Scientific Research Publishing

Journal of Mathematical Finance, 2024, 14, 18-33
https://www.scirp.org/journal/jmf
ISSN Online: 2162-2442
ISSN Print: 2162-2434

Analyzing Bankruptcy Probability under Partial Shareholder Payments and Dependent Claims via Spearman Copula

Kiswendsida Mahamoudou Ouedraogo[1], Delwendé Abdoul-Kabir Kafando[1,2]*, Lassané Sawadogo[1], François Xavier Ouedraogo[1], Pierre Clovis Nitiema[3]

[1]Department of Mathematics, Université Joseph KI ZERBO, Ouagadougou, Burkina Faso
[2]Department of Mathematics, Université Ouaga 3S, Ouagadougou, Burkina Faso
[3]Department of Mathematics, Université Thomas SANKARA, Ouagadougou, Burkina Faso
Email: mahouera2000@yahoo.fr, *kafandokabir92@gmail.com, slassane70@gmail.com, fxavierouedraogo@gmail.com, pnitiema@gmail.com

How to cite this paper: Ouedraogo, K.M., Kafando, D.A.-K., Sawadogo, L., Ouedraogo, F.X. and Nitiema, P.C. (2024) Analyzing Bankruptcy Probability under Partial Shareholder Payments and Dependent Claims via Spearman Copula. *Journal of Mathematical Finance*, **14**, 18-33.
https://doi.org/10.4236/jmf.2024.141002

Received: December 8, 2023
Accepted: January 14, 2024
Published: January 17, 2024

Abstract

This paper is an extension of the compound poisson risk model with a strategy of partial dividend payment to shareholders, constant threshold b and dependence between claim amounts and inter-claim times via the Spearman copula. We study the probability of ultimate ruin associated with this risk model.

Keywords

Gerber-Shiu Functions, Dependence, Spearman Copula, Dividends, Integro-Differential Equation

1. Introduction

Risk management is a major issue for financial companies. Mathematical models are constantly being developed to provide a better understanding of risks and their evolution, with the simplifying assumption of independence between the random variables involved in risk modelling (see, for example, references [1] [2]). However, in certain practical contexts, this assumption is inappropriate and too restrictive. In flood insurance, for example, the occurrence of several floods in a short space of time can generate large amounts of damage, and therefore large claims, due to the accumulation of water. In earthquake insurance, it's the other way around: in a high-risk zone, the longer the time between two earthquakes, the greater the impact of the second earthquake, due to the accumulation of energy.

To make up for this shortcoming, many works include in the risk model the dependence between certain dependence between certain random variables, in particular the variables claim amount and inter-claim time, thanks to the Farlie Gumbel Morgenstern copula [3]-[8]. Although this copula is commonly used in the literature, encounters certain limitations. It fails to model tail dependencies [9] [10] [11] [12] [13].

To remedy the inadequacy of the Farlie Gumbel Morgenstern copula, while taking into account the reality of insurance companies, we consider in this article, the Compound Poisson risk model in which we integrate not only the dependence between the variables claim amounts and interclaim times via the Spearman copula, with also a strategy of partial payment of dividends to shareholders of constant threshold *b*.

In this model, when the surplus process reaches the constant threshold barrier *b* set, bonuses are partially granted to shareholders at a contant rate θ such that $0<\theta<1$. Noting by $U_b(t)$ the surplus process in the presence of the threshold dividend barrier *b* (with $U_b(0)=u$), the model follows the following dynamics:

$$\mathrm{d}U_b(t)=\begin{cases} c\mathrm{d}t-\mathrm{d}S(t) & if\ U_b(t)<b \\ (1-\theta)c\mathrm{d}t-\mathrm{d}S(t) & if\ U_b(t)=b \end{cases} \tag{1.1}$$

where:

- $U_b(t)$ is the surplus process in the presence of a *b* threshold dividend barrier *b* (with $U_b(0)=u$ the initial surplus and $0<u\le b$);
- *c* is the constant rate of premium received by the insurer per unit of time;
- t_b is the first instant when the surplus reaches the horizontal barrier *b* so $t_b=\frac{b-u}{c}$.
- $S(t)=\sum_{i=1}^{N(t)}X_i$ s the aggregate Poisson loss process composed of:
  - ➢ $\{N(t),t\ge 0\}$ the total number of claims recorded up to time *t*, which follows a Poisson process of intensity $\lambda>0$; (Note that $S(t)=0$ if $N(t)=0$);
  - ➢ $\{X_i,i\ge 1\}$ a sequence of random representing the individual amounts of claims with common density function *f* and distribution function *F* and assumed to have an exponential distribution with parameter β.

The interclaim times $\{V_i,i\ge 1\}$ form a sequence of random variables with exponential law of parameter λ, probability density function $k(t)=\lambda t\mathrm{e}^{-\lambda t}$ and distribution function $K(t)=1-\mathrm{e}^{-\lambda t}$.

The aim of this work is to determine the probability of ultimate ruin in the risk model defined by relations (1.1). The rest of the article is structured as follows: In section 2, we discuss the preliminaries of the risk model defined by the relation (1.1). In Section 3, we study the integro-differential equation satisfied by the Gerber Shiu function in the risk model defined by relation (1.1).

Section 4 deals with the study of the Laplace transforms of the Gerber Shiu functions and the probability of ultimate ruin in the risk model defined by relation (1.1).

In section 5, we discuss the probability of ultimate ruin in the risk model defined by relations (1.1).

2. Preliminaries

2.1. Ruin measures

The insurer's probability of ruin is the probability of ruin occurring either over a finite horizon or over an infinite horizon. In the latter case, we speak of the ultimate probability of risk.

Let τ be the insurance company's instant of ruin. τ is defined by:

$$\tau = \inf\{t \geq 0, U(t) < 0\} \tag{2.1}$$

When the probability of ruin is always zero, by convention we note $\tau = \infty$ in this case

$$U(t) \geq 0 \quad \forall t \geq 0.$$

The probability of ultimate failure is defined by:

$$\psi(u,t) = Pr\left[\tau \in [0,t], U(t) < 0 \mid U(0) = u\right] \tag{2.2}$$

Similarly, the probability of ultimate failure is defined by:

$$\psi(u) = \psi(u,\infty) = Pr\left[\tau < \infty, U(t) < 0 \mid U(0) = u\right] \tag{2.3}$$

2.2. Gerber-Shiu Discounted Penalty Function

The Gerber-Shiu expected penalty function or Gerber-Shiu function appeared in 1998 in the work of Gerber and Shiu (see [1]). Nowadays, this function is of great interest for research. Its analysis remains a central issue in both insurance and finance, as it is a valuable tool not only in the study of the probability of ruin, but also in the calculation of pension and reinsurance premiums, the pricing of options and so on. It is defined by:

$$\phi(u) = E\left[e^{-\delta\tau} w\left(U_{(\tau^-)}, |U_\tau|\right) I(\tau < \infty) \mid U(0) = u\right] \tag{2.4}$$

where:

- τ is the instant of failure defined by the relation (2.1);
- τ^- is the moment just before ruin;
- δ is a force of interest;
- The penalty function $w(x,y)$ is a positive function of the surplus just before ruin $U_{(\tau^-)}$ and of the ruin deficit $|U_\tau|, \forall x, y \geq 0$;
- I is the indicator function which is worth 1 if event A occurs and 0 otherwise.

2.3. Dependency Structure

In 1959, Abe Sklar introduced the copula function, which was not widely recognized by financial experts until the 1990s [14]. As a method for studying associated structures of random variables, the copula possesses unique properties, such as the ability to describe the multivariate distribution function using univariate marginal functions and multivariate correlation structure functions. Copulas are mathematical tools used to model the structure of dependence between multiple random variables, regardless of their marginal distributions [15] [16] [17] [18] [19].

2.3.1. Tail Dependence

The concept of tail dependence is essential for analysing the asymptotic dependence between two random variables. It allows us to describe the level of dependence at the extremes of the distribution, which makes it an appropriate tool for studying the dependence between strong values (higher tail dependence) and weak values (lower tail dependence). This measure is of great importance for extreme value copulas. There are two tail dependence coefficients which are defined as follows:

Definition: Let X; Y be two continuous random variables with respective distribution functions F and G. The lower tail dependence coefficient λ_L is defined by:

$$\lambda_L(X,Y) = \lim_{u \to 0+} Pr\left(X \le F^{-1}(\alpha) \mid Y \le G^{-1}(\alpha)\right)$$

and the upper dependency coefficient λ_U is defined by:

$$\lambda_U(X,Y) = \lim_{u \to 0+} Pr\left(X \le F^{-1}(\alpha) \mid Y \le G^{-1}(\alpha)\right)$$

These measurements can be defined in terms of a copula C.

Definition: Let X; Y be two continuous random variables of copula C, then we have

$$\lambda_L(X,Y) = \lim_{u \to 0+} \frac{C(u,u)}{u};$$

and

$$\lambda_U(X,Y) = \lim_{u \to 1^-} \frac{1-2u+C(u,u)}{1-u}$$

Remark

- When $\lambda_L \in]0,1]$; then C has a lower tail dependency.
- When $\lambda_L = 0$; then C has no lower tail dependency.
- When $\lambda_U \in]0,1]$; then C has an upper tail dependency.
- When $\lambda_U = 0$; then C has no upper tail dependency.

Many authors [20] [21] [22] [23] [24] to name but a few, have used the Farlie-Gumbel-Morgenstern (FGM) copula to define the dependency structure between the size of demand and the delay between requests. The FGM copula is given by

$$C_\alpha(u,v) = uv + \alpha uv(1-u)(1-v);\ 0 \le u,v \le 1.$$

It is not suitable for modelling dependencies on extreme values because $\lambda_L = 0$ and $\lambda_U = 0$.

2.3.2. Dependency Model Based on Spearman's Copula

In this work, the dependency structure is provided by the Spearman copula defined by: $\forall (u,v) \in [0,1]^2$ and $\alpha \in [0,1]$ par:

$$C_\alpha(u,v) = (1-\alpha)C_I(u,v) + \alpha C_M(u,v) \tag{2.5}$$

where: $C_I(u,v) = uv$; $C_M(u,v) = \min(u,v)$; α is dependency parameter.

It is suitable for modelling dependence on extreme values because $\lambda_L = \alpha$ and $\lambda_U = \alpha$.

Spearman's copula can be used to express positive dependencies and also tail dependencies in many situations in many situations. Using formula (3.1), the random vector claims amount and inter-claim times (*X*, *V*) has the joint distribution function given by:

$$\begin{aligned} F_{X,V}(x,t) &= C_\alpha(F_X(x), F_V(t)) \\ &= (1-\alpha)C_I(F_X(x), F_V(t)) + \alpha C_M(F_X(x), F_V(t)) \\ &= (1-\alpha)F_I(x,t) + \alpha F_M(x,t) \end{aligned} \tag{2.6}$$

where: F_X, F_V are the respective marginals of the random variables *X* and *V*.

3. Integro-Differential Equation Satisfied by the Gerber Shiu Function

The aim of this section is to determine the differential equation satisfied by the function $\phi_b(u)$ in a risk model with constant threshold dividend payment *b* and dependence between the random variables claim amount and inter-claim time via Spearman's copula. In this risk model [9] [23] [24], the Gerber Shiu function $\phi_b(u)$ is given by:

$$\phi_b(u) = (1-\alpha)\left(I_{b,1}(u) + I_{b,2}(u)\right) + \alpha\left(I_{b,3}(u) + I_{b,4}(u)\right) \tag{3.1}$$

where:

$$I_{b,1}(u) = \int_0^\infty \int_0^{u+ct} \mathrm{e}^{-\delta t} \phi_b(u+ct-x)\,\mathrm{d}F_I(x,t);$$

$$I_{b,2}(u) = \int_0^\infty \int_{u+ct}^\infty \mathrm{e}^{-\delta t} W(u+ct, x-u-ct)\,\mathrm{d}F_I(x,t);$$

$$I_{b,3}(u) = \int_0^\infty \int_0^{u+ct} \mathrm{e}^{-\delta t} \phi_b(u+ct-x)\,\mathrm{d}F_M(x,t);$$

$$I_{b,4}(u) = \int_0^\infty \int_{u+ct}^\infty \mathrm{e}^{-\delta t} W(u+ct, x-u-ct)\,\mathrm{d}F_M(x,t)$$

To determine the integro-differential equation satisfied by the Gerber Shiu function in the risk model defined by relation (1.1), we adopt the following approach:

- The first loss occurs at time *t* before the surplus process reaches the barrier *b* ($t < \frac{b-u}{c}$). The amount *x* is such that $x < u + ct$.
- The first loss occurs at time *t* before the surplus process reaches the barrier *b* ($t < \frac{b-u}{c}$). The amount *x* is such that $x > u + ct$.

- The first loss occurs at time t after the surplus process has crossed the barrier b ($t > \frac{b-u}{c}$). The amount x is such that $x < b + (1-\theta)c(t-t_b)$.
- The first loss occurs at time t after the surplus process has crossed the barrier b ($t > \frac{b-u}{c}$). The amount x is such that $x > b + (1-\theta)c(t-t_b)$.

By conditioning on the time and amount of the first claim, and taking into account the different scenarios above, we have:

$$
\begin{aligned}
I_{b,1}(u) &= \int_0^{t_b}\int_0^{u+ct} \mathrm{e}^{-\delta t}\phi_b(u+ct-x)\,\mathrm{d}F_I(x,t) \\
&\quad + \int_{t_b}^{\infty}\int_0^{b+\varepsilon_1 c(t-t_b)} \mathrm{e}^{-\delta t}\phi_b\left(b+\varepsilon_1 c(t-t_b)-x\right)\mathrm{d}F_I(x,t)
\end{aligned}
\tag{3.2}
$$

where: $t_b = \frac{b-u}{c}$

$$
\begin{aligned}
I_{b,2}(u) &= \int_0^{t_b}\int_{u+ct}^{\infty} \mathrm{e}^{-\delta t}W(u+ct, x-u-ct)\,\mathrm{d}F_I(x,t) \\
&\quad + \int_{t_b}^{\infty}\int_{b+\varepsilon_1 c(t-t_b)}^{\infty} \mathrm{e}^{-\delta t}W\left(b+\varepsilon_1 c(t-t_b), x-b-\varepsilon_1 c(t-t_b)\right)\mathrm{d}F_I(x,t)
\end{aligned}
\tag{3.3}
$$

where: $\varepsilon_1 = 1-\theta$;

The copula C_I being the independent part of the Spearman copula, we have:

$$
\mathrm{d}F_I(x,t) = \lambda \mathrm{e}^{-\lambda t} f_X(x)\,\mathrm{d}x\mathrm{d}t \tag{3.4}
$$

By posing $I_b(u) = I_{b,1}(u) + I_{b,2}(u)$, and with using the relations (3.2)-(3.4), we have:

$$
\begin{aligned}
I_b(u) &= \int_0^{t_b}\int_0^{u+ct} \mathrm{e}^{-\delta t}\phi_b(u+ct-x)\lambda \mathrm{e}^{-\lambda t} f_X(x)\,\mathrm{d}x\mathrm{d}t \\
&\quad + \int_{t_b}^{\infty}\int_0^{b+\varepsilon_1 c(t-t_b)} \mathrm{e}^{-\delta t}\phi_b\left(b+\varepsilon_1 c(t-t_b)-x\right)\lambda \mathrm{e}^{-\lambda t} f_X(x)\,\mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^{t_b}\int_{u+ct}^{\infty} \mathrm{e}^{-\delta t}W(u+ct, x-u-ct)\lambda \mathrm{e}^{-\lambda t} f_X(x)\,\mathrm{d}x\mathrm{d}t \\
&\quad + \int_{t_b}^{\infty}\int_{b+\varepsilon_1 c(t-t_b)}^{\infty} \mathrm{e}^{-\delta t}W\left(b+\varepsilon_1 c(t-t_b), x-b-\varepsilon_1 c(t-t_b)\right)\lambda \mathrm{e}^{-\lambda t} f_X(x)\,\mathrm{d}x\mathrm{d}t
\end{aligned}
\tag{3.5}
$$

To simplify the notation of relation (3.4), we pose:

$$
\omega(u) = \int_u^{\infty} w(u, x-u) f(x)\,\mathrm{d}x;\ \sigma_b(u) = \int_0^u \phi_b(u-x) f(x)\,\mathrm{d}x + \omega(u) \tag{3.6}
$$

The relation (3.5) becomes:

$$
I_b(u) = \lambda \int_0^{\frac{b-u}{c}} \mathrm{e}^{-(\delta+\lambda)t}\sigma_b(u+ct)\,\mathrm{d}t + \lambda \int_{\frac{b-u}{c}}^{\infty} \mathrm{e}^{-(\delta+\lambda)t}\sigma_b\left(b+c\varepsilon_1\left(t-\frac{b-u}{c}\right)\right)\mathrm{d}t \tag{3.7}
$$

Let's move on to calculating integrals $I_{b,3}(u)$ and $I_{b,4}(u)$ in the relation

(3.1).

The copula support C_M is $D=\left\{(u,v)\in[0,1]^2: u=v\right\}$.

On the domain $[0,1]^2\setminus D$, $\frac{\partial^2 C_M}{\partial u\partial v}=0$; and on *D*, C_M is uniformly distributed.

Since the dependency structure is described by the copula C_M then they are monotonic and there is almost certainly an increasing function *l*, such that $X=l(V)$ (See Nelsen 2006 [6], page 27).

The distribution function of X is:

$$
\begin{aligned}
&F_X(x)=F_V\left(l^{-1}(x)\right)\\
&\Leftrightarrow 1-\mathrm{e}^{-\beta x}=1-\mathrm{e}^{-\lambda l^{-1}(x)}\\
&\Leftrightarrow -\lambda l^{-1}(x)=-\beta x\\
&\Leftrightarrow l^{-1}(x)=\frac{\beta x}{\lambda}
\end{aligned}
\tag{3.8}
$$

From relation (3.2) we deduce that: $l(t)=\frac{\lambda}{\beta}t$.

The joint distribution $F_{X,V}(x,t)$ of the random vector (X,V) is singular, whose support is the domain $D'=\left\{(x,t): F_X(x)=F_V(t)\right\}=\left\{(x,t): x=l(t)\right\}$.

Its distribution is $G(t)=F_M\left(l(t),t\right)=1-\mathrm{e}^{-\lambda t}$ on the domain $D'=\left\{(x,t): x=\frac{\lambda}{\beta}t\right\}$.

$$
\begin{aligned}
I_{b,3}(u)&=\int_0^{t_b}\int_0^{u+ct}\mathrm{e}^{-\delta t}\phi_b(u+ct-x)\mathrm{d}F_M(x;t)+\int_0^{t_b}\int_{u+ct}^{\infty}\mathrm{e}^{-\delta t}W(u+ct,x-u-ct)\mathrm{d}F_M(x;t)\\
&=\int_K \mathrm{e}^{-\delta t}\phi_b(u+ct-x)\mathrm{d}G(t)+\int_J \mathrm{e}^{-\delta t}W(u+ct,x-u-ct)\mathrm{d}G(t)
\end{aligned}
\tag{3.9}
$$

where:

$$
\begin{aligned}
K&=\left\{t\in\mathbb{R}^+: 0\le t\le\frac{b-u}{c}\ and\ 0\le x=\frac{\lambda}{\beta}t\le u+ct\right\}\\
&=\left\{t\in\mathbb{R}^+: 0\le t\le\frac{b-u}{c}\ and\ \left(c-\frac{\lambda}{\beta}\right)t\ge -u\right\}\\
&=\left\{t\in\mathbb{R}^+: 0\le t\le\frac{b-u}{c}\ and\ t\in\mathbb{R}^+\right\}\ car\ c>\frac{\lambda}{\beta}; t\ge 0\ and\ u>0
\end{aligned}
$$

Hence:

$$
K=\left[0;\frac{b-u}{c}\right]
\tag{3.10}
$$

$$
\begin{aligned}
J&=\left\{t\in\mathbb{R}^+: 0\le t\le\frac{b-u}{c}\ and\ u+ct\le x=\frac{\lambda}{\beta}t\right\}\\
&=\left\{t\in\mathbb{R}^+: 0\le t\le\frac{b-u}{c}\ and\ \left(\frac{\lambda}{\beta}-c\right)t\ge u\right\}
\end{aligned}
$$

$$\frac{\lambda}{\beta}-c<0; t\geq 0 \text{ and } u>0 \Rightarrow \left\{t\in\mathbb{R}^+ \text{ and } \left(\frac{\lambda}{\beta}-c\right)t\geq u\right\}=\varnothing$$

Hence:

$$J=\varnothing \tag{3.11}$$

By injecting relations (3.10) and (3.11) in the relation (3.9), we obtain:

$$I_{b,3}(u)=\int_0^{t_b}\mathrm{e}^{-\delta t}\phi_b(u+ct-x)\mathrm{d}G(t)$$

$$=\lambda\int_0^{t_b}\mathrm{e}^{-(\delta+\lambda)t}\phi_b\left(u+ct-\frac{\lambda}{\beta}t\right)\mathrm{d}t$$

$$I_{b,3}(u)=\lambda\int_0^{\frac{b-u}{c}}\mathrm{e}^{-(\delta+\lambda)t}\phi_b\left(u+\left(c-\frac{\lambda}{\beta}\right)t\right)\mathrm{d}t \tag{3.12}$$

where:

$$K'=\left\{t\in\mathbb{R}^+ : t\geq\frac{b-u}{c} \text{ and } 0\leq x=\frac{\lambda}{\beta}t\leq b+c\varepsilon_1(t-t_b)\right\}$$

$$=\left\{t\in\mathbb{R}^+ : t\geq\frac{b-u}{c} \text{ and } \left(\frac{\lambda}{\beta}-c\varepsilon_1\right)t\leq b-\varepsilon_1(b-u)\right\}$$

To guarantee solvency, it is assumed that: $c\varepsilon_1\geq\frac{\lambda}{\beta}\Rightarrow\frac{\lambda}{\beta}-c\varepsilon_1\leq 0$;

We also have: $t\geq 0$; $b-\varepsilon_1(b-u)\geq 0$ because $0<\varepsilon_1\leq 1$ and $b-u<b$.
Hence:

$$K'=\left\{t\in\mathbb{R}^+ : t\geq\frac{b-u}{c} \text{ and } t\in\mathbb{R}^+\right\}$$

Subsequently:

$$K'=\left[\frac{b-u}{c};+\infty\right[\tag{3.13}$$

$$J'=\left\{t\in\mathbb{R}^+ : t\geq\frac{b-u}{c} \text{ and } x=\frac{\lambda}{\beta}t\geq b+c\varepsilon_1(t-t_b)\right\}$$

$$=\left\{t\in\mathbb{R}^+ : t\geq\frac{b-u}{c} \text{ and } \left(\frac{\lambda}{\beta}-c\varepsilon_1\right)t\geq b-\varepsilon_1(b-u)\right\}$$

We have: $\frac{\lambda}{\beta}-c\varepsilon_1<0$; $t\geq 0$; $b-\varepsilon_1(b-u)$ hence:
$\left\{t\in\mathbb{R}^+ : \left(\frac{\lambda}{\beta}-c\varepsilon_1\right)t\geq b-\varepsilon_1(b-u)\right\}=\varnothing$.

We also have:

$$J'=\varnothing \tag{3.14}$$

By injecting relations (3.14) and (3.15) into relation (3.13), we have:

$$I_{b,4}(u)=\lambda\int_{\frac{b-u}{c}}^{\infty}\mathrm{e}^{-(\delta+\lambda)t}\phi_b\left(b+c\varepsilon_1\left(t-\frac{b-u}{c}\right)-\frac{\lambda}{\beta}t\right)\mathrm{d}t \tag{3.15}$$

By posing: $I_b^{\star}(u)=I_{b,3}(u)+I_{b,4}(u)$.

$$\begin{aligned}I_b^{\star}(u)&=\lambda\int_0^{\frac{b-u}{c}}\mathrm{e}^{-(\delta+\lambda)t}\phi_b\left(u+\left(c-\frac{\lambda}{\beta}\right)t\right)\mathrm{d}t\\&+\lambda\int_{\frac{b-u}{c}}^{\infty}\mathrm{e}^{-(\delta+\lambda)t}\phi_b\left(b+c\varepsilon_1\left(t-\frac{b-u}{c}\right)-\frac{\lambda}{\beta}t\right)\mathrm{d}t\end{aligned} \tag{3.16}$$

By relations (3.7) and (3.17) relation (4.1) becomes:

$$\begin{aligned}\phi_b(u)&=(1-\alpha)\left(\lambda\int_0^{\frac{b-u}{c}}\mathrm{e}^{-(\delta+\lambda)t}\sigma_b(u+ct)\mathrm{d}t+\lambda\int_{\frac{b-u}{c}}^{\infty}\mathrm{e}^{-(\delta+\lambda)t}\sigma_b\left(b+c\varepsilon_1\left(t-\frac{b-u}{c}\right)\right)\mathrm{d}t\right)\\&+\alpha\left(\lambda\int_0^{\frac{b-u}{c}}\mathrm{e}^{-(\delta+\lambda)t}\phi_b\left(u+\left(c-\frac{\lambda}{\beta}\right)t\right)\mathrm{d}t\right.\\&\left.+\lambda\int_{\frac{b-u}{c}}^{\infty}\mathrm{e}^{-(\delta+\lambda)t}\phi_b\left(b+c\varepsilon_1\left(t-\frac{b-u}{c}\right)-\frac{\lambda}{\beta}t\right)\mathrm{d}t\right)\end{aligned} \tag{3.17}$$

The relation (3.18) can be written as:

$$\begin{aligned}\phi_b(u)&=\lambda(1-\alpha)\int_0^{\infty}\mathrm{e}^{-(\delta+\lambda)t}\sigma_b\left((u+ct)\wedge\left(b+c\varepsilon_1\left(t-\frac{b-u}{c}\right)\right)\right)\mathrm{d}t\\&+\alpha\lambda\int_0^{\infty}\mathrm{e}^{-(\delta+\lambda)t}\phi_b\left(\left(u+\left(c-\frac{\lambda}{\beta}\right)t\right)\wedge\left(b+c\varepsilon_1\left(t-\frac{b-u}{c}\right)-\frac{\lambda}{\beta}t\right)\right)\mathrm{d}t\end{aligned} \tag{3.18}$$

By changing the variable to $s=b+c\varepsilon_1\left(t-\frac{b-u}{c}\right)$ *and* $s=u+\left(c-\frac{\lambda}{\beta}\right)t$ in the relation (3.19), we obtain:

$$\begin{aligned}\phi_b(u)&=\frac{\lambda(1-\alpha)}{c\varepsilon_1}\times\int_{b-\varepsilon_1(b-u)}^{\infty}\mathrm{e}^{-\left(\frac{\delta+\lambda}{c}\right)\left(\frac{s-b}{\varepsilon_1}+b-u\right)}\sigma_b\left(\left(b+\frac{s-b}{\varepsilon_1}\right)\wedge s\right)\mathrm{d}s\\&+\frac{\alpha\beta\lambda}{\beta c-\lambda}\times\int_u^{\infty}\mathrm{e}^{-\beta\left(\frac{\delta+\lambda}{\beta c-\lambda}\right)(s-u)}\phi_b\left(s\wedge\left(b+(\beta c\varepsilon_1-\lambda)\left(\frac{s-u}{\beta c-\lambda}\right)-\varepsilon_1(b-u)\right)\right)\mathrm{d}s\end{aligned} \tag{3.19}$$

Lemma 3.1: *The Gerber Shiu function in the risk model defined by relation* (1.1) *satisfies the following integro-differential equation*:

$$\begin{aligned}&\left(\mathcal{D}-\frac{\beta(\delta+\lambda)}{\beta c-\lambda}\ell\right)\left(\mathcal{D}-\frac{\delta+\lambda}{c}\ell\right)\phi_b(u)\\&=\left(\frac{\alpha\beta\lambda(\delta+\lambda)}{c(\beta c-\lambda)}\ell-\frac{\alpha\beta\lambda}{\beta c-\lambda}\mathcal{D}\right)\phi_b(u)+\left(\frac{\beta\lambda(1-\alpha)(\delta+\lambda)}{c(\beta c-\lambda)}\ell-\frac{\lambda(1-\alpha)}{c}\mathcal{D}\right)\sigma_b(u)\end{aligned} \tag{3.20}$$

Proof:

We derive $\phi_b(u)$ in relation (3.20) with respect to *u*.

$$\phi_b'(u)=\frac{\lambda(1-\alpha)}{c\varepsilon_1}\left(\frac{\delta+\lambda}{c}\right)\int_{b-\varepsilon_1(b-u)}^{\infty}\mathrm{e}^{-\left(\frac{\delta+\lambda}{c}\right)\left(\frac{s-b}{\varepsilon_1}+b-u\right)}\sigma_b\left(\left(b+\frac{s-b}{\varepsilon_1}\right)\wedge s\right)\mathrm{d}s$$
$$+\frac{\alpha\beta^2\lambda}{\beta c-\lambda}\left(\frac{\delta+\lambda}{\beta c-\lambda}\right)$$
$$\times\int_u^{\infty}\mathrm{e}^{-\beta\left(\frac{\delta+\lambda}{\beta c-\lambda}\right)(s-u)}\phi_b\left(s\wedge\left(b+(\beta c\varepsilon_1-\lambda)\left(\frac{s-u}{\beta c-\lambda}\right)-\varepsilon_1(b-u)\right)\right)\mathrm{d}s$$
$$-\frac{\lambda(1-\alpha)}{c}\sigma_b(u)-\frac{\alpha\beta\lambda}{\beta c-\lambda}\phi_b(u) \tag{3.21}$$

Noting by $\mathcal{D}$ and ℓ the respective differentiation and identity operators, let's calculate (3.20) and (3.22), let's calculate $g(u)=\left(\mathcal{D}-\frac{\delta+\lambda}{c}\ell\right)\phi_b(u)$.

$$g(u)=-\frac{\lambda(1-\alpha)}{c}\sigma_b(u)-\frac{\alpha\beta\lambda}{\beta c-\lambda}\phi_b(u)+\frac{\alpha\beta\lambda(\delta+\lambda)}{\beta c-\lambda}\left(\frac{\beta}{\beta c-\lambda}-\frac{1}{c}\right)$$
$$\times\int_u^{\infty}\mathrm{e}^{-\beta\left(\frac{\delta+\lambda}{\beta c-\lambda}\right)(s-u)}\phi_b\left(s\wedge\left(b+(\beta c\varepsilon_1-\lambda)\left(\frac{s-u}{\beta c-\lambda}\right)-\varepsilon_1(b-u)\right)\right)\mathrm{d}s \tag{3.22}$$

Let's derive $g(u)$ in relation (3.23) with respect to *u*.

$$g'(u)=-\frac{\lambda(1-\alpha)}{c}\sigma_b'(u)-\frac{\alpha\beta\lambda}{\beta c-\lambda}\phi_b'(u)+\frac{\alpha\beta^2\lambda(\delta+\lambda)}{\beta c-\lambda}\left(\frac{\beta}{\beta c-\lambda}-\frac{1}{c}\right)\left(\frac{\delta+\lambda}{\beta c-\lambda}\right)$$
$$\times\int_u^{\infty}\mathrm{e}^{-\beta\left(\frac{\delta+\lambda}{\beta c-\lambda}\right)(s-u)}\phi_b\left(s\wedge\left(b+(\beta c\varepsilon_1-\lambda)\left(\frac{s-u}{\beta c-\lambda}\right)-\varepsilon_1(b-u)\right)\right)\mathrm{d}s \tag{3.23}$$
$$-\frac{\alpha\beta\lambda(\delta+\lambda)}{\beta c-\lambda}\left(\frac{\beta}{\beta c-\lambda}-\frac{1}{c}\right)\phi_b(u)$$

Using relations (3.23) and (3.24), let's calculate
$h(u)=\left(\mathcal{D}-\frac{\beta(\delta+\lambda)}{\beta c-\lambda}\ell\right)\phi_b(u)$.

$$h(u)=-\frac{\lambda(1-\alpha)}{c}\sigma_b'(u)-\frac{\alpha\beta\lambda}{\beta c-\lambda}\phi_b'(u)+\frac{\beta\lambda(1-\alpha)(\delta+\lambda)}{c(\beta c-\lambda)}\sigma_b(u)$$
$$+\frac{\alpha\beta\lambda(\delta+\lambda)}{c(\beta c-\lambda)}\phi_b(u) \tag{3.24}$$

From relations (3.20), (3.22), (3.23), (3.24) and (3.25), we deduce relation (3.21) ■

4. Laplace Transforms of Gerber Shiu Functions $\phi_b(u)$ and the Probability of Ultimate Ruin

The aim of this section is to determine the Laplace transform of the Gerber Shiu functions $\phi_b(u)$ and the probability of ultimate failure in the risk model defined by relationship (1.1)

Lemma 4.1: *The Gerber Shiu fgunction* $\phi_b(u)$ *in the risk model defined by the relation* (1.1) *has Laplace transform* $\hat{\phi}_b(s)$ *given by:*

$$\hat{\phi}_b(s) = \frac{N_1(s)+N_2(s)}{D_1(s)+D_2(s)} \tag{4.1}$$

where:

$$N_1(s) = \left(\frac{\beta\lambda(1-\alpha)(\delta+\lambda)}{c(\beta c-\lambda)} - \frac{\lambda(1-\alpha)}{c}s\right)\hat{\omega}(s) + \frac{\lambda(1-\alpha)}{c}\omega(0)$$

$$N_2(s) = \left(s + \frac{\alpha\beta\lambda c-(\delta+\lambda)(2\beta c-\lambda)}{c(\beta c-\lambda)}\right)\phi_b(0) + \phi_b'(0)$$

$$D_1(s) = s^2 + \frac{\alpha\beta\lambda c-(\delta+\lambda)(2\beta c-\lambda)}{c(\beta c-\lambda)}s + \frac{\beta(\delta+\lambda)^2-\alpha\beta\lambda(\delta+\lambda)}{c(\beta c-\lambda)}$$

$$D_2(s) = \frac{\beta\lambda(1-\alpha)(\beta c-\lambda)s}{c(\beta+s)(\beta c-\lambda)} - \frac{\beta^2\lambda(1-\alpha)(\delta+\lambda)}{c(\beta+s)(\beta c-\lambda)}$$

Proof:

By posing:

$$K_1(u) = \left(\mathcal{D} - \frac{\beta(\delta+\lambda)}{\beta c-\lambda}\ell\right)\left(\mathcal{D} - \frac{\delta+\lambda}{c}\ell\right)\phi_b(u)$$

$$K_2(u) = \left(\frac{\alpha\beta\lambda(\delta+\lambda)}{c(\beta c-\lambda)}\ell - \frac{\alpha\beta\lambda}{\beta c-\lambda}\mathcal{D}\right)\phi_b(u) + \left(\frac{\beta\lambda(1-\alpha)(\delta+\lambda)}{c(\beta c-\lambda)}\ell - \frac{\lambda(1-\alpha)}{c}\mathcal{D}\right)\sigma_b(u)$$

Taking the Laplace transform of the two members of Equation (3.21), we have:

$$\int_0^\infty e^{-su}K_1(u)\,du = \left(s^2 - \frac{(\delta+\lambda)(2\beta c-\lambda)}{c(\beta c-\lambda)}s + \frac{\beta(\delta+\lambda)^2}{c(\beta c-\lambda)}\right)\hat{\phi}_b(s) + \left(\frac{(\delta+\lambda)(2\beta c-\lambda)}{c(\beta c-\lambda)} - s\right)\phi_b(0) - \phi_b'(0) \tag{4.2}$$

$$\begin{aligned}\int_0^\infty e^{-su}K_2(u)\,du = {} & \frac{\alpha\beta\lambda(\delta+\lambda)}{c(\beta c-\lambda)}\hat{\phi}_b(s) - s\frac{\alpha\beta\lambda}{\beta c-\lambda}\hat{\phi}_b(s) + \frac{\alpha\beta\lambda}{\beta c-\lambda}\phi_b(0) \\ & + \left(\frac{\beta\lambda(1-\alpha)(\delta+\lambda)}{c(\beta c-\lambda)} - \frac{\lambda(1-\alpha)}{c}s\right)\left(\frac{\beta}{\beta+s}\right)\hat{\phi}_b(s) \\ & + \left(\frac{\beta\lambda(1-\alpha)(\delta+\lambda)}{c(\beta c-\lambda)} - \frac{\lambda(1-\alpha)}{c}s\right)\hat{\omega}(s) + \frac{\lambda(1-\alpha)}{c}\omega(0)\end{aligned} \tag{4.3}$$

From relations (4.2) and (4.3), we deduce relation (4.1) ■

Theorem 4.1: *The Laplace transform of the probability of ultimate ruin in the risk model defined by relation* (1.1) *is given by:*

$$\hat{\psi}_b(s) = \frac{N_3(s)+N_4(s)}{D_3(s)+D_4(s)} \tag{4.4}$$

where:

$$N_3(s)=\frac{\beta\lambda^2(1-\alpha)}{c(\beta c-\lambda)(\beta+s)}-\frac{\lambda(1-\alpha)}{c(\beta+s)}s+\frac{\lambda(1-\alpha)}{c}\omega(0)$$

$$N_4(s)=\left(s+\frac{\alpha\beta\lambda c-\lambda(2\beta c-\lambda)}{c(\beta c-\lambda)}\right)\phi_b(0)+\phi_b'(0)$$

$$D_3(s)=s^2+\frac{\alpha\beta\lambda c-\lambda(2\beta c-\lambda)}{c(\beta c-\lambda)}s+\frac{\beta\lambda^2(1-\alpha)}{c(\beta c-\lambda)}$$

$$D_4(s)=\frac{\beta\lambda(1-\alpha)(\beta c-\lambda)s}{c(\beta+s)(\beta c-\lambda)}-\frac{(\beta\lambda)^2(1-\alpha)}{c(\beta+s)(\beta c-\lambda)}$$

Proof:

By posing $w(x,y)=1$, we have: $\hat{\omega}(s)=\dfrac{1}{\beta+s}$. By setting $\delta=0$ and $w(x,y)=1$, in the relation (4.1), we have the relation (4.4) ■

5. Probability of Ultimate Ruin

Lemma 5.1: *The Laplace transform of the ultimate probability of ruin can be written as*

$$\hat{\psi}_b(s)=\frac{k_1s^2+k_2s+k_3}{sd(s)} \tag{5.1}$$

where:

$$k_1=c(\beta c-\lambda)\phi_b(0)$$

$$k_2=\lambda(1-\alpha)(\beta c-\lambda)(\omega(0)-1)+\left(\alpha\beta\lambda c-\beta\lambda c+(\beta c-\lambda)^2\right)\phi_b(0)+c(\beta c-\lambda)\phi_b'(0)$$

$$k_3=\beta\lambda(1-\alpha)(\beta c-\lambda)\omega(0)+\left(\alpha\beta^2\lambda c-\beta\lambda(2\beta c-\lambda)\right)\phi_b(0)+\beta c(\beta c-\lambda)\phi_b'(0)$$

$$d(s)=cs^2(\beta c-\lambda)+s\left(\alpha\beta\lambda c-\beta\lambda c+(\beta c-\lambda)^2\right)-\beta\lambda(\beta c-\lambda)$$

Proof:

By multiplying the numerator and denominator of the relations (4.4) by $c(\beta+s)(\beta c-\lambda)$, on we obtain after simplification the desired result. ■

Theorem 5.1: *The probability of ultimate ruin in the risk model defined by relation* (1.1) *is explicitly expressed as follows*:

$$\psi_b(u)=\frac{\lambda(1-\alpha)(\beta c-\lambda)}{cR_2+\beta c}e^{R_1u} \tag{5.2}$$

where:

$$R_1=\frac{\beta c\lambda-\alpha\beta\lambda c-(\beta c-\lambda)^2-\sqrt{\left(\alpha\beta\lambda c-\beta\lambda c+(\beta c-\lambda)^2\right)^2+4\beta\lambda c(\beta c-\lambda)^2}}{2c(\beta c-\lambda)} \tag{5.3}$$

$$R_2 = \frac{\beta c\lambda - \alpha\beta\lambda c - (\beta c - \lambda)^2 + \sqrt{\left(\alpha\beta\lambda c - \beta\lambda c + (\beta c - \lambda)^2\right)^2 + 4\beta\lambda c(\beta c - \lambda)^2}}{2c(\beta c - \lambda)} \quad (5.4)$$

Proof:

The polynomial $d(s)$ in relation (5.1) is clearly a polynomial of degree 2 in s with discriminant $\Delta = \left(\alpha\beta\lambda c - \beta\lambda c + (\beta c - \lambda)^2\right)^2 + 4\beta\lambda c(\beta c - \lambda)^2 > 0$ and poles R_1 and R_2 given, by relations (5.3) and (5.4). (To note that $R_1 < 0$ and $R_2 > 0$; [9]).

The denominator of relation (5.1) is clearly a polynomial of degree 3 in s, while its numerator is a polynomial of degree 2. By simple element decomposition, the Laplace transform of the ultimate ruin probability $\hat{\psi}_b(s)$ in the relation (5.1) can therefore be expressed as:

$$\hat{\psi}_b(s) = \frac{c(A+B+C)s^2 + c(-AR_1 - AR_2 - BR_2 - CR_1)s + AcR_1R_2}{cs(s-R_1)(s-R_2)} \quad (5.5)$$

where: $A, B, C \in \mathbb{R}$; R_1 and R_2 given by the relations (5.3) and (5.4).

By identifying relations (5.1) and (5.5), we find:

$$A = \frac{k_3}{cR_1R_2} \quad (5.6)$$

$$C = \frac{cAR_1 + k_1R_2 + k_2}{c(R_2 - R_1)} \quad (5.7)$$

$$B = \frac{k_1}{c} - A - C \quad (5.8)$$

where: k_1, k_2 and k_3 are given in the relation (5.1).

Using the properties of the inverse Laplace transform, the probability of ultimate failure can therefore be expressed as:

$$\psi_b(u) = A + Be^{R_1u} + Ce^{R_2u} \quad (5.9)$$

where: A, B and C are respectively given by the relations (5.6), (5.7) and (5.8).

Since $\lim_{u\to+\infty}\psi_b(u) = 0$ (natural condition), we deduce that:

$$C = 0 \quad (5.10)$$

$$A = 0 \quad (5.11)$$

$$B = (\beta c - \lambda)\phi_b(0) \quad (5.12)$$

$$\phi_b(0) = \frac{\lambda(1-\alpha)}{\beta c + cR_2} \quad (5.13)$$

By injecting relations (5.10); (5.11); (5.12) and (5.13) into relation (5.9), we obtain the desired result. ■

Example. By fixing the parameters $c = 0.5$; $\lambda = 0.3$; $\beta = 1$; $b = 10$; using MATLAB we present the curves associated with the probabilities of failure associated with various values of the dependency parameter α (See **Figure 1**).

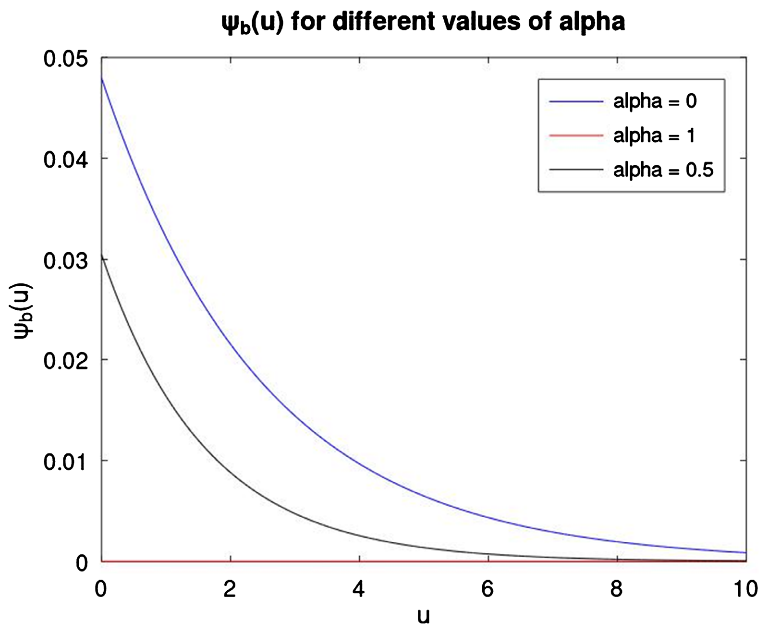

Figure 1. $\psi_b(u)$ for different values of alpha.

The probability of ruin $\psi_b(u)$ is the decreasing function of the dependence parameter α.

6. Conclusion

In this paper, we have determined the probability of ultimate ruin in a compound Poisson risk model with a partial shareholder dividend policy and a dependency between claim amounts and inter-claim times via the Spearman copula. In the remainder of this work, we will look at the applications of our results to insurance.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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