsymbol{\varepsilon}$ are then

$$\begin{aligned} \langle\boldsymbol{\varepsilon}\rangle &= \varepsilon_1 + 2\varepsilon_2 = \varepsilon_e + 2\varepsilon_o, \\ [\boldsymbol{\varepsilon}] &= \varepsilon_2\left(\varepsilon_2 + 2\varepsilon_1\right) = \varepsilon_o\left(\varepsilon_o + 2\varepsilon_e\right), \\ |\boldsymbol{\varepsilon}| &= \varepsilon_1\varepsilon_2^2 = \varepsilon_e\varepsilon_o^2. \end{aligned} \tag{10.9}$$

Furthermore we introduce the notations

$$n_\parallel^2 = n_1^2, \quad n_\perp^2 \equiv n_2^2 + n_3^2, \tag{10.10}$$

The invariants of the operator $\mathsf{L}$ are the same as in (10.4). The invariant $[\mathsf{L}]$ can be written in the same two forms as in (10.5). The decomposition in product form is the same as in (10.6).

The cone in the cone approximation of refraction vectors near to an optic axis in biaxial crystals degenerates in uniaxial media to a tangential plane and there is then no difference between ordinary and extraordinary waves in this approximation. To get differences one would have go to the next higher-order approximation that, however, is not necessary since it is not very difficult to treat uniaxial media in full generality.

## 11. Optic Axes in the Domain of Complex Refraction Vectors

The optic axes which we considered up to now are present for real wave vectors $\boldsymbol{n}$ and they are the only ones in this domain. If we suppose real non-negative principal values $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ of the permittivity tensor $\boldsymbol{\varepsilon}$ and preserve the ordering (4.4) that means $0 \le \varepsilon_1 \le \varepsilon_2 \le \varepsilon_3$ then in analogy to (5.1) we can may write down the following two solutions of the dispersion equations for the plane $n_3 = 0$

$$
\begin{aligned}
n_3 = 0: \quad \Rightarrow \quad & 0 = \varepsilon_1\varepsilon_2\left(n_1^2 + n_2^2 - \varepsilon_3\right), \\
& 0 = \varepsilon_3\left(\varepsilon_1 n_1^2 + \varepsilon_2 n_2^2 - \varepsilon_1\varepsilon_2\right).
\end{aligned} \tag{11.1}
$$

and for the plane $n_1 = 0$

$$
\begin{aligned}
n_1 = 0: \quad \Rightarrow \quad & 0 = \varepsilon_2\varepsilon_3\left(n_2^2 + n_3^2 - \varepsilon_1\right), \\
& 0 = \varepsilon_1\left(\varepsilon_2 n_2^2 + \varepsilon_3 n_3^2 - \varepsilon_2\varepsilon_3\right).
\end{aligned} \tag{11.2}
$$

If we form the difference of both equations we find in case of (11.1)

$$
n_3 = 0: \quad \varepsilon_1\left(\varepsilon_3 - \varepsilon_2\right) n_1^2 = -\varepsilon_2\left(\varepsilon_3 - \varepsilon_1\right) n_2^2, \quad \Rightarrow \quad \frac{n_2}{n_1} = \sqrt{-\frac{\varepsilon_1\left(\varepsilon_3 - \varepsilon_2\right)}{\varepsilon_2\left(\varepsilon_3 - \varepsilon_1\right)}}, \tag{11.3}
$$

and in case of (11.2)

$$
n_1 = 0: \quad \varepsilon_2\left(\varepsilon_3 - \varepsilon_1\right) n_2^2 = -\varepsilon_3\left(\varepsilon_2 - \varepsilon_1\right) n_3^2 \quad \Rightarrow \quad \frac{n_2}{n_3} = \sqrt{-\frac{\varepsilon_3\left(\varepsilon_2 - \varepsilon_1\right)}{\varepsilon_2\left(\varepsilon_3 - \varepsilon_2\right)}}. \tag{11.4}
$$

Contrary to the case $n_2 = 0$ dealt with from Section 5 on, in the here considered two additional cases we cannot have at the same time only real components of the refraction vectors for the intersection points. These are the only cases with self-intersection and where at least one of the component $(n_1, n_2, n_3)$ is vanishing.

We discuss now the more general problem to determine the simultaneous solutions of the two equations $|\mathsf{L}| = 0$ and $[\mathsf{L}] = 0$ in the complex domain and try to solve the question whether or not the case $\overline{\mathsf{L}} \neq 0$ but $|\mathsf{L}| = 0$ and $[\mathsf{L}] = 0$ is possible. For this purpose we admit also that the principal values $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ may possess complex values since this does not make the derivations more difficult. Self-intersections of the solutions of $|\mathsf{L}| = 0$ are only possible if one of the three components $(n_1, n_2, n_3)$ is vanishing and it seems that these cases are dealt with exhaustively by this. So we may suppose that in the searched case all three components $(n_1, n_2, n_3)$ are non-vanishing.

The dispersion equation $|\mathsf{L}| = 0$ for the refraction vectors $\boldsymbol{n}$ given in (4.8) in the principal-axis form resolved to the component, e.g., $n_2$ in dependence on $n_1$ and $n_3$ becomes the biquadratic equation

$$\left(n_2^2\right)^2 - 2p_1 n_2^2 + q_1^2 = 0, \tag{11.5}$$

with the abbreviations

$$\begin{aligned} 2p_1 &\equiv \varepsilon_1 + \varepsilon_3 - n_1^2 - n_3^2 - \frac{\varepsilon_1 n_1^2 + \varepsilon_3 n_3^2}{\varepsilon_2}, \\ q_1^2 &\equiv \frac{\left(\varepsilon_2 - n_1^2 - n_3^2\right)\left(\varepsilon_1\varepsilon_3 - \varepsilon_1 n_1^2 - \varepsilon_3 n_3^2\right)}{\varepsilon_2}, \end{aligned} \tag{11.6}$$

and the equation for vanishing $[\mathsf{L}] = 0$ of the second invariant of $\mathsf{L}$ in (4.10) the biquadratic equation

$$\left(n_2^2\right)^2 - 2p_2 n_2^2 + q_2^2 = 0, \tag{11.7}$$

with the abbreviations

$$\begin{aligned} 2p_2 &\equiv \varepsilon_1 + \varepsilon_3 + 2\left(\varepsilon_2 - n_1^2 - n_3^2\right), \\ q_2^2 &\equiv \varepsilon_1\varepsilon_3 + \left(\varepsilon_1 + \varepsilon_3\right)\varepsilon_2 - \left(2\varepsilon_1 + \varepsilon_2 + \varepsilon_3\right)n_1^2 - \left(\varepsilon_1 + \varepsilon_2 + 2\varepsilon_3\right)n_3^2 + \left(n_1^2 + n_3^2\right)^2. \end{aligned} \tag{11.8}$$

The solutions of a biquadratic equation of the form

$$z^4 - 2pz^2 + q^2 = 0, \tag{11.9}$$

may be represented in the following ways

$$z_\pm^2 = p \pm \sqrt{p^2 - q^2}, \quad z_\pm^{(\pm)} = (\pm)\left(\sqrt{\frac{p+q}{2}} \pm \sqrt{\frac{p-q}{2}}\right). \tag{11.10}$$

However, it is not necessary to solve immediately the special biquadratic equations (11.5) and (11.7). One may first form the difference of both these equations and obtains the following quadratic equation for $n_2$

$$2\left(p_1 - p_2\right)n_2^2 - \left(q_1^2 - q_2^2\right) = 0, \tag{11.11}$$

which expresses $n_2^2$ in dependence of $n_1^2$ and $n_3^2$ and on the permittivities. The result for $n_2^2$ can be inserted in one of the biquadratic Equations (11.5) or (11.7) which provide then relations where $n_2$ is eliminated. Effectively this becomes a bicubic equation which has to be solved. It is hardly possible to make this without a computer. Alternatively, one can also solve by computer both the Equations (5) and (7) with respect to $n_2$ that provides 4 solutions (2 solutions for $n_2^2$) for both equations with 4 possible combinations of establishing equalities to eliminate $n_2^2$. Due to voluminous coefficients in these equations the arising bicubic equation possesses solutions which even by computer are given in an extremely long form which we do not write down. The trials to simplify the expressions or to find interesting special cases were without success up to now. Therefore, we can say that, apparently, exist searched cases $|\mathsf{L}| = 0$ and $[\mathsf{L}] = 0$ but $\overline{\mathsf{L}} \neq 0$ in the complex domain but due to their difficulty they are probably little interesting. This problem is not solved. For real components $(n_1, n_2, n_3)$ and symmetry $\mathsf{L} = \mathsf{L}^{\mathrm{T}}$ such solutions are absent due to $\tilde{\boldsymbol{e}} = \boldsymbol{e} = \boldsymbol{e}^*$ for left-hand and right-hand eigenvectors and $\boldsymbol{e}^*\boldsymbol{e} \neq 0$.

## 12. Remarks to a Duality between Refraction Vectors and Ray Vectors

In the more general consideration of spatial dispersion one may start from the wave-equation operator $\mathsf{L}(\boldsymbol{k}, \omega)$ and the vectorial wave equation

$$\mathsf{L}(\boldsymbol{k}, \omega) \equiv \frac{c^2}{\omega^2}\left(\boldsymbol{k} \cdot \boldsymbol{k} - \left(\boldsymbol{k}^2\right)\mathsf{I}\right) + \boldsymbol{\varepsilon}(\boldsymbol{k}, \omega), \quad \mathsf{L}(\boldsymbol{k}, \omega)\boldsymbol{e} = \boldsymbol{0}, \quad \tilde{\boldsymbol{e}}\mathsf{L}(\boldsymbol{k}, \omega) = \boldsymbol{0}, \tag{12.1}$$

which in case of neglect of dispersion that means with constant $\boldsymbol{\varepsilon}$

$$\boldsymbol{\varepsilon}(\boldsymbol{k}, \omega) = \boldsymbol{\varepsilon}, \tag{12.2}$$

may be reduced using refraction vectors $\boldsymbol{n} \equiv \frac{c}{\omega}\boldsymbol{k}$ to the operator $\mathsf{L}(\boldsymbol{n})$ according to

$$\mathsf{L}(\boldsymbol{n}) \equiv \boldsymbol{n} \cdot \boldsymbol{n} - \left(\boldsymbol{n}^2\right)\mathsf{I} + \boldsymbol{\varepsilon}, \tag{12.3}$$

as was assumed for the treatment of optic axes in present article.

In the concept (12.1) we may resolve the dispersion equation $|\mathsf{L}(\boldsymbol{k}, \omega)| = 0$ to

$$\omega \equiv \omega(\boldsymbol{k}), \quad \Rightarrow \quad |\mathsf{L}(\boldsymbol{k}, \omega)| \equiv |\mathsf{L}(\boldsymbol{k}, \omega(\boldsymbol{k}))| = 0. \tag{12.4}$$

The group velocity $\boldsymbol{v} \equiv \frac{\partial \omega}{\partial \boldsymbol{k}}$ can be obtained by differentiation of the last identity $|\mathsf{L}(\boldsymbol{k}, \omega(\boldsymbol{k}))| = 0$ which depends only on the wave vector $\boldsymbol{k}$ and under neglect of the dispersion and with introduction of refraction vectors $\boldsymbol{n}$ one obtains

$$\begin{aligned}
\boldsymbol{v} &= -\frac{\frac{\partial}{\partial \boldsymbol{k}}|\mathsf{L}(\boldsymbol{k}, \omega)|}{\frac{\partial}{\partial \omega}|\mathsf{L}(\boldsymbol{k}, \omega)|} \\
&= \omega \frac{\frac{c^2}{\omega^2}\left(2(\boldsymbol{k}\boldsymbol{\varepsilon}\boldsymbol{k})\boldsymbol{k} + \boldsymbol{k}^2(\boldsymbol{\varepsilon}\boldsymbol{k} + \boldsymbol{n}\boldsymbol{k})\right) - \left(\langle\boldsymbol{\varepsilon}\rangle(\boldsymbol{\varepsilon}\boldsymbol{k} + \boldsymbol{n}\boldsymbol{k}) - \left(\boldsymbol{\varepsilon}^2\boldsymbol{k} + \boldsymbol{k}\boldsymbol{\varepsilon}^2\right)\right)}{4\frac{c^2}{\omega^2}\left(\boldsymbol{k}^2\right)\boldsymbol{k}\boldsymbol{\varepsilon}\boldsymbol{k} - 2\left(\langle\boldsymbol{\varepsilon}\rangle\boldsymbol{k}\boldsymbol{\varepsilon}\boldsymbol{k} - \boldsymbol{k}\boldsymbol{\varepsilon}^2\boldsymbol{k}\right)} \\
&= c\frac{\left(2(\boldsymbol{n}\boldsymbol{\varepsilon}\boldsymbol{n})\boldsymbol{n} + \boldsymbol{n}^2(\boldsymbol{\varepsilon}\boldsymbol{n} + \boldsymbol{n}\boldsymbol{\varepsilon})\right) - \left(\langle\boldsymbol{\varepsilon}\rangle(\boldsymbol{\varepsilon}\boldsymbol{n} + \boldsymbol{n}\boldsymbol{\varepsilon}) - \left(\boldsymbol{\varepsilon}^2\boldsymbol{n} + \boldsymbol{n}\boldsymbol{\varepsilon}^2\right)\right)}{4\left(\boldsymbol{n}^2\right)\boldsymbol{n}\boldsymbol{\varepsilon}\boldsymbol{n} - 2\left(\langle\boldsymbol{\varepsilon}\rangle\boldsymbol{n}\boldsymbol{\varepsilon}\boldsymbol{n} - \boldsymbol{n}\boldsymbol{\varepsilon}^2\boldsymbol{n}\right)}, \quad \boldsymbol{n} \equiv \frac{c}{\omega}\boldsymbol{k}.
\end{aligned} \tag{12.5}$$

From this follows with introduction of ray vectors $\boldsymbol{s} \equiv \frac{\boldsymbol{v}}{c}$ (e.g., [2], §97 and [6], §81)

$$\boldsymbol{k}\boldsymbol{v}=\omega,\quad \boldsymbol{n}\boldsymbol{v}=\frac{c}{\omega}\boldsymbol{k}\boldsymbol{v}=c,\quad \Rightarrow\quad \boldsymbol{n}\frac{\boldsymbol{v}}{c}=\boldsymbol{n}\boldsymbol{s}=1,\quad \boldsymbol{s}\equiv\frac{\boldsymbol{v}}{c}. \tag{12.6}$$

In the concept (12.3) with the refraction vectors $\boldsymbol{n}$ and the operator $\mathsf{L}(\boldsymbol{n})$ we find

$$\begin{aligned}
&\frac{\partial}{\partial\boldsymbol{n}}\left|\mathsf{L}(\boldsymbol{n})\right|=\left(2(\boldsymbol{n}\boldsymbol{\varepsilon}\boldsymbol{n})\boldsymbol{n}+\boldsymbol{n}^2(\boldsymbol{\varepsilon}\boldsymbol{n}+\boldsymbol{n}\boldsymbol{\varepsilon})\right)-\left(\langle\boldsymbol{\varepsilon}\rangle(\boldsymbol{\varepsilon}\boldsymbol{n}+\boldsymbol{n}\boldsymbol{\varepsilon})-\left(\boldsymbol{\varepsilon}^2\boldsymbol{n}+\boldsymbol{n}\boldsymbol{\varepsilon}^2\right)\right),\\
&\boldsymbol{n}\frac{\partial}{\partial\boldsymbol{n}}\left|\mathsf{L}(\boldsymbol{n})\right|=4\boldsymbol{n}^2(\boldsymbol{n}\boldsymbol{\varepsilon}\boldsymbol{n})-2\left(\langle\boldsymbol{\varepsilon}\rangle\boldsymbol{n}\boldsymbol{\varepsilon}\boldsymbol{n}-\boldsymbol{n}\boldsymbol{\varepsilon}^2\boldsymbol{n}\right),
\end{aligned} \tag{12.7}$$

and the ray vectors $\boldsymbol{s}$ can be directly determined by the operator $\mathsf{L}(\boldsymbol{n})$ in the following way

$$\begin{aligned}
\boldsymbol{s}&\equiv\frac{\frac{\partial}{\partial\boldsymbol{n}}\left|\mathsf{L}(\boldsymbol{n})\right|}{\boldsymbol{n}\frac{\partial}{\partial\boldsymbol{n}}\left|\mathsf{L}(\boldsymbol{n})\right|}\\
&=\frac{\left(2(\boldsymbol{n}\boldsymbol{\varepsilon}\boldsymbol{n})\boldsymbol{n}+\boldsymbol{n}^2(\boldsymbol{\varepsilon}\boldsymbol{n}+\boldsymbol{n}\boldsymbol{\varepsilon})\right)-\left(\langle\boldsymbol{\varepsilon}\rangle(\boldsymbol{\varepsilon}\boldsymbol{n}+\boldsymbol{n}\boldsymbol{\varepsilon})-\left(\boldsymbol{\varepsilon}^2\boldsymbol{n}+\boldsymbol{n}\boldsymbol{\varepsilon}^2\right)\right)}{4\boldsymbol{n}^2(\boldsymbol{n}\boldsymbol{\varepsilon}\boldsymbol{n})-2\left(\langle\boldsymbol{\varepsilon}\rangle\boldsymbol{n}\boldsymbol{\varepsilon}\boldsymbol{n}-\boldsymbol{n}\boldsymbol{\varepsilon}^2\boldsymbol{n}\right)}.
\end{aligned} \tag{12.8}$$

This is part of the well-known duality between treatment of crystal optics in the space of refraction vectors and the treatment by ray vectors in the real vector space (e.g., [2], Eq. (97.19))

$$\boldsymbol{E}\leftrightarrow\boldsymbol{D},\quad \boldsymbol{\varepsilon}\leftrightarrow\boldsymbol{\varepsilon}^{-1},\quad \boldsymbol{n}\leftrightarrow\boldsymbol{s}. \tag{12.9}$$

In the cone approximation in the neighborhood of an optic axis from the dispersion equation

$$0=\delta\boldsymbol{n}\mathsf{C}_0\delta\boldsymbol{n}=(\boldsymbol{n}-\boldsymbol{n}_0)\mathsf{C}_0(\boldsymbol{n}-\boldsymbol{n}_0), \tag{12.10}$$

follows for the ray vectors $\boldsymbol{s}$ using the symmetry of the cone tensor $\mathsf{C}_0=\mathsf{C}_0^{\mathrm{T}}$

$$\boldsymbol{s}=\frac{\frac{\partial}{\partial\boldsymbol{n}}(\boldsymbol{n}-\boldsymbol{n}_0)\mathsf{C}_0(\boldsymbol{n}-\boldsymbol{n}_0)}{\boldsymbol{n}\frac{\partial}{\partial\boldsymbol{n}}(\boldsymbol{n}-\boldsymbol{n}_0)\mathsf{C}_0(\boldsymbol{n}-\boldsymbol{n}_0)}=\frac{\mathsf{C}_0(\boldsymbol{n}-\boldsymbol{n}_0)}{\boldsymbol{n}\mathsf{C}_0(\boldsymbol{n}-\boldsymbol{n}_0)}=\frac{\mathsf{C}_0(\boldsymbol{n}-\boldsymbol{n}_0)}{\boldsymbol{n}_0\mathsf{C}_0(\boldsymbol{n}-\boldsymbol{n}_0)}=\frac{\mathsf{C}_0\delta\boldsymbol{n}}{\boldsymbol{n}_0\mathsf{C}_0\delta\boldsymbol{n}}. \tag{12.11}$$

Thus from the cone approximation for the refraction vectors $\boldsymbol{n}$ in the neighborhood of the optic axis

$$0=(\boldsymbol{n}-\boldsymbol{n}_0)\mathsf{C}_0\mathsf{C}_0^{-1}\mathsf{C}_0(\boldsymbol{n}-\boldsymbol{n}_0)=\underbrace{\left(\boldsymbol{n}\mathsf{C}_0(\boldsymbol{n}-\boldsymbol{n}_0)\right)^2}_{\neq 0}\boldsymbol{s}\mathsf{C}_0^{-1}\boldsymbol{s}, \tag{12.12}$$

follows the equation for the ray vectors $\boldsymbol{s}$ in the neighborhood of an optic axis (see also [2], Eq. (97.14))

$$\boldsymbol{s}\mathsf{C}_0^{-1}\boldsymbol{s}=0,\quad \boldsymbol{n}\boldsymbol{s}=\boldsymbol{n}_0\boldsymbol{s}=1,\quad \Rightarrow\quad (\boldsymbol{n}-\boldsymbol{n}_0)\boldsymbol{s}=(\delta\boldsymbol{n}\boldsymbol{s})=0. \tag{12.13}$$

For the direction of an optic axis $\delta\boldsymbol{n}=\boldsymbol{n}-\boldsymbol{n}_0=\boldsymbol{0}$ the ray vectors $\boldsymbol{s}$ become indeterminate on the ray cone but the relation $\boldsymbol{n}_0\boldsymbol{s}=1$ remains true in the limit.

One must not forget that the derived relations are only true under neglect of the dispersion of the permittivity tensor $\boldsymbol{\varepsilon}$ and here additionally concern the cone approximation in the neighborhood of an optic axis. In other case it is usually more difficult to derive a scalar equation describing a surface, for example, for the group velocity. We did not intend to consider in this article in detail the mentioned known duality.

## 13. Conclusion

Our main result for crystal optics is the coordinate-invariant derivation of the cone equation (6.18) with the cone tensor (6.17) as approximation of the dispersion equation in the neighborhood of optic axes and of the eigenvalues and eigenvectors of this cone tensor with the basic results in (7.5) and (7.9). This cone proved to be an elliptic cone. Although the differences of the experimentally determined principal values of the permittivities for some crystals which we found in different sources are small and the elliptic cone is then very near to a circular cone it is a basic question of its form since they can be enlarged by external fields. Polarization vectors for the electric field to the refraction vectors in the neighborhood of optic axes are calculated. They depend mainly on the small deviations $\delta\boldsymbol{n} = \boldsymbol{n} - \boldsymbol{n}_0$ of the refraction vectors from the refraction vector $\boldsymbol{n}_0$ of the optic axis. Furthermore, the presence of optic axes as degeneration points of the dispersion equation for the refraction vectors is connected with the necessary vanishing of the second invariant of the three-dimensional wave-equation operator. This vanishing is not sufficient for the presence of an optic axis and involves also the case where the scalar product of right-hand and left-hand polarization vectors of the electric field is vanishing and cannot be normalized. Such cases cannot be present for real-valued refraction vectors. This problem was investigated but not fully solved. In Appendix A we calculate the eigenvalues and eigenvectors of a special three-dimensional operator. In a further Appendix B we consider shortly the introduction of axes in the sense of Fyodorov for an arbitrary second-rank symmetric tensor.

## Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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## Appendix A. Operator $\mathsf{B} \equiv \boldsymbol{x}\cdot\boldsymbol{y}+\boldsymbol{y}\cdot\boldsymbol{x}-2(\boldsymbol{xy})\mathsf{I}$ and Its Invariants and Complementary Operator

We consider the three-dimensional symmetrical operator $\mathsf{B}$ defined by

$$\mathsf{B} \equiv \boldsymbol{x}\cdot\boldsymbol{y}+\boldsymbol{y}\cdot\boldsymbol{x}-2(\boldsymbol{xy})\mathsf{I}, \tag{A.1}$$

with general three-dimensional vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ and calculate its invariants and its complementary operator. First, we have to calculate its second and third powers for which one easily finds

$$\begin{aligned}\mathsf{B}^2 &= -3(\boldsymbol{xy})(\boldsymbol{x}\cdot\boldsymbol{y}+\boldsymbol{y}\cdot\boldsymbol{x})+(\boldsymbol{y}^2)\boldsymbol{x}\cdot\boldsymbol{x}+(\boldsymbol{x}^2)\boldsymbol{y}\cdot\boldsymbol{y}+4(\boldsymbol{xy})^2\mathsf{I},\\ \mathsf{B}^3 &= \left((\boldsymbol{x}^2)(\boldsymbol{y}^2)+7(\boldsymbol{xy})^2\right)(\boldsymbol{x}\cdot\boldsymbol{y}+\boldsymbol{y}\cdot\boldsymbol{x})\\ &\quad -4(\boldsymbol{xy})\left((\boldsymbol{y}^2)\boldsymbol{x}\cdot\boldsymbol{x}+(\boldsymbol{x}^2)\boldsymbol{y}\cdot\boldsymbol{y}\right)-8(\boldsymbol{xy})^3\mathsf{I},\end{aligned} \tag{A.2}$$

and from which follows for their traces

$$\begin{aligned}\langle\mathsf{B}\rangle &= -4(\boldsymbol{xy}),\\ \langle\mathsf{B}^2\rangle &= 2\left(3(\boldsymbol{xy})^2+(\boldsymbol{x}^2)(\boldsymbol{y}^2)\right),\\ \langle\mathsf{B}^3\rangle &= -2(\boldsymbol{xy})\left(3(\boldsymbol{x}^2)(\boldsymbol{y}^2)+5(\boldsymbol{xy})^2\right).\end{aligned} \tag{A.3}$$

Using the general formulae (2.3) for three-dimensional operators one also easily finds the second invariant $[\mathsf{B}]$ and the determinant $|\mathsf{B}|$

$$\begin{aligned}\langle\mathsf{B}\rangle &= -4(\boldsymbol{xy}),\\ [\mathsf{B}] &= 5(\boldsymbol{xy})^2-(\boldsymbol{x}^2)(\boldsymbol{y}^2) = 4(\boldsymbol{xy})^2-[\boldsymbol{x},\boldsymbol{y}]^2,\\ |\mathsf{B}| &= 2(\boldsymbol{xy})\left((\boldsymbol{x}^2)(\boldsymbol{y}^2)-(\boldsymbol{xy})^2\right) = 2(\boldsymbol{xy})[\boldsymbol{x},\boldsymbol{y}]^2,\end{aligned} \tag{A.4}$$

with $[\boldsymbol{x},\boldsymbol{y}]$ the vector product of $\boldsymbol{x}$ and $\boldsymbol{y}$. We see immediately that the determinant $|\mathsf{B}|$ vanishes in the two special cases $\boldsymbol{x}=\boldsymbol{y}$ and $\boldsymbol{xy}=0$ where the operator $\mathsf{B}$ specializes to $\mathsf{B}=2\left(\boldsymbol{x}\cdot\boldsymbol{x}-(\boldsymbol{x}^2)\mathsf{I}\right)$ and to $\mathsf{B}=\boldsymbol{x}\cdot\boldsymbol{y}+\boldsymbol{y}\cdot\boldsymbol{x}$, respectively, and which can be calculated in more easier way. For the complementary operator $\overline{\mathsf{B}}$ to $\mathsf{B}$, generally defined in (2.4), we find

$$\begin{aligned}\overline{\mathsf{B}} &\equiv \mathsf{B}^2-\langle\mathsf{B}\rangle\mathsf{B}+[\mathsf{B}]\mathsf{I}\\ &= (\boldsymbol{xy})(\boldsymbol{x}\cdot\boldsymbol{y}+\boldsymbol{y}\cdot\boldsymbol{x})+(\boldsymbol{y}^2)\boldsymbol{x}\cdot\boldsymbol{x}+(\boldsymbol{x}^2)\boldsymbol{y}\cdot\boldsymbol{y}+\left((\boldsymbol{xy})^2-(\boldsymbol{x}^2)(\boldsymbol{y}^2)\right)\mathsf{I}\\ &= (\boldsymbol{x}\cdot\boldsymbol{y}+\boldsymbol{y}\cdot\boldsymbol{x})^2-[\boldsymbol{x},\boldsymbol{y}]^2\mathsf{I},\quad \langle\overline{\mathsf{B}}\rangle = 4(\boldsymbol{xy})^2-[\boldsymbol{x},\boldsymbol{y}]^2=[\mathsf{B}],\quad |\overline{\mathsf{B}}|=|\mathsf{B}|^2.\end{aligned} \tag{A.5}$$

Using the Hamilton-Cayley identity for the operator $\boldsymbol{x}\cdot\boldsymbol{y}+\boldsymbol{y}\cdot\boldsymbol{x}$ and the definition (A.1) one may check that the relation $\overline{\mathsf{B}}\mathsf{B}=|\mathsf{B}|\mathsf{I}$ is satisfied and, furthermore, $\langle\overline{\mathsf{B}}\rangle=[\mathsf{B}]$.

Alternatively, in this and similar cases one may calculate the complementary operator $\overline{\mathsf{B}}$ to $\mathsf{B}$ also from the eigenvalues $\beta$ and then the eigenvectors (or their projection operators $\Pi$ in the sense $\mathsf{B}\Pi=\Pi\mathsf{B}=\beta\Pi$) of $\mathsf{B}$ for which follows from (A.4)

$$\beta_0 = -2(\boldsymbol{xy}), \quad \Pi_0 = \frac{[\boldsymbol{x},\boldsymbol{y}]\cdot[\boldsymbol{x},\boldsymbol{y}]}{[\boldsymbol{x},\boldsymbol{y}]^2},$$

$$\beta_\pm = -\boldsymbol{xy} \pm \sqrt{(\boldsymbol{x}^2)(\boldsymbol{y}^2)}, \quad \Pi_\pm = \frac{\left(\sqrt{\boldsymbol{y}^2}\boldsymbol{x} \pm \sqrt{\boldsymbol{x}^2}\boldsymbol{y}\right)\cdot\left(\sqrt{\boldsymbol{y}^2}\boldsymbol{x} \pm \sqrt{\boldsymbol{x}^2}\boldsymbol{y}\right)}{2\sqrt{\boldsymbol{x}^2\boldsymbol{y}^2}\left(\sqrt{\boldsymbol{x}^2\boldsymbol{y}^2} \pm \boldsymbol{xy}\right)}, \tag{A.6}$$

$$\Pi_0^2 = \Pi_0, \quad \Pi_\pm^2 = \Pi_\pm, \quad \Pi_0\Pi_\pm = \Pi_+\Pi_- = 0, \quad \langle\Pi_0\rangle = \langle\Pi_\pm\rangle = 1.$$

Then one gets the representations

$$\mathsf{B} = \beta_0\Pi_0 + \beta_+\Pi_+ + \beta_-\Pi_-, \quad \overline{\mathsf{B}} = \beta_+\beta_-\Pi_0 + \beta_0(\beta_-\Pi_+ + \beta_+\Pi_-). \tag{A.7}$$

One may also write $\mathsf{B} = \mathsf{C} + \mathsf{D}$ with $\mathsf{C} \equiv \boldsymbol{x}\cdot\boldsymbol{y} + \boldsymbol{y}\cdot\boldsymbol{x}$ and $\mathsf{D} \equiv -2(\boldsymbol{xy})\mathsf{I}$ and then one may use all formulae derived for a sum of two three-dimensional operators $\mathsf{C}$ and $\mathsf{D}$.

The above coordinate-invariant calculations for the operator (A.1) represent also a good example illustrating the advantages to use coordinate-invariant methods in such and similar cases.

## Appendix B. Fyodorov's Introduction of Axes of a Symmetrical Three-Dimensional Tensor

We consider here according to Fyodorov [7] [8] the introduction of axes $\boldsymbol{c}_\pm$ for a three-dimensional second-rank real symmetrical tensor $\mathsf{A} = \mathsf{A}^* = \mathsf{A}^{\mathrm{T}}$ with the (trivial) extension to a Hermitean tensor $\mathsf{A} = \mathsf{A}^\dagger \equiv \mathsf{A}^{*\mathrm{T}}$

$$\begin{aligned}\mathsf{A} &= \alpha_1\boldsymbol{c}_1\cdot\boldsymbol{c}_1^* + \alpha_2\boldsymbol{c}_2\cdot\boldsymbol{c}_2^* + \alpha_3\boldsymbol{c}_3\cdot\boldsymbol{c}_3^* \\ &= \alpha_2\mathsf{I} - (\alpha_2 - \alpha_1)\boldsymbol{c}_1\cdot\boldsymbol{c}_1^* + (\alpha_3 - \alpha_2)\boldsymbol{c}_3\cdot\boldsymbol{c}_3^*, \quad \alpha_1 \le \alpha_2 \le \alpha_3, \quad \boldsymbol{c}_i^*\boldsymbol{c}_j = \delta_{ij},\end{aligned} \tag{B.1}$$

with the real eigenvalues $\alpha_i = \alpha_i^*$ and make the following transformation of the vectors $\boldsymbol{c}_1$ and $\boldsymbol{c}_3$ to new vectors $\boldsymbol{c}_\pm$ (denoted by $(\boldsymbol{c}', \boldsymbol{c}'')$ in [7] [8])

$$\boldsymbol{c}_\pm = \sqrt{\frac{\alpha_2 - \alpha_1}{\alpha_3 - \alpha_1}}\boldsymbol{c}_1 \pm \sqrt{\frac{\alpha_3 - \alpha_2}{\alpha_3 - \alpha_1}}\boldsymbol{c}_3, \quad \boldsymbol{c}_\pm^*\boldsymbol{c}_\pm = 1, \quad \boldsymbol{c}_\pm^*\boldsymbol{c}_\mp = \frac{2\alpha_2 - (\alpha_1 + \alpha_3)}{\alpha_3 - \alpha_1}, \tag{B.2}$$

with the inversion

$$\boldsymbol{c}_1 = \sqrt{\frac{\alpha_3 - \alpha_1}{\alpha_2 - \alpha_1}}\frac{\boldsymbol{c}_+ + \boldsymbol{c}_-}{2}, \quad \boldsymbol{c}_3 = \sqrt{\frac{\alpha_3 - \alpha_1}{\alpha_3 - \alpha_2}}\frac{\boldsymbol{c}_+ - \boldsymbol{c}_-}{2}. \tag{B.3}$$

Then we find for the $n$-th powers of $\mathsf{A}$

$$\begin{aligned}\mathsf{A}^n &= \alpha_1^n\boldsymbol{c}_1\cdot\boldsymbol{c}_1^* + \alpha_2^n\boldsymbol{c}_2\cdot\boldsymbol{c}_2^* + \alpha_3^n\boldsymbol{c}_3\cdot\boldsymbol{c}_3^* \\ &= \alpha_2^n\mathsf{I} - \frac{\alpha_3 - \alpha_1}{4}\sum_{k=0}^{n-1}\alpha_2^{n-1-k}\Big\{\left(\alpha_3^k + \alpha_1^k\right)\left(\boldsymbol{c}_+\cdot\boldsymbol{c}_-^* + \boldsymbol{c}_-\cdot\boldsymbol{c}_+^*\right) \\ &\quad - \left(\alpha_3^k - \alpha_1^k\right)\left(\boldsymbol{c}_+\cdot\boldsymbol{c}_+^* + \boldsymbol{c}_-\cdot\boldsymbol{c}_-^*\right)\Big\}.\end{aligned} \tag{B.4}$$

This formula is also correct for negative $n$-th powers $\mathsf{A}$. The second sum term in braces vanishes only for $n = 1$ leading to

$$\mathsf{A} = \alpha_2\mathsf{I} - \frac{\alpha_3 - \alpha_1}{2}\left(\boldsymbol{c}_+\cdot\boldsymbol{c}_-^* + \boldsymbol{c}_-\cdot\boldsymbol{c}_+^*\right). \tag{B.5}$$

For all other powers of $n$ the formula for $\mathsf{A}^n$ becomes complicated and does not bring an advantage in comparison to the first representation in (B.4). Fyodorov has the opposite sign at the first sum term in braces because he defined our vector $\boldsymbol{c}_-$ with the opposite sign (§ 26 of [8]). Furthermore, Fyodorov considers the case of real symmetric tensors $\boldsymbol{c}_i = \boldsymbol{c}_i^*$ and calls the directions determined by the unit vectors $\boldsymbol{c}_\pm$ the axes of the tensor $\mathsf{A}$. In accordance with general use he calls real symmetric tensors with $\alpha_1 = \alpha_3$ isotropic, with $\alpha_1 \neq \alpha_3$ but $\boldsymbol{c}_+ = \boldsymbol{c}_-$ uniaxial and general ones biaxial. In a later § 28 he considers also complex tensors in similar way.

The optic axes for the refraction vectors in the dispersion equation equivalent to the Fresnel equation are the axes (in the sense of F.) of the inverse permittivity tensor $\boldsymbol{\varepsilon}^{-1}$. The application of the axes of a second-rank symmetric tensor according to Fyodorov is often of restricted use since all powers of the permittivity tensor involved, e.g. in the dispersion equation, have different axes and the same is the case in many other equations but the equation for the magnetic field contains only the inverse permittivity tensor that is used by Fyodorov in his monograph [7] in § 22.

## Appendix C. Remarks to a Plagiarism of Fyodorov's Coordinate-Invariant Methods

This Appendix reports in form of an example about a fraud in science by Hollis C. Chen from Ohio University in U.S.A. who published in 1983 a book with the title "*Theory of Electromagnetic Waves, A Coordinate-Free Approach*" at McGraw-Hill Book Company, New York. This book is a plagiarism of two books of Fyodor Ivanovich Fyodorov from Minsk with titles "*Optika anisotropnykh sred* (Optics of Anisotropic Media), Minsk 1958 (2000 exemplars) and of "*Teoriya girotropii*" (Theory of Gyrotropy), Minsk 1976 (1050 exemplars). Some of the chapters of the book of Chen are almost a literal translation of chapters of the two books of Fyodorov and of one of my papers about radiation in uniaxial media in two chapters without any citation of the genuine authors but with citations which have nothing to do with coordinate-invariant representation. Often he uses totally the same letters for notations without trying to conceal this. What Chen made is the change of notations in the algebra of vectors and tensors from that of Fyodorov (which partially is an older but very good one) to that of Gibbs later introduced but less appropriate for many purposes, in particular, anisotropic media. The first chapter of Chen "Linear analysis" (pp. 1-57, with omissions) is almost the rewritten last chapter of Fyodorov's secondly mentioned book (pp. 362-440).

Due to present paper about degenerate and peculiar cases, in particular, optic axes I made a comparison of Fyodorov's work of three-dimensional algebra with the similar in mentioned chapter (of my copy) of Chen's book. From a general symmetric second-rank tensor of the form

$$\alpha = \lambda_1 \boldsymbol{e}_1 \cdot \boldsymbol{e}_1 + \lambda_2 \boldsymbol{e}_2 \cdot \boldsymbol{e}_1 + \lambda_3 \boldsymbol{e}_3 \cdot \boldsymbol{e}_3, \quad \left(\lambda_1 < \lambda_2 < \lambda_3, \boldsymbol{e}_i \boldsymbol{e}_j = \delta_{ij}\right), \tag{C.1}$$

with $1 = \boldsymbol{e}_1 \cdot \boldsymbol{e}_1 + \boldsymbol{e}_2 \cdot \boldsymbol{e}_1 + \boldsymbol{e}_3 \cdot \boldsymbol{e}_3$ it is easy to come to the representation (Fyodorov, Gyrotropy, p. 394) by the formula (Fyodorov usually omits identity operators)

$$\alpha = \lambda_2 + (\lambda_3 - \lambda_2)\boldsymbol{e}_3 \cdot \boldsymbol{e}_3 - (\lambda_2 - \lambda_1)\boldsymbol{e}_1 \cdot \boldsymbol{e}_1, \tag{C.2}$$

The same formula of Chen (p. 35) is written (dyadic products without a point between two vectors)

$$\mathbf{A} = \lambda_2 \bar{\mathbf{I}} + (\lambda_3 - \lambda_2)\hat{\mathbf{e}}_3\hat{\mathbf{e}}_3 - (\lambda_2 - \lambda_1)\hat{\mathbf{e}}_1\hat{\mathbf{e}}_1. \tag{C.3}$$

Now follows a linear combination of the vectors $\boldsymbol{e}$ to new non-orthogonal vectors which leads to a result (Fyodorov, p. 395, Eq. (26.108) and the same Ch., p. 35, Eq. 1.179 a little differently written). H. Ch. used that likely the Minsk colleagues and also I could not travel at that time to Conferences in the West and that the mentioned misused books of Fyodorov were difficult to get (even in libraries in East-Berlin) and that they are now a rarity.

To the overall-plagiarism by Chen I want to mention in addition the following case. In the effort to rationalize his notations Fyodorov does not introduce a special symbol for the second invariant of a three-dimensional operator $\mathsf{A}$ which I denote by $[\mathsf{A}] \equiv \frac{1}{2}\left(\langle \mathsf{A}\rangle^2 - \langle \mathsf{A}^2\rangle\right)$ and which he denotes by $\bar{\alpha}_c$ in [7] (from Russian "sled" which means "trace"; "c" is Russian letter for Latin "s") and $\bar{\alpha}_t$ in [8]. Apart from visibility and unfavorable distinction from vector indices this is not favorable since the complementary operator $\bar{\alpha}$ to $\alpha$ must then be directly determined by $\bar{\alpha} \equiv \alpha^2 - \alpha_t \alpha + \bar{\alpha}_t$ with $\bar{\alpha}$ on both sides and the Hamilton-Cayley identity is then $\alpha^3 - \alpha_t \alpha^2 + \bar{\alpha}_t \alpha - |\alpha| \equiv 0$ but in such and similar cases F. writes then in detail $\bar{\alpha}_t \equiv \frac{1}{2}\left[(\alpha_t)^2 - (\alpha^2)_t\right]$. Ch. makes exactly the same without citation of Fyodorov in his book only with some other letter types.