Journal of Modern Physics, 2022, 13, 442-494
https://www.scirp.org/journal/jmp
ISSN Online: 2153-120X
ISSN Print: 2153-1196

Nonlinear Conformal Electromagnetism

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How to cite this paper: Pommaret, J.F. (2022) Nonlinear Conformal Electromagnetism. *Journal of Modern Physics*, **13**, 442-494.
https://doi.org/10.4236/jmp.2022.134031

Received: February 25, 2022
Accepted: April 19, 2022
Published: April 22, 2022

Abstract

In 1909 the brothers E. and F. Cosserat discovered a new nonlinear group theoretical approach to elasticity (EL), with the only experimental need to measure the EL constants. In a modern language, their idea has been to use the nonlinear Spencer sequence instead of the nonlinear Janet sequence for the Lie groupoid defining the group of rigid motions of space. Following H. Weyl, our purpose is to compute for the first time the nonlinear Spencer sequence for the Lie groupoid defining the conformal group of space-time in order to provide the mathematical foundations of electromagnetism (EM), with the only experimental need to measure the EM constant in vacuum. With a manifold of dimension *n*, the difficulty is to deal with the *n* nonlinear transformations that have been called "elations" by E. Cartan in 1922. Using the fact that dimension $n=4$ has very specific properties for the computation of the Spencer cohomology, we prove that there is thus no conceptual difference between the Cosserat EL field or induction equations and the Maxwell EM field or induction equations. As a byproduct, the well known field/matter couplings (piezzoelectricity, photoelasticity, streaming birefringence, …) can be described abstractly, with the only experimental need to measure the corresponding coupling constants. The main consequence of this paper is the need to revisit the mathematical foundations of gauge theory (GT) because we have proved that EM was depending on the conformal group and not on $U(1)$, with a shift by one step to the left in the physical interpretation of the differential sequence involved.

Keywords

Nonlinear Differential Sequences, Linear Differential Sequences, Lie Groupoids, Lie Algebroids, Conformal Group, Spencer Cohomology, Maxwell Equations, Cosserat Equations

1. Introduction

Let us start this paper with a personal but meaningful story that has oriented my research during the last forty years or so, since the French "*Grandes Ecoles*" created their own research laboratories. Being a fresh permanent researcher of Ecole Nationale des Ponts et Chaussées in Paris, the author of this paper has been asked to become the scientific adviser of a young student in order to introduce him to research. As General Relativity was far too much difficult for somebody without any specific mathematical knowledge while remembering his own experience at the same age, he asked the student to collect about 50 books of Special Relativity and classify them along the way each writer was avoiding the use of the conformal group of space-time implied by the Michelson and Morley experiment, only caring about the Poincaré or Lorentz subgroups. After six months, the student (like any reader) arrived at the fact that most books were almost copying each other and could be nevertheless classified into three categories:

- 30 books, including the original 1905 paper ([1]) by Einstein, were at once, as a working assumption, deciding to restrict their study to a linear group reducing to the Galilée group when the speed of light was going to infinity. It must be noticed that people did believe that Einstein had not been influenced in 1905 by the Michelson and Morley experiment of 1887 till the discovery of hand written notes taken during lectures given by Einstein in Chicago (1921) and Kyoto (1922).
- 15 books were trying to "*prove*" that the conformal factor was indeed reduced to a constant equal to 1 when space-time was supposed to be homogeneous and isotropic.
- 5 books *only* were claiming that the conformal factor could eventually depend on the property of space-time, adding however that, if there was no surrounding electromagnetism or gravitation, the situation should be reduced to the preceding one but nothing was said otherwise.

The student was so disgusted by such a state of affair that he decided to give up on research and to become a normal civil engineer. As a byproduct, if group theory must be used, the underlying group of transformations of space-time *must* be related to the propagation of light *by itself* rather than by considering tricky signals between observers, thus *must* have to do with the biggest group of invariance of Maxwell Equations ([2] [3]). However, at the time we got the solution of this problem with the publication of ([4]) in 1988 (See [5] for recent results), a deep confusion was going on which is still not acknowledged though it can be explained in a few lines ([6]). Using standard notations of differential geometry, if the 2-form $F \in \wedge^2 T^*$ describing the EM field is satisfying the *first set* of Maxwell equations, it amounts to say that it is closed, that is killed by the exterior derivative $d : \wedge^2 T^* \to \wedge^3 T^*$. The EM field can be thus (locally) parametrized by the EM potential 1-form $A \in T^*$ with $dA = F$ where $d : T^* \to \wedge^2 T^*$ is again the exterior derivative, because $d^2 = d \circ d = 0$. Now, if E is a vector bundle over a manifold X of dimension n, then we may define its *adjoint* vector bundle $ad(E) = \wedge^n T^* \otimes E^*$ where E^* is obtained from E by inverting the transition rules, like T^* is obtained from $T=T(X)$ and such a construction can be extended to linear partial differential operators between (sections of) vector bundles. When $n=4$, it follows that the *second set* of Maxwell equations for the EM induction is just described by $ad(d):\wedge^4T^*\otimes\wedge^2T\to\wedge^4T^*\otimes T$, *independently of any Minkowski constitutive relation between field and induction*. Using *Hodge duality* with respect to the volume form $dx=dx^1\wedge\cdots\wedge dx^4$, this operator is isomorphic to $d:\wedge^2T^*\to\wedge^3T^*$. It follows that *both the first set and second set of Maxwell equations are invariant by any diffeomorphism* and that the conformal group of space-time is the biggest group of transformations preserving the Minkowski constitutive relations *in vacuum* where the speed of light is truly c as a universal constant. It was thus natural to believe that the mathematical structure of electromagnetism and gravitation had only to do with such a group having:

$$4\text{ translations}+6\text{ rotations}+1\text{ dilatation}+4\text{ elations}=15\text{ parameters}$$

the main difficulty being to deal with these later non-linear transformations. Of course, such a challenge could not be solved without the help of the non-linear theory of partial differential equations and Lie pseudogroups combined with homological algebra, that is before 1995 *at least* ([7]).

From a purely physical point of view, these new nonlinear methods have been introduced for the first time in 1909 by the brothers E. and F. Cosserat for studying the mathematical foundations of EL ([8]-[14]). We have presented their link with the *nonlinear Spencer differential sequences* existing in the formal theory of Lie pseudogroups at the end of our book "Differential Galois Theory" published in 1983 ([15]). Similarly, the conformal methods have been introduced by H. Weyl in 1918 for revisiting the mathematical foundations of EM ([3]). We have presented their link with the above approach through a *unique* differential sequence only depending on the structure of the *conformal group* in our book "Lie Pseudogroups and Mechanics" published in 1988 ([4]). However, the Cosserat brothers were only dealing with *translations* and *rotations* while Weyl was only dealing with *dilatation* and *elations*. Also, as an additional condition not fulfilled by the classical Einstein-Fokker-Nordström theory ([16]), if the conformal factor has to do with gravitation, it *must* be defined everywhere but at the central attractive mass as we already said.

From a purely mathematical point of view, the concept of a finite length *differential sequence*, now called *Janet sequence*, has been first described as a footnote by M. Janet in 1920 ([17]). Then, the work of D. C. Spencer in 1970 has been the first attempt to use the formal theory of systems of partial differential equations that he developed himself in order to study the formal theory of Lie pseudogroups ([18] [19] [20]). However, the *nonlinear Spencer sequences for Lie pseudogroups*, though never used in physics, largely supersede the "*Cartan structure equations*" introduced by E.Cartan in 1905 ([21] [22]) and are different from the "*Vessiot structure equations*" introduced by E. Vessiot in 1903 ([23]) or 1904 ([24]) for the same purpose but still not known today after more than a century because they have never been acknowledged by Cartan himself or even by his successors.

The purpose of the present difficult paper is to apply these new methods for studying the common nonlinear conformal origin of electromagnetism *and* gravitation, *in a purely mathematical way*, by constructing explicitly the corresponding nonlinear Spencer sequence. All the physical consequences will be presented in another paper.

2. Groupoids and Algebroids

Let us now turn to the clever way proposed by Vessiot in 1903 ([23]) and 1904 ([24]). Our purpose is only to sketch the main results that we have obtained in many books ([4] [7] [13] [15], we do not know other references) and to illustrate them by a series of specific examples, asking the reader to imagine any link with what has been said. We break the study into 8 successive steps.

1) If $\mathcal{E} = X \times X$, we shall denote by $\Pi_q = \Pi_q(X,X)$ the open sub-fibered manifold of $J_q(X \times X)$ defined independently of the coordinate system by $det(y_i^k) \neq 0$ with *source projection* $\alpha_q : \Pi_q \to X : (x, y_q) \to (x)$ and *target projection* $\beta_q : \Pi_q \to X : (x, y_q) \to (y)$. We shall sometimes introduce a copy Y of X with local coordinates (y) in order to avoid any confusion between the source and the target manifolds. In order to construct another nonlinear sequence, we need a few basic definitions on *Lie groupoids* and *Lie algebroids* that will become substitutes for Lie groups and Lie algebras. The first idea is to use the chain rule for derivatives $j_q(g \circ f) = j_q(g) \circ j_q(f)$ whenever $f, g \in aut(X)$ can be composed and to replace both $j_q(f)$ and $j_q(g)$ respectively by f_q and g_q in order to obtain the new section $g_q \circ f_q$. This kind of "composition" law can be written in a symbolic way by introducing another copy Z of X with local coordinates (z) as follows:

$$\gamma_q : \Pi_q(Y,Z) \times_Y \Pi_q(X,Y) \to \Pi_q(X,Z) :$$
$$\left(y, z, \frac{\partial z}{\partial y}, \cdots\right), \left(x, y, \frac{\partial y}{\partial x}, \cdots\right) \to \left(x, z, \frac{\partial z}{\partial y}\frac{\partial y}{\partial x}, \cdots\right)$$

We may also define $j_q(f)^{-1} = j_q(f^{-1})$ and obtain similarly an "inversion" law.

DEFINITION 2.1: A fibered submanifold $\mathcal{R}_q \subset \Pi_q$ is called a *system of finite Lie equations* or a *Lie groupoid* of order q if we have an induced *source projection* $\alpha_q : \mathcal{R}_q \to X$, *target projection* $\beta_q : \mathcal{R}_q \to X$, *composition* $\gamma_q : \mathcal{R}_q \times_X \mathcal{R}_q \to \mathcal{R}_q$, *inversion* $\iota_q : \mathcal{R}_q \to \mathcal{R}_q$ and *identity* $j_q(id) = id_q : X \to \mathcal{R}_q$. In the sequel we shall only consider *transitive* Lie groupoids such that the map $(\alpha_q, \beta_q) : \mathcal{R}_q \to X \times X$ is an epimorphism and we shall denote by $\mathcal{R}_q^0 = id^{-1}(\mathcal{R}_q)$ the *isotropy Lie group bundle* of $\mathcal{R}_q$. Also, one can prove that the new system $\rho_r(\mathcal{R}_q) = \mathcal{R}_{q+r}$ obtained by differentiating r times all the defining equations of $\mathcal{R}_q$ is a Lie groupoid of order $q+r$.

Let us start with a Lie pseudogroup $\Gamma \subset aut(X)$ defined by a system

$\mathcal{R}_q \subset \Pi_q$ of order *q*. Roughly speaking, if $f, g \in \Gamma \Rightarrow g \circ f, f^{-1} \in \Gamma$ but such a definition is totally meaningless in actual practice as it cannot be checked most of the time. In all the sequel we shall suppose that the system is involutive ([4] [7] [13] [15] [25]) and that Γ is *transitive* that is $\forall x, y \in X, \exists f \in \Gamma, y = f(x)$ or, equivalently, the map $(\alpha_q, \beta_q): \mathcal{R}_q \to X \times X : (x, y_q) \to (x, y)$ is surjective.

2) The Lie algebra $\Theta \subset T$ of infinitesimal transformations is then obtained by linearization, setting $y = x + t\xi(x) + \cdots$ and passing to the limit $t \to 0$ in order to obtain the linear involutive system $R_q = id_q^{-1}(V(\mathcal{R}_q)) \subset J_q(T)$ by reciprocal image with $\Theta = \{\xi \in T \mid j_q(\xi) \in R_q\}$. We define the *isotropy Lie algebra bundle* $R_q^0 \subset J_q^0(T)$ by the short exact sequence $0 \to R_q^0 \to R_q \xrightarrow{\pi_0^q} T \to 0$.

3) Passing from source to target, we may *prolong* the vertical infinitesimal transformations $\eta = \eta^k(y)\frac{\partial}{\partial y^k}$ to the jet coordinates up to order *q* in order to obtain:

$$\eta^k(y)\frac{\partial}{\partial y^k} + \left(\frac{\partial \eta^k}{\partial y^r} y_i^r\right)\frac{\partial}{\partial y_i^k} + \left(\frac{\partial^2 \eta^k}{\partial y^r \partial y^s} y_i^r y_j^s + \frac{\partial \eta^k}{\partial y^r} y_{ij}^r\right)\frac{\partial}{\partial y_{ij}^k} + \cdots$$

where we have replaced $j_q(f)(x)$ by y_q, each component being the "formal" derivative of the previous one .

4) As $[\Theta, \Theta] \subset \Theta$, we may use the Frobenius theorem in order to find a generating fundamental set of *differential invariants* $\{\Phi^\tau(y_q)\}$ up to order *q* which are such that $\Phi^\tau(\bar{y}_q) = \Phi^\tau(y_q)$ by using the chain rule for derivatives whenever $\bar{y} = g(y) \in \Gamma$ acting now on *Y*. Specializing the Φ^τ at $id_q(x)$ we obtain the *Lie form* $\Phi^\tau(y_q) = \omega^\tau(x)$ of $\mathcal{R}_q$.

Of course, in actual practice *one must use sections of* R_q *instead of solutions* and we now prove why the use of the Spencer operator becomes crucial for such a purpose. Indeed, using the *algebraic bracket* $\{j_{q+1}(\xi), j_{q+1}(\eta)\} = j_q([\xi, \eta]), \forall \xi, \eta \in T$, we may obtain by bilinearity a *differential bracket* on $J_q(T)$ extending the bracket on *T*:

$$[\xi_q, \eta_q] = \{\xi_{q+1}, \eta_{q+1}\} + i(\xi) D\eta_{q+1} - i(\eta) D\xi_{q+1}, \forall \xi_q, \eta_q \in J_q(T)$$

which does not depend on the respective lifts ξ_{q+1} and η_{q+1} of ξ_q and η_q in $J_{q+1}(T)$. This bracket on sections satisfies the Jacobi identity and we set ([4] [7] [13] [25]):

DEFINITION 2.2: We say that a vector subbundle $R_q \subset J_q(T)$ is a *system of infinitesimal Lie equations* or a *Lie algebroid* if $[R_q, R_q] \subset R_q$, that is to say $\xi_q, \eta_q \in R_q \Rightarrow [\xi_q, \eta_q] \in R_q$. Such a definition can be tested by means of computer algebra. We shall also say that R_q is *transitive* if we have the short exact sequence $0 \to R_q^0 \to R_q \xrightarrow{\pi_0^q} T \to 0$. In that case, a *splitting* of this sequence, namely a map $\chi_q : T \to R_q$ such that $\pi_0^q \circ \chi_q = id_T$ or equivalently a section $\chi_q \in T^* \otimes R_q$ over $id_T \in T^* \otimes T$, is called a R_q-*connection* and its *curvature* $\kappa_q \in \wedge^2 T^* \otimes R_q^0$ is defined by $\kappa_q(\xi, \eta) = [\chi_q(\xi), \chi_q(\eta)] - \chi_q([\xi, \eta]), \forall \xi, \eta \in T$.

PROPOSITION 2.3: If $\left[R_q, R_q\right] \subset R_q$, then $\left[R_{q+r}, R_{q+r}\right] \subset R_{q+r}, \forall r \geq 0$.

Proof: When $r = 1$, we have $\rho_1\left(R_q\right) = R_{q+1} = \left\{\xi_{q+1} \in J_{q+1}(T) \mid \xi_q \in R_q, D\xi_{q+1} \in T^* \otimes R_q\right\}$ and we just need to use the following formulas showing how D acts on the various brackets (See [7] and [25] for more details):

$$i(\zeta) D\left\{\xi_{q+1}, \eta_{q+1}\right\} = \left\{i(\zeta) D\xi_{q+1}, \eta_q\right\} + \left\{\xi_q, i(\zeta) D\eta_{q+1}\right\}, \quad \forall \zeta \in T$$

$$i(\zeta) D\left[\xi_{q+1}, \eta_{q+1}\right] = \left[i(\zeta) D\xi_{q+1}, \eta_q\right] + \left[\xi_q, i(\zeta) D\eta_{q+1}\right] + i\left(L(\eta_1)\zeta\right) D\xi_{q+1} - i\left(L(\xi_1)\zeta\right) D\eta_{q+1}$$

because the right member of the second formula is a section of R_q whenever $\xi_{q+1}, \eta_{q+1} \in R_{q+1}$. The first formula may be used when R_q is formally integrable.

□

EXAMPLE 2.4: With $n = 1, q = 3, X = \mathbb{R}$ and evident notations, the components of $[\xi_3, \eta_3]$ at order zero, one, two and three are defined by the totally unusual successive formulas:

$$[\xi, \eta] = \xi\partial_x \eta - \eta\partial_x \xi$$

$$([\xi_1, \eta_1])_x = \xi\partial_x \eta_x - \eta\partial_x \xi_x$$

$$([\xi_2, \eta_2])_{xx} = \xi_x \eta_{xx} - \eta_x \xi_{xx} + \xi\partial_x \eta_{xx} - \eta\partial_x \xi_{xx}$$

$$([\xi_3, \eta_3])_{xxx} = 2\xi_x \eta_{xxx} - 2\eta_x \xi_{xxx} + \xi\partial_x \eta_{xxx} - \eta\partial_x \xi_{xxx}$$

For affine transformations, $\xi_{xx} = 0, \eta_{xx} = 0 \Rightarrow ([\xi_2, \eta_2])_{xx} = 0$ and thus $[R_2, R_2] \subset R_2$.

For projective transformations, $\xi_{xxx} = 0, \eta_{xxx} = 0 \Rightarrow ([\xi_3, \eta_3])_{xxx} = 0$ and thus $[R_3, R_3] \subset R_3$.

THEOREM 2.5: (*prolongation/projection* (PP) *procedure*) If an arbitrary system $R_q \subseteq J_q(E)$ is given, one can *effectively* find two integers $r, s \geq 0$ such that the system $R_{q+r}^{(s)}$ is formally integrable or even involutive.

COROLLARY 2.6: The bracket is compatible with the PP procedure:

$$\left[R_q, R_q\right] \subset R_q \quad \Rightarrow \quad \left[R_{q+r}^{(s)}, R_{q+r}^{(s)}\right] \subset R_{q+r}^{(s)}, \forall r, s \geq 0$$

EXAMPLE 2.7: With $n = m = 2$ and $q = 1$, let us consider the Lie pseudodogroup $\Gamma \subset aut(X)$ of finite transformations $y = f(x)$ such that $y^2 dy^1 = x^2 dx^1 = \omega = \left(x^2, 0\right) \in T^*$. Setting $y = x + t\xi(x) + \cdots$ and linearizing, we get the Lie operator $\mathcal{D}\xi = \mathcal{L}(\xi)\omega$ where $\mathcal{L}$ is the Lie derivative because it is well known that $\left[\mathcal{L}(\xi), \mathcal{L}(\eta)\right] = \mathcal{L}(\xi) \circ \mathcal{L}(\eta) - \mathcal{L}(\eta) \circ \mathcal{L}(\xi) = \mathcal{L}([\xi, \eta])$ in the operator sense. The system $R_1 \subset J_1(T)$ of linear infinitesimal Lie equations is:

$$x^2 \partial_1 \xi^1 + \xi^2 = 0, \quad \partial_2 \xi^1 = 0$$

Replacing $j_1(\xi)$ by a section $\xi_1 \in J_1(T)$, we have:

$$\xi_1^1 = -\frac{1}{x^2} \xi^2, \quad \xi_2^1 = 0$$

Let us choose the two sections:

$$\xi_1 = \left\{ \xi^1 = 0, \xi^2 = -x^2, \xi_1^1 = 1, \xi_2^1 = 0, \xi_1^2 = 0, \xi_2^2 = 0 \right\} \in R_1$$

$$\eta_1 = \left\{ \eta^1 = x^2, \eta^2 = 0, \eta_1^1 = 0, \eta_2^1 = -x^2, \eta_1^2 = 0, \eta_2^2 = 1 \right\} \in R_1$$

We let the reader check that $[\xi_1, \eta_1] \in R_1$. However, we have the strict inclusion $R_1^{(1)} \subset R_1$ defined by the additional equation $\xi_1^1 + \xi_2^2 = 0$ and thus $\xi_1, \eta_1 \notin R_1^{(1)}$ though we have indeed $\left[R_1^{(1)}, R_1^{(1)} \right] \subset R_1^{(1)}$, a result not evident at all because the sections ξ_1 and η_1 have *nothing to do* with solutions. The reader may proceed in the same way with $x^2 dx^1 - x^1 dx^2$ and compare.

5) The main discovery of Vessiot, as early as in 1903 and thus fifty years in advance, has been to notice that the prolongation at order q of any horizontal vector field $\xi = \xi^i(x)\frac{\partial}{\partial x^i}$ commutes with the prolongation at order q of any vertical vector field $\eta = \eta^k(y)\frac{\partial}{\partial y^k}$, exchanging therefore the differential invariants. Keeping in mind the well known property of the Jacobian determinant while passing to the finite point of view, any (local) transformation $y = f(x)$ can be lifted to a (local) transformation of the differential invariants between themselves of the form $u \to \lambda(u, j_q(f)(x))$ allowing to introduce a *natural bundle* $\mathcal{F}$ over X by patching changes of coordinates $\bar{x} = \varphi(x), \bar{u} = \lambda(u, j_q(\varphi)(x))$. A section ω of $\mathcal{F}$ is called a *geometric object* or *structure* on X and transforms like $\bar{\omega}(f(x)) = \lambda(\omega(x), j_q(f)(x))$ or simply $\bar{\omega} = j_q(f)(\omega)$. This is a way to generalize vectors and tensors ($q = 1$) or even connections ($q = 2$). As a byproduct we have $\Gamma = \left\{ f \in aut(X) \mid \Phi_\omega\left(j_q(f)\right) = j_q(f)^{-1}(\omega) = \omega \right\}$ as a new way to write out the Lie form and we may say that Γ *preserves* ω. We also obtain $\mathcal{R}_q = \left\{ f_q \in \Pi_q \mid f_q^{-1}(\omega) = \omega \right\}$. Coming back to the infinitesimal point of view and setting $f_t = exp(t\xi) \in aut(X), \forall \xi \in T$, we may define the *ordinary Lie derivative* with value in $F = \omega^{-1}(V(\mathcal{F}))$ by introducing the *vertical bundle* of $\mathcal{F}$ as a vector bundle over $\mathcal{F}$ and the formula:

$$\mathcal{D}\xi = \mathcal{L}(\xi)\omega = \frac{d}{dt} j_q(f_t)^{-1}(\omega)\Big|_{t=0} \Rightarrow \Theta = \left\{ \xi \in T \mid \mathcal{L}(\xi)\omega = 0 \right\}$$

while we have $x \to x + t\xi(x) + \cdots \Rightarrow u^\tau \to u^\tau + t\partial_\mu \xi^k L_k^{\tau\mu}(u) + \cdots$ where $\mu = (\mu_1, \cdots, \mu_n)$ is a multi-index as a way to write down the system $R_q \subset J_q(T)$ of infinitesimal Lie equations in the *Medolaghi form*:

$$\Omega^\tau \equiv \left(\mathcal{L}(\xi)\omega\right)^\tau \equiv -L_k^{\tau\mu}\left(\omega(x)\right)\partial_\mu \xi^k + \xi^r \partial_r \omega^\tau(x) = 0$$

EXAMPLE 2.8: With $n = 1$, let us consider the Lie group of projective transformations $y = (ax + b)/(cx + d)$ as a lie pseudogroup. Differentiating three times in order to eliminate the parameters, we obtain the third order *Schwarzian* OD equation and its linearization over $y = x$:

$$\mathcal{R}_3 \subset \Pi_3, \quad \Psi \equiv \frac{y_{xxx}}{y_x} - \frac{3}{2}\left(\frac{y_{xx}}{y_x}\right)^2 = 0$$

$$R_3 \subset J_3(T), \quad \xi_{xxx} = 0$$

Accordingly, the prolongation $\#(\eta_3)$ of any $\eta_3 \in J_3(T(Y))$ over Y such that $\eta_{yyy} = 0$ becomes:

$$\eta(y)\frac{\partial}{\partial y} + \eta_y(y)\left(y_x\frac{\partial}{\partial y_x} + y_{xx}\frac{\partial}{\partial y_{xx}} + y_{xxx}\frac{\partial}{\partial y_{xxx}}\right)$$
$$+\eta_{yy}(y)\left((y_x)^2\frac{\partial}{\partial y_{xx}} + 3y_x y_{xx}\frac{\partial}{\partial y_{xxx}}\right)$$

It follows that Ψ is a generating third order differential invariant and R_3 is in Lie form.

Now, we have:

$$\bar{x} = \varphi(x) \Rightarrow y_x = y_{\bar{x}}\partial_x\varphi, y_{xx} = y_{\bar{x}\bar{x}}(\partial_x\varphi)^2 + y_{\bar{x}}\partial_{xx}\varphi,$$
$$y_{xxx} = y_{\bar{x}\bar{x}\bar{x}}(\partial_x\varphi)^3 + 3y_{\bar{x}\bar{x}}\partial_x\varphi\partial_{xx}\varphi + y_{\bar{x}}\partial_{xxx}\varphi$$

and the natural bundle $\mathcal{F}$ is thus defined by the transition rules:

$$\bar{x} = \varphi(x), \; u = \bar{u}(\partial_x\varphi)^2 + \left(\frac{\partial_{xxx}\varphi}{\partial_x\varphi} - \frac{3}{2}\left(\frac{\partial_{xx}\varphi}{\partial_x\varphi}\right)^2\right)$$

The general Lie form of $\mathcal{R}_3$ is:

$$\frac{y_{xxx}}{y_x} - \frac{3}{2}\left(\frac{y_{xx}}{y_x}\right)^2 + \omega(y)(y_x)^2 = \omega(x)$$

We obtain $R_3 \subset J_3(T)$ in Medolaghi form as follows:

$$\Omega \equiv \mathcal{L}(\xi)\omega \equiv \partial_{xxx}\xi + 2\omega(x)\partial_x\xi + \xi\partial_x\omega(x) = 0$$

Using a section $\xi_3 \in J_3(T)$, we finally get the formal Lie derivative:

$$\Omega \equiv L(\xi_3)\omega \equiv \xi_{xxx} + 2\omega(x)\xi_x + \xi\partial_x\omega(x) = 0$$

and let the reader ckeck directly that $[L(\xi_3), L(\eta_3)] = L([\xi_3, \eta_3])$, $\forall \xi_3, \eta_3 \in J_3(T)$, a result absolutely not evident at first sight.

6) By analogy with "special" and "general" relativity, we shall call the given section *special* and any other arbitrary section *general*. The problem is now to study the formal properties of the linear system just obtained with coefficients only depending on $j_1(\omega)$, exactly like L.P. Eisenhart did for $\mathcal{F} = S_2T^*$ when finding the constant Riemann curvature condition for a metric ω with $det(\omega) \neq 0$ ([7], Example 10, p 246 to 256, [26]). Indeed, if any expression involving ω and its derivatives is a scalar object, it must reduce to a constant because Γ is assumed to be transitive and thus cannot be defined by any zero order equation. Now one can prove that the CC for $\bar{\omega}$, thus for ω too, only depend on the Φ and take the quasi-linear symbolic form $v \equiv I(u_1) \equiv A(u)u_x + B(u) = 0$, allowing to define an affine subfibered manifold $\mathcal{B}_1 \subset J_1(\mathcal{F})$ over $\mathcal{F}$. Now, if one has two sections ω and $\bar{\omega}$ of $\mathcal{F}$, the *equivalence problem* is to look for $f \in aut(X)$ such that $j_q(f)^{-1}(\omega) = \bar{\omega}$. When the two sections satisfy the same CC, the problem is sometimes locally

possible (Lie groups of transformations, Darboux problem in analytical mechanics, …) but sometimes not ([25], p. 333).

7) Instead of the CC for the equivalence problem, let us look for the *integrability conditions* (IC) for the system of infinitesimal Lie equations and suppose that, for the given section, all the equations of order $q+r$ are obtained by differentiating *r* times only the equations of order *q*, then it was claimed by Vessiot ([23] with no proof, see [7], pp. 207-211) that such a property is held if and only if there is an equivariant section $c:\mathcal{F}\to\mathcal{F}_1:(x,u)\to(x,u,v=c(u))$ where $\mathcal{F}_1=J_1(\mathcal{F})/\mathcal{B}_1$ is a natural vector bundle over $\mathcal{F}$ with local coordinates (x,u,v). Moreover, any such equivariant section depends on a finite number of constants *c* called *structure constants* and the IC for the *Vessiot structure equations* $I(u_1)=c(u)$ are of a polynomial form $J(c)=0$.

EXAMPLE 2.9: Comig back to Example 2.7 first considered by Vessiot as early as in 1903 ([23]), the geometric object $\omega=(\alpha,\beta)\in T^*\otimes_X\wedge^2T^*$ must satisfy the Vessiot structure equation $d\alpha=c\beta$ with a single Vessiot structure constant $c=-1$ in the situation considered where $\alpha=x^2dx^1$ and $\beta=dx^1\wedge dx^2$ (See ([27]) for other examples and applications). As a byproduct, there is no conceptual difference between such a constant and the constant appearing in the constant Riemannian curvature condition of Eisenhart ([26]).

8) Finally, when *Y* is no longer a copy of *X*, a system $\mathcal{A}_q\subset J_q(X\times Y)$ is said to be an *automorphic system* for a Lie pseudogroup $\Gamma\subset aut(Y)$ if, whenever $y=f(x)$ and $\bar{y}=\bar{f}(x)$ are two solutions, then there exists one and only one transformation $\bar{y}=g(y)\in\Gamma$ such that $\bar{f}=g\circ f$. Explicit tests for checking such a property formally have been given in ([15]) and can be implemented on computer in the differential algebraic framework.

3. Nonlinear Sequences

Contrary to what happens in the study of Lie pseudogroups and in particular in the study of the algebraic ones that can be found in mathematical physics, *nonlinear operators do not in general admit* CC, unless they are defined by differential polynomials, as can be seen by considering the two following examples with $m=1,n=2,q=2$. With standard notations from differential algebra, if we are dealing with a ground differential field *K*, like $\mathbb{Q}$ in the next examples, we denote by $K\{y\}$ the ring (which is even an integral domain) of differential polynomials in *y* with coefficients in *K* and by $K\langle y\rangle=Q(K\{y\})$ the corresponding quotient field of differential rational functions in *y*. Then, if $u,v\in K\langle y\rangle$, we have the two towers $K\subset K\langle u\rangle\subset K\langle y\rangle$ and $K\subset K\langle v\rangle\subset K\langle y\rangle$ of extensions, thus the tower $K\subset K\langle u,v\rangle\subset K\langle y\rangle$. Accordingly, the differential extension $K\langle u,v\rangle/K$ is a finitely generated differential extension. If we consider *u* and *v* as new indeterminates, then $K\langle u\rangle$ and $K\langle v\rangle$ are both differential transcendental extensions of *K* and the kernel of the canonical differential morphism $K\{u\}\otimes_K K\{v\}\to K\langle y\rangle$ is a prime differential ideal in the differential integral domain $K\langle y\rangle\otimes_K K\langle v\rangle$, a way to describe by residue the smallest differential field containing $K\langle u\rangle$ and $K\langle v\rangle$ in $K\langle y\rangle$. Of course, the true difficulty is to find out such a prime differential ideal.

EXAMPLE 3.1: First of all, let us consider the following nonlinear system in y with second member (u,v):

$$P \equiv y_{22} - \frac{1}{3}(y_{11})^3 = u,\ Q \equiv y_{12} - \frac{1}{2}(y_{11})^2 = v \Rightarrow y_{11} = \frac{u_1 - v_2}{v_1}$$

The differential ideal $\mathfrak{a}$ generated by P and Q in $\mathbb{Q}\{y\}$ is prime because $d_2Q + d_1P - y_{11}d_1Q = 0$ and thus $\mathbb{Q}\{y\}/\{P,Q\} \simeq \mathbb{Q}[y, y_1, y_2, y_{11}, y_{111}, \cdots]$ is an integral domain.

We may consider the following nonlinear involutive system with two equations:

$$\begin{cases} y_{22} - \frac{1}{3}(y_{11})^3 = 0 \\ y_{12} - \frac{1}{2}(y_{11})^2 = 0 \end{cases} \quad \boxed{\begin{matrix} 1 & 2 \\ 1 & \bullet \end{matrix}}$$

We have also the linear inhomogeneous finite type second order system with three equations:

$$\begin{cases} y_{22} = u + \frac{1}{3}\left(\frac{u_1 - v_2}{v_1}\right)^3 \\ y_{12} = v + \frac{1}{2}\left(\frac{u_1 - v_2}{v_1}\right)^2 \\ y_{11} = \frac{u_1 - v_2}{v_1} \end{cases} \quad \boxed{\begin{matrix} 1 & 2 \\ 1 & \bullet \\ 1 & \bullet \end{matrix}}$$

Though we have *a priori* two CC, we let the reader prove, as a delicate exercise, that there is only the single nonlinear second order CC obtained from the bottom dot:

$$\boxed{d_2\left(\frac{u_1 - v_2}{v_1}\right) - d_1\left(v + \frac{1}{2}\left(\frac{u_1 - v_2}{v_1}\right)^2\right) = 0}$$

EXAMPLE 3.2: On the contrary, if we consider the following new nonlinear system:

$$P \equiv y_{22} - \frac{1}{2}(y_{11})^2 = u,\ Q \equiv y_{12} - y_{11} = v \Rightarrow (y_{11} - 1)y_{111} = v_2 + v_1 - u_1 = w$$

we obtain successively:

$$d_2Q + d_1Q - d_1P \equiv (y_{11} - 1)y_{111}$$

$$y_{111}(d_{12}Q + d_{11}Q - d_{11}P) - y_{1111}(d_2Q + d_1Q - d_1P) = (y_{111})^3$$

The symbol at order 3 is thus not a vector bundle and no direct study as above can be used because the differential ideal generated by (P,Q) is not perfect as it contains $(y_{111})^3$ without containing y_{111} (See [15] and [28] for more details). The following nonlinear system is *not* involutive:

$$\left\{\begin{array}{l} y_{22}-\frac{1}{2}(y_{11})^2=0 \\ y_{12}-y_{11}=0 \end{array}\right. \quad \boxed{\begin{array}{cc} 1 & 2 \\ 1 & \bullet \end{array}}$$

We have the following four generic nonlinear additional finite type third order equations:

$$\left\{\begin{array}{l} y_{222}-y_{11}\left(v_1+\dfrac{w}{y_{11}-1}\right)=u_2 \\ y_{122}-y_{11}\dfrac{w}{y_{11}-1}=u_1 \\ y_{112}-\dfrac{w}{y_{11}-1}=v_1 \\ y_{111}-\dfrac{w}{y_{11}-1}=0 \end{array}\right. \quad \boxed{\begin{array}{cc} 1 & 2 \\ 1 & \bullet \\ 1 & \bullet \\ 1 & \bullet \end{array}}$$

Though we have now *a priori* three CC and thus three additional equations because the system is not involutive, setting $y_{11}-1=z \Rightarrow y_{112}=z_2, y_{111}=z_1$, there is only the single additional nonlinear second order equation:

$$\boxed{v_{11}z^2+(w_1-w_2)z+v_1w=0}$$

Differentiating once and using the relation $zz_1=w$, we get:

$$\boxed{v_{111}z^3+(w_{11}-w_{12})z^2+(v_1w_1+3v_{11}w)z+(w_1-w_2)w=0}$$

a result leading to a tricky resultant providing a third order differential polynomial in (u,v).

However, the kernel of a linear operator $\mathcal{D}:E\to F$ is always taken with respect to the zero section of *F*, while it must be taken with respect to a prescribed section by a *double arrow* for a nonlinear operator. Keeping in mind the linear Janet sequence and the examples of Vessiot structure equations already presented, one obtains:

THEOREM 3.3: There exists a *nonlinear Janet sequence* associated with the Lie form of an involutive system of finite Lie equations:

$$0\to\Gamma\to aut(X)\ \underset{\omega\circ\alpha}{\overset{\Phi\circ j_q}{\rightrightarrows}}\ \mathcal{F}\ \underset{0}{\overset{I\circ j_1}{\rightrightarrows}}\ \mathcal{F}_1$$

where the kernel of the first operator $f\to\Phi\circ j_q(f)=\Phi(j_q(f))=j_q(f)^{-1}(\omega)$ is taken with respect to the section ω of $\mathcal{F}$ while the kernel of the second operator $\omega\to I(j_1(\omega))\equiv A(\omega)\partial_x\omega+B(\omega)$ is taken with respect to the zero section of the vector bundle $\mathcal{F}_1$ over $\mathcal{F}$.

COROLLARY 3.4: By linearization at the identity, one obtains the involutive *Lie operator* $\mathcal{D}:T\to F_0:\xi\to\mathcal{L}(\xi)\omega$ with kernel $\Theta=\{\xi\in T\mid\mathcal{L}(\xi)\omega=0\}\subset T$ satisfying $[\Theta,\Theta]\subset\Theta$ and the corresponding *linear Janet sequence*:

$$0 \to \Theta \to T \xrightarrow{\mathcal{D}} F_0 \xrightarrow{\mathcal{D}_1} F_1$$

where $F_0 = F = \omega^{-1}(V(\mathcal{F}))$ and $F_1 = \omega^{-1}(\mathcal{F}_1)$.

Now we notice that T is a natural vector bundle of order 1 and $J_q(T)$ is thus a natural vector bundle of order $q+1$. Looking at the way a vector field and its derivatives are transformed under any $f \in aut(X)$ while replacing $j_q(f)$ by f_q, we obtain:

$$\eta^k(f(x)) = f_r^k(x)\xi^r(x) \Rightarrow \eta_u^k(f(x)) f_i^u(x) = f_r^k(x)\xi_i^r(x) + f_{ri}^k(x)\xi^r(x)$$

and so on, a result leading to:

LEMMA 3.5: $J_q(T)$ is *associated* with $\Pi_{q+1} = \Pi_{q+1}(X,X)$ that is we can obtain a new section $\eta_q = f_{q+1}(\xi_q)$ from any section $\xi_q \in J_q(T)$ and any section $f_{q+1} \in \Pi_{q+1}$ by the formula:

$$d_\mu \eta^k \equiv \eta_r^k f_\mu^r + \cdots = f_r^k \xi_\mu^r + \cdots + f_{\mu+1_r}^k \xi^r, \forall 0 \le |\mu| \le q$$

where the left member belongs to $V(\Pi_q)$. Similarly $R_q \subset J_q(T)$ is associated with $\mathcal{R}_{q+1} \subset \Pi_{q+1}$.

More generally, looking now for transformations "close" to the identity, that is setting $y = x + t\xi(x) + \cdots$ when $t \ll 1$ is a small constant parameter and passing to the limit $t \to 0$, we may linearize any (nonlinear) *system of finite Lie equations* in order to obtain a (linear) *system of infinitesimal Lie equations* $R_q \subset J_q(T)$ for vector fields. Such a system has the property that, if ξ, η are two solutions, then $[\xi,\eta]$ is also a solution. Accordingly, the set $\Theta \subset T$ of its solutions satisfies $[\Theta,\Theta] \subset \Theta$ and can therefore be considered as the Lie algebra of Γ.

More generally, the next definition will extend the *classical Lie derivative*:

$$\mathcal{L}(\xi)\omega = (i(\xi)d + di(\xi))\omega = \frac{d}{dt} j_q(\exp t\xi)^{-1}(\omega)\Big|_{t=0}.$$

DEFINITION 3.6: We say that a vector bundle F is *associated* with R_q if there exists a first order differential operator $L(\xi_q): F \to F$ called *formal Lie derivative* and such that:

1) $L(\xi_q + \eta_q) = L(\xi_q) + L(\eta_q) \quad \forall \xi_q, \eta_q \in R_q$.
2) $L(f\xi_q) = fL(\xi_q) \quad \forall \xi_q \in R_q, \forall f \in C^\infty(X)$.
3) $[L(\xi_q), L(\eta_q)] = L(\xi_q) \circ L(\eta_q) - L(\eta_q) \circ L(\xi_q) = L([\xi_q, \eta_q]) \quad \forall \xi_q, \eta_q \in R_q$.
4) $L(\xi_q)(f\eta) = fL(\xi_q)\eta + (\xi f)\eta \quad \forall \xi_q \in R_q, \forall f \in C^\infty(X), \forall \eta \in F$.

LEMMA 3.7: If E and F are associated with R_q, we may set on $E \otimes F$:

$$L(\xi_q)(\eta \otimes \zeta) = L(\xi_q)\eta \otimes \zeta + \eta \otimes L(\xi_q)\zeta \quad \forall \xi_q \in R_q, \forall \eta \in E, \forall \zeta \in F$$

If $\Theta \subset T$ denotes the solutions of R_q, then we may set $\mathcal{L}(\xi) = L(j_q(\xi)), \forall \xi \in \Theta$ but no explicit computation can be done when Θ is infinite dimensional. However, we have:

PROPOSITION 3.8: $J_q(T)$ is associated with $J_{q+1}(T)$ if we define:

$$L(\xi_{q+1})\eta_q = \{\xi_{q+1}, \eta_{q+1}\} + i(\xi)D\eta_{q+1} = [\xi_q, \eta_q] + i(\eta)D\xi_{q+1}$$

and thus R_q is associated with R_{q+1}.

Proof: It is easy to check the properties 1, 2, 4 and it only remains to prove property 3 as follows.

$$\begin{aligned}
&\left[L\left(\xi_{q+1}\right),L\left(\eta_{q+1}\right)\right]\zeta_q\\
&=L\left(\xi_{q+1}\right)\left(\left\{\eta_{q+1},\zeta_{q+1}\right\}+i(\eta)D\zeta_{q+1}\right)-L\left(\eta_{q+1}\right)\left(\left\{\xi_{q+1},\zeta_{q+1}\right\}+i(\xi)D\zeta_{q+1}\right)\\
&=\left\{\xi_{q+1},\left\{\eta_{q+2},\zeta_{q+2}\right\}\right\}-\left\{\eta_{q+1},\left\{\xi_{q+2},\zeta_{q+2}\right\}\right\}+\left\{\xi_{q+1},i(\eta)D\zeta_{q+2}\right\}\\
&\quad-\left\{\eta_{q+1},i(\xi)D\zeta_{q+2}\right\}+i(\xi)D\left\{\eta_{q+2},\zeta_{q+2}\right\}-i(\eta)D\left\{\xi_{q+2},\zeta_{q+2}\right\}\\
&\quad+i(\xi)D\left(i(\eta)D\zeta_{q+2}\right)-i(\eta)D\left(i(\xi)D\zeta_{q+2}\right)\\
&=\left\{\left\{\xi_{q+2},\eta_{q+2}\right\},\zeta_{q+1}\right\}+\left\{i(\xi)D\eta_{q+2},\zeta_{q+1}\right\}-\left\{i(\eta)D\xi_{q+2},\zeta_{q+1}\right\}+i([\xi,\eta])D\zeta_{q+1}\\
&=\left\{\left[\xi_{q+1},\eta_{q+1}\right],\zeta_{q+1}\right\}+i([\xi,\eta])D\zeta_{q+1}
\end{aligned}$$

by using successively the Jacobi identity for the algebraic bracket and the last proposition.

□

EXAMPLE 3.9: T and T^* both with any tensor bundle are associated with $J_1(T)$. For T we may define $L(\xi_1)\eta=[\xi,\eta]+i(\eta)D\xi_1=\{\xi_1,j_1(\eta)\}$. We have $\xi^r\partial_r\eta^k-\eta^s\partial_s\xi^k+\eta^s\left(\partial_s\xi^k-\xi_s^k\right)=-\eta^s\xi_s^k+\xi^r\partial_r\eta^k$ and the four properties of the formal Lie derivative can be checked directly. Of course, we find back $\mathcal{L}(\xi)\eta=[\xi,\eta],\forall\xi,\eta\in T$. We let the reader treat similarly the case of T^*.

PROPOSITION 3.10: There is a *first nonlinear Spencer sequence*:

$$0\to aut(X)\xrightarrow{j_{q+1}}\Pi_{q+1}(X,X)\xrightarrow{\bar{D}}T^*\otimes J_q(T)\xrightarrow{\bar{D}'}\wedge^2T^*\otimes J_{q-1}(T)$$

with $\bar{D}f_{q+1}\equiv f_{q+1}^{-1}\circ j_1\left(f_q\right)-id_{q+1}=\chi_q$
$\Rightarrow\bar{D}'\chi_q(\xi,\eta)\equiv D\chi_q(\xi,\eta)-\left\{\chi_q(\xi),\chi_q(\eta)\right\}=0$. Moreover, setting $\chi_0=A-id\in T^*\otimes T$, this sequence is locally exact if $det(A)\neq0$.

Proof: There is a canonical inclusion $\Pi_{q+1}\subset J_1\left(\Pi_q\right)$ defined by $y_{\mu,i}^k=y_{\mu+1_i}^k$ and the composition $f_{q+1}^{-1}\circ j_1\left(f_q\right)$ is a well defined section of $J_1\left(\Pi_q\right)$ over the section $f_q^{-1}\circ f_q=id_q$ of Π_q like id_{q+1}. The difference $\chi_q=f_{q+1}^{-1}\circ j_1\left(f_q\right)-id_{q+1}$ is thus a section of $T^*\otimes V\left(\Pi_q\right)$ over id_q and we have already noticed that $id_q^{-1}\left(V\left(\Pi_q\right)\right)=J_q(T)$. For $q=1$ we get with $g_1=f_1^{-1}$:

$$\chi_{,i}^k=g_l^k\partial_if^l-\delta_i^k=A_i^k-\delta_i^k,\quad\chi_{j,i}^k=g_l^k\left(\partial_if_j^l-A_i^rf_{rj}^l\right)$$

We also obtain from Lemma 3.5 the useful formula $f_r^k\chi_{\mu,i}^r+\cdots+f_{\mu+1_r}^k\chi_{,i}^r=\partial_if_\mu^k-f_{\mu+1_i}^k$ allowing to determine χ_q inductively.

We refer to ([7], p 215-216) for the inductive proof of the local exactness, providing the only formulas that will be used later on and can be checked directly by the reader:

$$\partial_i\chi_{,j}^k-\partial_j\chi_{,i}^k-\chi_{i,j}^k+\chi_{j,i}^k-\left(\chi_{,i}^r\chi_{r,j}^k-\chi_{,j}^r\chi_{r,i}^k\right)=0\qquad(1)$$

$$\partial_i\chi_{l,j}^k-\partial_j\chi_{l,i}^k-\chi_{li,j}^k+\chi_{lj,i}^k-\left(\chi_{,i}^r\chi_{lr,j}^k+\chi_{l,i}^r\chi_{r,j}^k-\chi_{l,j}^r\chi_{r,i}^k-\chi_{,j}^r\chi_{lr,i}^k\right)=0\qquad(2)$$

$$\begin{aligned}&\partial_i \chi^k_{lr,j} - \partial_j \chi^k_{lr,i} - \chi^k_{lri,j} + \chi^k_{lrj,i} - \left(\chi^s_{,i}\chi^k_{lrs,j} + \chi^s_{r,i}\chi^k_{ls,j} + \chi^s_{l,i}\chi^k_{rs,j} \right. \\ &\left. + \chi^s_{lr,i}\chi^k_{s,j} - \chi^s_{,j}\chi^k_{lrs,i} - \chi^s_{r,j}\chi^k_{ls,i} - \chi^s_{l,j}\chi^k_{rs,i} - \chi^s_{lr,j}\chi^k_{s,i} \right) = 0\end{aligned} \quad (3)$$

There is no need for double-arrows in this framework as the kernels are taken with respect to the zero section of the vector bundles involved. We finally notice that the main difference with the gauge sequence is that *all the indices range from* 1 *to n* and that the condition $det(A) \neq 0$ amounts to $\Delta = det\left(\partial_i f^k\right) \neq 0$ because $det\left(f_i^k\right) \neq 0$ by assumption.

□

COROLLARY 3.11: There is a *restricted first nonlinear Spencer sequence*:

$$0 \to \Gamma \xrightarrow{j_{q+1}} \mathcal{R}_{q+1} \xrightarrow{\bar{D}} T^* \otimes R_q \xrightarrow{\bar{D}'} \wedge^2 T^* \otimes J_{q-1}(T)$$

DEFINITION 3.12: A *splitting* of the short exact sequence

$0 \to R_q^0 \to R_q \xrightarrow{\pi_0^q} T \to 0$ is a map $\chi_q' : T \to R_q$ such that $\pi_0^q \circ \chi_q' = id_T$ or equivalently a section of $T^* \otimes R_q$ over $id_T \in T^* \otimes T$ and is called a R_q-*connection*. Its *curvature* $\kappa_q' \in \wedge^2 T^* \otimes R_q^0$ is defined by $\kappa_q'(\xi,\eta) = \left[\chi_q'(\xi), \chi_q'(\eta)\right] - \chi_q'([\xi,\eta])$. We notice that $\chi_q' = -\chi_q$ is a connection with $\bar{D}'\chi_q' = \kappa_q'$ if and only if $A = 0$. In particular $\left(\delta_i^k, -\gamma_{ij}^k\right)$ is the only existing symmetric connection for the Killing system.

REMARK 3.13: Rewriting the previous local formulas with A instead of χ_0 we get:

$$\partial_i A_j^k - \partial_j A_i^k - A_i^r \chi^k_{r,j} + A_j^r \chi^k_{r,i} = 0 \quad (1^*)$$

$$\partial_i \chi^k_{l,j} - \partial_j \chi^k_{l,i} - A_i^r \chi^k_{lr,j} + A_j^r \chi^k_{lr,i} - \chi^r_{l,i}\chi^k_{r,j} + \chi^r_{l,j}\chi^k_{r,i} = 0 \quad (2^*)$$

$$\begin{aligned}&\partial_i \chi^k_{lr,j} - \partial_j \chi^k_{lr,i} - A_i^s \chi^k_{lrs,j} + A_j^s \chi^k_{lrs,i} \\ &-\left(\chi^s_{r,i}\chi^k_{ls,j} + \chi^s_{l,i}\chi^k_{rs,j} + \chi^s_{lr,i}\chi^k_{s,j} - \chi^s_{r,j}\chi^k_{ls,i} - \chi^s_{l,j}\chi^k_{rs,i} - \chi^s_{lr,j}\chi^k_{s,i} \right) = 0\end{aligned} \quad (3^*)$$

When $q = 1, g_2 = 0$ and though surprising it may look like, we find back *exactly* all the formulas presented by E. and F. Cosserat in ([10], p 123). Even more strikingly, *in the case of a Riemann structure, the last two terms disappear but the quadratic terms are left while, in the case of screw and complex structures, the quadratic terms disappear but the last two terms are left*. We finally notice that $\chi_q' = -\chi_q$ is a R_q-connection if and only if $A = 0$, *a result contradicting the use of connections in physics*. However, when $A = 0$, we have $\chi_0'(\xi) = \xi$ and thus:

$$\begin{aligned}\bar{D}'\chi_{q+1} &= \left(D\chi_{q+1}\right)(\xi,\eta) - \left(\left[\chi_q(\xi), \chi_q(\eta)\right] + i(\xi) D\left(\chi_{q+1}(\eta)\right) - i(\eta) D\left(\chi_{q+1}(\xi)\right)\right) \\ &= -\left[\chi_q(\xi), \chi_q(\eta)\right] - \chi_q([\xi,\eta]) \\ &= -\left(\left[\chi_q'(\xi), \chi_q'(\eta)\right] - \chi_q'([\xi,\eta])\right) \\ &= -\kappa_q'(\xi,\eta)\end{aligned}$$

does not depend on the lift of χ_q.

COROLLARY 3.14: When $det(A) \neq 0$ there is a *second nonlinear Spencer sequence* stabilized at order *q*:

$$0 \to aut(X) \xrightarrow{j_q} \Pi_q \xrightarrow{\bar{D}_1} C_1(T) \xrightarrow{\bar{D}_2} C_2(T)$$

where $\bar{D}_1$ and $\bar{D}_2$ are involutive and a *restricted second nonlinear Spencer sequence*:

$$0 \to \Gamma \xrightarrow{j_q} \mathcal{R}_q \xrightarrow{\bar{D}_1} C_1 \xrightarrow{\bar{D}_2} C_2$$

such that $\bar{D}_1$ and $\bar{D}_2$ are involutive whenever $\mathcal{R}_q$ is involutive.

Proof: With $|\mu| = q$ we have $\chi^k_{\mu,i} = -g^k_l A^r_i f^l_{\mu+1_r} + terms(order \le q)$. Setting $\chi^k_{\mu,i} = A^r_i \tau^k_{\mu,r}$, we obtain $\tau^k_{\mu,r} = -g^k_l f^l_{\mu+1_r} + terms(order \le q)$ and $\bar{D} : \Pi_{q+1} \to T^* \otimes J_q(T)$ restricts to $\bar{D}_1 : \Pi_q \to C_1(T)$.

Finally, setting $A^{-1} = B = id - \tau_0$, we obtain successively:

$$\partial_i \chi^k_{\mu,j} - \partial_j \chi^k_{\mu,i} + terms(\chi_q) - \left(A^r_i \chi^k_{\mu+1_r,j} - A^r_j \chi^k_{\mu+1_r,i}\right) = 0$$

$$B^i_r B^j_s \left(\partial_i \chi^k_{\mu,j} - \partial_j \chi^k_{\mu,i}\right) + terms(\chi_q) - \left(\tau^k_{\mu+1_r,s} - \tau^k_{\mu+1_s,r}\right) = 0$$

We obtain therefore $D\tau_{q+1} + terms(\tau_q) = 0$ and $\bar{D}' : T^* \otimes J_q(T) \to \wedge^2 T^* \otimes J_{q-1}(T)$ restricts to $\bar{D}_2 : C_1(T) \to C_2(T)$.

In the case of Lie groups of transformations, the symbol of the involutive system R_q *must* be $g_q = 0$ providing an isomorphism $\mathcal{R}_{q+1} \simeq \mathcal{R}_q \Rightarrow R_{q+1} \simeq R_q$ and we have therefore $C_r = \wedge^r T^* \otimes R_q$ for $r = 1, \cdots, n$ like in the linear Spencer sequence.

□

REMARK 3.15: In the case of the (local) action of a Lie group G on X, we may consider the graph of this action, that is the morphism $X \times G \to X \times X : (x,a) \to (x, y = f(x,a))$. If q is large enough, then there is an isomorphism $X \times G \to \mathcal{R}_q \subset \Pi_q : (x,a) \to j_q(f)(x,a)$ obtained by eliminating the parameters and $C_r = \wedge^r T^* \otimes R_q$. If $\{\theta_\tau\}$ with $1 \le \tau \le dim(G)$ is a basis of infinitesimal generators of this action, there is a morphism of Lie algebroids over X, namely $X \times \mathcal{G} \to R_q : \lambda^\tau(x) \to \lambda^\tau(x) j_q(\theta_\tau)$ when q is large enough and the linear Spencer sequence $R_q \xrightarrow{D_1} T^* \otimes R_q \xrightarrow{D_2} \wedge^2 T^* \otimes R_q \xrightarrow{D_3} \cdots$ is locally exact because it is locally isomoprphic to the tensor product by $\mathcal{G}$ of the Poincaré sequence $\wedge^0 T^* \xrightarrow{d} \wedge^1 T^* \xrightarrow{d} \wedge^2 T^* \xrightarrow{d} \cdots$ where d is the exterior derivative ([7]).

We may also consider similarly $dy = dax = daa^{-1}y$ and $dx = dbb^{-1}dx = -a^{-1}dax$, depending on the choice of the independent variable among the *source* x or the *target* y.

Surprisingly, in the case of Lie pseudogroups or Lie groupoids, the situation is quite different. We recall the way to introduce a groupoid structure on $\Pi_{q,1} \subset J_1(\Pi_q)$ from the groupoid structure on Π_q when $\Delta = det(\partial_i f^k(x)) \ne 0$, that is how to define $j_1(h_q) = j_1(g_q \circ f_q) = j_1(g_q) \circ j_1(f_q)$. We get successively with $y = f(x)$:

$$h(x) = (g \circ f)(x) = g(f(x)) \Rightarrow \frac{\partial h^r}{\partial x^i} = \frac{\partial g^r}{\partial y^k} \frac{\partial f^k}{\partial x^i} \Rightarrow h^r_i(x) = g^r_k(f(x)) f^k_i(x)$$

$$\frac{\partial h_i^r}{\partial x^j} = \frac{\partial g_k^r}{\partial y^l} f_i^k \frac{\partial f^l}{\partial x^j} + g_k^r \frac{\partial f_i^k}{\partial x^j}$$

$$\Rightarrow h_{ij}^r(x) = g_{kl}^r(f(x)) f_i^k(x) f_j^l(x) + g_k^r(f(x)) f_{ij}^k(x)$$

$$\frac{\partial h_{ij}^r}{\partial x^s} = \frac{\partial g_{ki}^r}{\partial y^u} f_i^k f_j^l \frac{\partial f^u}{\partial x^s} + g_{kl}^r \left(\frac{\partial f_i^k}{\partial x^s} f_j^l + f_i^k \frac{\partial f_j^l}{\partial x^s} \right) + \frac{\partial g_k^r}{\partial y^u} \frac{\partial f^u}{\partial x^s} f_{ij}^k + g_k^r \frac{\partial f_{ij}^k}{\partial x^s}$$

$$\Rightarrow h_{ijs}^r = g_{klu}^r f_i^k f_j^l f_s^u + g_{kl}^r \left(f_{is}^k f_j^l + f_i^k f_{js}^l \right) + g_{ku}^r f_s^u f_{ij}^k + g_k^r f_{ijs}^k$$

and so on with more and more involved formulas.

Now, if we want to obtain objects over the *source* x according to the nonlinear Spencer sequence, we have only two possibilities in actual practice, namely:

$$\chi_q = f_{q+1}^{-1} \circ j_1(f_q) - id_{q+1} \in T^* \otimes J_q(T)$$

$$\leftrightarrow \bar{\chi}_q = j_1(f_q)^{-1} \circ f_{q+1} - id_{q+1} \in T^* \otimes J_q(T)$$

As we have already considered the first, we have now only to study the second. In $J_1(\Pi_q)$, we have:

$$\chi_q + id_{q+1} = \left(A_r^k, \chi_{i,r}^k, \chi_{ij,r}^k, \cdots\right) \text{ and } \bar{\chi}_q + id_{q+1} = \left(\bar{A}_r^k, \bar{\chi}_{i,r}^k, \bar{\chi}_{ij,r}^k, \cdots\right)$$

over $(x, x, \delta, 0, \cdots)$

LEMMA 3.16: $\bar{\chi}_q$ is a quasi-linear rational function of χ_q, $\forall q \geq 0$. With more details, when $q = 0$, we have $\bar{\chi}_0 = \bar{A} - id$ and $\chi_0 = A - id$ with $\bar{A} = A^{-1} = B$ and when $q \geq 1$, we have $\bar{\chi}_q \circ A = -\chi_q$, that is to say $\bar{\chi}_q = -\tau_q$.

Proof: In the groupoid framework, we have:

$$\left(\bar{\chi}_q + id_{q+1}\right) \circ \left(\chi_q + id_{q+1}\right) = id_{q+1} \in J_1(\Pi_q)$$

Doing the substitutions:

$$\frac{\partial g^r}{\partial y^k} \to \bar{A}_k^r, \ \frac{\partial g_k^r}{\partial y^l} \to \bar{\chi}_{k,l}^r, \ \frac{\partial g_{kl}^r}{\partial y^u} \to \bar{\chi}_{kl,u}^r$$

$$\frac{\partial f^k}{\partial x^i} \to A_i^k, \ \frac{\partial f_i^k}{\partial x^j} \to \chi_{i,j}^k, \ \frac{\partial f_{ij}^k}{\partial x^s} \to \chi_{ij,s}^k$$

while using the fact that $f_i^k = \delta_i^k, f_{ij}^k = 0, \cdots$ and $g_k^r = \delta_k^r, g_{kl}^r = 0, \cdots$, we obtain at once:

$$\bar{A}_k^r A_i^k = \delta_i^r, \bar{\chi}_{k,l}^r A_j^l + \chi_{i,j}^k = 0, \bar{\chi}_{ij,u}^r A_s^u + \chi_{ij,s}^r = 0, \cdots$$

Proceeding by induction, we finally obtain:

$$\bar{\chi}_{\mu,r}^k A_s^r + \chi_{\mu,i}^k = 0$$

that is to say $\bar{\chi}_{\mu,i}^k + \tau_{\mu,i}^k = 0$ because $\Delta \neq 0 \Rightarrow det(A) \neq 0$, thus $\bar{\chi}_q \circ A = -\chi_q$ or, equivalently, $\bar{\chi}_q = -\tau_q$.

□

REMARK 3.17: The passage from χ_q to τ_q is *exactly* the one done by E. and F. Cosserat in ([10], p 190), even though it is based on a *subtle misunderstanding* that we shall correct later on.

REMARK 3.18: According to the previous results, the "*field*" must be a section of the natural bundle $\mathcal{F}$ of geometric objects if we use the nonlinear Janet sequence or a section of the first Spencer bundle C_1 if we use the nonlinear Spencer sequence. The aim of this paper is to prove that the second choice is by far more convenient for mathematical physics.

4. Variational Calculus

It remains to graft a variational procedure adapted to the previous results. Contrary to what happens in analytical mechanics or elasticity for example, *the main idea is to vary sections but not points*. Hence, we may introduce the variation $\delta f^k(x)=\eta^k(f(x))$ and set $\eta^k(f(x))=\xi^i\partial_i f^k(x)(x)$ along the "*vertical machinery*" but *notations like* $\delta x^i=\xi^i$ *or* $\delta y^k=\eta^k$ *have no meaning at all*.

As a major result first discovered in specific cases by the brothers Cosserat in 1909 and by Weyl in 1916, we shall prove and apply the following key result:

THE PROCEDURE ONLY DEPENDS ON THE LINEAR SPENCER OPERATOR AND ITS FORMAL ADJOINT.

In order to prove this result, if $f_{q+1},g_{q+1},h_{q+1}\in\Pi_{q+1}$ can be composed in such a way that $g'_{q+1}=g_{q+1}\circ f_{q+1}=f_{q+1}\circ h_{q+1}$, we get:

$$\begin{aligned}\bar{D}g'_{q+1}&=f_{q+1}^{-1}\circ g_{q+1}^{-1}\circ j_1(g_q)\circ j_1(f_q)-id_{q+1}=f_{q+1}^{-1}\circ\bar{D}g_{q+1}\circ j_1(f_q)+\bar{D}f_{q+1}\\&=h_{q+1}^{-1}\circ f_{q+1}^{-1}\circ j_1(f_q)\circ j_1(h_q)-id_{q+1}=h_{q+1}^{-1}\circ\bar{D}f_{q+1}\circ j_1(h_q)+\bar{D}h_{q+1}\end{aligned}$$

Using the local exactness of the first nonlinear Spencer sequence or ([25], p 219), we may state:

LEMMA 4.1: For any section $f_{q+1}\in\mathcal{R}_{q+1}$, the *finite gauge transformation*:

$$\chi_q\in T^*\otimes R_q\to\chi'_q=f_{q+1}^{-1}\circ\chi_q\circ j_1(f_q)+\bar{D}f_{q+1}\in T^*\otimes R_q$$

exchanges the solutions of the *field equations* $\bar{D}'\chi_q=0$.

Introducing the *formal Lie derivative* on $J_q(T)$ by the formulas:

$$L(\xi_{q+1})\eta_q=\{\xi_{q+1},\eta_{q+1}\}+i(\xi)D\eta_{q+1}=[\xi_q,\eta_q]+i(\eta)D\xi_{q+1}$$

$$\left(L\left(j_1(\xi_{q+1})\right)\chi_q\right)(\zeta)=L(\xi_{q+1})\left(\chi_q(\zeta)\right)-\chi_q([\xi,\zeta])$$

LEMMA 4.2: Passing to the limit *over the source* with $h_{q+1}=id_{q+1}+t\xi_{q+1}+\cdots$ for $t\to 0$, we get an *infinitesimal gauge transformation* leading to the *infinitesimal variation*:

$$\delta\chi_q=D\xi_{q+1}+L\left(j_1(\xi_{q+1})\right)\chi_q \tag{3}$$

which *does not depend on the parametrization* of χ_q. Setting $\bar{\xi}_{q+1}=\xi_{q+1}+\chi_{q+1}(\xi)$, we get:

$$\delta\chi_q=D\bar{\xi}_{q+1}-\{\chi_{q+1},\bar{\xi}_{q+1}\} \tag{3*}$$

LEMMA 4.3: Passing to the limit *over the target* with $\chi_q=\bar{D}f_{q+1}$ and $g_{q+1}=id_{q+1}+t\eta_{q+1}+\cdots$, we get the other *infinitesimal variation* where $D\eta_{q+1}$ is *over the target*:

$$\delta\chi_q=f_{q+1}^{-1}\circ D\eta_{q+1}\circ j_1(f_q) \tag{4}$$

which *depends on the parametrization* of χ_q.

EXAMPLE 4.4: We obtain for $q=1$:

$$\delta\chi_{,i}^{k}=\left(\partial_i\xi^k-\xi_i^k\right)+\left(\xi^r\partial_r\chi_{,i}^k+\chi_{,r}^k\partial_i\xi^r-\chi_{,i}^r\xi_r^k\right)$$
$$=\left(\partial_i\bar{\xi}^k-\bar{\xi}_i^k\right)+\left(\chi_{r,i}^k\bar{\xi}^r-\chi_{,i}^r\bar{\xi}_r^k\right)$$
$$\delta\chi_{j,i}^{k}=\left(\partial_i\xi_j^k-\xi_{ij}^k\right)+\left(\xi^r\partial_r\chi_{j,i}^k+\chi_{j,r}^k\partial_i\xi^r+\chi_{r,i}^k\xi_j^r-\chi_{j,i}^r\xi_r^k-\chi_{,i}^r\xi_{jr}^k\right)$$
$$=\left(\partial_i\bar{\xi}_j^k-\bar{\xi}_{ij}^k\right)+\left(\chi_{rj,i}^k\bar{\xi}^r+\chi_{r,i}^k\bar{\xi}_j^r-\chi_{j,i}^r\bar{\xi}_r^k-\chi_{,i}^r\bar{\xi}_{jr}^k\right)$$

Introducing the inverse matrix $B=A^{-1}$, we obtain therefore equivalently:

$$\delta A_i^k=\xi^r\partial_rA_i^k+A_r^k\partial_i\xi^r-A_i^r\xi_r^k\Leftrightarrow\delta B_k^i=\xi^r\partial_rB_k^i-B_k^r\partial_r\xi^i+B_r^i\xi_k^r$$

both with:

$$\delta\chi_{j,i}^k=\left(\partial_i\xi_j^k-A_i^r\xi_{jr}^k\right)+\left(\xi^r\partial_r\chi_{j,i}^k+\chi_{j,r}^k\partial_i\xi^r+\chi_{r,i}^k\xi_j^r-\chi_{j,i}^r\xi_r^k\right)$$

For the Killing system $R_1\subset J_1(T)$ with $g_2=0$, these variations are *exactly* the ones that can be found in ([10], (50) + (49), p 124 with a printing mistake corrected on p 128) when replacing a 3 × 3 skew-symmetric matrix by the corresponding vector. *The last unavoidable Proposition is thus essential in order to bring back the nonlinear framework of finite elasticity to the linear framework of infinitesimal elasticity that only depends on the linear Spencer operator.*

For the conformal Killing system $\hat{R}_1\subset J_1(T)$ (see next section) we obtain:

$$\delta\chi_{r,i}^r=\left(\partial_i\xi_r^r-\xi_{ri}^r\right)+\left(\xi^r\partial_r\chi_{s,i}^s+\chi_{s,r}^s\partial_i\xi^r-\chi_{,i}^s\xi_{rs}^r\right)$$

but $\chi_{r,i}^r(x)dx^i$ is *far* from being a 1-form. However, $\left(\chi_{j,i}^k+\gamma_{js}^k\chi_{,i}^s\right)\in T^*\otimes T^*\otimes T$ and thus $\left(\alpha_i=\chi_{r,i}^r+\gamma_{rs}^r\chi_{,i}^s\right)\in T^*$ is a pure 1-form if we replace $\left(\chi_{r,i}^r,\chi_{,i}^r\right)$ by $(\alpha_i,0)$. Hence, $\alpha(\zeta)$ is a scalar for any $\zeta\in T$ and we have $L(\xi_1)(\alpha(\zeta))-\alpha([\xi,\zeta])=\left(\alpha_r\partial_i\xi^r+\xi^r\partial_r\alpha_i\right)\zeta^i$. As we shall see in section *V.A*, we have $\left(L(\xi_2)\gamma\right)_{ij}^k=\xi_{ij}^k$ for any section $\xi_2\in J_2(T)$ and we obtain therefore successively:

$$\delta\alpha_i=\left(\partial_i\xi_r^r-\xi_{ri}^r\right)+\left(\alpha_r\partial_i\xi^r+\xi^r\partial_r\alpha_i\right)$$
$$\varphi_{ij}=\partial_i\alpha_j-\partial_j\alpha_i\Rightarrow\delta\varphi_{ij}=\left(\partial_j\xi_{ri}^r-\partial_i\xi_{rj}^r\right)+\left(\varphi_{rj}\partial_i\xi^r+\varphi_{ir}\partial_j\xi^r+\xi^r\partial_r\varphi_{ij}\right)$$

These are *exactly* the variations obtained by Weyl ([3], (76), p. 289) who was assuming implicitly $A=0$ when setting $\bar{\xi}_r^r=0\Leftrightarrow\xi_r^r=-\alpha_i\xi^i$ by introducing a connection. Accordingly, ξ_{ri}^r is the variation of the EM potential itself, that is the δA_i of engineers used in order to exhibit the Maxwell equations from a variational principle ([3], p. 26) but the introduction of the Spencer operator is new in this framework.

The explicit general formulas of the two lemma cannot be found somewhere else (The reader may compare them to the ones obtained in [19] by means of the so-called "diagonal" method that cannot be applied to the study of explicit examples). The following unusual difficult proposition generalizes well known variational techniques used in continuum mechanics and will be crucially used for applications:

PROPOSITION 4.5: The same variation is obtained whenever $\eta_q=f_{q+1}\left(\xi_q+\chi_q(\xi)\right)$ with $\chi_q=\bar{D}f_{q+1}$, a transformation only depending on

$j_1(f_q)$ and invertible if and only if $det(A)\neq 0$.

Proof: First of all, setting $\bar{\xi}_q = \xi_q + \chi_q(\xi)$, we get $\bar{\xi} = A(\xi)$ for $q=0$, a transformation which is invertible if and only if $det(A)\neq 0$. In the nonlinear framework, we have to keep in mind that there is no need to vary the object ω which is given but only the need to vary the section f_{q+1} as we already saw, using $\eta_q \in R_q(Y)$ *over the target* or $\xi_q \in R_q$ *over the source*. With $\eta_q = f_{q+1}(\xi_q)$, we obtain for example:

$$\delta f^k = \eta^k = f_r^k \xi^r$$
$$\delta f_i^k = \eta_u^k f_i^u = f_r^k \xi_i^r + f_{ri}^k \xi^r$$
$$\delta f_{ij}^k = \eta_{uv}^k f_i^u f_j^v + \eta_u^k f_{ij}^u = f_r^k \xi_{ij}^r + f_{ri}^k \xi_j^r + f_{rj}^k \xi_i^r + f_{rij}^k \xi^r$$

and so on. Introducing the formal derivatives d_i for $i=1,\cdots,n$, we have:

$$\delta f_\mu^k = \zeta_\mu^k(f_q,\eta_q) = d_\mu \eta^k = \eta_u^k f_\mu^u + \cdots = f_r^k \xi_\mu^r + \cdots + f_{\mu+1_r}^k \xi^r$$

We shall denote by $\#(\eta_q) = \zeta_\mu^k(y_q,\eta_q)\frac{\partial}{\partial y_\mu^k} \in V(\mathcal{R}_q)$ with $\zeta^k = \eta^k$ the corresponding vertical vector field, namely:

$$\#(\eta_q) = 0\frac{\partial}{\partial x^i} + \eta^k(y)\frac{\partial}{\partial y^k} + \left(\eta_u^k(y)y_i^u\right)\frac{\partial}{\partial y_i^k}$$
$$+\left(\eta_{uv}^k(y)y_i^u y_j^v + \eta_u^k(y)y_{ij}^u\right)\frac{\partial}{\partial y_{ij}^k} + \cdots$$

However, the standard prolongation of an infinitesimal change of source coordinates described by the horizontal vector field ξ, obtained by replacing all the derivatives of ξ by a section $\xi_q \in R_q$ over $\xi \in T$, is the vector field:

$$\flat(\xi_q) = \xi^i(x)\frac{\partial}{\partial x^i} + 0\frac{\partial}{\partial y^k} - \left(y_r^k \xi_i^r(x)\right)\frac{\partial}{\partial y_i^k}$$
$$-\left(y_r^k \xi_{ij}^r(x) + y_{rj}^k \xi_i^r(x) + y_{ri}^k \xi_j^r(x)\right)\frac{\partial}{\partial y_{ij}^k} + \cdots$$

It can be proved that $\left[\flat(\xi_q),\flat(\xi_q')\right] = \flat\left(\left[\xi_q,\xi_q'\right]\right), \forall \xi_q,\xi_q' \in R_q$ *over the source*, with a similar property for $\#(.)$ *over the target* ([25]). However, $\flat(\xi_q)$ *is not a vertical vector field and cannot therefore be compared to* $\#(\eta_q)$. The solution of this problem explains a strange comment made by Weyl in ([3], p 289 + (78), p 290) and which became a founding stone of classical gauge theory. Indeed, ξ_r^r is *not* a scalar because ξ_i^k is *not* a 2-tensor. However, when $A=0$, then $-\chi_q$ is a R_q-connection and $\bar{\xi}_r^r = \xi_r^r + \chi_{r,i}^r \xi^i$ is a true scalar that may be set equal to zero in order to obtain $\xi_r^r = -\chi_{r,i}^r \xi^i$, a fact explaining why the EM-potential is considered as a connection in quantum mechanics instead of using the second order jets ξ_{ri}^r of the conformal system, with a *shift by one step in the physical interpretation of the Spencer sequence* (See [4] for more historical details).

The main idea is to consider the vertical vector field $T(f_q)(\xi) - \flat(\xi_q) \in V(\mathcal{R}_q)$ whenever $y_q = f_q(x)$. Passing to the limit $t \to 0$ in the formula $g_q \circ f_q = f_q \circ h_q$, we first get $g \circ f = f \circ h \Rightarrow f(x) + t\eta(f(x)) + \cdots = f(x + t\xi(x) + \cdots)$. Using the chain rule for derivatives and substituting jets, we get successively:

$$\delta f^k(x)=\xi^r\partial_r f^k,\quad \delta f_i^k=\xi^r\partial_r f_i^k+f_r^k\xi_i^r,$$
$$\delta f_{ij}^k=\xi^r\partial_r f_{ij}^k+f_{rj}^k\xi_i^r+f_{ri}^k\xi_j^r+f_r^k\xi_{ij}^r$$

and so on, replacing $\xi^r f_{\mu+1_r}^k$ by $\xi^r\partial_r f_\mu^k$ in $\eta_q=f_{q+1}(\xi_q)$ in order to obtain:

$$\delta f_\mu^k=\eta_r^k f_\mu^r+\cdots=\xi^i\left(\partial_i f_\mu^k-f_{\mu+1_i}^k\right)+f_{\mu+1_r}^k\xi^r+\cdots+f_r^k\xi_\mu^r$$

where the right member only depends on $j_1(f_q)$ when $|\mu|=q$.

Finally, we may write the symbolic formula $f_{q+1}(\chi_q)=j_1(f_q)-f_{q+1}=Df_{q+1}\in T^*\otimes V(\mathcal{R}_q)$ in the explicit form:

$$f_r^k\chi_{\mu,i}^r+\cdots+f_{\mu+1_r}^k\chi_{,i}^r=\partial_i f_\mu^k-f_{\mu+1_i}^k$$

Substituting in the previous formula provides $\eta_q=f_{q+1}(\xi_q+\chi_q(\xi))$ and we just need to replace q by $q+1$ in order to achieve the proof.

Checking directly the proposition is not evident even when $q=0$ as we have:

$$\left(\frac{\partial\eta^k}{\partial y^u}-\eta_u^k\right)\partial_i f^u=f_r^k\left[\left(\partial_i\bar{\xi}^r-\bar{\xi}_i^r\right)-\left(\chi_{,i}^s\bar{\xi}_s^r-\chi_{s,i}^r\bar{\xi}^s\right)\right]$$

but cannot be done by hand when $q\ge 1$.

□

For an arbitrary vector bundle E and involutive system $R_q\subseteq J_q(E)$, we may define the *r-prolongations* $\rho_r(R_q)=R_{q+r}=J_r(R_q)\cap J_{q+r}(E)\subset J_r(J_q(E))$ and their respective *symbols* $g_{q+r}=\rho_r(g_q)$ defined from $g_q\subseteq S_qT^*\otimes E$ where S_qT^* is the vector bundle of q-symmetric covariant tensors. Using the Spencer δ-map, we now recall the definition of the *Spencer bundles*:

$$C_r=\wedge^r T^*\otimes R_q\big/\delta\left(\wedge^{r-1}T^*\otimes g_{q+1}\right)$$
$$\subseteq\wedge^r T^*\otimes J_q(E)\big/\delta\left(\wedge^{r-1}T^*\otimes S_{q+1}\right)T^*\otimes E=C_r(E)$$

and of the *Janet bundles*:

$$F_r=\wedge^r T^*\otimes J_q(E)\big/\left(\wedge^r T^*\otimes R_q+\delta\wedge^{r-1}T^*\otimes S_{q+1}T^*\otimes E\right)$$

When $D=\Phi\circ j_q$, we may obtain by induction on r the following *fundamental diagram I* relating the *second linear Spencer sequence* to the *linear Janet sequence* with epimorphisms $\Phi=\Phi_0,\cdots,\Phi_n$:

$$\begin{array}{ccccccccccccccc}
 & & & & & & 0 & & 0 & & 0 & & 0 & & \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \to & \Theta & \xrightarrow{j_q} & C_0 & \xrightarrow{D_1} & C_1 & \xrightarrow{D_2} & C_2 & \xrightarrow{D_3}\cdots\xrightarrow{D_n} & C_n & \to & 0\\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \to & E & \xrightarrow{j_q} & C_0(E) & \xrightarrow{D_1} & C_1(E) & \xrightarrow{D_2} & C_2(E) & \xrightarrow{D_3}\cdots\xrightarrow{D_n} & C_n(E) & \to & 0\\
 & & & & \| & & \downarrow\Phi_0 & & \downarrow\Phi_1 & & \downarrow\Phi_2 & & \downarrow\Phi_n & & \\
0\to & \Theta & \to & & E & \xrightarrow{\mathcal{D}} & F_0 & \xrightarrow{\mathcal{D}_1} & F_1 & \xrightarrow{\mathcal{D}_2} & F_2 & \xrightarrow{\mathcal{D}_3}\cdots\xrightarrow{\mathcal{D}_n} & F_n & \to & 0\\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & & & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

Chasing in the above diagram, the Spencer sequence is locally exact at C_1 if and only if the Janet sequence is locally exact at F_0 because the central sequence is locally exact (See [7] [13] [25] for more details). In the present situation, we shall always have $E = T$. *The situation is much more complicate in the nonlinear framework and we provide details for a later use*.

Let $\bar{\omega}$ be a section of $\mathcal{F}$ satisfying the same CC as ω, namely $I\left(j_1(\omega)\right) = 0$. As $\mathcal{F}$ is a quotient of Π_q, we may find a section $f_q \in \Pi_q$ such that:

$$\Phi \circ f_q \equiv f_q^{-1}(\omega) = \bar{\omega}$$

$$\Rightarrow \rho_1(\Phi) \circ j_1(f_q) \equiv j_1(f_q^{-1})(j_1(\omega)) = j_1(f_q^{-1}(\omega)) = j_1(\bar{\omega})$$

Similarly, as $\mathcal{F}$ is a natural bundle of order q, then $J_1(\mathcal{F})$ is a natural bundle of order $q+1$ and we can find a section $f_{q+1} \in \Pi_{q+1}$ such that:

$$\rho_1(\Phi) \circ f_{q+1} \equiv f_{q+1}^{-1}(j_1(\omega)) = j_1(\bar{\omega})$$

and we are facing two possible but quite different situations:

- Eliminating $\bar{\omega}$, we obtain:

$$j_1(f_q^{-1})(j_1(\omega)) = f_{q+1}^{-1}(j_1(\omega))$$

$$\Rightarrow \left(f_{q+1} \circ j_1(f_q^{-1})\right)^{-1}(j_1(\omega)) - j_1(\omega) = L(\sigma_q)\omega = 0$$

and thus $\sigma_q = \bar{D}f_{q+1}^{-1} \in T^* \otimes R_q = -f_{q+1} \circ \chi_q \circ j_1(f)^{-1}$ *over the target* if we set $\chi_q = \bar{D}f_{q+1} = f_{q+1}^{-1} \circ j_1(f_q) - id_{q+1}$ *over the source*, even if f_{q+1} may not be a section of $\mathcal{R}_{q+1}$. As σ_q is killed by $\bar{D}'$, we have related cocycles at $\mathcal{F}$ in the Janet sequence *over the source* with cocycles at $T^* \otimes R_q$ or C_1 *over the target*.

- Eliminating ω, we obtain successively:

$$\left(f_{q+1}^{-1} \circ j_1(f_q)\right)(j_1(\bar{\omega})) - j_1(\bar{\omega})$$

$$= -\left(f_{q+1}^{-1} \circ j_1(f_q)\right)\left[\left(f_{q+1}^{-1} \circ j_1(f_q)\right)^{-1}(j_1(\bar{\omega})) - j_1(\bar{\omega})\right]$$

$$= -\left(f_{q+1}^{-1} \circ j_1(f_q)\right)L(\chi_q)\bar{\omega}$$

where we have *over the source*:

$$L(\chi_q)\bar{\omega} = \left(\bar{\Omega}_i^\tau \equiv -L_k^{\tau\mu}(\bar{\omega}(x))\chi_{\mu,i}^k + \chi_{,i}^r \partial_r \bar{\omega}^\tau(x)\right) \in T^* \otimes F_0$$

However, we know that F_0 is associated with R_q and is thus not affected by $f_{q+1}^{-1} \circ j_1(f_q)$ which projects onto $f_q^{-1} \circ f_q = id_q$. Hence, only T^* is affected by $f_1^{-1} \circ j_1(f) = A$ in a covariant way and we obtain therefore *over the source*:

$$\left(f_{q+1}^{-1} \circ j_1(f_q)\right)(j_1(\bar{\omega})) - j_1(\bar{\omega}) = -BL(\chi_q)\bar{\omega} = -L(\tau_q)\bar{\omega} = 0$$

where $B = A^{-1}$. It follows that $\chi_q \in T^* \otimes R_q(\bar{\omega})$ with $\bar{D}'\chi_q = 0$ in the first non-linear Spencer sequence for $R_q(\bar{\omega}) \subset J_q(T)$.

We invite the reader to follow all the formulas involved in these technical results on the next examples. Of course, whenever $\mathcal{R}_q$ is formally integrable and $f_{q+1} \in \mathcal{R}_{q+1}$ is a lift of $f_q \in \mathcal{R}_q$, then we have $\bar{\omega} = \omega$ and $\xi_q \in T^* \otimes R_q$ because $R_q(\omega)=R_q$.

EXAMPLE 4.6: In the case of Riemannian structures, we have $\mathcal{F}\in S_2T^*$ because we deal with a non-degenerate metric $\omega=(\omega_{ij})\in S_2T^*$ with $det(\omega)\neq 0$ and may introduce $\omega^{-1}=(\omega^{ij})\in S_2T$. We have by definition $\omega_{kl}(f(x))f_i^k(x)f_j^l(x)=\bar{\omega}_{ij}(x)$ that we shall simply write $\omega_{kl}(f)f_i^kf_j^l=\bar{\omega}_{ij}(x)$ and obtain therefore:

$$\omega_{kl}(f)f_j^l\partial_rf_i^k+\omega_{kl}(f)f_i^k\partial_rf_j^l+\frac{\partial\omega_{kl}}{\partial y^u}(f)f_i^kf_j^l\partial_rf^u-\partial_r\bar{\omega}_{ij}(x)=0$$

Our purpose is now to compute the expression:

$$\omega_{kl}(f)f_j^lf_{ir}^k+\omega_{kl}(f)f_i^kf_{jr}^l+\frac{\partial\omega_{kl}}{\partial y^u}(f)f_i^kf_j^lf_r^u-\partial_r\bar{\omega}_{ij}(x)\neq 0$$

In order to eliminate the derivatives of ω over te target we may multiply the first equation by B and substract from the second while using the fact that $\omega_{kl}(f)=\bar{\omega}_{ij}(x)g_k^ig_l^j$ with $\chi_0=A-id_T\Rightarrow\tau_0=B\chi_0=id_T-B$ in order to get:

$$-\left(\bar{\omega}_{sj}\tau_{i,r}^s+\bar{\omega}_{is}\tau_{j,r}^s+\tau_{,r}^s\partial_s\bar{\omega}_{ij}\right)=-\left(L(\tau_1)\bar{\omega}\right)_{ij,r}$$

These results can be extended at once to any tensorial geometric object but the conformal case needs more work and we let the reader treat it as an exercise. He will discover that the standard elimination of a conformal factor is not the best way to use in order to understand the conformal structure which has to do with a tensor density and no longer with a tensor.

In the non-linear case, the non-linear CC of the system $\mathcal{R}_q$ defined by $\Phi(y_q)=\bar{\omega}(x)$ only depend on the differential invariants and are exactly the ones satisfied by ω in the sense that they have the same Vessiot structure constants whenever $\mathcal{R}_q$ is formally integrable, in particular involutive as shown in Example 2.7. Accordingly, we can always find f_{q+1} over f_q. In the linear case, the procedure is similar but slightly simpler. Indeed, if $\mathcal{D}:T\to F_0$ is an involutive Lie operator, we may consider only the initial part of the *fundamental diagram I*:

$$\begin{array}{ccccccccccc}
 & & & & & & 0 & & 0 & & 0\\
 & & & & & & \downarrow & & \downarrow & & \downarrow\\
 & & 0 & \to & \Theta & \xrightarrow{j_q} & C_0 & \xrightarrow{D_1} & C_1 & \xrightarrow{D_2} & C_2\\
 & & & & & & \downarrow & & \downarrow & & \downarrow\\
 & & 0 & \to & T & \xrightarrow{j_q} & C_0(T) & \xrightarrow{D_1} & C_1(T) & \xrightarrow{D_2} & C_2(T)\\
 & & & & \| & & \downarrow\Phi_0 & & \downarrow\Phi_1 & & \\
0\to & \Theta & & \to & T & \xrightarrow{\mathcal{D}} & F_0 & \xrightarrow{\mathcal{D}_1} & F_1 & & \\
 & & & & & & \downarrow & & \downarrow & & \\
 & & & & & & 0 & & 0 & &
\end{array}$$

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & 0 & \to & g_{q+1} & \xrightarrow{-\delta} & \delta\left(g_{q+1}\right) & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & \Theta & \xrightarrow{j_{q+1}} & R_{q+1} & \xrightarrow{D} & T^{*}\otimes R_q & & \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & \Theta & \xrightarrow{j_q} & R_q & \xrightarrow{D_1} & C_1 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

and study the linear inhomogeneous involutive system $\mathcal{D}\xi = \Omega$ with $\Omega \in F_0$ and $\mathcal{D}_1\Omega = 0$. If we pick up any lift $\xi_q \in C_0(T) = J_q(T)$ of Ω and chase, we notice that $X_1 = D_1\xi_q \in C_1 \subset C_1(T)$ is such that $D_2X_1 = 0$.

EXAMPLE 4.7: In the Example 2.7, using the involutive system $R_1' = R_1^{(1)} \subset R_1 \subset J_1(T)$, we have $m = n = 2, q = 1$ and the fundamental diagram I:

$$\begin{array}{ccccccccccccc} & & & & & & 0 & & 0 & & 0 & & \\ & & & & & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & \to & \Theta & \xrightarrow{j_1} & 3 & \xrightarrow{D_1} & 5 & \xrightarrow{D_2} & 2 & \to & 0 \\ & & & & & & \downarrow & & \downarrow & & \| & & \\ & & 0 & \to & 2 & \xrightarrow{j_1} & 6 & \xrightarrow{D_1} & 6 & \xrightarrow{D_2} & 2 & \to & 0 \\ & & & & \| & & \downarrow \Phi_0 & & \downarrow \Phi_1 & & \downarrow & & \\ 0 \to & & \Theta & \to & 2 & \xrightarrow{\mathcal{D}} & 3 & \xrightarrow{\mathcal{D}_1} & 1 & \to & 0 & & \\ & & & & & & \downarrow & & \downarrow & & & & \\ & & & & & & 0 & & 0 & & & & \end{array}$$

with fiber dimensions:

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & 0 & \to & 1 & \xrightarrow{-\delta} & 1 & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & \Theta & \xrightarrow{j_2} & 4 & \xrightarrow{D} & 6 & & \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & \Theta & \xrightarrow{j_1} & 3 & \xrightarrow{D_1} & 5 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

It is important to point out the importance of formal integrability and involution in this case. For this, let us start with a 1-form $\alpha = (\alpha_1, \alpha_2)$, denote its variation by $A = (A_1, A_2)$ and consider only the linear inhomogeneous system

$\mathcal{D}\xi = \mathcal{L}(\xi)\alpha = A$ with no CC for A. If the ground differential field is $K = \mathbb{Q}(x^1, x^2)$ with commuting derivations (d_1, d_2), let us choose $\alpha = x^2 dx^1 = (x^2, 0)$, $A = (x^2, x^1)$. As a lift $\xi_1 \in J_1(T)$ of A, we let the reader check that we may choose in K:

$$\xi^1 = 0, \xi^2 = 0, \xi_1^1 = 1, \xi_2^1 = \frac{x^1}{x^2}, \xi_1^2 = 0, \xi_2^2 = 0$$

Using one prolongation, we have:

$$d_1 A_1 \equiv x^2 \xi_{11}^1 + \xi_1^2 = 0, d_2 A_1 \equiv x^2 \xi_{12}^1 + \xi_1^1 + \xi_2^2 = 1,$$
$$d_1 A_2 \equiv x^2 \xi_{12}^1 = 1, d_2 A_2 \equiv x^2 \xi_{22}^1 + \xi_2^1 = 0$$

If $\beta = -d\alpha = dx^1 \wedge dx^2$, we may denote its variation by B and get at once $B = d_2 A_1 - d_1 A_2 \equiv \xi_1^1 + \xi_2^2 = 0$. Such a result is contradicting our initial choice $1 + 0 = 1$ and we cannot therefore find a lift ξ_2 of $j_1(A)$. Hence, we have to introduce the new geometric object $\omega = (\alpha, \beta)$ with $\Omega = (A, B)$ and CC $d\alpha + \beta = 0$ leading to $d_1 A_2 - d_2 A_1 + B = 0$ while using the previous diagrams. We can therefore lift $\Omega = (A, B)$ to $\xi_1 \in J_1(T)$ by choosing in K:

$$\xi^1 = 0, \xi^2 = 0, \xi_1^1 = 1, \xi_2^1 = \frac{x^1}{x^2}, \xi_1^2 = 0, \xi_2^2 = -1$$

However, we have now to add:

$$d_1 B \equiv \xi_{11}^1 + \xi_{12}^2 = 0, d_2 B \equiv \xi_{12}^1 + \xi_{22}^2 = 0$$

and lift $j_1(\Omega)$ to $\xi_2 \in J_2(T)$ over $\xi_1 \in J_1(T)$ by choosing in K:

$$\xi_{11}^1 = 0, \xi_{12}^1 = \frac{1}{x^2}, \xi_{22}^1 = -\frac{x^1}{(x^2)^2}, \xi_{11}^2 = 0, \xi_{12}^2 = 0, \xi_{22}^2 = -\frac{1}{x^2}$$

The image of the Spencer operator is $X_1 = D\xi_2 = j_1(\xi_1) - \xi_2$ that is to say:

$$X_{,1}^1 = -1, X_{,2}^1 = -\frac{x^1}{x^2}, X_{,1}^2 = 0, X_{,2}^2 = 1,$$

$$X_{1,1}^1 = 0, X_{2,1}^1 = 0, X_{1,2}^1 = -\frac{1}{x^2}, X_{2,2}^1 = 0, X_{1,1}^2 = 0, X_{2,1}^2 = 0, X_{1,2}^2 = 0, X_{2,2}^2 = \frac{1}{x^2}$$

and we check that $X_1 \in T^* \otimes R_1$, namely:

$$x^2 X_{1,i}^1 + X_{,i}^2 = 0, X_{2,i}^1 = 0, X_{1,i}^1 + X_{2,i}^2 = 0, \forall i = 1, 2$$

a result which is not evident at first sight and has no meaning in any classical approach because we use *sections* and not *solutions*.

Now, if $f_{q+1}, f'_{q+1} \in \Pi_{q+1}$ are such that $f_{q+1}^{-1}(j_1(\omega)) = f'^{-1}_{q+1}(j_1(\omega)) = j_1(\bar{\omega})$, it follows that $(f'_{q+1} \circ f_{q+1}^{-1})(j_1(\omega)) = j_1(\omega) \Rightarrow \exists g_{q+1} \in \mathcal{R}_{q+1}$ such that $f'_{q+1} = g_{q+1} \circ f_{q+1}$ and the new $\sigma'_q = \bar{D} f'^{-1}_{q+1}$ differs from the initial $\sigma_q = \bar{D} f_{q+1}^{-1}$ by a gauge transformation.

Conversely, let $f_{q+1}, f'_{q+1} \in \Pi_{q+1}$ be such that $\sigma_q = \bar{D} f_{q+1}^{-1} = \bar{D} f'^{-1}_{q+1} = \sigma'_q$. It follows that $\bar{D}(f_{q+1}^{-1} \circ f'_{q+1}) = 0$ and one can find $g \in aut(X)$ such that $f'_{q+1} = f_{q+1} \circ j_{q+1}(g)$ providing $\bar{\omega}' = f'^{-1}_q(\omega) = (f_q \circ j_q(g))^{-1}(\omega) = j_q(g)^{-1}(f_q^{-1}(\omega)) = j_q(g)^{-1}(\bar{\omega})$.

PROPOSITION 4.8: Natural transformations of $\mathcal{F}$ *over the source* in the nonlinear Janet sequence correspond to gauge transformations of $T^* \otimes R_q$ or C_1 *over the target* in the nonlinear Spencer sequence. Similarly, the Lie derivative $\mathcal{D}\xi = \mathcal{L}(\xi)\omega \in F_0$ in the linear Janet sequence corresponds to the Spencer operator $D\xi_{q+1} \in T^* \otimes R_q$ or $D_1\xi_q \in C_1$ in the linear Spencer sequence.

With a slight abuse of language $\delta f = \eta \circ f \Leftrightarrow \delta f \circ f^{-1} = \eta \Leftrightarrow f^{-1} \circ \delta f = \xi$ when $\eta = T(f)(\xi)$ and we get $j_q(f)^{-1}(\omega) = \bar{\omega} \Rightarrow j_q(f+\delta f)^{-1}(\omega) = \bar{\omega} + \delta\bar{\omega}$ that is $j_q\left(f^{-1} \circ (f+\delta f)\right)^{-1}(\bar{\omega}) = \bar{\omega} + \delta\bar{\omega} \Rightarrow \delta\bar{\omega} = \mathcal{L}(\xi)\bar{\omega}$ and
$j_q\left((f+\delta f)\circ f^{-1} \circ f\right)^{-1}(\omega) = j_q(f)^{-1}\left(j_q\left((f+\delta f)\circ f^{-1}\right)^{-1}(\omega)\right)$
$\Rightarrow \delta\bar{\omega} = j_q(f)^{-1}\left(\mathcal{L}(\eta)\omega\right)$.

Passing to the infinitesimal point of view, we obtain the following generalization of Remark 3.12 which is important for applications.

COROLLARY 4.9:
$\bar{\Omega} = \delta\bar{\omega} = L(\xi_q)\bar{\omega} = f_q^{-1}\left(L(\eta_q)\omega\right) \Rightarrow \delta\bar{\omega} = \mathcal{L}(\xi)\bar{\omega} = j_q(f)^{-1}\left(\mathcal{L}(\eta)\omega\right)$.

Recapitulating the results so far obtained concerning the links existing between the source and the target points of view, we may set in a symbolic way:

$$\delta f_q \overset{(f_q)}{\leftrightarrow} \eta_q \overset{(f_{q+1})}{\leftrightarrow} \bar{\xi}_q \overset{(\chi_q)}{\leftrightarrow} \xi_q$$

In order to help the reader maturing the corresponding nontrivial formulas, we compute explicitly the case $n=1, q=1,2$ and let the case n arbitrary left to the reader as a difficult exercise that cannot be achieved by hand when $q \ge 3$:

EXAMPLE 4.10: Using the previous formulas, we have $\delta f(x) = \eta(f(x))$, $\delta f_x(x) = \eta_y(f(x))f_x(x)$ and:

$$\eta_1 = f_2(\bar{\xi}_1) \Rightarrow \left(\eta(f(x)) = f_x(x)\bar{\xi}(x),\right.$$
$$\left.\eta_y(f(x))f_x(x) = f_x(x)\bar{\xi}_x(x) + f_{xx}(x)\bar{\xi}(x)\right)$$

The delicate point is that we have successively:

$$\chi_{,x} = \frac{\partial_x f}{f_x} - 1 = A - 1, \quad \chi_{x,x} = \frac{1}{f_x}\left(\partial_x f_x - \frac{\partial_x f}{f_x} f_{xx}\right)$$

$$\bar{\xi} = \xi + \chi_{,x}(\xi) = \frac{\partial_x f}{f_x}\xi = A\xi, \quad \bar{\xi}_x = \xi_x + \chi_{x,x}\xi$$

$$\Rightarrow \quad \eta = \partial_x f \xi, \quad \eta_y = \xi_x + \frac{\partial_x f_x}{f_x}\xi$$

$$f_x \eta_{yy} = \xi_{xx} + \frac{f_{xx}}{f_x}\xi_x + \left(\frac{\partial_x f_{xx}}{f_x} - \frac{f_{xx}}{(f_x)^2}\partial_x f_x\right)\xi$$

When $z = g(y), y = f(x) \Rightarrow z = (g \circ f)(x) = h(x)$, we obtain therefore the simple groupoid composition formulas $h_x(x) = g_y(f(x))f_x(x)$ and thus:

$$\zeta = \partial_x h \xi = \partial_y g \eta = \partial_y g \partial_x f \xi,$$

$$\zeta_z = \eta_y + \frac{\partial_y g_y}{g_y}\eta = \xi_x + \left(\frac{\partial_y g_y}{g_y}\partial_x f + \frac{\partial_x f_x}{f_x}\right)\xi = \xi_x + \frac{\partial_x h_x}{h_x}\xi$$

Using indices in arbitrary dimension, we get successively:

$$\eta^k = f_r^k \bar{\xi}^r, \eta_u^k f_i^u = f_r^k \bar{\xi}_i^r + f_{ri}^k \bar{\xi}^r \eta^k$$
$$\Rightarrow \eta^k = \xi^r \partial_r f^k, \eta_u^k f_i^u = f_s^k \left(\xi_i^s + g_u^s \left(\partial_r f_i^u - A_r^t f_{ti}^u\right)\xi^r\right) + f_{ti}^k A_r^t \xi^r$$

$$\eta_u^k = g_u^i f_s^k \xi_i^s + \xi^r g_u^i \partial_r f_i^k \Rightarrow \eta_k^k = \xi_r^r + \xi^r g_u^i \partial_r f_i^u$$

As a very useful application, we obtain successively:

$$\Delta(x) = det\left(\partial_i f^k(x)\right) \Rightarrow \delta\Delta = \Delta\frac{\partial \eta^k}{\partial y^k} = \Delta\partial_r \xi^r + \xi^r \partial_r \Delta = \partial_r\left(\xi^r \Delta\right)$$

$$\delta det(A) = det(A)\left(\frac{\partial \eta^k}{\partial y^k} - \eta_k^k\right) = det(A)\left(\partial_r \xi^r - \xi_r^r\right) + \xi^r \partial_r det(A)$$

where sections of jet bundles are used in an essential way, and the important lemma:

LEMMA 4.11: When the transformation $y = f(x)$ is invertible with inverse $x = g(y)$, we have the *fundamental identity* over the source or over the target:

$$\frac{\partial}{\partial x^i}\left(\Delta(x)\frac{\partial g^i}{\partial y^k}(f(x))\right) \equiv 0, \quad \forall x \in X$$

$$\Leftrightarrow \frac{\partial}{\partial y^k}\left(\frac{1}{\Delta(g(y))}\frac{\partial f^k}{\partial x^i}(g(y))\right) \equiv 0, \quad \forall y \in Y$$

EXAMPLE 4.12: We proceed the same way for studying the links existing between $\chi_q = \bar{D} f_{q+1}$ *over the source*, $\chi_q^{-1} = \sigma_q = \bar{D} f_{q+1}^{-1}$ *over the target* and the nonlinear Spencer operator. First of all, we notice that:

$$\sigma_q = f_{q+1} \circ j_1\left(f_q^{-1}\right) - id_{q+1} = f_{q+1} \circ \left(id_{q+1} - f_{q+1}^{-1} j_1\left(f_q\right)\right) \circ j_1\left(f_q\right)^{-1}$$
$$= -f_{q+1} \circ \chi_q \circ j_1\left(f_q\right)^{-1}$$

and the components of σ_q thus factor through linear combinations of the components of χ_q. After tedious computations, we get successively when $m = n = 1$:

$$\chi_{,x} = \frac{\partial_x f}{f_x} - 1 = A - 1 = \frac{1}{f_x}\left(\partial_x f - f_x\right)$$

$$\chi_{x,x} = \frac{1}{f_x}\left(\partial_x f_x - \frac{\partial_x f}{f_x} f_{xx}\right) = \frac{1}{f_x}\left(\partial_x f_x - f_{xx}\right) - \frac{f_{xx}}{\left(f_x\right)^2}\left(\partial_x f - f_x\right)$$

$$\chi_{xx,x} = \frac{1}{f_x}\left(\partial_x f_{xx} - \frac{\partial_x f}{f_x} f_{xxx}\right) - 2\frac{f_{xx}}{\left(f_x\right)^2}\left(\partial_x f_x - \frac{\partial_x f}{f_x} f_{xx}\right)$$
$$= \frac{1}{f_x}\left(\partial_x f_{xx} - f_{xxx}\right) - 2\frac{f_{xx}}{\left(f_x\right)^2}\left(\partial_x f_x - f_{xx}\right) + \left(2\frac{\left(f_{xx}\right)^2}{\left(f_x\right)^3} - \frac{f_{xxx}}{\left(f_x\right)^2}\right)\left(\partial_x f - f_x\right)$$

These formulas agree with the successive constructive/inductive identities:

$$\begin{cases} \chi_{,x} f_x = \partial_x f - f_x \\ \chi_{x,x} f_x + \chi_{,x} f_{xx} = \partial_x f_x - f_{xx} \\ \chi_{xx,x} f_x + 2\chi_{x,x} f_{xx} + \chi_{,x} f_{xxx} = \partial_x f_{xx} - f_{xxx} \end{cases}$$

showing that χ_q is linearly depending on Df_{q+1} and we finally get:

$$\begin{cases} \sigma_{,y} = -(\partial_x f - f_x)\dfrac{1}{\partial_x f} = \dfrac{f_x}{\partial_x f} - 1 = -f_x \chi_{,x} \dfrac{1}{\partial_x f} \\ \sigma_{y,y} = -\dfrac{1}{f_x}(\partial_x f_x - f_{xx})\dfrac{1}{\partial_x f} = -\left(\chi_{x,x} + \dfrac{f_{xx}}{f_x}\chi_{,x}\right)\dfrac{1}{\partial_x f} \\ \sigma_{yy,y} = -\left(\dfrac{1}{(f_x)^2}(\partial_x f_{xx} - f_{xxx}) - \dfrac{f_{xx}}{(f_x)^3}(\partial_x f_x - f_{xx})\right)\dfrac{1}{\partial_x f} \\ \qquad = -\left(\dfrac{1}{f_x}\chi_{xx,x} + \dfrac{f_{xx}}{(f_x)^2}\chi_{x,x} + \left(\dfrac{f_{xxx}}{(f_x)^2} - \dfrac{(f_{xx})^2}{(f_x)^3}\right)\chi_{,x}\right)\dfrac{1}{\partial_x f} \end{cases}$$

while using successively the relations $g_y f_x = 1$, $\partial_y g \partial_x f = 1$, $g_{yy}(f_x)^2 + g_y f_{xx} = 0$ and so on when $x = g(y)$ is the inverse of $y = f(x)$, in a coherent way with the action of f_3 on $J_2(T)$ which is described as follows:

$$\begin{cases} \eta = f_x \xi \\ \eta_y = \xi_x + \dfrac{f_{xx}}{f_x}\xi \\ \eta_{yy} = \dfrac{1}{f_x}\xi_{xx} + \dfrac{f_{xx}}{(f_x)^2}\xi_x + \left(\dfrac{f_{xxx}}{(f_x)^2} - \dfrac{(f_{xx})^2}{(f_x)^3}\right)\xi \end{cases}$$

Restricting these formulas to the affine case defined by $y_{xx} = 0 \Rightarrow \xi_{xx} = 0$, we have thus $y_{xx} = 0, y_{xxx} = 0 \Rightarrow f_{xx} = 0, f_{xxx} = 0$. It follows that $\eta = f_x \xi, \eta_y = \xi_x, \eta_{yy} = \dfrac{1}{f_x}\xi_{xx} = 0$ on one side and $\chi_{xx,x} = 0 \Leftrightarrow \sigma_{yy,y} = 0$ in a coherent way. It is finally important to notice that these results are not evident, even when $m = n = 1$, as soon as second order jets are involved.

We shall use *all* the preceding formulas in the next example showing that, contrary to what happens in elasticity theory where the source is usually identified with the Lagrange variables, in both the Vessiot/Janet and the Cartan/Spencer approaches, *the source must be identified with the Euler variables without any possible doubt*.

EXAMPLE 4.13: With $n = 1, q = 1, \mathcal{F} = T^*$ and the finite OD Lie equation $\omega(y) y_x = \omega(x)$ with $\omega \in T^*$ and corresponding Lie operator $\mathcal{D}\xi \equiv \mathcal{L}(\xi)\omega = \omega(x)\partial_x \xi + \xi \partial_x \omega(x)$ *over the source*, we have:

$$\omega(f(x)) f_x(x) = \bar{\omega}(x), \quad \omega(f(x)) f_{xx}(x) + \partial_y \omega(f(x)) f_x^2(x) = \partial_x \bar{\omega}(x)$$

Differentiating once the first equation and substracting the second, we obtain therefore:

$$\omega \sigma_{y,y} + \sigma_{,y}\partial_y \omega \equiv -\omega(1/f_x)(\partial_x f_x - f_{xx})(1/\partial_x f) + ((f_x/\partial_x f) - 1)\partial_y \omega = 0$$

whenever $y = f(x)$. Finally, setting $\omega(f(x))\partial_x f(x) = \bar{\omega}(x)$, we get *over the target*:

$$\delta\bar{\omega} = \omega(f(x))\frac{\partial \eta}{\partial y}\partial_x f(x) + \partial_x f(x)\frac{\partial \omega}{\partial y}(f(x))\eta = \partial_x f \mathcal{L}(\eta)\omega$$

Differentiating $\eta = \xi\partial_x f$ in order to obtain $\frac{\partial\eta}{\partial y} = \partial_x\xi + \xi\left(\partial_{xx} f/\partial_x f\right)$, we get *over the source*:

$$\delta\bar{\omega} = \bar{\omega}\partial_x\xi + \xi\partial_x\bar{\omega} = \mathcal{L}(\xi)\bar{\omega}$$

We may summarize these results as follows:

$$\delta\bar{\omega} = \mathcal{L}(\xi)\bar{\omega} \overset{(j_1(f))}{\rightarrow} \delta\bar{\omega} = \partial_x f\,\mathcal{L}(\eta)\omega$$

We invite the reader to extend this result to an arbitrary dimension $n \geq 2$.

EXAMPLE 4.14: The case of an affine stucture needs more work with $n = m = 1, q = 2$. Indeed, let us consider the action of the affine Lie group of transformations $\bar{y} = ay + b$ with $a, b = cst$ acting on the target $y \in Y$ considered as a copy of the real line *X*. We obtain the prolongations up to order 2 of the 2 infinitesimal generators of the action:

$$a \rightarrow y\frac{\partial}{\partial y} + y_x\frac{\partial}{\partial y_x} + y_{xx}\frac{\partial}{\partial y_{xx}}, \quad b \rightarrow \frac{\partial}{\partial y} + 0\frac{\partial}{\partial y_x} + 0\frac{\partial}{\partial y_{xx}}$$

There cannot be any differential invariant of order 1 and the only generating one of order 2 can be $\Phi \equiv y_{xx}/y_x$. When $\bar{x} = \varphi(x)$ we get successively $y_x = y_{\bar{x}}\partial_x\varphi$, $y_{xx} = y_{\bar{x}\bar{x}}\left(\partial_x\varphi\right)^2 + y_{\bar{x}}\partial_{xx}\varphi$ and Φ transforms like $u = \partial_x\varphi\bar{u} + \frac{\partial_{xx}\varphi}{\partial_x\varphi}$ a result providing the bundle of geometric objects $\mathcal{F}$ with local coordinates (x, u) and corresponding transition rules. For any section γ, we get the Vessiot general system $\mathcal{R}_2 \subset \Pi_2$ of second order finite Lie equations $\frac{y_{xx}}{y_x} + \gamma(y)y_x = \gamma(x)$ which is already in Lie form and relates the jet coordinates (x, y, y_x, y_{xx}) of order 2. The special section is $\gamma = 0$ and we may consider the automorphic system $\Phi \equiv \frac{y_{xx}}{y_x} = \bar{\gamma}(x)$ obtained by introducing *any second order section* $f_2(x) = \left(f(x), f_x(x), f_{xx}(x)\right)$, for example $f_2 = j_2(f)$ providing $\left(f(x), \partial_x f(x), \partial_{xx} f(x)\right)$. It is not at all evident, *even on such an elementary example*, to compute the variation $\bar{\Gamma} = \delta\bar{\gamma}$ induced by the previous formulas and to prove that, like any *field* quantity, it only depends on $\bar{\gamma}$ on the condition to use *only* source quantities. For this, setting $\frac{f_{xx}(x)}{f_x(x)} = \bar{\gamma}(x)$, varying and substituting, we obtain:

$$\bar{\Gamma} = \delta\bar{\gamma} = \frac{\delta f_{xx}}{f_x} - \frac{f_{xx}}{\left(f_x\right)^2}\delta f_x = f_x\eta_{yy} = \xi_{xx} + \bar{\gamma}\xi_x + \xi\partial_x\bar{\gamma}$$

Now, linearizing the preceding Lie equation over the identity transformation $y = x$, we get the Medolaghi equation:

$$L(\xi_2)\gamma \equiv \xi_{xx} + \gamma(x)\xi_x + \xi\partial_x\gamma(x) = 0, \forall \xi_2 \in R_2 \subset J_2(T)$$

and the striking formula $\bar{\Gamma} = \delta\bar{\gamma} = L(\xi_2)\bar{\gamma}$ over the source for an arbitrary $\xi_2 \in J_2(T)$. We finally point out the fact that, as we have just shown above and contrary to what the brothers Cosserat had in mind, the first order operators involved in the nonlinear Spencer sequence have *strictly nothing to do* with the operators involved in the nonlinear Janet sequence whenever $q \geq 2$. For example, in the present situation, $\chi_{,x} = \frac{\partial_x f}{f_x} - 1$ has nothing to do with $\Phi \equiv \frac{f_{xx}}{f_x}$. Similarly, using the comment before example 4.7 in the linear framework, we have the first order Spencer operator $D_1 : (\xi, \xi_x) \rightarrow (\partial_x \xi - \xi_x, \partial_x \xi_x)$ on one side and the second order Lie operator $\mathcal{D} : \xi \rightarrow \partial_{xx} \xi$ on the other side.

The next delicate example proves nevertheless that target quantities may also be used.

EXAMPLE 4.15: In the last example, depending on the way we use $\bar{\gamma}(x)$ on the source or $\gamma(y)$ on the target, we may consider the two (very different) Medolaghi equations:

$$\xi_{xx} + \bar{\gamma}(x)\xi_x + \xi \partial_x \bar{\gamma}(x) = 0 \quad \leftrightarrow \quad \eta_{yy} + \gamma(y)\eta_y + \eta \partial_y \gamma(y) = 0$$

Now, starting from the single OD equation $\frac{f_{xx}}{f_x} = \bar{\gamma}(x)$ in sectional notations, we may successively *differentiate* and *prolongate* once in order to get:

$$\frac{\partial_x f_{xx}}{f_x} - \frac{f_{xx}}{(f_x)^2} \partial_x f_x = \partial_x \bar{\gamma}(x) \quad \leftrightarrow \quad \frac{f_{xxx}}{f_x} - \left(\frac{f_{xx}}{f_x}\right)^2 = \partial_x \bar{\gamma}(x)$$

Substracting the second from the first as a way to eliminate $\bar{\gamma}$, we obtain a linear relation involving only the components of the nonlinear Spencer operator in a coherent way with the theory of nonlinear systems, namely:

$$\frac{1}{f_x}(\partial_x f_{xx} - f_{xxx}) - \frac{f_{xx}}{(f_x)^2}(\partial_x f_x - f_{xx}) = 0$$

At first sight it does not seem possible to know whether we have a linear combination of the components of χ_2 or of the components of σ_2. However, if we come back to the original situation $f_q^{-1}(\omega) = \bar{\omega}$, we have eliminated $j_1(\bar{\gamma})$ over the source and we are thus only left with $j_1(\gamma)$ over the target. Hence it can only depend on σ_2 and we find indeed the striking relation:

$$-\frac{1}{f_x}\left[\frac{1}{f_x}(\partial_x f_{xx} - f_{xxx}) - \frac{f_{xx}}{(f_x)^2}(\partial_x f_x - f_{xx})\right]\frac{1}{\partial_x f} = \sigma_{yy,y} = 0$$

provided by the simple second order Medolaghi equation $\gamma = 0 \Rightarrow \eta_{yy} = 0$ over the target. It is essential to notice that no classical technique can provide these results which are essentially depending on the nonlinear Spencer operator and are thus not known today.

EXAMPLE 4.16: The above methods can be applied to any explicit example. The reader may treat as an exercise the case of the pseudogroup of isometries of a non degenerate metric which can be found in any textbook of continuum mechanics or elasticity theory, though in a very different framework with methods only valid for tensors. With the previous notations, let $\omega \in S_2 T^*$ with $det(\omega) \neq 0$

and consider the following nonlinear system $\omega_{kl}\left(f(x)\right)\partial_i f^k(x)\partial_j f^l(x)=\bar{\omega}_{ij}(x)$ with $1\le i,j,k,l\le n$. One obtains therefore:

$$\delta\bar{\omega}_{ij}=\bar{\omega}_{rj}\partial_i\xi^r+\bar{\omega}_{ir}\partial_i\xi^u+\xi^r\partial_r\bar{\omega}_{ij}=\partial_i f^k\partial_j f^l\left(\omega_{ul}\frac{\partial\eta^u}{\partial y^k}+\omega_{ku}\frac{\partial\eta^u}{\partial y^l}+\eta^u\frac{\partial\omega_{kl}}{\partial y^u}\right)$$

and thus the same recapitulating formulas linking the source and target variations:

$$\delta\bar{\omega}=\mathcal{L}(\xi)\bar{\omega}\overset{(j_1(f))}{\rightarrow}\delta\bar{\omega}=\partial_i f^k\partial_j f^l\left(\mathcal{L}(\eta)\omega\right)_{kl}$$

It is also difficult to compute or compare the variational formulas over the source and target in the nonlinear Spencer sequence, even when $m=n=1$ and $q=0,1$ ([29]).

EXAMPLE 4.17: Let us prove that the explicit computation of the gauge transformation is at the limit of what can be done with the hand, even when $m=n=1,q=1$. We have successively:

$$\chi_{,x}=\frac{\partial_x f}{f_x}-1,\ \chi_{x,x}=\frac{1}{f_x}\left(\partial_x f_x-\frac{\partial_x f}{f_x}f_{xx}\right)$$

$$f'(x)=g\left(f(x)\right)\Rightarrow f'_x=g_y f_x,\ f'_{xx}=g_{yy}\left(f_x\right)^2+g_y f_{xx}$$

and thus:

$$\chi'_{,x}=\frac{\partial_x f'}{f'_x}-1=\frac{\partial_y g\partial_x f}{g_y f_x}-1=\left(\chi_{,y}+1\right)\frac{\partial_x f}{f_x}-1=\frac{\partial_x f}{f_x}\chi_{,y}+\left(\frac{\partial_x f}{f_x}-1\right)$$

$$\begin{aligned}\chi'_{x,x}&=\frac{1}{f'_x}\left(\partial_x f'_x-\frac{\partial_x f'}{f'_x}f'_{xx}\right)\\&=\frac{1}{g_y f_x}\left(\partial_y g_y\left(\partial_x f\right)f_x+g_y\partial_x f_x\right)-\frac{\partial_y g\partial_x f}{g_y f_x}\left(g_{yy}\left(f_x\right)^2+g_y f_{xx}\right)\\&=\frac{1}{g_y}\partial_y g_y\partial_x f+\frac{\partial_x f_x}{f_x}-\frac{\partial_y g\partial_x fg_{yy}}{\left(g_y\right)^2}-\frac{\partial_y g\partial_x ff_{xx}}{g_y\left(f_x\right)^2}\\&=\left(\partial_x f\chi_{y,y}-\frac{\partial_x ff_{xx}}{\left(f_x\right)^2}\chi_{,y}\right)+\frac{1}{f_x}\left(\partial_x f_x-\frac{\partial_x f}{f_x}f_{xx}\right)\end{aligned}$$

Setting $f_2=id_2+t\xi_2+\cdots$ and passing to the limit when $t\to 0$, we finally obtain:

$$\begin{aligned}\delta\chi_{,x}&=\left(\partial_x\xi-\xi_x\right)+\left(\xi\partial_x\chi_{,x}+\chi_{,x}\partial_x\xi-\chi_{,x}\xi_x\right)\\\delta\chi_{x,x}&=\left(\partial_x\xi_x-\xi_{xx}\right)+\left(\xi\partial_x\chi_{x,x}+\chi_{x,x}\partial_x\xi-\chi_{,x}\xi_{xx}\right)\end{aligned}$$

If we use the standard euclidean metric $\omega=1\Rightarrow\gamma=0$, we may thus introduce the pure 1-form $\alpha=\chi_{x,x}+\gamma\chi_{,x}$. We should consider the defining formula $\chi'_1=f_2^{-1}\circ\chi_1\circ j_1\left(f_1\right)+\bar{D}f_2$ and have to introduce the additional term $f_2^{-1}(\gamma)\chi_{,x}$ which is only leading to the additional infinitesimal term $\left(L\left(\xi_2\right)\gamma\right)\chi_{,x}=\xi_{xx}\chi_{,x}$ because $\gamma=0$. We finally obtain:

$$\delta\alpha = \delta\chi_{x,x} + \xi_{xx}\chi_{,x} + \gamma\delta\chi_{,x} = \left(\partial_x\xi_x - \xi_{xx}\right) + \left(\xi\partial_x\alpha + \alpha\partial_x\xi\right)$$

and this result can be easily extended to an arbitrary dimension with the formula:

$$\alpha_i = \chi^r_{r,i} + \gamma^s_{sr}\chi^r_{,i} \Rightarrow (\delta\alpha)_i = \left(\partial_i\xi^r_r - \xi^r_{ri}\right) + \left(\xi^r\partial_r\alpha_i + \alpha_r\partial_i\xi^r\right)$$

Comparing this procedure with the one we have adopted in the previous exampes, we have:

$$\chi_{,x} = \frac{\partial_x f}{f_x} - 1 = A - 1 \Rightarrow \delta\chi_{,x} = \frac{\partial_x\delta f}{f_x} - \frac{\partial_x f}{(f_x)^2}\delta f_x = \frac{1}{f_x}\left(\frac{\partial\eta}{\partial y} - \eta_y\right)\partial_x f$$

However, taking into account the formulas $\eta = \xi\partial_x f$ and $\eta_y = \xi_x + \frac{\partial_x f_x}{f_x}\xi$, we also get:

$$\begin{aligned}\delta\chi_{,x} &= \frac{1}{f_x}\left(\partial_x\xi\partial_x f + \xi\partial_{xx} f\right) - \frac{\partial_x f}{(f_x)^2}\left(\xi_x f_x + \xi\partial_x f_x\right)\\ &= A\left(\partial_x\xi - \xi_x\right) + \xi\partial_x\chi_{,x}\\ &= \left(\partial_x\xi - \xi_x\right) + \left(\xi\partial_x\chi_{,x} + \chi_{,x}\partial_x\xi - \chi_{,x}\xi_x\right)\end{aligned}$$

Working over the target is more difficult. Indeed, we have successively (*care to the first step*):

$$\sigma_{,y} = \frac{f_x}{\partial_x f} - 1 \Rightarrow \delta\sigma_{,y} + \eta\frac{\partial\sigma_{,y}}{\partial y} = \frac{\delta f_x}{\partial_x f} - \frac{f_x}{(\partial_x f)^2}\partial_x\delta f = -\frac{f_x}{(\partial_x f)^2}\left(\frac{\partial\eta}{\partial y} - \eta_y\right)$$

$$\begin{aligned}\delta\sigma_{,y} &= -\left[\frac{f_x}{\partial_x f}\left(\frac{\partial\eta}{\partial y} - \eta_y\right) + \eta\frac{\partial\sigma_{,y}}{\partial y}\right]\\ &= -\left[\left(\frac{\partial\eta}{\partial y} - \eta_y\right) + \left(\eta\frac{\partial\sigma_{,y}}{\partial y} + \sigma_{,y}\frac{\partial\eta}{\partial y} - \sigma_{,y}\eta_y\right)\right]\end{aligned}$$

More generaly, we let the reader prove that the variation of σ_q over the target (respectively the source) is described by "*minus*" the same formula as the variation of χ_q over the source (respectively the target). In any case, the reader must not forget that the word "variation" just means that the section f_{q+1} is changed, *not* that the source is moved. Accordingly, getting in mind this example and for simplicity, we shall always prefer to work with vertical bundles over the source, closely following the purely mathematical definitions, contrary to Weyl ([3], §28, formulas (17) to (27), p 233-236). The reader must be now ready for comparing the variations of $\chi_{x,x}$ and $\sigma_{y,y}$.

In order to conclude this section, we provide without any proof two results and refer the reader to ([7]) for details.

PROPOSITION 4.18: Changing slightly the notation while setting $\sigma_{q-1} = \bar{D}'\chi_q$, we have:

$$\chi'_q = f^{-1}_{q+1}\circ\chi_q\circ j_1(f) + \bar{D}f_{q+1} \Rightarrow \sigma'_{q-1} = f^{-1}_q\circ\sigma_{q-1}\circ j_1(f)$$

where f^{-1}_q acts on $J_{q-1}(T)$ and $j_1(f)$ acts on $\wedge^2 T^*$. It follows that gauge transformations exchange the solutions of $\bar{D}'$ among themselves.

COROLLARY 4.19: Denoting by $\mathcal{C}(\ )$ the cyclic sum, we have the so-called *Bianchi identity*:

$$D\sigma_{q-1}(\xi,\eta,\zeta)+\mathcal{C}(\xi,\eta,\zeta)\{\sigma_{q-1}(\xi,\eta),\chi_{q-1}(\zeta)\}=0$$

5. Applications

Before studying in a specific way electromagnetism and gravitation, we shall come back to Example 4.10 and provide a technical result which, though looking like evident at first sight, is at the origin of a deep misunderstanding done by the brothers Cosserat and Weyl on the variational procedure used in the study of physical problems (Compare to [14]).

Setting $dx = dx^1 \wedge \cdots \wedge dx^n$ for simplicity while using Lemma 4.11 and the fact that the standard Lie derivative is commuting with any diffeomorphism, we obtain at once:

$$y = f(x) \Rightarrow dy = det\left(\partial_i f^k(x)\right)dx = \Delta(x)dx$$

$$\eta = T(f)\xi \Rightarrow \mathcal{L}(\eta)dy = \mathcal{L}(\xi)\left(\Delta(x)dx\right)$$

$$\Rightarrow \delta\Delta = \Delta div_y(\eta) = \Delta div_x(\xi) + \xi^r\partial_r\Delta$$

The interest of such a presentation is to provide the right correspondence between the source/target and the Euler/Lagrange choices. Indeed, if we use the way followed by most authors up to now in continuum mechanics, we should have source = Lagrange, target = Euler, a result leading to the conservation of mass $dm = \rho dy = \rho_0 dx = dx$ when ρ_0 is the original initial mass per unit volume. We may set $\rho_0 = 1$ and obtain therefore $\rho(f(x)) = 1/\Delta(x)$, a choice leading to:

$$\delta\rho + \eta^k\frac{\partial\rho}{\partial y^k} = -\frac{1}{\Delta^2}\delta\Delta \Rightarrow \delta\rho = -\rho\frac{\partial\eta^k}{\partial y^k} - \eta^k\frac{\partial\rho}{\partial y^k} = -\rho\frac{\partial\xi^r}{\partial x^r} \Rightarrow \delta\rho = -\frac{\partial\left(\rho\eta^k\right)}{\partial y^k}$$

but the concept of "*variation*" is not mathematically well defined, though this result is coherent with the classical formulas that can be found for example in ([4] [9]) or ([3], (17) and (18) p 233, (20) to (21) p 234, (76) p 289, (78) p 290) where "*points are moved*".

On the contrary, if we adopt the *unusual* choice source = Euler, target = Lagrange, we should get $\rho(x) = \Delta(x)$, a choice leading to $\delta\rho = \delta\Delta$ and thus:

$$\delta\rho = \rho\frac{\partial\eta^k}{\partial y^k} = \rho\frac{\partial\xi^r}{\partial x^r} + \xi^r\partial_r\rho = \partial_r\left(\rho\xi^r\right)$$

which is the right choice agreeing, up to the sign, with classical formulas but with the important improvement that this result becomes a purely mathematical one, obtained from a well defined variational procedure involving only the so-called "*vertical*" machinery. This result fully explains why we had doubts about the sign involved in the variational formulas of ([4], p. 383) but without being able to correct them at that time. We may finally revisit Lemma 4.11 in order to obtain the *fundamental identity over the source*:

$$\frac{\partial}{\partial x^i}\left(\Delta(x)\frac{\partial g^i}{\partial y^k}(f(x))\right)\equiv 0,\quad \forall x\in X$$

which becomes the conservation of mass when $n=4$ and $k=4$.

In addition, as many chases will be used through many diagrams in the sequel, we invite the reader not familiar with these technical tools to consult the books ([30] [31]) that we consider as the best references for learning about homological algebra. A more elementary approach can be found in ([32]) that has been used during many intensive courses on the applications of homological algebra to control theory. As for *differential homological algebra*, one of the most difficult tools existing in mathematics today, and its link with applications, we refer the reader to the various references provided in ([33]).

Finally, for the reader interested by a survey on more explicit applications, we particularly refer to ([2] [34] [35] [36]) for analytical mechanics and hydrodynamics, ([5] [37] [38]) for coupling phenomenas, ([36] [39] [40]) for the foundations of Gauge Theory, ([36] [41]) for the foundations of General Relativity.

A) POINCARE, WEYL AND CONFORMAL GROUPS

When constructing inductively the Janet and Spencer sequences for an involutive system $R_q\subset J_q(E)$, we have to use the following commutative and exact diagrams where we have set $F_0=J_q(E)/R_q$ and used a diagonal chase:

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
0\rightarrow & \delta\left(\wedge^{r-1}T^*\otimes g_{q+1}\right) & \rightarrow & \wedge^r T^*\otimes R_q & \rightarrow & C_r & \rightarrow 0\\
 & \downarrow & & \downarrow & & \downarrow & \\
0\rightarrow & \delta\left(\wedge^{r-1}T^*\otimes S_{q+1}T^*\otimes T\right) & \rightarrow & \wedge^r T^*\otimes J_q(E) & \rightarrow & C_r(E) & \rightarrow 0\\
 & \downarrow & & \downarrow & \searrow & \downarrow & \\
0\rightarrow & \wedge^r T^*\otimes R_q+\delta\left(\wedge^{r-1}T^*\otimes S_{q+1}T^*\otimes E\right) & \rightarrow & \wedge^r T^*\otimes F_0 & \rightarrow & F_r & \rightarrow 0\\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
\end{array}$$

It follows that the short exact sequences $0\rightarrow C_r\rightarrow C_r(E)\xrightarrow{\Phi_r}F_r\rightarrow 0$ are allowing to define the Janet and Spencer bundles inductively. If we consider *two* involutive systems $0\subset R_q\subset \hat{R}_q\subset J_q(E)$, it follows that the *kernels* of the induced canonical epimorphisms $F_r\rightarrow \hat{F}_r\rightarrow 0$ are isomorphic to the *cokernels* of the canonical monomorphisms $0\rightarrow C_r\rightarrow \hat{C}_r\subset C_r(E)$ and we may say that *Janet and Spencer play at see-saw* because we have the formula

$dim(C_r)+dim(F_r)=dim(C_r(E))$.

When dealing with applications, we have set $E=T$ and considered systems of finite type Lie equations determined by Lie groups of transformations. Accordingly, we have obtained in particular $C_r=\wedge^r T^*\otimes R_2\subset \wedge^r T^*\otimes \hat{R}_2=\hat{C}_r\subset C_r(T)$ when comparing the classical and conformal Killing systems, but *these bundles have never been used in physics*. However, instead of the classical Killing system $R_1 \subset J_1(T)$ defined by the infinitesimal first order PD Lie equations $\Omega \equiv \mathcal{L}(\xi)\omega = 0$ *and* its first prolongations $R_2 \subset J_2(T)$ defined by the infinitesimal additional second order PD Lie equations $\Gamma \equiv \mathcal{L}(\xi)\gamma = 0$ or the conformal Killing system $\hat{R}_2 \subset J_2(T)$ defined by $\Omega \equiv \mathcal{L}(\xi)\omega = 2A(x)\omega$ and $\Gamma \equiv \mathcal{L}(\xi)\gamma = \left(\delta_i^k A_j(x) + \delta_j^k A_i(x) - \omega_{ij}\omega^{ks} A_s(x)\right) \in S_2T^* \otimes T$ but we may also consider the formal Lie derivatives for geometric objects:

$$\Omega_{ij} \equiv \left(L(\xi_1)\omega\right)_{ij} \equiv \omega_{rj}\xi_i^r + \omega_{ir}\xi_j^r + \xi^r \partial_r \omega_{ij} = 0$$

$$\Gamma_{ij}^k \equiv \left(L(\xi_2)\gamma\right)_{ij}^k \equiv \xi_{ij}^k + \gamma_{rj}^k \xi_i^r + \gamma_{ir}^k \xi_j^r - \gamma_{ij}^r \xi_k^r + \xi^r \partial_r \gamma_{ij}^k = 0$$

We may now introduce the *intermediate differential system* $\tilde{R}_2 \subset J_2(T)$ defined by $\mathcal{L}(\xi)\omega = 2A(x)\omega$ and $\Gamma \equiv \mathcal{L}(\xi)\gamma = 0$, for the *Weyl group* obtained by adding the only dilatation with infinitesimal generator $x^i\partial_i$ to the Poincaré group. We have the relations $R_1 \subset \tilde{R}_1 = \hat{R}_1$ and the strict inclusions $R_2 \subset \tilde{R}_2 \subset \hat{R}_2$ when $R_2 = \rho_1(R_1)$, $\tilde{R}_2 = \rho_1(\tilde{R}_1)$, $\hat{R}_2 = \rho_1(\hat{R}_1)$ but we have to notice that we must have $\partial_i A - A_i = 0$ for the conformal system and thus $A_i = 0 \Rightarrow A = cst$ if we do want to deal again with an involutive second order system $\tilde{R}_2 \subset J_2(T)$. However, we must not forget that the comparison between the Spencer and the Janet sequences can only be done for involutive operators, that is we can indeed use the involutive systems $R_2 \subset \tilde{R}_2$ but we have to use $\hat{R}_3$ even if it is isomorphic to $\hat{R}_2$. Finally, as $\hat{g}_2 \simeq T^*$ and $\hat{g}_3 = 0, \forall n \geq 3$, the first Spencer operator $\hat{R}_2 \xrightarrow{D_1} T^* \otimes \hat{R}_2$ is induced by the usual Spencer operator $\hat{R}_3 \xrightarrow{D} T^* \otimes \hat{R}_2 : \left(0, 0, \xi_{rj}^r, \xi_{rij}^r = 0\right) \to \left(0, \partial_i 0 - \xi_{ri}^r, \partial_i \xi_{rj}^r - 0\right)$ and thus projects by cokernel onto the induced operator $T^* \to T^* \otimes T^*$. Composing with δ, it projects therefore onto $T^* \xrightarrow{d} \wedge^2 T^* : A \to dA = F$ as in EM and so on by using the fact that D_1 *and d are both involutive* or the composite epimorphisms $\hat{C}_r \to \hat{C}_r / \tilde{C}_r \simeq \wedge^r T^* \otimes \left(\hat{R}_2 / \tilde{R}_2\right) \simeq \wedge^r T^* \otimes \hat{g}_2 \simeq \wedge^r T^* \otimes T^* \xrightarrow{\delta} \wedge^{r+1} T^*$. The main result we have obtained is thus to be able to increase the order and dimension of the underlying jet bundles and groups as we have ([29]):

$$POINCARE \quad GROUP \subset WEYL \quad GROUP \subset CONFORMAL \quad GROUP$$

that is $10 < 11 < 15$ when $n = 4$ and our aim is now to prove that *the mathematical structures of electromagnetism and gravitation only depend on the second order jets.*

With more details, the *Killing system* $R_2 \subset J_2(T)$ is defined by the infinitesimal Lie equations in *Medolaghi form* with the well known *Levi-Civita isomorphism* $(\omega, \gamma) \simeq j_1(\omega)$ for geometric objects:

$$\begin{cases} \Omega_{ij} \equiv \omega_{rj}\xi_i^r + \omega_{ir}\xi_j^r + \xi^r \partial_r \omega_{ij} = 0 \\ \Gamma_{ij}^k \equiv \gamma_{rj}^k \xi_i^r + \gamma_{ir}^k \xi_j^r - \gamma_{ij}^r \xi_r^k + \xi^r \partial_r \gamma_{ij}^k = 0 \end{cases}$$

We notice that $R_2(\bar{\omega}) = R_2(\omega) \Leftrightarrow \bar{\omega} = a\omega, a = cst, \bar{\gamma} = \gamma$ and refer the reader to ([27]) for more details about the link between this result and the deformation theory of algebraic structures. We also notice that R_1 is formally integrable and thus R_2 is involutive if and only if ω has constant Riemannian curvature along the results of L. P. Eisenhart ([26]). The only structure constant c appearing in the corresponding Vessiot structure equations is such that $\bar{c} = c/a$ and the normalizer of R_1 is R_1 if and only if $c \neq 0$. Otherwise R_1 is of codimension 1 in its normalizer $\tilde{R}_1$ as we shall see below by adding the only dilatation. In all what follows, ω is assumed to be flat with $c = 0$ and vanishing Weyl tensor.

The *Weyl system* $\tilde{R}_2 \subset J_2(T)$ is defined by the infinitesimal Lie equations:

$$\begin{cases} \omega_{rj}\xi_i^r + \omega_{ir}\xi_j^r + \xi^r\partial_r\omega_{ij} = 2A(x)\omega_{ij} \\ \xi_{ij}^k + \gamma_{rj}^k\xi_i^r + \gamma_{ri}^k\xi_j^r - \gamma_{ij}^r\xi_r^k + \xi^r\partial_r\gamma_{ij}^k = 0 \end{cases}$$

and is involutive if and only if $\partial_i A = 0 \Rightarrow A = cst$. Introducing for convenience the *metric density* $\hat{\omega}_{ij} = \omega_{ij} \Big/ \left(|det(\omega)|\right)^{\frac{1}{n}}$, we obtain the *Medolaghi form* for $(\hat{\omega}, \gamma)$ with $|det(\hat{\omega})| = 1$:

$$\begin{cases} \hat{\Omega}_{ij} \equiv \hat{\omega}_{rj}\xi_i^r + \hat{\omega}_{ir}\xi_j^r - \frac{2}{n}\hat{\omega}_{ij}\xi_r^r + \xi^r\partial_r\hat{\omega}_{ij} = 0 \\ \Gamma_{ij}^k \equiv \xi_{ij}^k + \gamma_{rj}^k\xi_i^r + \gamma_{ri}^k\xi_j^r - \gamma_{ij}^r\xi_r^k + \xi^r\partial_r\gamma_{ij}^k = 0 \end{cases}$$

Finally, the *conformal system* $\hat{R}_2 \subset J_2(T)$ is defined by the following infinitesimal Lie equations:

$$\begin{cases} \omega_{rj}\xi_i^r + \omega_{ir}\xi_j^r + \xi^r\partial_r\omega_{ij} = 2A(x)\omega_{ij} \\ \xi_{ij}^k + \gamma_{rj}^k\xi_i^r + \gamma_{ri}^k\xi_j^r - \gamma_{ij}^r\xi_r^k + \xi^r\partial_r\gamma_{ij}^k = \delta_i^k A_j(x) + \delta_j^k A_i(x) - \omega_{ij}\omega^{kr}A_r(x) \end{cases}$$

and is involutive if and only if $\partial_i A - A_i = 0$ or, equivalently, if ω has vanishing Weyl tensor.

However, introducing again the *metric density* $\hat{\omega}$ while substituting, we obtain after prolongation and division by $\left(|det(\omega)|\right)^{\frac{1}{n}}$ the second order system $\hat{R}_2 \subset J_2(T)$ in *Medolaghi form* and the Levi-Civita isomorphim $(\omega, \gamma) \simeq j_1(\omega)$ *restricts* to an isomorphism $(\hat{\omega}, \hat{\gamma}) \simeq j_1(\hat{\omega})$ if we set:

$$\hat{\gamma}_{ij}^k = \gamma_{ij}^k - \frac{1}{n}\left(\delta_i^k\gamma_{rj}^r + \delta_j^k\gamma_{ri}^r - \omega_{ij}\omega^{ks}\gamma_{rs}^r\right) \Rightarrow \hat{\gamma}_{ri}^r = 0 \left(tr(\hat{\gamma}) = 0\right)$$

$$\begin{cases} \hat{\Omega}_{ij} \equiv \hat{\omega}_{rj}\xi_i^r + \hat{\omega}_{ir}\xi_j^r - \frac{2}{n}\hat{\omega}_{ij}\xi_r^r + \xi^r\partial_r\hat{\omega}_{ij} = 0 \Rightarrow \omega^{ij}\bar{\Omega}^{ij} = 0 \\ \hat{\Gamma}_{ij}^k \equiv \xi_{ij}^k - \frac{1}{n}\left(\delta_i^k\xi_{rj}^r + \delta_j^k\xi_{ri}^r - \hat{\omega}_{ij}\hat{\omega}^{kr}\xi_{rs}^s\right) + \hat{\gamma}_{rj}^k\xi_i^r + \hat{\gamma}_{ri}^k\xi_j^r - \hat{\gamma}_{ij}^r\xi_r^k + \xi^r\partial_r\hat{\gamma}_{ij}^k = 0 \Rightarrow \hat{\Gamma}_{ri}^r = 0 \end{cases}$$

Contracting the first equations by $\hat{\omega}^{ij}$ we notice that ξ_r^r is *no longer vanishing* while, contracting in k and j the second equations, we now notice that ξ_{ri}^r is *no longer vanishing*. It is also essential to notice that the symbols $\hat{g}_1$ and $\hat{g}_2$ only depend on ω and not on any conformal factor.

The following Proposition does not seem to be known:

PROPOSITION 5.A.1: $(id, -\hat{\gamma})$ is the only symmetric $\hat{R}_1$-connection with vanishing trace.

Proof: Using a direct substitution, we have to study:

$$-\hat{\omega}_{ir}\hat{\gamma}^r_{jt} - \hat{\omega}_{rj}\hat{\gamma}^r_{it} + \frac{2}{n}\hat{\omega}_{ij}\hat{\gamma}^r_{rt} + \partial_t\hat{\omega}_{ij}$$

Multiplying by $\left(\left|det(\omega)\right|\right)^{\frac{1}{n}}$, we have to study:

$$-\omega_{ir}\hat{\gamma}^r_{jt} - \omega_{rj}\hat{\gamma}^r_{it} + \frac{2}{n}\omega_{ij}\hat{\gamma}^r_{rt} + \left(\left|det(\omega)\right|\right)^{\frac{1}{n}}\partial_t\hat{\omega}_{ij}$$

or equivalently:

$$-\omega_{ir}\hat{\gamma}^r_{jt} - \omega_{rj}\hat{\gamma}^r_{it} + \frac{2}{n}\omega_{ij}\hat{\gamma}^r_{rt} + \partial_t\omega_{ij} - \frac{1}{n}\omega_{ij}\left(\left|det(\omega)\right|\right)^{-1}\partial_t\left(\left|det(\omega)\right|\right)$$

that is to say:

$$-\omega_{ir}\hat{\gamma}^r_{jt} - \omega_{rj}\hat{\gamma}^r_{it} + \partial_t\omega_{ij} - \frac{2}{n}\omega_{ij}\gamma^s_{st}$$

Now, we have:

$$-\omega_{ir}\left(\gamma^r_{jt} - \frac{1}{n}\left(\delta^r_j\gamma^s_{st} + \delta^r_t\gamma^s_{sj} - \omega_{jt}\omega^{ru}\gamma^s_{su}\right)\right) = -\omega_{ir}\gamma^r_{jt} + \frac{1}{n}\omega_{ij}\gamma^s_{st} + \frac{1}{n}\omega_{it}\gamma^s_{sj} - \frac{1}{n}\omega_{jt}\gamma^s_{si}$$

Finally, taking into account that $(id,-\gamma)$ is a R_1-connection, we have:

$$-\omega_{ir}\gamma^r_{jt} - \omega_{rj}\gamma^r_{it} + \partial_t\omega_{ij} = 0$$

Hence, collecting all the remaining terms, we are left with $\frac{2}{n}\omega_{ij}\gamma^s_{st} - \frac{2}{n}\omega_{ij}\gamma^s_{st} = 0$.

As for the unicity, it comes from a chase in the commutative and exact diagram:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 \rightarrow & & \hat{g}_2 & \xrightarrow{\delta} & T^*\otimes\hat{g}_1 & \xrightarrow{\delta} & \wedge^2T^*\otimes T & \rightarrow 0 & \\ & & \downarrow & & \downarrow & & \| & & \\ 0 \rightarrow & & S_2T^*\otimes T & \xrightarrow{\delta} & T^*\otimes T^*\otimes T & \xrightarrow{\delta} & \wedge^2T^*\otimes T & \rightarrow 0 & \\ & & & & & & \downarrow & & \\ & & & & & & 0 & & \end{array}$$

obtained by counting the respective dimensions with $dim(\hat{g}_1) = (n(n-1)/2)+1 = (n^2-n+2)/2$ and $dim(\hat{g}_2) = n$ while checking that $-n+n(n^2-n+2)-n^2(n-1)/2 = 0$. The lower sequence splits because the short exact δ-sequence $0 \rightarrow S_2T^* \xrightarrow{\delta} T^*\otimes T^* \xrightarrow{\delta} \wedge^2 T^* \rightarrow 0$ splits and the upper sequence also splits because we have a composite monomorphism $\wedge^2T^*\otimes T \simeq T^*\otimes g_1 \rightarrow T^*\otimes\hat{g}_1$.

□

COROLLARY 5.A.2: The R_1-connection $(id,-\gamma)$ is also a $\hat{R}_1$-connection.

Proof: This result first follows from the fact that $(id,-\gamma)\in T^*\otimes R_1$ is over $id\in T^*\otimes T$ and $R_1\subset\hat{R}_1$. However, we may also check such a property directly. Indeed, mutiplying $-\hat{\omega}_{rj}\gamma^r_{it} - \hat{\omega}_{ir}\gamma^r_{rt} + \frac{2}{n}\hat{\omega}_{ij}\gamma^r_{rt} + \partial_t\hat{\omega}_{ij}$ by $\left(\left|det(\omega)\right|\right)^{\frac{1}{n}}$ as in the last Proposition, we obtain:

$$-\omega_{rj}\gamma_{it}^{r}-\omega_{ir}\gamma_{jt}^{r}+\frac{2}{n}\omega_{ij}\gamma_{rt}^{r}+\partial_t\hat{\omega}_{ij}=-\omega_{rj}\gamma_{it}^{r}-\omega_{rj}\gamma_{jt}^{r}+\partial_t\omega_{ij}=0$$

because $(id,-\gamma)$ is a R_1-connection.

□

REMARK 5.A.3: If one is using $(id,-\gamma)$, then $\left(L(\xi_2)\gamma\right)_{ij}^{k}\xi_{ij}^{k}$ when $\gamma=0$ locally and we have $(\delta\alpha)_i=\left(\partial_i\xi_r^r-\xi_{ri}^r\right)+\left(\alpha_r\partial_i\xi^r+\xi^r\partial_r\alpha-i\right)$ as the simplest variation. However, we have $f_2^{-1}(\gamma)=\bar{\gamma}\neq\gamma$ and we cannot thus split the Spencer operator over the target by means of a pull-back. Nevertheless, if one is using $(id,-\hat{\gamma})$, then $L(\xi_2)\hat{\gamma}=0$ when $\xi_2\in\hat{R}_2$ and the variation $(\delta\alpha)_i$ contains an additional term $\xi_{sr}^{s}\chi_{,i}^{r}$ but $f_2^{-1}(\hat{\gamma})=\hat{\gamma}$ and one can split the Spencer operator over the source and over the target with the help of $\hat{\gamma}$ but we have to point out that $\gamma=0\Rightarrow\hat{\gamma}=0$ *locally*.

□

We let the reader exhibit similarly the finite *Lie forms* of the previous equations that will be presented when needed. We have to distinguish the strict inclusions $\Gamma\subset\tilde{\Gamma}\subset\hat{\Gamma}\subset aut(X)$ with:

- The Lie pseudogroup $\Gamma\subset aut(X)$ of isometries which is preserving the metric $\omega\in S_2T^*$ with $det(\omega)\neq 0$ *and thus also* γ.
- The Lie pseudogroup $\tilde{\Gamma}$ which is preserving $\hat{\omega}$ *and* γ.
- The Lie pseudogroup $\hat{\Gamma}$ of conformal isometries which is preserving $\hat{\omega}$ *and thus also* $\hat{\gamma}$ with:

$$\begin{aligned}&g_l^k(x)\left(f_{ij}^l(x)+\gamma_{rs}^l\left(f(x)\right)f_i^r(x)f_j^s(x)\right)\\&=\bar{\gamma}_{ij}^k(x)=\gamma_{ij}^k(x)+\delta_i^k a_j(x)+\delta_i^k a_j(x)-\omega_{ij}(x)\omega^{kr}(x)a_r(x)\end{aligned}$$

where $a_i(x)dx^i\in T^*$ and thus $\bar{\gamma}-\gamma\in\hat{g}_2\subset S_2T^*\otimes T^*\otimes T$.

B) ELECTROMAGNETISM

The key idea, still never acknowledged, is that, even if we shall prove that *electromagnetism only depends on the elations of the conformal group* which are clearly non-linear transformations, we shall see that *electromagnetism has* “*by chance*” *a purely linear behaviour*.

Indeed, setting as we already did $\chi_0=A-id$ and defining $\chi_{lr,j}^k=A_j^s\tau_{lr,s}^k$, we may rewrite the defining equation of the second non-linear Spencer operator $\bar{D}'$ in the form:

$$\begin{cases}\partial_i A_j^k-\partial_j A_i^k=A_i^r\chi_{r,j}^k-A_j^r\chi_{r,i}^k=A_i^rA_j^s\left(\tau_{r,s}^k-\tau_{s,r}^k\right)\\ \partial_i\chi_{l,j}^k-\partial_j\chi_{l,i}^k-\chi_{l,i}^r\chi_{r,j}^k+\chi_{l,j}^r\chi_{r,i}^k=A_i^r\chi_{lr,j}^k-A_j^r\chi_{lr,i}^k=A_i^rA_j^s\left(\tau_{lr,s}^k-\tau_{ls,r}^k\right)\end{cases}$$

Hence, contracting in *k* and *l*, *the quadratic terms in* χ *disappear* and we get:

$$\partial_i\chi_{r,j}^r-\partial_j\chi_{r,i}^r=A_i^rA_j^s\left(\tau_{kr,s}^k-\tau_{ks,r}^k\right)$$

By analogy with EM it should be tempting to introduce $\alpha_i=\chi_{r,i}^r$ and denote by φ_{ij} the right member of the last formula but the relation $\partial_i \alpha_j - \partial_j \alpha_i = \varphi_{ij}$ thus obtained has no intrinsic meaning because α is far from being a 1-form while φ is far from being a 2-form.

REMARK 5.B.1: The *target* "*y*" could be called "*hidden variable*" as it is just used in order to construct objects over the *source* "*x*". As a byproduct, the changes of local coordinates are of the form $\bar{x} = \varphi(x), \bar{y} = \psi(y)$ but the second one does not appear through the implicit summations over the target because the first order transition rules are:

$$\bar{y}_j^l \frac{\partial \varphi^j}{\partial x^i}(x) = \frac{\partial \psi^l}{\partial y^k}(y) y_i^k \Rightarrow \bar{f}_j^l(\varphi(x)) \frac{\partial \varphi^j}{\partial x^i}(x) = \frac{\partial \psi^l}{\partial y^k}(f(x)) f_i^k(x)$$

It follows therefore that $A \in T^* \otimes T$ indeed and is thus a well defined object over the source.

LEMMA 5.B.2: The short exact δ-sequence $0 \to S_2 T^* \xrightarrow{\delta} T^* \otimes T^* \xrightarrow{\delta} \wedge^2 T^* \to 0$ admits a canonical splitting, that is a splitting coherent with the tensor nature of the vector bundles involved.

Proof: The splitting of the above sequence is obtained by setting

$\left(\tau_{i,j}\right) \in T^* \otimes T^* \to \left(\frac{1}{2}\left(\tau_{i,j} + \tau_{j,i}\right)\right) \in S_2 T^*$ in such a way that

$\left(\tau_{i,j} = \tau_{j,i} = \tau_{ij}\right) \in S_2 T^* \Rightarrow \frac{1}{2}\left(\tau_{ij} + \tau_{ji}\right) = \tau_{ij}$.

Similarly, $\left(\varphi_{ij} = -\varphi_{ji}\right) \in \wedge^2 T^* \to \left(\frac{1}{2}\varphi_{ij}\right) \in T^* \otimes T^*$ and

$\left(\frac{1}{2}\varphi_{ij} - \frac{1}{2}\varphi_{ji}\right) = \left(\varphi_{ij}\right) \in \wedge^2 T^*$.

□

We shall revisit the previous results by showing that, *in fact*, all the maps and splittings existing for the Killing operator are coming from maps and splittings existing for the conformal Killing operator, *though surprising it may look like*. As these results are based on a systematic use of the Spencer δ-map, they are neither known nor acknowledged.

We now recall the commutative diagrams allowing to define the (analogue) of the first Janet bundle and their dimensions when $n = 4$:

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \to & g_3 & \to & S_3 T^* \otimes T & \to & S_2 T^* \otimes F_0 & \to & F_1 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \to & T^* \otimes g_2 & \to & T^* \otimes S_2 T^* \otimes T & \to & T^* \otimes T^* \otimes F_0 & \to & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \to & \wedge^2 T^* \otimes g_1 & \to & \underline{\wedge^2 T^* \otimes T^* \otimes T} & \to & \wedge^2 T^* \otimes F_0 & \to & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \to & \wedge^3 T^* \otimes T & = & \wedge^3 T^* \otimes T & \to & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & & & \\
 & & 0 & & 0 & & & & & &
\end{array}$$

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \rightarrow & 80 & \rightarrow & 100 & \rightarrow & 20 & \rightarrow 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \rightarrow & 160 & \rightarrow & 160 & \rightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
0 \rightarrow & & 36 & \rightarrow & 96 & \rightarrow & 60 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \rightarrow & & 16 & = & 16 & \rightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

PROPOSITION 5.B.3: Recalling that we have $F_1 = H^2(g_1) = Z^2(g_1)$ in the Killing case and $\hat{F}_1 = H^2(\hat{g}_1) \neq Z^2(\hat{g}_1)$ in the conformal Killing case, we have the unusual commutative diagram:

$$\begin{array}{ccccccccc}
 & 0 & & 0 & & 0 & & 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 \rightarrow & Z^2(g_1) & \rightarrow & Z^2(\hat{g}_1) & \subset & Z^2(T^* \otimes T) & \rightarrow & S_2T^* \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \delta \\
0 \rightarrow & \wedge^2T^* \otimes g_1 & \rightarrow & \wedge^2T^* \otimes \hat{g}_1 & \subset & \wedge^2T^* \otimes T^* \otimes T & \rightarrow & T^* \otimes T^* \\
 & \downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \downarrow \delta \\
0 \rightarrow & \wedge^3T^* \otimes T & = & \wedge^3T^* \otimes T & = & \wedge^3T^* \otimes T & \rightarrow & \wedge^2T^* \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & 0 & & 0 & & 0 & & 0
\end{array}$$

Proof: First of all, we must point out that the surjectivity of the bottom δ in the central column is well known from the exactness of the δ-sequence for S_3T^* and thus also after tensoring by T. However, the surjectivity of the bottom δ in the left column is *not evident at all* as it comes from a delicate circular chase in the preceding diagram, using the fact that the Riemann and Weyl operators are second order operators. Then, setting $\varphi_{ij} = \rho^r_{r,ij} = -\varphi_{ji}$ and $\rho_{ij} = \rho^r_{i,rj} \neq \rho_{ji}$, we may define the right central horizontal map by $\rho^k_{l,ij} \rightarrow \rho_{ij} - \frac{1}{2}\varphi_{ij}$ and the right bottom horizontal map by $\omega \otimes \xi \rightarrow -i(\xi)\omega$ by introducing the interior product $i(\ )$. We obtain at once:

$$-\left(\rho^r_{r,ij} + \rho^r_{i,jr} + \rho^r_{j,ri}\right) = -\varphi_{ij} + \rho_{ij} - \rho_{ji} = \left(\rho_{ij} - \frac{1}{2}\varphi_{ij}\right) - \left(\rho_{ji} - \frac{1}{2}\varphi_{ji}\right)$$

and the right bottom diagram is commutative, clearly inducing the upper map. If we restrict to the Killing symbol, then $\varphi_{ij} = 0$ and we obtain
$\rho_{ij} - \rho_{ji} = 0 \Rightarrow (\rho_{ij} = \rho_{ji}) \in S_2T^*$, that is the classical contraction allowing to obtain the Ricci tensor from the Riemann tensor but *there is no way to go backwards with a canonical lift*. A similar comment may be done for the conformal Killing symbol and the $\frac{1}{2}$ coefficient.

□

Using the previous diagram allowing to define both $F_1 = H^2(g_1) = Z^2(g_1)$ if we use ω or $\hat{F}_1 = H^2(\hat{g}_1) = Z^2(\hat{g}_1)/\delta(T^* \otimes \hat{g}_2)$ if we use $\hat{\omega}$ while taking into account that $dim(\hat{g}_1/g_1) = 1$ and $\hat{g}_2 \simeq T^*$, we obtain the crucial theorem *which is in fact only depending on* ω:

THEOREM 5.B.4: We have the commutative and exact "*fundamental diagram II*":

$$
\begin{array}{ccccccccccc}
 & & & & & & & & 0 & & \\
 & & & & & & & & \downarrow & & \\
 & & & & & & 0 & & S_2T^* & & \\
 & & & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & \rightarrow & Z^2(g_1) & \rightarrow & H^2(g_1) & \rightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \rightarrow & T^*\otimes\hat{g}_2 & \stackrel{\delta}{\rightarrow} & Z^2(\hat{g}_1) & \rightarrow & H^2(\hat{g}_1) & \rightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
0 \rightarrow & & S_2T^* & \stackrel{\delta}{\rightarrow} & T^*\otimes T^* & \stackrel{\delta}{\rightarrow} & \wedge^2 T^* & \rightarrow & 0 & & \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & 0 & & 0 & & & &
\end{array}
$$

The following theorem will provide *all* the classical formulas of both Riemannian and conformal geometry in a totally unusual framework *not depending on any conformal factor*.

THEOREM 5.B.5: All the short exact sequences of the preceding diagram split in a canonical way, that is in a way compatible with the underlying tensorial properties of the vector bundles involved. With more details:

$$
\begin{aligned}
T^* \otimes T^* \simeq S_2T^* \oplus \wedge^2 T^* &\Rightarrow Z^2(\hat{g}_1) \simeq Z^2(g_1) + \delta(T^* \otimes \hat{g}_2) \simeq Z^2(g_1) \oplus \wedge^2 T^* \\
&\Rightarrow H^2(g_1) \simeq H^2(\hat{g}_1) \oplus S_2T^* \\
&\Rightarrow F_1 \simeq \hat{F}_1 \oplus S_2T^*
\end{aligned}
$$

Proof: First of all, we recall that:

$$
\begin{aligned}
g_1 &= \left\{ \xi_i^k \in T^* \otimes T \mid \omega_{rj}\xi_i^r + \omega_{ir}\xi_j^r = 0 \right\} \\
\subset \hat{g}_1 &= \left\{ \xi_i^k \in T^* \otimes T \mid \omega_{rj}\xi_i^r + \omega_{ir}\xi_j^r - \frac{2}{n}\omega_{ij}\xi_r^r = 0 \right\}
\end{aligned}
$$

$$
\Rightarrow \quad 0 = g_2 \subset \hat{g}_2 = \left\{ \xi_{ij}^k \in S_2T^* \otimes T \mid n\xi_{ij}^k = \delta_i^k \xi_{rj}^r + \delta_j^k \xi_{ri}^r - \omega_{ij}\omega^{ks}\xi_{rs}^r \right\}
$$

Now, if $(\tau_{li,j}^k) \in T^* \otimes \hat{g}_2$, then we have:

$$
n\tau_{li,j}^k = \delta_l^k \tau_{ri,j}^r + \delta_i^k \tau_{rl,j}^r - \omega_{li}\omega^{ks}\tau_{rs,j}^r
$$

and we may set $\tau_{ri,j}^r = \tau_{i,j} \neq \tau_{j,i}$ with $(\tau_{i,j}) \in T^* \otimes T$ and such a formula does not depend on any conformal factor. Taking into account Proposition 4.B.5, we have:

$$
\delta(\tau_{li,j}^k) = (\tau_{li,j}^k - \tau_{lj,i}^k) = (\rho_{l,ij}^k) \in B^2(\hat{g}_1) \subset Z^2(\hat{g}_1)
$$

with:

$$Z^2(\hat{g}_1) = \left\{ \left(\rho^k_{l,ij}\right) \in \wedge^2 T^* \otimes \hat{g}_1 \mid \delta\left(\rho^k_{l,ij}\right) = 0 \right\} \Rightarrow \varphi_{ij} = \rho^r_{r,ij} \neq 0$$

$$\delta\left(\rho^k_{l,ji}\right) = \left(C_{(l,i,j)}\rho^k_{l,ij} = \rho^k_{l,ij} + \rho^k_{i,jl} + \rho^k_{j,li}\right) \in \wedge^3 T^* \otimes T$$

- The splitting of the central vertical column is obtained from a lift of the epimorphism $Z^2(\hat{g}_1) \to \wedge^2 T^* \to 0$ which is obtained by lifting $(\varphi_{ij}) \in \wedge^2 T^*$ to $\left(\frac{1}{2}\varphi_{ij}\right) \in T^* \otimes T^*$, setting $\tau^r_{ri,j} = \frac{1}{2}\varphi_{ij}$ and applying δ to obtain $\left(\tau^r_{ri,j} - \tau^r_{rj,i} = \frac{1}{2}\varphi_{ij} - \frac{1}{2}\varphi_{ji} = \varphi_{ij}\right) \in B^2(\hat{g}_1) \subset Z^2(\hat{g}_1)$.
- Now, let us define $\left(\rho_{i,j} = \rho^r_{i,rj} \neq \rho_{j,i}\right) \in T^* \otimes T^*$. Hence, elements of $Z^2(g_1)$ are such that:

$$\varphi_{ij} = \rho^r_{r,ij} = 0, \varphi_{ij} - \rho_{i,j} + \rho_{j,i} = 0 \Rightarrow \left(\rho_{ij} = \rho_{i,j} = \rho_{j,i} = \rho_{ji}\right) \in S_2 T^*$$

while elements of $Z^2(\hat{g}_1)$ are such that:

$$\left(\rho^r_{r,ij} = \varphi_{ij} = \rho_{i,j} - \rho_{j,i} = \tau_{i,j} - \tau_{j,i} \neq 0\right) \in \wedge^2 T^*$$

Accordingly, $\left(\rho_{i,j} - \frac{1}{2}\varphi_{ij} = \rho_{j,i} - \frac{1}{2}\varphi_{ji}\right) \in S_2 T^*$. More generally, we may consider $\rho^k_{l,ij} - \left(\tau^k_{li,j} - \tau^k_{lj,i}\right)$ with $\tau^r_{ri,j} = \frac{1}{2}\varphi_{ij}$. Such an element is killed by δ and thus belongs to $Z^2(\hat{g}_1)$ because each member of the difference is killed by δ. However, we have $\rho^r_{r,ij} - \left(\tau^r_{ri,j} - \tau^r_{rj,i}\right) = \varphi_{ij} - \varphi_{ij} = 0$ and the element does belong indeed to $Z^2(g_1)$, providing a lift $Z^2(\hat{g}_1) \to Z^2(g_1) \to 0$.

- Of course, *the most important result is to split the right column*. As this will be the hard step, we first need to describe the monomorphism $0 \to S_2 T^* \to H^2(g_1)$ which is in fact produced by a north-east diagonal snake type chase. Let us choose $\left(\tau_{ij} = \tau_{i,j} = \tau_{j,i} = \tau_{ji}\right) \in S_2 T^* \subset T^* \otimes T^*$. Then, we may find $\left(\tau^k_{li,j}\right) \in T^* \otimes \hat{g}_2$ by deciding that $\tau^r_{ri,j} = \tau_{i,j} = \tau_{j,i} = \tau^r_{rj,i}$ in $Z^2(\hat{g}_1)$ and apply δ in order to get $\rho^k_{l,ij} = \tau^k_{li,j} - \tau^k_{k,lj,i}$ such that $\rho^r_{r,ij} = \varphi_{ij} = 0$ and thus $\left(\rho^k_{l,ij}\right) \in Z^2(g_1) = H^2(g)_1$. We obtain:

$$\begin{aligned} n\rho^k_{l,ij} &= \delta^k_l \tau^r_{ri,j} - \delta^k_l \tau^r_{rj,i} + \delta^k_i \tau^r_{rl,j} - \delta^k_j \tau^r_{rli} - \omega^{ks}\left(\omega_{li}\tau^r_{rs,j} - \omega_{lj}\tau^r_{rs,i}\right) \\ &= \left(\delta^k_i \tau_{lj} - \delta^k_j \tau_{li}\right) - \omega^{ks}\left(\omega_{li}\tau_{sj} - \omega_{lj}\tau_{si}\right) \end{aligned}$$

Contracting in k and i while setting simply $tr(\tau) = \omega^{ij}\tau_{ij}, tr(\rho) = \omega^{ij}\rho_{ij}$, we get:

$$\begin{aligned} &n\rho_{ij} = n\tau_{ij} - \tau_{ij} - \tau_{ij} + \omega_{ij} tr(\tau) = (n-2)\tau_{ij} + \omega_{ij} tr(\tau) = n\rho_{ji} \\ &\Rightarrow ntr(\rho) = 2(n-1) tr(\tau) \end{aligned}$$

Substituting, we finally obtain $\tau_{ij} = \frac{n}{n-2}\rho_{ij} - \frac{n}{2(n-1)(n-2)}\omega_{ij} tr(\rho)$ and thus the tricky formula:

$$\begin{aligned} \rho^k_{l,ij} = &\frac{1}{n-2}\left(\left(\delta^k_i \rho_{lj} - \delta^k_j \rho_{li}\right) - \omega^{ks}\left(\omega_{li}\rho_{sj} - \omega_{lj}\rho_{si}\right)\right) \\ &- \frac{1}{(n-1)(n-2)}\left(\delta^k_i \omega_{lj} - \delta^k_j \omega_{li}\right) tr(\rho) \end{aligned}$$

Contracting in k and i, we check that $\rho_{ij} = \rho_{ij}$ indeed, obtaining therefore the desired canonical lift $H^2(g_1) \to S_2T^* \to 0 : \rho^k_{l,lj} \to \rho^r_{i,rj} = \rho_{ij}$. Finally, using again Proposition 3.4, the epimorphism $H^2(g_1) \to H^2(\hat{g}_1) \to 0$ is just described by the formula:

$$\sigma^k_{l,ij} = \rho^k_{l,ij} - \frac{1}{n-2}\left(\left(\delta^k_i \rho_{lj} - \delta^k_j \rho_{li}\right) - \omega^{ks}\left(\omega_{li}\rho_{sj} - \omega_{lj}\rho_{si}\right)\right) + \frac{1}{(n-1)(n-2)}\left(\delta^k_i \omega_{lj} - \delta^k_j \omega_{li}\right) tr(\rho)$$

which is just the way to define the Weyl tensor. We notice that $\sigma^r_{r,ij} = \rho^r_{r,ij} = 0$ and $\sigma^r_{i,rj} = 0$ by using indices or a circular chase showing that $Z^2(\hat{g}_1) = Z^2(g_1) + \delta(T^* \otimes \hat{g}_2)$. This purely algebraic result only depends on the metric ω and does not depend on any conformal factor. In actual practice, the lift $H^2(g_1) \to S_2T^*$ is described by $\rho^k_{l,ij} \to \rho^r_{i,rj} = \rho_{ij} = \rho_{ji}$ but it is not evident at all that the lift $H^2(\hat{g}_1) \to H^2(g_1)$ is described by the strict inclusion $\sigma^k_{l,ij} \to \rho^k_{l,ij} = \sigma^k_{l,ij}$ providing a short exact sequence as in Proposition 3.4 because $\rho_{ij} = \rho^r_{i,rj} = \sigma^r_{i,rj} = 0$ by composition.

□

PROPOSITION 5.B.6: We have the following commutative and exact diagram made by splitting sequences according to a circular chase in the right upper commutative square:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \to & \hat{g}_2 & \to & T^* & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & \tilde{R}_2 & \to & \hat{R}_2 & \to & T^* & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \tilde{R}_1 & = & \hat{R}_1 & \to & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

This diagram is thus leading to the short exact sequence:

$$0 \to T^* \otimes \tilde{R}_2 \to T^* \otimes \hat{R}_2 \to T^* \otimes T^* \to 0$$

with a canonical splitting $T^* \otimes T^* \simeq S_2T^* \oplus \wedge^2 T^*$.

Proof: According to the definition of the Christoffel symbols γ for the metric ω, we have:

$$2\omega_{rk}\gamma^k_{ij} = \partial_i \omega_{rj} + \partial_j \omega_{ri} - \partial_r \omega_{ij} \Leftrightarrow \omega_{kj}\gamma^k_{ir} + \omega_{ik}\gamma^k_{jr} - \partial_r \omega_{ij} = 0$$

It follows that $-\gamma$ (*care*) is the unique symmetric R_1-connection, that is a map $T \to R_1$ considered as an element of $T^* \otimes R_1$ projecting onto $id_T \in T^* \otimes T$. Accordingly, any $\chi_1 \in T^* \otimes J_1(T)$ provides $\left(\chi^k_{j,i} + \gamma^k_{jr}\chi^r_{,i}\right) \in T^* \otimes T^* \otimes T$ and thus a true 1-form $\left(\alpha_i = \chi^r_{r,i} + \gamma^r_{r,s}\chi^s_{,i}\right) \in T^*$. However, such an approach cannot be extended to higher orders and we prefer to consider half of the morphism defining the Killing operator, namely the morphism $J_1(T) \to S_2T^* : \xi_1 \to \frac{1}{2}L(\xi_1)\omega$, tensor it by T^* and contract it by ω^{-1} in order to get:

$$\frac{1}{2}\omega^{st}\left(\omega_{rt}\chi^r_{s,i}+\omega_{sr}\chi^r_{t,i}+\chi^r_{,i}\partial_r\omega_{st}\right)=\chi^r_{r,i}+\frac{1}{2}\chi^r_{,i}\omega^{st}\partial_r\omega_{st}=\alpha_i$$

where we notice that:

$$2\gamma^r_{ri}=\omega^{st}\partial_r\omega_{st}=\left(1/det(\omega)\right)\partial_i det(\omega)\Rightarrow \partial_i\gamma^r_{rj}-\partial_j\gamma^r_{r,i}=0$$

Similarly, there is a well defined map $J_2(T)\to S_2T^*\otimes T:\xi_2\to L(\xi_2)\gamma$ that can be tensored by T^* and restricted to $T^*\otimes\hat{R}_2$ in order to obtain a well defined map $T^*\otimes\hat{R}_2\to T^*\otimes S_2T^*\otimes T$ that can be contracted to $T^*\otimes T^*$ according to the following local formulas:

$$\beta^k_{lr,s}=\tau^k_{lr,s}+\gamma^k_{ur}\tau^u_{l,s}+\gamma^k_{lu}\tau^u_{r,s}-\gamma^u_{lr}\tau^k_{u,s}+\tau^u_{,s}\partial_u\gamma^k_{lr}$$
$$\beta^k_{kr,s}=\tau^k_{kr,s}+\gamma^k_{ku}\tau^u_{r,s}+\tau^u_{,s}\partial_u\gamma^k_{kr}$$

We can "*twist*" by A and apply $\delta:T^*\otimes S_2T^*\otimes T\to\wedge^2T^*\otimes T^*\otimes T$ that can be contracted to $\wedge^2T^*$ according to the following local formulas:

$$\varphi^k_{l,ij}=A^r_iA^s_j\left(\beta^k_{lr,s}-\beta^k_{ls,r}\right)\Rightarrow\varphi_{ij}=\varphi^r_{r,ij}=A^r_iA^s_j\left(\beta^k_{kr,s}-\beta^k_{ks,r}\right)$$

□

As $\varphi\in\wedge^2T^*$ though $\chi_2\in T^*\otimes\hat{R}_2$, we obtain the following crucial theorem ([4] [8]):

THEOREM 5.B.7: The non-linear Spencer sequence for the conformal group of transformations projects onto a part of the Poincaré sequence for the exterior derivative according to the following commutative and locally exact diagram:

$$\begin{array}{ccccccccc}
0\to & \hat{\Gamma} & \xrightarrow{j_2} & \hat{\mathcal{R}}_2 & \xrightarrow{\bar{D}_1} & T^*\otimes\hat{R}_2 & \xrightarrow{\bar{D}_2} & \wedge^2T^*\otimes\hat{R}_2 \\
 & & & \downarrow & \swarrow & \downarrow & & \downarrow \\
 & & & T^* & \xrightarrow{d} & \wedge^2T^* & \xrightarrow{d} & \wedge^3T^* \\
 & & & \alpha & & d\alpha=\varphi & & d\varphi=0
\end{array}$$

Accordingly, *this purely mathematical result contradicts classical gauge theory*.

Proof: Substituting the previous results in the last formula, we obtain successively:

$$\begin{aligned}
\varphi_{ij}&=A^r_iA^s_j\left(\tau^k_{kr,s}-\tau^k_{ks,r}\right)+\gamma^k_{ku}A^r_iA^s_j\left(\tau^u_{r,s}-\tau^u_{s,r}\right)+A^r_iA^s_j\left(\tau^u_{,s}\partial_u\gamma^k_{kr}-\tau^u_{,r}\partial_u\gamma^k_{ks}\right)\\
&=\left(\partial_i\chi^r_{r,j}-\partial_j\chi^r_{r,i}\right)+\gamma^r_{ru}\left(\partial_iA^u_j-\partial_jA^u_i\right)+\left(\delta^r_i+\chi^r_{,i}\right)\chi^s_{,j}\partial_r\gamma^k_{ks}-\left(\delta^s_j+\chi^s_{,j}\right)\chi^r_{,i}\partial_s\gamma^k_{kr}\\
&=\left(\partial_i\chi^r_{r,j}-\partial_j\chi^r_{r,i}\right)+\gamma^r_{rs}\left(\partial_i\chi^s_{,j}-\partial_j\chi^s_{,i}\right)+\left(\chi^s_{,j}\partial_i\gamma^r_{rs}-\chi^s_{,i}\partial_j\gamma^r_{rs}\right)\\
&=\partial_i\alpha_j-\partial_j\alpha_i
\end{aligned}$$

because $\partial_i\gamma^r_{rj}-\partial_j\gamma^r_{ri}=0$. It follows that $d\alpha=\varphi\in\wedge^2T^*$ and thus $d\varphi=0$, that is $\partial_i\varphi_{jk}+\partial_j\varphi_{ki}+\partial_k\varphi_{ij}=0$, has an intrinsic meaning in $\wedge^3T^*$. It is important to notice that the corresponding EM Lagrangian is defined on sections of $\hat{C}_1$ killed by $\bar{D}_2$ but *not* on $\hat{C}_2$, *contrary to gauge theory*. Finally, the south west arrow in the left square is the composition:

$$f_2 \in \hat{\mathcal{R}}_2 \xrightarrow{\bar{D}_1} \chi_2 \in T^* \otimes \hat{R}_2 \xrightarrow{\pi_1^2} \chi_1 \in T^* \otimes \hat{R}_1 \xrightarrow{(\gamma)} \alpha \in T^*$$

Accordingly, though α is a potential for φ, it can also be considered as a part of the *field* but the important fact is that the first set of (*linear*) Maxwell equations $d\varphi = 0$ is induced by the (*nonlinear*) operator $\bar{D}_2$. The linearized framework will explain this point.

One of the most important but difficult result of this paper will be the following direct proof of the existence of the right square in the previous diagram.

Supposing for simplicity that ω is a (locally) constant metric (in fact the Minkowski metric!) and thus $\gamma = 0$. When we are considering the conformal group of space-time, it first follows that the jets of order three vanish and the Formula (3*) can be now written:

$$\partial_i \chi^k_{lr,j} - \partial_j \chi^k_{lr,i} - \left(\chi^s_{r,i}\chi^k_{ls,j} + \chi^s_{l,i}\chi^k_{rs,j} + \chi^s_{lr,i}\chi^k_{s,j} - \chi^s_{r,j}\chi^k_{ls,i} - \chi^s_{l,j}\chi^k_{rs,i} - \chi^s_{lr,j}\chi^k_{s,i}\right) = 0$$

Contracting in $k = l = u$ and replacing r by t, we obtain the simple formula:

$$\partial_i \chi^u_{ut,j} - \partial_j \chi^u_{ut,i} - \chi^s_{t,i}\chi^u_{us,j} + \chi^s_{t,j}\chi^u_{us,i} = 0$$

Multiplyig by A^t_k the *two last terms* and replacing χ by τ, we get *for these terms only*:

$$A^r_i A^s_j A^t_k \left(\tau^v_{t,s}\tau^u_{uv,r} - \tau^v_{t,r}\tau^u_{uv,s}\right)$$

Now, denoting by $\mathcal{C}(i,j,k)$ the cyclic sum on the permutation $(i,j,k) \to (j,k,i) \to (k,i,j)$ and proceeding in this way on the last formula, we obtain easily:

$$\mathcal{C}(i,j,k) A^r_i A^s_j A^t_k \left(\tau^v_{t,s} - \tau^v_{s,t}\right)\tau^u_{uv,r}$$

or, equivalently:

$$A^r_i A^s_j A^t_k \mathcal{C}(r,s,t)\left(\tau^v_{t,s} - \tau^v_{s,t}\right)\tau^u_{uv,r} = A^r_i A^s_j A^t_k \mathcal{C}(r,s,t)\left(\tau^v_{s,r} - \tau^v_{r,s}\right)\tau^u_{uv,t}$$

Let us now similarly consider only the *two first terms*. After multiplication by A^t_k and integration by part, we get for the first:

$$A^t_k\left(\partial_i\left(A^s_j \tau^u_{ut,s}\right)\right) = \partial_i\left(A^s_j A^t_k \tau^u_{ut,s}\right) - A^s_j \tau^u_{ut,s}\partial_i A^t_k$$

Applying the same procedure to the second term and considering the sum $\mathcal{C}(i,j,k)$ while rearranging the six terms of the summation two by two, we obtain:

$$\mathcal{C}(i,j,k)\left(\partial_k\left(A^t_j A^s_i \tau^u_{ut,s} - A^t_i A^r_j \tau^u_{ut,r}\right) + A^t_k \tau^u_{ur,t}\partial_j A^r_i - A^t_k \tau^u_{ut,r}\partial_i A^r_j\right)$$

Exchanging the dumb indices between themselves, we finally obtain:

$$\mathcal{C}(i,j,k)\left(\partial_k\left(A^r_i A^s_j\left(\tau^u_{us,r} - \tau^u_{ur,s}\right)\right) + A^t_k \tau^u_{ur,t}\left(\partial_i A^r_j - \partial_j A^r_i\right)\right)$$

that is to say, taking into account the Equations (1*) and changing the signs:

$$\mathcal{C}(i,j,k)\left(\partial_k \varphi_{ij}\right) - \mathcal{C}(i,j,k)\left(A^r_i A^s_j A^t_k\left(\tau^v_{r,s} - \tau^v_{s,r}\right)\tau^u_{uv,t}\right)$$

or, equivalently:

$$\mathcal{C}(i,j,k)(\partial_k\varphi_{ij}) - A_i^r A_j^s A_k^t \mathcal{C}(r,s,t)(\tau_{r,s}^v - \tau_{s,r}^v)\tau_{uv,t}^u$$

Collecting all the results, we are only left with $\mathcal{C}(i,j,k)(\partial_k\varphi_{ij}) = 0$ as we wished.

□

COROLLARY 5.B.8: The linear Spencer sequence for the conformal group of transformations projects onto a part of the Poincaré sequence for the exterior derivative according to the following commutative and locally exact diagram:

$$\begin{array}{cccccccccc}
0\rightarrow & \hat{\Theta} & \xrightarrow{j_2} & \hat{R}_2 & \xrightarrow{D_1} & T^*\otimes\hat{R}_2 & \xrightarrow{D_2} & \wedge^2T^*\otimes\hat{R}_2 \\
 & & & \downarrow & \swarrow & \downarrow & & \downarrow \\
 & & & T^* & \xrightarrow{d} & \wedge^2T^* & \xrightarrow{d} & \wedge^3T^* \\
 & & & A & & dA=F & & dF=0
\end{array}$$

Accordingly, *this purely mathematical result also contradicts classical gauge theory* because it proves that EM only depends on the structure of the conformal group of space-time but not on $U(1)$.

Proof: Considering ω and γ as geometric objects, we obtain at once the formulas:

$$\bar{\omega}_{ij} = e^{2a(x)}\omega_{ij} \quad\Rightarrow\quad \bar{\gamma}_{ri}^r = \gamma_{ri}^r + \partial_i a$$

Though looking like the key Formula (69) in ([3], p 286), this transformation is quite different because the sign is not coherent and the second object has nothing to do with a 1-form. Moreover, if we use $n=2$ and set $\mathcal{L}(\xi)\omega = 2A\omega$ for the standard euclidean metric, we should have $(\partial_{11}+\partial_{22})A = 0$, contrary to the assumption that A is arbitrary which is *only* agreeing with the following jet formulas improving the ones already provided in ([29] [36] [40]) in order to point out the systematic use of the Spencer operator:

$$L(\xi_1)\omega = 2A\omega \quad\Rightarrow\quad (\xi_r^r + \gamma_{ri}^r\xi^i) = nA, \quad (L(\xi_2)\gamma)_{ri}^r = nA_i, \quad \forall\xi_2\in\hat{R}_2$$

Now, if we make a change of coordinates $\bar{x} = \varphi(x)$ on a function $a\in\wedge^0T^*$, we get:

$$\bar{a}(\varphi(x)) = a(x) \quad\Rightarrow\quad \frac{\partial\bar{a}}{\partial\bar{x}^j}\frac{\partial\varphi^j}{\partial x^i} = \frac{\partial a}{\partial x^i}$$

We obtain therefore an isomorphism $J_1(\wedge^0T^*) \simeq \wedge^0T^*\times_X T^*$, a result leading to the following commutative diagram:

$$\begin{array}{ccccccc}
0\rightarrow & R_2 & \rightarrow & \hat{R}_2 & \rightarrow & J_1(\wedge^0T^*) & \rightarrow 0 \\
 & \downarrow D & & \downarrow D & & \downarrow D & \\
0\rightarrow & T^*\otimes R_1 & \rightarrow & T^*\otimes\hat{R}_1 & \rightarrow & T^* & \rightarrow 0
\end{array}$$

where the rows are exact by counting the dimensions. The operator $D:(A,A_i)\rightarrow(\partial_iA - A_i)$ on the right is induced by the central Spencer operator, a result that could not have been even imagined by Weyl and followers. This result provides a good transition towards the conformal origin of electromagnetism.

As the nonlinear aspect has been already presented, we restrict our study to the linear framework. A first problem to solve is to construct vector bundles from the components of the image of D_1. Using the corresponding capital letter for denoting the linearization, let us introduce:

$$\left(B^k_{l,i} = X^k_{l,i} + \gamma^k_{ls} X^s_{,i}\right) \in T^* \otimes T^* \otimes T \Rightarrow \left(B^r_{r,i} = B_i\right) \in T^*$$

$$\left(B^k_{lj,i} = X^k_{lj,i} + \gamma^k_{sj} X^s_{l,i} + \gamma^k_{ls} X^s_{j,i} - \gamma^s_{lj} X^k_{s,i} + X^r_{,i}\partial_r \gamma^k_{lj}\right) \in T^* \otimes S_2 T^* \otimes T$$

$$\Rightarrow \left(B^r_{ri,j} - B^r_{rj,i} = F_{ij}\right) \in \wedge^2 T^*$$

We obtain from the relations $\partial_i \gamma^r_{rj} = \partial_j \gamma^r_{ri}$ and the previous results:

$$\begin{aligned} F_{ij} &= B^r_{ri,j} - B^r_{rj,i} = X^r_{ri,j} - X^r_{rj,i} + \gamma^r_{rs} X^s_{i,j} - \gamma^r_{rs} X^s_{j,i} + X^r_{,j}\partial_r \gamma^s_{si} - X^r_{,i}\partial_r \gamma^s_{sj} \\ &= \partial_i X^r_{r,j} - \partial_j X^r_{r,i} + \gamma^r_{rs}\left(X^s_{i,j} - X^s_{j,i}\right) + X^r_{,j}\partial_i \gamma^s_{sr} - X^r_{,i}\partial_j \gamma^s_{sr} \\ &= \partial_i\left(X^r_{r,j} + \gamma^r_{rs} X^s_{,j}\right) - \partial_j\left(X^r_{r,i} + \gamma^r_{rs} X^s_{s,i}\right) \\ &= \partial_i B_j - \partial_j B_i \end{aligned}$$

Now, using the contracted formula $\xi^r_{ri} + \gamma^r_{rs}\xi^s_i + \xi^s \partial_s \gamma^r_{ri} = nA_i$ from section *A*, we obtain:

$$\begin{aligned} B_i &= \left(\partial_i \xi^r_r - \xi^r_{ri}\right) + \gamma^r_{rs}\left(\partial_i \xi^s - \xi^s_i\right) \\ &= \partial_i \xi^r_r + \gamma^r_{rs}\partial_i \xi^s + \xi^s \partial_s \gamma^r_{ri} - nA_i \\ &= \partial_i\left(\xi^r_r + \gamma^r_{rs}\xi^s\right) - nA_i \\ &= n\left(\partial_i A - A_i\right) \end{aligned}$$

and we finally get $F_{ij} = n\left(\partial_j A_i - \partial_i A_j\right)$ *which is no longer depending on A*, a result fully solving the dream of Weyl. Of course, when $n = 4$ and ω is the Minkowski metric, then we have $\gamma = 0$ in actual practice and the previous formulas become particularly simple.

It follows that $dB = F \Leftrightarrow -ndA = F$ in $\wedge^2 T^*$ and thus $dF = 0$, that is $\partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0$, has an intrinsic meaning in $\wedge^3 T^*$. It is finally important to notice that the usual EM Lagrangian is defined on sections of $\hat{C}_1$ killed by D_2 but *not* on $\hat{C}_2$. Finally, the south west arrow in the left square is the composition:

$$\xi_2 \in \hat{R}_2 \xrightarrow{D_1} X_2 \in T^* \otimes \hat{R}_2 \xrightarrow{\pi^2_1} X_1 \in T^* \otimes \hat{R}_1 \xrightarrow{(\gamma)} \left(B_i\right) \in T^* \Leftrightarrow \xi_2 \in \hat{R}_2 \rightarrow \left(nA_i\right) \in T^*$$

Accordingly, though A and B are potentials for F, then B can also be considered as a part of the *field* but the important fact is that the first set of (*linear*) Maxwell equations $dF = 0$ is induced by the (*linear*) operator D_2 because we are only dealing with involutive and thus formally integrable operators, a *fact* justifying the commutativity of the square on the left of the diagram.

□

REMARK 5.B.9: Taking the determinant of each term of the non-linear second order PD equations defining $\hat{\Gamma}$, we obtain successively:

$$det(\omega)\left(det\left(f^k_i(x)\right)\right)^2 = \mathrm{e}^{2na(x)} det(\omega) \Rightarrow det\left(f^k_i(x)\right) = \mathrm{e}^{na(x)}$$

in such a way that we may define $b(f(x)) = a(x) \Leftrightarrow b(y) = a(g(y))$ and set $\Theta(y) = e^{-b(y)} > 0$ over the target when caring only about the connected component $0 \rightarrow 1 \rightarrow \infty$ of the dilatation group. The problem is thus to change at the same time the numerical value of the section and /or the nature of the geometric object cosifered, passing therefore from a (metric) tensor to a (metric) tensor density, exactly what also happens with the contact structure when it was necessary to pass from a 1-form to a 1-form density ([4] [7]). In a more specific way, the idea has been to consider successively the two non-linear systems of finite defining Lie equations:

$$\omega_{kl}(y) y_i^k y_j^l = \omega_{ij}(x) \rightarrow \hat{\omega}_{kl} y_i^k y_j^l \left(det\left(y_i^k\right)\right)^{\frac{-2}{n}} = \hat{\omega}_{ij}(x)$$

Now, with $\gamma = 0$ we have $\chi_{r,i}^r = g_k^s \left(\partial_i f_s^k - A_i^r f_{rs}^k\right)$ and:

$$g_k^s \partial_i f_s^k = \left(1/det\left(f_i^k\right)\right) \partial_i det\left(f_i^k\right) = n\partial_i a,\ g_k^s f_{rs}^k = na_r(x)$$

Finally, we have the jet compositions and contractions:

$$g_k^r f_i^k = \delta_i^r \Rightarrow g_k^r f_{ij}^k = -g_{kl}^r f_i^k f_j^l$$
$$\Rightarrow na_i(x) = g_k^s f_{is}^k = -f_i^k f_r^l g_{kl}^r = -n f_i^k(x) b_k(f(x))$$

It follows that $\alpha_i = n\left(\partial_i a(x) - A_i^r a_r(x)\right)$ but we may also set $a_i(x) = f_i^k(x) b_k(f(x))$ in order to obtain $\alpha_i = n\left(\frac{\partial b}{\partial y^k} - b_k\right)\partial_i f^k$ as a way to pass from source to target. We have:

PROPOSITION 5.B.10: EM does not depend on the choice between source and target.

Proof: Replacing the groupoid by its inverse in each formula, we may introduce:

$$\alpha = \alpha_i(x) dx^i,\ \alpha_i = n\left(\partial_i a - A_i^r a_r\right) \Leftrightarrow \beta = \beta_k(y) dy^k,\ \beta_k = n\left(\frac{\partial b}{\partial y^k} - b_k\right)$$

and compare:

$$x \xrightarrow{a} (\alpha, \varphi) \Leftrightarrow y \xrightarrow{b} (\beta, \psi)$$

while setting $\psi_{kl} = \frac{\partial \beta_l}{\partial y^k} - \frac{\partial \beta_k}{\partial y^l}$. We have successively:

$$\begin{aligned}
\varphi_{ij} &= \partial_i \alpha_j - \partial_j \alpha_i = -n\left(\partial_i\left(A_j^s a_s\right) - \partial_j\left(A_i^r a_r\right)\right) \\
&= -n\left(\partial_i\left(b_l \partial_j f^l\right) - \partial_j\left(b_k \partial_i f^k\right)\right) \\
&= -n\left(\frac{\partial b_l}{\partial y^k} - \frac{\partial b_k}{\partial y^l}\right)\partial_i f^k \partial_j f^l \\
&= \left(\frac{\partial \beta_l}{\partial y^k} - \frac{\partial \beta_k}{\partial y^l}\right)\partial_i f^k \partial_j f^l \\
&= \psi_{kl} \partial_i f^k \partial_j f^l
\end{aligned}$$

and we notice that φ *does not depend any longer on a* while ψ *does not de-*

pend any longer on b. Accordingly, we have the equivalences:

$$NO\ EM \Leftrightarrow \varphi = 0 \Leftrightarrow \psi = 0 \Leftrightarrow \frac{\partial b_l}{\partial y^k} - \frac{\partial b_k}{\partial y^l} = 0 \Leftrightarrow \partial_i\left(A_j^r a_r\right) - \partial_j\left(A_i^r a_r\right) = 0$$

□

REMARK 5.B.11: If we use *only* the conformal group, we must use the metric density $\hat{\omega}$ instead of the metric ω. However, if we can define $\hat{\omega}$ from ω by setting $\hat{\omega}_{ij} = \omega_{ij} \Big/ \left(\left|det(\omega)\right|\right)^{\frac{1}{n}}$, we cannot recover ω from $\hat{\omega}$. The way to escape from such a situation is to notice that:

$$\begin{aligned} &\omega \to e^{2a(x)}\omega \Rightarrow \gamma_{ij}^k \to \gamma_{ij}^k + \delta_i^k \partial_j a(x) + \delta_j^k \partial_i a(x) - \omega_{ij}\omega^{kr}\partial_r a(x) \\ &\Rightarrow \gamma_{ri}^r \to \gamma_{ri}^r + n\partial_i a(x) \end{aligned}$$

a result showing that the conformal symbols $\hat{g}_1$ and $\hat{g}_2$ do not depend on any conformal factor.

REMARK 5.B.12: In fact, our purpose is quite different now though it is also based on the combined use of group theory and the Spencer operator. The idea is to notice that the brothers are *always* dealing with the same group of rigid motions because the lines, surfaces or media they consider are all supposed to be in the same 3-dimensional background/surrounding space which is acted on by the group of rigid motions, namely a group with 6 parameters (3 *translations* + 3 *rotations*). In 1909 it should have been *strictly impossible* for the two brothers to extend their approach to bigger groups, in particular to include the only additional *dilatation* of the Weyl group that will provide the virial theorem and, *a fortiori*, the *elations* of the conformal group considered later on by H. Weyl ([29]). In order to explain the reason for using Lie equations, we provide the explicit form of the n finite elations and their infinitesimal counterpart with $1 \le r,s,t \le n$:

$$\boxed{y = \frac{x - x^2 b}{1 - 2(bx) + b^2 x^2} \Rightarrow \theta_s = -\frac{1}{2}x^2\partial_s + \omega_{st}x^t x^r \partial_r \Rightarrow \partial_r \theta_s^r = n\omega_{st}x^t, \left[\theta_s, \theta_t\right] = 0}$$

where the underlying metric is used for the scalar products x^2, bx, b^2 involved. It is easy to check that $\xi_2 \in S_2T^* \otimes T$ defined by $\xi_{ij}^k(x) = \lambda^s(x)\partial_{ij}\theta_s^k(x)$ belongs to $\hat{g}_2$ with $A_i = \omega_{si}\lambda^s$. In view of these local formulas, we understand how important it is to use "*equations*" rather than "*solutions*".

REMARK 5.B.13: Setting $\sigma_{q-1} = \bar{D}'\chi_q \in \wedge^2 T^* \otimes J_{q-1}(T)$, we let the reader prove, as an exercise, that the following so-called *Bianchi identities* hold ([7], p 221):

$$D\sigma_{q-1}(\xi,\eta,\zeta) + C(\xi,\eta,\zeta)\left\{\sigma_{q-1}(\xi,\eta), \chi_{q-1}(\zeta)\right\} = 0,\ \forall \xi,\eta,\zeta \in T$$

In the nonlinear conformal framework, it follows that the first set of Maxwell equations has only to do with $\bar{D}'$ in the nonlinear Spencer sequence and thus nothing to do with the Bianchi identities, contrary to what happens with $U(1)$ in classical gauge theory. Similarly, in the linear conformal framework, the first set of Maxwell equations has only to do with D_2 and thus nothing to do with

D_3 in the linear Spencer sequence. Indeed, the EM potential A is a section of $\hat{C}_0$ while the EM field F is a section of $\hat{C}_1$ killed by D_2. This "*shift by one step to the left*" is the most important result of this section and could not be even imagined with any other approach.

6. Conclusions

This paper is part of the achievement of a lifetime research work on the common conformal origin of electromagnetism and gravitation. Roughly speaking, the Cosserat brothers have only been dealing with the 3 translations and 3 rotations of the group of rigid motions of space with 6 parameters while Weyl has only been dealing with the dilatation and the 4 elations of the conformal group of space-time having now $4+6+1+4=15$ parameters ([29]). Among the most striking results obtained from this conformal extension, we successively notice:

- The generating nonlinear *first order* (care) compatibility conditions (CC) for the Cosserat fields are *exactly* described by the *first order* nonlinear second Spencer operator $\bar{D}_2$. Accordingly, there is no conceptual difference between these nonlinear CC and the first set $d:\wedge^2 T^* \to \wedge^3 T^* : F \to dF = 0$ of Maxwell equations where d is the exterior derivative, which is parametrized by $d:T^* \to \wedge^2 T^* : A \to dA = F$. However, the classical CC of elasticity are described by the *nonlinear second order* (care) *Riemann* operator existing in the nonlinear Janet sequence but this different canonical nonlinear differential sequence could not explain the existence of field-matter couplings like piezzoelectricity, photoelasticity or even streaming birefringence ([5] [37]). On the contrary, in the conformal approach, it is essential to notice that *the elastic and electromagnetic fields are both specific sections of* $\hat{C}_1 = T^* \otimes \hat{R}_2$ *killed by* $\bar{D}_2$ and parametrized by $\bar{D}_1$. They can thus be coupled in a natural way but *cannot be associated to the concept of curvature* described by $\hat{C}_2$. Meanwhile, we insist on the fact that the phenomenological laws of these quoted couplings have been discovered by… Maxwell himself. This *shift by one step to the left*, even in the nonlinear framework, can be considered as the main novelty of this paper.
- The linear Cosserat equations are *exactly* described by the formal adjoint $ad(D_1)$ of the *linear* first Spencer operator $D_1:\hat{C}_0 \to \hat{C}_1$ which is a first order operator ([38]). Accordingly, there is no conceptual difference between these equations and the second set $ad(d)$ of Maxwell equations where $d:T^* \to \wedge^2 T^*$. This result explains why the *Cosserat* equations are quite different from the *Cauchy* equations which are described by the formal adjoint of the *Killing* operator in the Janet sequence used in classical elasticity, that is $Cauchy = ad(Killing)$ in the language of operators. It follows that the elastic and electromagnetic inductions are both specific sections of $\wedge^4 T^* \otimes \hat{C}_1^* \simeq \wedge^3 T^* \otimes \hat{R}_2^*$, *independently of any constitutive relation*. The specific use of the 1-dimensional *dilatation subgroup* allows to understand the mathematical origin of *thermoelectricity* and the so-called *virial theorem* through the trace of the Cauchy tensor ([36] [40]).

- Combining the two previous comments, respectively related to "*geometry*" and to "*physics*" according to H. Poincaré ([34]), there is no conceptual difference between the elastic constitutive constants of elasticity and the magnetic constant μ or rather $1/\mu$ of electromagnetism in the case of homogeneous isotropic materials on one side (*space*) or between the mass per unit volume and the dielectric constant ε on the other side (*time*), a result confirmed by the speeds of the various elastic or electromagnetic existing waves ([5] [37]). In general one has $\varepsilon\mu c^2 = n^2$ where n is the index of refraction but in vacuum we have $\varepsilon_0\mu_0 c^2 = 1$ and we have thus only one electromagnetic constant involved in the corresponding Minkowski constitutive law of vacuum ([2]).
- As for gravitation and the possibility to exhibit a conformal factor defined everywhere but at the origin, we may simply say that we needed 25 years in order to correct the result we already obtained in 1994 ([7] [41]). Such a possibility highly depends on the new mathematical tools involved in the construction of the Janet or Spencer nonlinear differential sequences for the conformal group of space-time because, *in this case*, the Spencer δ-cohomology has very specific properties for the dimension $n = 4$ *only*. This will be the subject of a forthcoming companion paper (arXiv: 2007.01710).

We end this paper with the French proverb "AUTRES TEMPS, AUTRES MOEURS" adapted from the famous Latin sentence "AUT TEMPORA, AUT MORES" as we do believe that a modern scientific translation could be "NEW MATHEMATICS, NEW PHYSICS".

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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List of the Main Notations

$\mu=(\mu_1,\cdots,\mu_n)$ multi-index, $|\mu|=\mu_1+\cdots+\mu_n$, $\mu+1_i=(\mu_1,\cdots,\mu_{i-1},\mu_i+1,\mu_{i+1},\cdots,\mu_n)$.

$\Pi_q=\Pi_q(X,X)\subset J_q(X\times X)$ Lie groupoid of order q over X.

$(x,y_q)=(x,y_\mu^k)$ local coordinates with $0\le|\mu|\le q$ and such that $det(y_i^k)\neq 0$.

$\alpha_q:\Pi_q\to X:(x,y_q)\to x$ source projection, $\beta_q:\Pi_q\to X=Y:(x,y_q)\to y$ target projection.

$f:X\to X\times X:(x)\to(x,f^k(x))$ section of $X\times X$ $(id:(x)\to(x,x))$.

$f_q:X\to\Pi_q:(x)\to(x,f_\mu^k(x))$ section of Π_q with $j_q(f):X\to\Pi_q:(x)\to(x,\partial_\mu f^k(x))$.

$E\to X$ vector bundle over X with section ξ, $J_q(E)$ q-jet bundle of E with section ξ_q over ξ.

$\pi_q^{q+r}:J_{q+r}(E)\to J_q(E):\xi_{q+r}\to\xi_q$ canonical projection.

$D\xi_{q+1}=j_1(\xi_q)-\xi_{q+1}:(\partial_i\xi^k(x)-\xi_i^k(x),\partial_i\xi_j^k(x)-\xi_{ij}^k(x),\cdots)\in T^*\otimes J_q(E)$ linear Spencer operator.

$d:\wedge^rT^*\to\wedge^{r+1}T^*:\alpha\to d\alpha$ exterior derivative with $d\circ d=0$.

$D(\alpha\otimes\xi_{q+1})=d\alpha\otimes\xi_q+(-1)^r\alpha\wedge D\xi_{q+1}\in\wedge^{r+1}T^*\otimes J_q(E),\forall\alpha\in\wedge^rT^*$ extension of D.

$\{\xi_{q+1},\eta_{q+1}\}=j_q([\xi,\eta]),\forall\xi,\eta\in T$ algebraic bracket.

$[\xi_q,\eta_q]=\{\xi_{q+1},\eta_{q+1}\}+i(\xi)D\eta_{q+1}-i(\eta)D\xi_{q+1},\forall\xi_q,\eta_q\in J_q(T)$ differential bracket.

$R_q\subset J_q(T)$ with $[R_q,R_q]\subset R_q$ system of infinitesimal Lie equations or Lie algebroid.

$0\to R_q^0\to R_q\xrightarrow{\pi_0^q}T\to 0$ exact $\Rightarrow$ transitive algebroid.

$\chi_q'\in T^*\otimes R_q$ over $id_T\in T^*\otimes T$ is called a R_q-connection.

$\kappa_q'(\xi,\eta)=[\chi_q'(\xi),\chi_q'(\eta)]-\chi_q'([\xi,\eta])\in R_q^0$ is called the curvature of χ_q'.

$\rho_1(R_q)=J_1(R_q)\cap J_{q+1}(T)\subset J_1(J_q(T))$ first prolongation of R_q.

$\rho_1(R_q)=\{\xi_{q+1}\in J_{q+1}(T)\mid\xi_q\in R_q,D\xi_{q+1}\in T^*\otimes R_q\}$ alternative definition.

$R_{q+r}^{(s)}=\pi_{q+r}^{q+r+s}(R_{q+r+s})\subseteq R_{q+r}$ prolongation/projection (PP) procedure.

$L(\xi_{q+1})\eta_q=\{\xi_{q+1},\eta_{q+1}\}+i(\xi)D\eta_{q+1}=[\xi_q,\eta_q]+i(\eta)D\xi_{q+1}$ formal Lie derivative.

$\bar{D}f_{q+1}=f_{q+1}^{-1}\circ j_1(f_q)\in T^*\otimes J_q(T)$ first nonlinear Spencer operator.

$\bar{D}'\chi_q(\xi,\eta)=D\chi_q(\xi,\eta)-\{\chi_q(\xi),\chi_q\eta\}\in J_{q-1}(T)$ second nonlinear Spencer operator.

$\delta:\wedge^rT^*\otimes S_{q+1}T^*\otimes E\to\wedge^{r+1}T^*\otimes S_qT^*\otimes E:\omega\to(\delta\omega)_\mu^k=dx^i\wedge\omega_{\mu+1_i}^k$ Spencer δ-map.

$F_r=\wedge^rT^*\otimes J_q(E)\big/\left(\wedge^rT^*\otimes R_q+\delta\left(\wedge^{r-1}T^*\otimes S_{q+1}T^*\otimes E\right)\right)$ Janet bundles.

$C_r=\wedge^rT^*\otimes R_q\big/\delta\left(\wedge^{r-1}T^*\otimes g_{q+1}\right)$ Spencer bundles.