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Quantization of the 1-D Forced Harmonic Oscillator in the Space (*x*, *v*)

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Abstract

The quantization of the forced harmonic oscillator is studied with the quantum variable $(x,\hat{v})$, with the commutation relation $[x,\hat{v}]=i\hbar/m$, and using a Schrödinger's like equation on these variable, and associating a linear operator to a constant of motion $K(x,v,t)$ of the classical system, The comparison with the quantization in the space (x,p) is done with the usual Schrödinger's equation for the Hamiltonian $H(x,p,t)$, and with the commutation relation $[x,\hat{p}]=i\hbar$. It is found that for the non-resonant case, both forms of quantization bring about the same result. However, for the resonant case, both forms of quantization are different, and the probability for the system to be in the exited state for the $(x,\hat{v})$ quantization has fewer oscillations than the $(x,\hat{p})$ quantization, the average energy of the system is higher in $(x,\hat{p})$ quantization than on the $(x,\hat{v})$ quantization, and the Boltzmann- Shannon entropy on the $(x,\hat{p})$ quantization is higher than on the $(x,\hat{v})$ quantization.

Keywords

Forced Harmonic Oscillator, $(x,\hat{v})$ Quantization, Constant of Motion

1. Introduction

The usual quantum mechanics formulation is done in the space $(x,\hat{p})$ [1], where $\hat{p}=-i\hbar\partial/\partial x$ is the linear operator associated to the classical generalized linear momentum of the motion of a particle of mass "m", where the commutation relation $[x,\hat{p}]=i\hbar$ [2] is satisfied. A linear operator is associated to the classical Hamiltonian, $\hat{H}(x,\hat{p},t)$, to form the so called Schrödinger's equation [3]

$$i\hbar\frac{\partial\Psi}{\partial t}=\hat{H}(x,\hat{p},t)\Psi, \tag{1}$$

where $\Psi = \Psi(x,t)$ is the wave function. This formulation has had enormous success to explain and to predict most microscopic behavior of the nature [4]. However, despite this enormous success, Hamiltonian-Lagrangian mathematical formulation has some details, even for 1-D problem where one knows that the Lagrangian (therefore the Hamiltonian) always exists [5]. First, from the expression to obtain the generalized linear momentum given the Lagragian, $L(x,\dot{x},t)$ for the system,

$$p(x,\dot{x},t) = \frac{\partial L}{\partial \dot{x}}, \tag{2}$$

it is not always possible to obtain explicitly $\dot{x} = \dot{x}(x,p,t)$ to be able to get the explicit expression for the Hamiltonian from the Legrandre's transformation [6],

$$H(x,p,t) = \dot{x}(x,p,t)\,p - L\left(x,\dot{x}(x,p,t),t\right). \tag{3}$$

Second, when one is dealing with classical dissipative systems [7],

$$\frac{\mathrm{d}(m\dot{x})}{\mathrm{d}t} = F(x,\dot{x}), \tag{4}$$

either it is not possible to find its Hamiltonian, or two different Hamiltonians are possible to find for the system [8] [9] [10] [11] [12]. Last one, for those problems of variable mass systems,

$$\frac{\mathrm{d}\left(m(x,\dot{x},t)\dot{x}\right)}{\mathrm{d}t} = F(x), \tag{5}$$

which are not invariant under Galileo's transformations and Sommerfeld modification is not consistent, to find the Hamiltonian for this system [13] requires to start from the "Inverse Problem of the Mechanics".

Therefore, one has the necessity to find some extension of the known quantization arised from the Hamilton-Lagrangian approach. In this way, there is already a proposition [14] [15] of using a function $K(x,v,t)$ that could be a constant of motion of the classical system, and to associate a linear operator to the velocity of the form

$$\hat{v} = -i\frac{\hbar}{m}\frac{\partial}{\partial x}, \tag{6}$$

such that $\left[x,\hat{v}\right] = i\hbar/m$, and to associate a linear operator

$$K(x,v,t) \to \hat{K}(x,\hat{v},t), \tag{7}$$

which can be used to form the Shrödinger's like equation

$$i\hbar\frac{\partial\Psi}{\partial t} = \hat{K}(x,\hat{v},t)\Psi. \tag{8}$$

The usual Quantum Mechanics is formulated through the Shrödinger's Equation (1), given in terms of the Hamiltonian associate to the system. However, in this paper, we intend to use the approach (8) as a possible alternative and extended way for the Quantum Mechanics, and this is done by studying the 1-D forced harmonic oscillator to determine whether or not there is a difference on

the quantization, and hopefully to see if the approach (8) could have with these result and experimental verification.

2. Analytical Approach for $K(x,v,t)$

The forced harmonic oscillator is classically characterized by Newton's equation

$$\frac{\mathrm{d}(m\dot{x})}{\mathrm{d}t} = -m\omega_0^2 x + \alpha\cos(\omega t+\varphi), \tag{9}$$

where "m" is the mass of the particle, ω_0 is the natural frequency of oscillation (when $\alpha = 0$), and α is the amplitude of the forced force. The well-known solution of this problem is

$$x(t) = \begin{cases} C_1\cos\omega_0 t + C_2\sin\omega_0 t + \dfrac{\alpha\cos(\omega t+\varphi)}{m(\omega_0^2-\omega^2)}, & \omega \neq \omega_0 \\ C_1\cos\omega_0 t + C_2\sin\omega_0 t + \dfrac{\alpha\sin(\omega_0 t+\varphi)}{2m\omega_0}t, & \omega = \omega_0 \end{cases} \tag{10}$$

where one has the non-resonant case ($\omega \neq \omega_0$) and the resonant case ($\omega = \omega_0$). The velocity is known by making the differentiation of (10) with respect the time, and the constants C_1 and C_2 are determined by the initial condition ($x(0), v(0)$). For the non-resonant case, these constants are

$$\begin{aligned} C_1 = {} & x\cos\omega_0 t - \frac{v}{\omega_0}\sin\omega_0 t \\ & - \frac{\alpha}{m(\omega_0^2-\omega^2)}\left\{\cos(\omega t+\varphi)\cos\omega_0 t + \frac{\omega}{\omega_0}\sin(\omega t+\varphi)\sin\omega_0 t\right\} \end{aligned} \tag{11a}$$

and

$$\begin{aligned} C_2 = {} & x\sin\omega_0 t + \frac{v}{\omega_0}\cos\omega_0 t \\ & - \frac{\alpha}{m(\omega_0^2-\omega^2)}\left\{\cos(\omega t+\varphi)\sin\omega_0 t - \frac{\omega}{\omega_0}\sin(\omega t+\varphi)\cos\omega_0 t\right\} \end{aligned} \tag{11b}$$

For the resonant case ($\omega = \omega_0$), one has

$$\begin{aligned} C_1 = {} & x\cos\omega_0 t - \frac{v}{\omega_0}\sin\omega_0 t + \frac{\alpha}{2m\omega_0}\left\{-t\sin(\omega_0 t+\varphi)\cos\omega_0 t\right. \\ & \left. + t\cos(\omega_0 t+\varphi)\sin\omega_0 t + \frac{1}{\omega_0}\sin(\omega_0 t+\varphi)\sin\omega_0 t\right\} \end{aligned} \tag{12a}$$

and

$$\begin{aligned} C_2 = {} & x\sin\omega_0 t + \frac{v}{\omega_0}\cos\omega_0 t - \frac{\alpha}{2m\omega_0}\left\{t\sin(\omega_0 t+\varphi)\sin\omega_0 t\right. \\ & \left. + t\cos(\omega_0 t+\varphi)\cos\omega_0 t + \frac{1}{\omega_0}\sin(\omega_0 t+\varphi)\cos\omega_0 t\right\} \end{aligned} \tag{12b}$$

Now, by choosing a constant of motion of the form

$$K_{\alpha}^{(nr,r)}(x,v,t)=\frac{1}{2}m\omega_0^2\left(C_1^2+C_2^2\right), \tag{13}$$

where "*nr*" means non-resonant and "*r*" means resonant, it follows that

$$\lim_{\alpha\to 0}K_{\alpha}^{(nr,r)}(x,v,t)=\frac{1}{2}mv^2+\frac{1}{2}m\omega_0^2x^2, \tag{14}$$

Which represents the usual energy of the harmonic oscillator, independently of the non-resonant case or resonant case. This constant of motion can be written as

$$K_{\alpha}^{(nr,r)}(x,v,t)=K_0(x,v)+W_{\alpha}^{(nr,r)}(x,v,t), \tag{15}$$

where K_0 and $W_{\alpha}^{(nr,r)}$ are defined as

$$K_0(x,v)=\frac{1}{2}mv^2+\frac{1}{2}m\omega_0^2x^2, \tag{16}$$

$$\begin{aligned}W_{\alpha}^{(nr)}(x,v,t)=\frac{1}{2}m\omega_0^2\Big[&A^2\cos^2(\omega t+\varphi)-2Ax\cos(\omega t+\varphi)\\&+B^2\sin^2(\omega t+\varphi)+\frac{2Bv}{\omega_0}\sin(\omega t+\varphi)\Big],\end{aligned} \tag{17}$$

and

$$\begin{aligned}W_{\alpha}^{(r)}(x,v,t)=\frac{1}{2}m\omega_0^2\Big[&a^2(t)-\frac{2a(t)v}{\omega_0}\cos(\omega_0t+\varphi)-\frac{2bv}{\omega_0}\sin(\omega_0t+\varphi)\\&+2a(t)b\cos(\omega_0t+\varphi)\sin(\omega_0t+\varphi)\\&-2a(t)x\sin(\omega_0t+\varphi)+b^2\sin^2(\omega_0t+\varphi)\Big],\end{aligned} \tag{18}$$

where one has made the definitions

$$A=\frac{\alpha}{m\left(\omega_0^2-\omega^2\right)},\quad B=\frac{\alpha\omega}{m\omega_0\left(\omega_0^2-\omega^2\right)} \tag{19a}$$

and

$$a(t)=\frac{\alpha t}{2m\omega_0},\quad b=\frac{\alpha}{2m\omega_0^2}. \tag{19b}$$

To solve Equation (8), one observes that the eigenvalues problem for the operator $\hat{K}_0$,

$$\hat{K}_0(x,\hat{v})\Phi=E\Phi, \tag{20}$$

has exactly the same solution of that one given by the Hamiltonian problem, $\left(\hat{p}^2/2m+m\omega_0^2x^2/m\right)\Phi=E\Phi$ where the solution is the set $\left\{E_m^{(0)},\Phi_n(x)\right\}_{n\geq 0}$,

$$E_n^{(0)}=\hbar\omega_0\left(n+1/2\right) \tag{21a}$$

and

$$\Phi_n(x)=A_n\mathrm{e}^{-\xi^2/2}H_n(x),\quad \xi=\sqrt{\frac{m\omega_0}{\hbar}}\,x,\quad A_n=\left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}. \tag{21b}$$

Using Dirac's notation [16], where $\Phi_n(x)=\langle x|n\rangle$, with $|n\rangle$ characterizing the nth-state, and then one has the eigenvalue problem written as

$$\hat{K}_0 \left| n \right\rangle = E_n^{(0)} \left| n \right\rangle. \tag{22}$$

Therefore, one can propose the solution of the Shrödinger's Equation (8) with the operator constant of motion $\hat{K}$,

$$i\hbar \frac{\partial \left| \Psi(t) \right\rangle}{\partial t} = \left\{ \hat{K}_0 \left(x, \hat{v} \right) + W_\alpha^{(nr,r)} \left(x, \hat{v}, t \right) \right\} \left| \Psi(t) \right\rangle, \tag{23}$$

of the form

$$\left| \Psi(t) \right\rangle = \sum_{n=0}^{\infty} C_n(t) \left| n \right\rangle. \tag{24}$$

Taking into consideration (22), the orthogonality of the states ($\left\langle m \middle| n \right\rangle = \delta_{mn}$), one obtains the following equation for the coefficients

$$i\hbar \dot{C}_m(t) = E_m^{(0)} C_m(t) + \sum_{n=0}^{\infty} C_n(t) W_{mn}^{(nr,r)}(t), \tag{25}$$

where $W_{mn}^{(nr,r)}(t)$ represents the matrix element

$$W_{mn}^{(nr,r)}(t) = \left\langle m \middle| W_\alpha^{(nr,r)} \middle| n \right\rangle. \tag{26}$$

The Equation (25) can be simplified using the new variable

$$C_k(t) = \mathrm{e}^{-iE_k^{(0)} t/\hbar} D_k(t). \tag{27}$$

The equations for these new coefficients are

$$i\hbar \dot{D}_m(t) = \sum_{n=0}^{\infty} \mathrm{e}^{i\omega_{mn} t} D_n(t) W_{mn}^{(nr,r)}(t), \tag{28}$$

where $C_k(0) = D_k(0)$ and the probability to find the system in the state $\left| k \right\rangle$ is $\left| C_k(t) \right|^2 = \left| D_k(t) \right|^2$. Matrix elements are much easier to calculate by using the non-Hermitian ascent "$a^\dagger$" and descent "a" operators,

$$a = \sqrt{\frac{m\omega_0}{2\hbar}}\, x + i\sqrt{\frac{m}{2\omega_0 \hbar}}\, \hat{v}, \quad a^\dagger = \sqrt{\frac{m\omega_0}{2\hbar}}\, x - i\sqrt{\frac{m}{2\omega_0 \hbar}}\, \hat{v}, \tag{29}$$

with the knows properties [17]

$$\left[a, a \right] = \left[a^\dagger, a^\dagger \right] = 0, \quad \left[a, a^\dagger \right] = 1, \tag{30a}$$

and

$$a^\dagger \left| n \right\rangle = \sqrt{n+1} \left| n+1 \right\rangle, \quad a \left| n \right\rangle = \sqrt{n} \left| n-1 \right\rangle. \tag{30b}$$

For non-resonant case (nr), after calculating the matrix elements, using the orthogonality of the states, and making some rearrangements, one gets the equations for the real and imaginary parts of the coefficients, $D_k(t) = X_k(t) + iY_k(t)$, as

$$\begin{aligned}
\dot{X}_k = & -c\sqrt{k}\left[\cos(\omega t + \varphi) \cos\omega_0 t Y_{k-1} + \cos(\omega t + \varphi) \sin\omega_0 t X_{k-1} \right] \\
& - c\sqrt{k+1}\left[\cos(\omega t + \varphi) \cos\omega_0 t Y_{k+1} - \cos(\omega t + \varphi) \sin\omega_0 t X_{k+1} \right] \\
& + d\sqrt{k}\left[\sin(\omega t + \varphi) \cos\omega_0 t X_{k-1} - \sin(\omega t + \varphi) \sin\omega_0 t Y_{k-1} \right] \\
& - d\sqrt{k+1}\left[\sin(\omega t + \varphi) \cos\omega_0 t X_{k+1} + \sin(\omega t + \varphi) \sin\omega_0 t Y_{k+1} \right] \\
& + a_1 \cos^2(\omega t + \varphi) Y_k + b_1 \sin^2(\omega t + \varphi) Y_k
\end{aligned} \tag{31a}$$

$$\begin{aligned}\dot{Y}_k &= +c\sqrt{k}\left[\cos(\omega t+\varphi)\cos\omega_0 tX_{k-1}-\cos(\omega t+\varphi)\sin\omega_0 tY_{k-1}\right]\\&+c\sqrt{k+1}\left[\cos(\omega t+\varphi)\cos\omega_0 tX_{k+1}+\cos(\omega t+\varphi)\sin\omega_0 tY_{k+1}\right]\\&+d\sqrt{k}\left[\sin(\omega t+\varphi)\cos\omega_0 tY_{k-1}+\sin(\omega t+\varphi)\sin\omega_0 tX_{k-1}\right]\\&-d\sqrt{k+1}\left[\sin(\omega t+\varphi)\cos\omega_0 tY_{k+1}-\sin(\omega t+\varphi)\sin\omega_0 tX_{k+1}\right]\\&+a_1\cos^2(\omega t+\varphi)X_k+b_1\sin^2(\omega t+\varphi)X_k,\end{aligned} \tag{31b}$$

where a_1, b_1, c and d have been defined as

$$a_1=\frac{\alpha^2\omega_0^2}{2m\hbar\left(\omega_0^2-\omega^2\right)^2},\quad b_1=\frac{\alpha^2\omega^2}{2m\hbar\left(\omega_0^2-\omega^2\right)^2} \tag{32b}$$

$$c=\frac{\alpha\omega_0^2}{\omega_0^2-\omega^2}\frac{1}{\sqrt{2m\hbar\omega_0}},\quad d=\frac{\alpha\omega}{\omega_0^2-\omega^2}\sqrt{\frac{\omega_0}{2m\hbar}}. \tag{32b}$$

For the resonant case (r), one can in addition make the following change of coefficients

$$D_k(t)=\mathrm{e}^{-i\alpha^2t^3/24m\hbar}\tilde{D}_k(t) \tag{33}$$

to eliminate the quadratic time dependence appearing in the expression (18) and (19b). Note that $D_k(0)=\tilde{D}_k(0)$ and $\left|D_k(t)\right|^2=\left|\tilde{D}_k(t)\right|^2$. Doing the same as it was done above, the real and imaginary parts of these new coefficients, $\tilde{D}_k=\tilde{X}_k+i\tilde{Y}_k$, obey the equations

$$\begin{aligned}\tilde{X}_k &= +f(t)\tilde{Y}_k-\sqrt{k}\left\{\left[h(t)\tilde{X}_{k-1}-g(t)\tilde{Y}_{k-1}\right]\sin\omega_0 t+\left[h(t)\tilde{Y}_{k-1}\right.\right.\\&\left.\left.+g(t)\tilde{X}_{k-1}\right]\cos\omega_0 t\right\}+\sqrt{k+1}\left\{\left[h(t)\tilde{X}_{k+1}+g(t)\tilde{Y}_{k+1}\right]\sin\omega_0 t\right.\\&\left.-\left[h(t)\tilde{Y}_{k+1}-g(t)\tilde{X}_{k+1}\right]\cos\omega_0 t\right\}\end{aligned} \tag{34a}$$

$$\begin{aligned}\tilde{Y}_k &= -f(t)\tilde{X}_k+\sqrt{k}\left\{\left[h(t)\tilde{X}_{k-1}-g(t)\tilde{Y}_{k-1}\right]\cos\omega_0 t-\left[h(t)\tilde{Y}_{k-1}\right.\right.\\&\left.\left.+g(t)\tilde{X}_{k-1}\right]\sin\omega_0 t\right\}+\sqrt{k+1}\left\{\left[h(t)\tilde{X}_{k+1}+g(t)\tilde{Y}_{k+1}\right]\cos\omega_0 t\right.\\&\left.+\left[h(t)\tilde{Y}_{k+1}-g(t)\tilde{X}_{k+1}\right]\sin\omega_0 t\right\}\end{aligned} \tag{34b}$$

where the functions f, h, and g have been defined as

$$f(t)=\frac{\alpha^2}{8m\hbar\omega_0}\sin^2\omega_0 t+\frac{\alpha^2 t}{4m\omega_0\hbar}\cos(\omega_0 t+\varphi)\sin(\omega_0 t+\varphi), \tag{35a}$$

$$g(t)=\frac{\alpha}{2\hbar}\left[\frac{1}{\omega_0}\sin(\omega_0 t+\varphi)+t\cos(\omega_0 t+\varphi)\right], \tag{35b}$$

and

$$h(t)=\frac{\alpha\omega_0 t}{2\hbar}\sin(\omega_0 t+\varphi). \tag{35c}$$

The dynamical systems (31) and (34) are solved by Runge-Kutta method a 4th-order.

3. Analytical Approach for $K(x,p,t)$

The Hamiltonian of the forced harmonic oscillator is [18]

$$H(x,p,t) = H_0(x,p) + \alpha x \cos(\omega t + \varphi), \tag{36}$$

where H_0 is given by

$$H_0(x,p) = \frac{p^2}{2m} + \frac{1}{2} m\omega_0^2 x^2. \tag{37}$$

The solution of the eigenvalue problem

$$H_0 \Phi = E\Phi \tag{38}$$

is well known [19], and its solution is the same as (21). Therefore, to solve the Shrödinger's Equation (1), one proposes a solution of the form

$$\left|\Psi(t)\right\rangle = \sum_{n=0}^{\infty} \mathrm{e}^{-iE_n^{(0)} t/\hbar} D_k(t) \left|n\right\rangle, \tag{39}$$

which, after substituting in the Shrödinger's equation, using the eigenvalues and the orthogonality between any two states, and making some rearranging, the following dynamical systems is brought about for the real and imaginary parts of the coefficients, $D_k(t) = x_k(t) + iy_k(t)$,

$$\begin{aligned} \dot{x}_k = & -\lambda\left[\sqrt{k}x_{k-1} - \sqrt{k+1}x_{k+1}\right]\cos(\omega t + \varphi)\sin\omega_0 t \\ & -\lambda\left[\sqrt{k}y_{k-1} + \sqrt{k+1}y_{k+1}\right]\cos(\omega t + \varphi)\cos\omega_0 t \end{aligned} \tag{40a}$$

$$\begin{aligned} \dot{y}_k = & -\lambda\left[\sqrt{k}y_{k-1} - \sqrt{k+1}y_{k+1}\right]\cos(\omega t + \varphi)\sin\omega_0 t \\ & +\lambda\left[\sqrt{k}x_{k-1} + \sqrt{k+1}x_{k+1}\right]\cos(\omega t + \varphi)\cos\omega_0 t, \end{aligned} \tag{40b}$$

where the constant λ has been defined as

$$\lambda = \alpha\sqrt{\frac{\hbar}{2m\omega_0}}. \tag{41}$$

These equations are also solved by using Runge-Kutta method at 4th-order.

4. Boltzmann-Shannon Entropy and Energy

Besides the probability to find the system in the state $\left|n\right\rangle$ at the time "t", $\left|D_k(t)\right|^2$, for the analysis of the dynamics of the system in the spaces $(x, \hat{v})$ and $(x, \hat{p})$, one can also consider the Boltzmann-Shannon entropy,

$$S(t) = -\sum_{k=0}^{I} \left|D_k(t)\right|^2 \ln\left|D_k(t)\right|^2, \tag{42}$$

and its average over an evolution time "T",

$$\overline{S} = \frac{1}{T}\int_0^T S(t)\,\mathrm{d}t, \tag{43}$$

as parameter which characterize the quantum dynamics of the system. This parameter gives us an indication of how many states enter in the dynamics evolution of the system. Therefore, it gives an indication of the information lost in the dynamics due to the increasing of the entropy in the quantum system. In addi-

tion, one can also consider the expectation value of the energy

$$\langle E\rangle(t)=\langle\Psi|\begin{pmatrix}\hat{H}_0\\ \hat{K}_0\end{pmatrix}|\Psi\rangle=\hbar\omega_0\sum_{n=0}^{l}n\left|D_n(t)\right|^2+\frac{1}{2}\hbar\omega_0, \tag{44}$$

and its average value over the evolution time of the system,

$$\bar{E}=\frac{1}{T}\int_0^T\langle E\rangle(t)\mathrm{d}t. \tag{45}$$

This parameter gives information about how the energy is distributed among the states and how many of them are involved in the quantum dynamics.

In this way, solving the dynamical systems (31), (34), and (40), the evolution of the probabilities $\left|D_k(t)\right|^2$'s are gotten. Thus, the Boltzmann-Shannon entropy (42), the expectation value of the energy (44) and their average values (43) and (45) can be calculated and can be compared for the quantization in the spaces (x,v) and (x,p).

5. Results

One considers a proton with mass $m=1.6726219\times10^{-27}\,\mathrm{kg}$ oscillating with a frequency $\omega_0=2\pi\times10^9\,\mathrm{Hz}$ on a one-dimensional line, and interacting with a periodic force of amplitude $a=10^{-13}$ Newtons with frequency ω and phase $\varphi=0$. The initial conditions of the system are

$$C_k(0)=D_k(0)=\tilde{D}_k(0)=\delta_{k0},\quad k=0,\cdots,11 \tag{46}$$

that is, the system is on the ground state, and one selects ten exited possible state of the system. For the non-resonant case ($\omega\neq\omega_0$), the resulting dynamics from expressions (1) and (8) are exactly the same. There is not excitation of the system at all since the system remains in the ground state in both cases. For the resonant case ($\omega=\omega_0$), Figure 1 shows the probabilities of having the system on the ground state ($k=0$) and on the first excited state ($k=1$) for the quantization on the space (x,v), solid lines, and the quantization on the space (x,p), dotted lines. As one can see, for the Hamiltonian quantization approach (H) there are much more oscillations of the probabilities than the quantization of the constant of motion approach (K), that is, there are more transitions per unit time in the H-approach case than in the K-approach case. One must note that the probability to have the system in the first excited state for the K-approach case is totally different from the H-approach case.

Figure 2 shows the average value of the energy as a function of the strength of the forced force (a). As one can see, this average value is always higher for H-approach case (the usual Quantum Mechanics approach) than for the K-approach case (our pretended extension for the Quantum Mechanics). However, the difference on the average energy value for both approaches is quite small and maybe out of experimental verification. This difference is expected since for the H-approach the Hamiltonian is not a constant of motion, but for the K-approach one has a quantization with a constant of motion of the system.

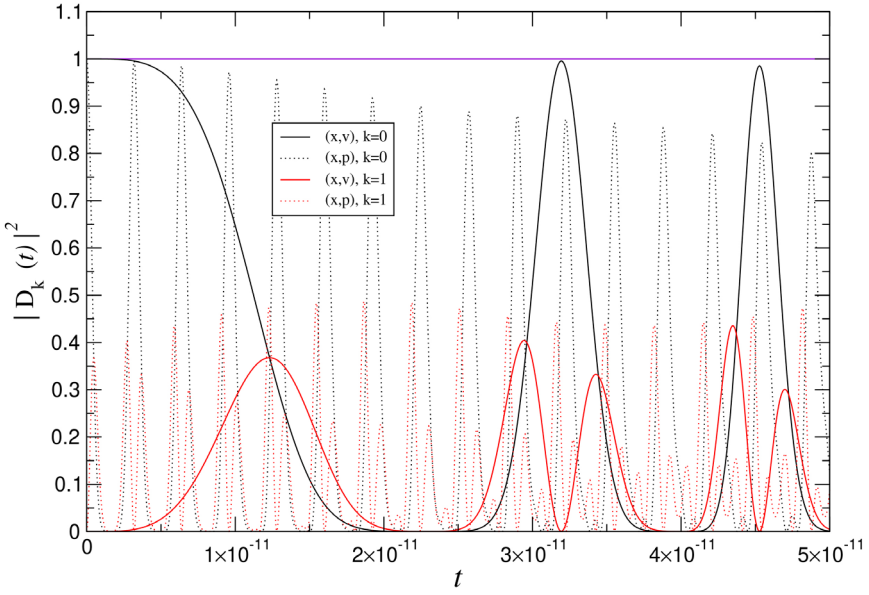

Figure 1. Ground state and first excited state evolution.

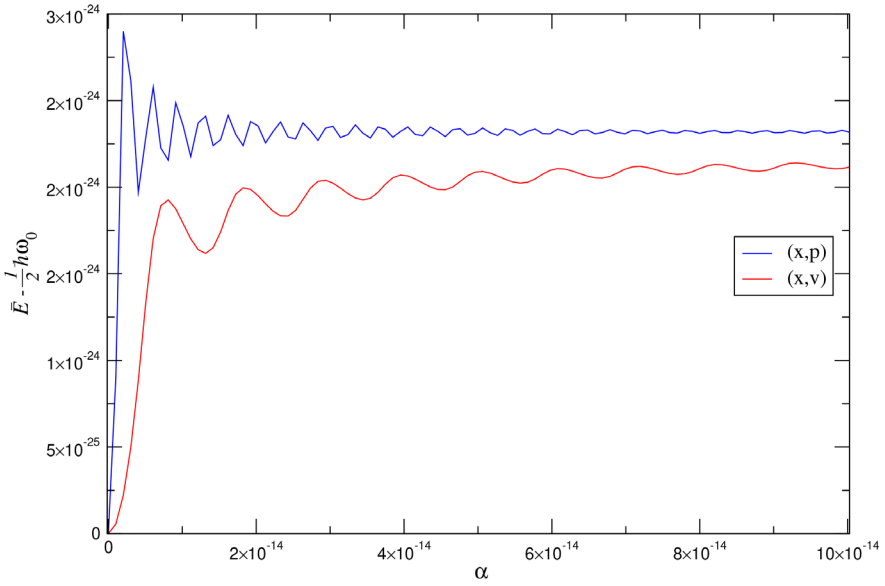

Figure 2. Average energy of the system.

Figure 3 shows the average value of the Boltzmann-Shannon entropy as a function of the strength of the forced force (α). Notice that, having total number of 11 states, the possible maximum entropy is 2.398. As the previous case, this parameter is always higher for the H-approach case than for the K-approach case due to the same reason that the Hamiltonian is not a constant of motion and K is indeed a constant of motion of the system. However, this difference is not so small and maybe could be used as a good parameter for experimental proposes. This difference means that the H-approach case brings about more complex behavior in the quantum dynamics than the K-approach case, and that the H-approach case losses more information than the K-approach case.

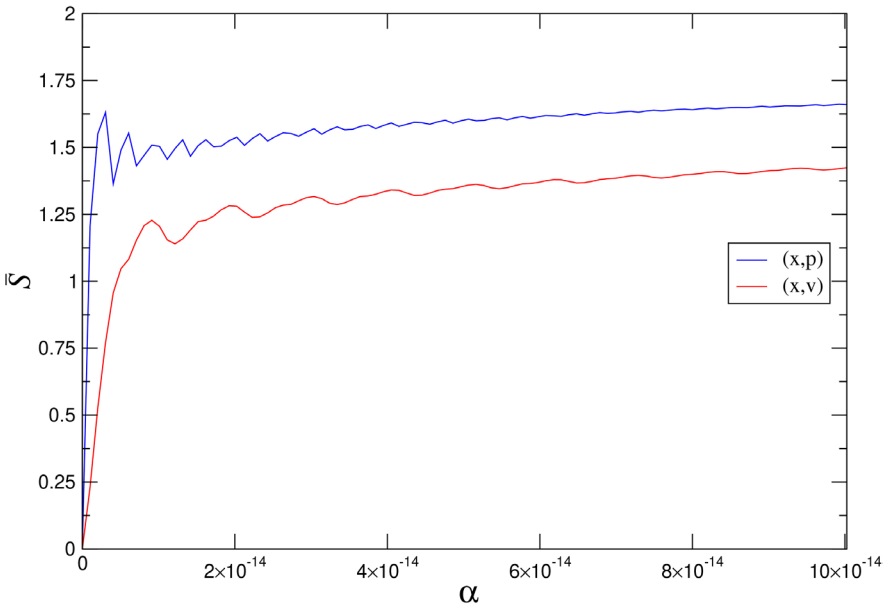

Figure 3. Avarage Boltzmann-Shannon entropy.

6. Conclusion

The quantization of the 1-D forced harmonic oscillator was carried out with the operators $(x,\hat{v})$ using the assigned linear operator to a constant of motion $K(x,v,t)$ of the classical case. The restriction imposed on this constant was that it must be reduced to the known energy expression when the forced force is zero. This quantization was compared with the usual quantization with the operators $(x,\hat{p})$ and the associated Hamiltonian $H(x,p,t)$ of the classical case. It was shown that the probabilities to find the system in the state $|n\rangle$, $|D_n(t)|^2$, has less oscillations in the K-quantization than in the H-quantization. In addition, the average values of the energy and the average value of the Boltzmann-Shannon entropy are lower in the K-quantization than in the H-quantization. Since the difference in the average value of the energy is quite small, this parameter does not look good to measure experimentally. However, the difference in the entropy is significant and it represents a good parameter to look experimentally.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Von Neumann, J. (1932) Mathematical Foundations of Quantum Mechanics, Volume 183. Princeton University Press, Princeton.

[2] Born, M. (1926) *Zeitschrift für Physik*, **38**, 803. https://doi.org/10.1007/BF01397184

[3] Schrödinger, E. (1926) *Physical Review*, **28**, 1049-1070. https://doi.org/10.1103/PhysRev.28.1049

[4] Spielman, I. (2017) *Nature*, **545**, 293-294. https://doi.org/10.1038/545293a

[5] Darboux, G. (1894) Lecons sur la Théorie Générale des Surfaces, volume 3. Gauthier-Villars, Paris.

[6] Goldstein, H. (1978) Classical Mechanics, Volume I, II. Addison-Wesley, Boston.

[7] López, P., López, G.V. and López, X.E. (2007) *International Journal of Theoretical Physics*, **46**, 1100. https://doi.org/10.1007/s10773-006-9260-7

[8] López, P., López, G.V. and López, X.E. (2011) *Advanced Studies in Theoretical Physics*, **5**, 253-268.

[9] Man'ko, V.I., Dodonov, V.V. and Skarzhinsky, V.D. (1981) *Hadronic Journal*, **4**, 1734.

[10] Man'ko, V.I., Dodonov, V.V. and Skarzhinsky, V.D. (1982) *Il Nuovo Cimento B*, **69**, 185. https://doi.org/10.1007/BF02721265

[11] Griselda, A., López, G.V. and Martínez-Prieto, R.M. (2018) *Journal of Applied Mathematical and Physics*, **6**, 1382. https://doi.org/10.4236/jamp.2018.66115

[12] Motensinos, M. and Torres del Castillo, G.F. (2004) *Physical Review A*, **70**, Article ID: 032104. https://doi.org/10.1103/PhysRevA.70.032104

[13] López, G.V. (1998) *Journal of Theoretical Physics*, **37**, 1617-1623. https://doi.org/10.1023/A:1026628221912

[14] Lopez, G.V. (2012) *Journal of Modern Physics*, **3**, 777-785. https://doi.org/10.4236/jmp.2012.38102

[15] López, G. and López, P. (2006) *International Journal of Theoretical Physics*, **45**, 753. https://doi.org/10.1007/s10773-006-9064-9

[16] Dirac, P. (1939) *Mathematical Proceedings of the Cambridge Philosophical Society*, **35**, 416-418. https://doi.org/10.1017/S0305004100021162

[17] Leventhal, J.J. and Burkhardt, C.E. (2008) Harmonic Oscillator Solution Using Operator Methods. https://doi.org/10.1007/978-0-387-77652-1_7

[18] Lifshitz, E.M. and Landau, L.D. (1970) Course of Theoretical Physics, Volume I.

[19] Messiah, A. (1966) Quatum Mechanics, Volume I. John Wiley & Sons, Hoboken, Ch. XII.