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Equity Value and Volatility

Moon Hoe Lee

Department of Finance and Management Science, Edwards School of Business, University of Saskatchewan, Saskatoon, Canada
Email: moonhlee@telus.net

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Abstract

As shown by continuous-time mathematics, a current stock price is the sum of the mean or equity value and the residual volatility of the current stock price. The residual volatility is a fraction of the volatility of the current stock price. Equity value is derived from the valuation of corporate and economic events. In a continuous-time first-order autoregressive process for a current demeaned stock price, valuation is completed when a lagged demeaned stock price is discounted. Volatility is present in a lagged demeaned stock price. Discounting a nominal lagged demeaned stock price converts it to equity value. A discounted model is a valuation model. The equity value from the valuation model is the sum of the mean stock price and the discounted lagged demeaned stock price. The valuation process starts from the process of mean reversion and ends at the process of autoregression. During mean reversion, the current demeaned stock price reacts to corporate and economic events. At autoregression, the lagged demeaned stock price is discounted completing valuation. My objective is to derive and test a valuation model under uncertainty. The residual volatility is produced by speculation. The residual volatility is a measure of stock market inefficiency, which is of topical interest. First-order autoregression of current demeaned stock prices was noticeably demonstrated at the start of the COVID-19 pandemic. The daily equity value represented 98.46% of the current S&P 500 in 2019. The proportion of daily equity value to the current S&P 500 was high. The inefficiency of a stock market is measured by the daily residual volatility of the current stock price. At the start of the COVID-19 pandemic, the S&P 500 market was 3.17% inefficient. The inefficiency was small in a stock market under great uncertainty.

Keywords

AR (1) Process, Continuous-Time, Demeaned Prices, Equity Value, Exponential Rate, Volatility

1. Introduction

I study the topics of equity value, discount rate, and volatility in the context of a current stock price. Equity value is created from the valuation of corporate and economic events. There is no known formula to calculate equity value in valuation under uncertainty. The present-value formula shows that valuation is a discounting process. Shiller [1] finds that the current stock price is volatile. It is not known how volatility is related to equity value or the current stock price. It is not known what is discounted.

In searching for a valuation process under uncertainty, I first define a demeaned stock price. Using continuous-time mathematics, I assume the differential of a current demeaned stock price is mean-reverting with volatility. The differential of a current demeaned stock price can be integrated to give a continuous-time first-order autoregressive process, denoted continuous-time AR (1) process. The continuous-time AR (1) process shows that a current stock price is the sum of the equity value and the residual volatility of a current stock price. The equity value includes a discount rate. Valuation under uncertainty is completed by discounting a lagged demeaned stock price. The outcome of valuation is equity value, which is equal to the mean stock price and the discounted lagged demeaned stock price.

The valuation process under uncertainty starts from the process of mean reversion and ends at the process of autoregression. Corporate and economic events move the current demeaned stock price during mean reversion. At autoregression, the lagged demeaned stock price is discounted. Discounting is valuation.

Since a current stock price is determined by valuation and speculation, deriving/testing a valuation model is like deriving/testing a pricing model. Speculation introduces the residual volatility of the current stock price.

The residual volatility of a current stock price is a measure of stock market inefficiency, which is of topical interest (Shiller [2]).

There is equity value in a current stock price only when a current demeaned stock price is first-order autoregressive, *i.e.*, when the exponential or discount rate is positive. *A positive discount rate for a lagged demeaned stock price is found in a continuous-time first-order autoregression of a current demeaned stock price. The autoregressive speed is a discount rate. The discount rate is endogenous. The lagged demeaned stock price is endogenous. The lagged demeaned stock price includes volatility. Discounting a nominal lagged demeaned stock price converts it to equity value.*

Shiller [1] tests the hypothesis that the S&P 500 is a forecast of the present value of the dividends on the S&P 500 stock portfolio. Shiller calls the forecast equation, expressed in real terms, the efficient markets model. The volatility test does not support the hypothesis.

By measuring the equity value using a continuous-time AR (1) process, I show that the equity value of a current stock price is a component of the current stock price. The equity value computed by the continuous-time AR (1) equation is in the present tense, the same as the current stock price. The equity value is compatible with the current stock price. An example of the calculation is given in Section 5.

Speculation on the future uncertain dividends on a firm's common stock leads to the volatility of the current stock price.

Interestingly, Fama and French [3] report that "average stock returns are related to the book-to-market equity ratio, B/M". Book value is a proxy for equity value.

Pricing of risky assets started in 1964 and has a long history. My survey of the literature is brief. Asset pricing has been based on models of expected or mean stock return. Asset pricing in the literature refers to the determination of mean stock returns and not the determination of stock prices. I model demeaned stock price instead of mean stock return. A current stock price with a discount rate and volatility of the current stock price can be obtained.

Sharpe [4] derives the first risky capital asset pricing model. The expected stock return in Sharpe's model is a function of the riskless interest rate and a market covariance risk premium. Several tests of the model follow. The model is not supported by U.S. data. However, Dessaint, *et al.* [5] report that Sharpe's [4] measure of a firm's expected stock return is commonly used to compute the firm's cost of equity, or the discount rate used in capital budgeting by the firm.

Black [6] modifies Sharpe's model by introducing restricted borrowing, and Ross [7] derives an arbitrage theory of asset pricing. Breeden [8] derives a consumption-based asset pricing model.

Among some other studies, Fama and French [3], on the other hand, extend Sharpe's [4] model by proposing empirical models. In the empirical models, Fama and French regress a reference stock portfolio return on additional factor stock portfolio returns.

Empirically, the volatility of stock prices, as measured by a continuous-time AR (1) process, is significant. The volatility of stock prices, measured differently, is recognized as important in the economic literature (Shiller [1], LeRoy and Porter [9]). The authors test models of pricing common stocks. The test models are present-value models. They are not models of expected stock return.

In the asset pricing literature, the volatility of residual stock return is suppressed (Sharpe [4]; Ross [7]; Breeden [8]; Fama and French [3]). In portfolio theory (mean-variance analysis), the volatility of residual stock return is assumed negligible when a common stock is held in a well-diversified stock portfolio. This is a property of portfolio theory. The models of expected stock return also belong to portfolio theory.

Estimates of the discount rate and the volatility at the start of the COVID-19 pandemic over a period of three months were obtained and tested for significance. The continuous-time AR (1) process was tested in an unsettled stock market with a small sample of data. Continuous-time mathematics show a current demeaned stock price is first-order autoregressive with volatility. Modeling explains the properties of U.S. stock prices. The two important properties of U.S. current demeaned stock prices are first-order autoregression and volatility.

Shiller [2] discusses volatility and speculation on the S&P 500. I derive a formula for the speculation risk premium, which depends on variance. The current stock price is speculative because a speculation risk premium is positive.

An AR (1) process has a half-life. The half-life is the time a current demeaned stock price halves its value to the stationary mean stock price. A first-order autoregressive demeaned price movement occurs during the half-life.

In a continuous-time AR (1) process, the implicit volatility $(1-\lambda)\nu$ is accounted for by a first-order autoregressive demeaned price movement due to changes in the valuation of a firm's common stock. Speculation on the current stock price S_t produces residual volatility $\lambda\nu$. Speculation risk is the residual volatility of the current stock price $\lambda\nu$. The speculation risk premium for a firm's common stock is $\gamma(\lambda\nu)^2$.

Black's [10] measure of a risk premium is a function of variance. The speculation risk premium is also a function of variance. In the estimation of the speculation risk premium in this paper, the parameter γ is a risk parameter. A numerical value is not assigned to the risk parameter γ.

My estimate of the annual volatility of the S&P 500 due to speculation is 16.44%. This finding provides empirical support for Shiller's [2] account of highly speculative stock prices. A counterbalance to speculation is substantial equity value, which is based on the valuation of a firm's common stock.

Demeaned S&P 500 is first-order autoregressive. The loss due to first-order autoregression is measured by volatility, $(1-\lambda)\nu$. The loss is due to changes in valuation.

Since current demeaned stock prices are normally distributed, current demeaned stock prices should be first-order autoregressive, so that the variance-covariance process for current demeaned stock prices is stationary with finite variance (Hamilton [11]). First-order autoregression is an equilibrating process, which ensures prices at any given point in time are equilibrium prices.

The current stock price is volatile. However, the current stock price will not be negative because there is a limit on the extent of market inefficiency. Market inefficiency is measured by the residual volatility of the current stock price. At the start of the COVID-19 pandemic, the daily S&P 500 market was 3.17% inefficient. The inefficiency was small in a stock market under great uncertainty.

An example of the price chart, which shows the volatility of the S&P 500, is given in the well-known papers by Shiller [1] [2]. The price chart shown in Shiller's [2] article is an annual plot of the S&P 500 from 1871 to 2013. I test the continuous-time AR (1) process on a more recent time series of the S&P 500.

Strong shocks to stock prices occur frequently. Strong shocks to stock prices can originate from trade tussles, imposition and lifting of tariffs, resolution of trade disputes, negotiations to free trade agreements, economic slowdown, economic growth, numerous corporate activities, domestic and foreign political unrests, threats of war, changes in Federal funds rate, changes in the prices of energy, changes in corporate income and personal income tax rates, budget standoffs, policy announcements, Federal Government shutdowns, U.S.-Mexico border crises, presidential elections, president's executive orders, congressional decisions, congressional elections, Supreme Court nominations and decisions, currency realignments, natural disasters, pandemics, and a host of other domestic and global shocks to the S&P 500.

External shocks have both positive and negative influences on stock prices and may explain the first-order autoregression of current demeaned stock prices.

I estimate the parameters of current demeaned stock prices as boundary-condition parameters and not as time-series parameters. Besides specifying the derived AR (1) process differently from the standard AR (1) process, problems, which are associated with the stability of time-series parameters, are avoided.

2. AR (1) Process with $\mathrm{e}^{-\theta}$

Decompose a current stock price S_t into a mean stock price μ and a current demeaned stock price X_t as follows:

$$S_t - \mu = X_t \tag{1}$$

The differential of X_t follows a process given by

$$\mathrm{d}X_t = -\theta X_t \mathrm{d}t + \sigma \mathrm{d}z \tag{2}$$

where $\theta > 0$ and $\sigma > 0$.

$$\mathrm{d}X_t \sim N\left\{-\theta X_t \mathrm{d}t, \sigma^2\right\}$$

In the above process (2), X_t is continuously mean reverting at a positive exponential rate θ. The mean of X_t is zero. The volatility is σ and $\mathrm{d}z \sim N(0,1)$.

The volatility σ is measured by the differential of X_t or the differential of S_t.

I derive a first-order autoregressive process from the differential of X_t given by Equation (2).

Change the variables to remove the drift.

$$Z_t = \mathrm{e}^{\theta t} X_t$$

Then

$$\mathrm{d}Z_t = \theta \mathrm{e}^{\theta t} X_t \mathrm{d}t + \mathrm{e}^{\theta t} \mathrm{d}X_t$$

$$\mathrm{d}Z_t = \theta \mathrm{e}^{\theta t} X_t \mathrm{d}t + \mathrm{e}^{\theta t}\left(-\theta X_t \mathrm{d}t + \sigma \mathrm{d}z\right)$$

$$\mathrm{d}Z_t = \mathrm{e}^{\theta t} \sigma \mathrm{d}z \tag{3}$$

The solution to the above stochastic differential equation is obtained by integrating both sides from s to t.

$$Z_t = Z_s + \sigma \int_s^t \mathrm{e}^{\theta q} \mathrm{d}z(q) \tag{4}$$

Reverse the change of variables.

$$X_t = \mathrm{e}^{-\theta t} Z_t$$

$$X_t = \mathrm{e}^{-\theta t} Z_s + \sigma \int_s^t \mathrm{e}^{-\theta(t-q)} \mathrm{d}z(q)$$

$$X_t = \mathrm{e}^{-\theta(t-s)} X_s + \sigma \int_s^t \mathrm{e}^{-\theta(t-q)} \mathrm{d}z(q) \tag{5}$$

Karatzas and Shreve [12] show an integration of the stochastic differential Equation (2) gives the solution (5), with $s = 0$. The stochastic integral (5) is not used in practice. However, the stochastic integral (5) is an estimable continuous-time AR (1) process with a discount rate and partial volatility. The autoregressive speed θ is a discount rate.

Set $s = t - 1$ and let the volatility ν be proportional to the current stock price S_t and $\nu > 0$. The volatility ν is dimensionless and serves as a dimensionless measure of volatility.

The volatility ν of the current stock price is measured by $\mathrm{d}X_t$ or $\mathrm{d}S_t$.

$$X_t = \mathrm{e}^{-\theta} X_{t-1} + \nu S_t \int_{t-1}^t \mathrm{e}^{-\theta(t-q)} \mathrm{d}z(q) \tag{6}$$

The choice of the stochastic differential Equation (2) is motivated by the fact that it can be integrated to give a continuous-time first-order autoregressive process. With a first-order autoregressive process, the variance-covariance process is stationary (Hamilton [11]). In this case, the estimate of volatility is finite. First-order autoregression ensures equilibrating demeaned prices, *i.e.*, prices at any given point in time are equilibrium prices. In addition, with a continuous-time AR (1) process, the autoregressive speed θ is a discount rate.

I evaluate the integral from Equation (6) to give the following continuous-time first-order autoregressive process:

$$X_t = \phi X_{t-1} + \lambda \nu S_t \mathrm{d}z \tag{7}$$

where $\phi = \mathrm{e}^{-\theta}$ and

$$\lambda = \int_{t-1}^t \mathrm{e}^{-\theta(t-q)} \mathrm{d}z(q)$$

$$\lambda = \sqrt{\frac{1 - \mathrm{e}^{-2\theta}}{2\theta}} \tag{8}$$

$$X_t \sim N\left\{\phi X_{t-1}, (\lambda \nu S_t)^2\right\}$$

The parameter, λ, given by Equation (8) is a number that falls within the range $0 < \lambda < 1$. A continuous-time AR (1) process is a process with a first-order autoregressive current demeaned price ($\lambda \neq 1$) and volatility ($\lambda \neq 0$).

Substitute X_t from Equation (7) for X_t in Equation (1) to give a continuous-time AR (1) process for ($S_t - \mu$). The continuous-time AR (1) process for ($S_t - \mu$) is given by the following:

$$S_t - \mu = \phi(S_{t-1} - \mu) + \lambda \nu S_t \mathrm{d}z \tag{9}$$

$$S_t - \mu \sim N\left\{\phi(S_{t-1} - \mu), (\lambda \nu S_t)^2\right\}$$

The parameter μ on both sides of Equation (9) is the mean of S_t from the same n observations. The sample size is *n*.

2.1. Equity Value

$$S_t = \mu + e^{-\theta}\left(S_{t-1} - \mu\right) + \lambda \nu S_t \mathrm{dz} \tag{10}$$

A current stock price is the sum of the equity value and the residual volatility of the current stock price. A mean stock price is equity value. A stock transaction adds an increment or a decrement to the equity value. The exponential factor, $e^{-\theta}$, discounts a lagged demeaned stock price to add equity value to a firm's common stock. The discount rate is endogenous. The residual volatility is a λ-fraction of the volatility of the current stock price.

The implicit volatility of the current stock price $(1-\lambda)\nu$ is due to changes in valuation. Residual volatility is noise and does not add equity value to a firm's common stock.

External economic shocks keep the equity value current by changing the discounted lagged demeaned stock price. The valuation of a firm's common stock is updated with every stock transaction. If it is assumed that at time *t*–1, X_{t-1} is a forecast of X_t and the forecast is discounted by $e^{-\theta}$, then the exponential factor $e^{-\theta}$ is a discount factor used in present-value calculations.

2.2. Estimation of Parameters

The boundary conditions of the continuous-time AR (1) process are $\theta > 0$ and $\nu > 0$. The two parameters θ and ν are restricted from being negative by transforming each parameter to e^{-x}. The exponential function is a continuous-time function. I estimate the exponential rate θ and the volatility ν in continuous time because I write the AR (1) process given by Equation (6) in continuous time.

The original parameters $\{\theta, \nu\}$ are exponentially transformed by letting $\theta = e^{-x_1}$ and $\nu = e^{-x_2}$. The original parameter $\mu = x_3$.

Maximum likelihood estimates of the three parameters $\{x_1, x_2, x_3\}$ are obtained by maximizing a log-likelihood function (see GAUSS [13]). The log-likelihood function is based on the following log-density function:

$$ld = \ln\left\{\frac{1}{\lambda \nu S_t \sqrt{2\pi}}\right\} - \left\{S_t - \mu - e^{-\theta}\left(S_{t-1} - \mu\right)\right\}^2 \Big/ 2\left(\lambda \nu S_t\right)^2 \tag{11}$$

The original parameters θ and ν can be recovered from their exponential functions. For example, the point estimate, $\breve{\theta} \cong e^{-\breve{x}}$. The numbering of x is suppressed. The maximum likelihood estimate of x is $\breve{x}$. The asymptotic standard error of $e^{-\breve{x}}$ is $\left|\frac{\mathrm{d}\left(e^{-\breve{x}}\right)}{\mathrm{d}\breve{x}}\right| \times \mathrm{S.E.}(\breve{x})$, where $\mathrm{S.E.}(\breve{x})$ is the standard error of $\breve{x}$. The t-statistic for $\breve{\theta}$ is $1/\mathrm{S.E.}(\breve{x})$. By eliminating the approximate mean $e^{-\breve{x}}$, the t-statistic for $\breve{\theta}$ depends on the standard error of the exponent $\breve{x}$ of the mean transformed parameter $e^{-\breve{x}}$. The value of the t-statistic is small if the value of the $\mathrm{S.E.}(\breve{x})$ is high.

The exponent x of e^{-x} has a normal distribution with mean $\bar{x}$ and variance

σ^2. In the derivation of a t-statistic for the point estimate $\breve{\theta}$, the numerical approximation of e^{-x} is used because it has a normal distribution with mean $e^{-\bar{x}}$ and variance $e^{-2\bar{x}}\sigma^2$. Note that $e^{-\bar{x}} < e^{-\bar{x}+0.5\sigma^2}$. The test is to show the lower bound is greater than zero because a t-statistic can be computed for the lower bound.

3. The Half-Life of an AR (1) Process

3.1. Standard AR (1) Process

Consider the following standard AR (1) process (Hamilton [11])

$$y_t = c + \phi y_{t-1} + \varepsilon_t \tag{12}$$

The variable y_t is a random variable, c is a constant, $0 < \phi < 1$, $\varepsilon_t \sim N\left(0, \sigma^2\right)$, and $t = 1, 2, \cdots, T$.

The stationary mean of an AR (1) process is $\bar{y} = c \div (1-\phi)$.

Replace c by $\bar{y}(1-\phi)$ in Equation (12) to give the following:

$$y_t - \bar{y} = \phi\left(y_{t-1} - \bar{y}\right) + \varepsilon_t \tag{13}$$

Re-write the above equation equivalently as follows:

$$x_t = \phi x_{t-1} + \varepsilon_t \tag{14}$$

The standard AR (1) process given by Equation (14) is different from the continuous-time AR (1) process given by Equation (6). The continuous-time AR (1) process has an exponential factor, $e^{-\theta}$, and an integral, $\int_{t-1}^{t} e^{-\theta(t-q)} dz(q)$. The integral will not sum to one if a demeaned stock price is first-order autoregressive. The standard AR (1) process is not robust.

3.2. Half-Life

The demeaned variable x_t is a measure of the value to the stationary mean $\bar{y}$. Assume at time $t+h$, the AR (1) process halves x_t to the stationary mean. Compute the half-life h so that

$$E_t\left(x_{t+h}\right) = \frac{1}{2} x_t$$

From Equation (14)

$$E_t\left(x_{t+h}\right) = \phi^h x_t$$

So that

$$\phi^h = \frac{1}{2}$$

Taking natural logarithm

$$h = -\ln(2)/\ln(\phi) \tag{15}$$

Setting $\phi = e^{-\theta}$ and $\theta > 0$, the half-life h varies inversely with θ, the autoregressive speed. The half-life h is an alternative measure of autoregressive speed. A longer half-life is equivalent to a slower autoregressive speed.

4. Data

I obtained the S&P 500 online from Yahoo! Finance. A sample of price data consists of the end of December closing S&P 500 of each year from 1990 to 2019. Yahoo! Finance adjusts the annual prices for stock splits.

A second sample of price data consists of the daily closing S&P 500. The prices are adjusted for stock splits. The daily price data are for two years, 2018 and 2019.

A quarterly sample of the daily closing S&P 500 is from March 2020 to May 2020. The sample period from March to May 2020 was in the early stage of the COVID-19 pandemic.

There were major shocks to market prices during the period. The major shocks to stock prices were the COVID-19 pandemic, the spread of the virus, the lockdown of the U.S. economy, and the passing of an initial U.S. congressional stimulus package of U.S. $2 trillion.

The ticker symbol for the S&P 500 is ^gspc. The websites for Yahoo! Finance [14] [15] are given in the references.

5. Empirical Evidence

According to the test Equation (9), the current demeaned stock price is normally distributed. The test equation shows that the exponential rate θ is estimated from the exponent of the autoregressive coefficient $e^{-\theta}$. The estimated ν is the volatility of the stock price.

The exponential rate θ estimated from 30 years of annual prices is an annual exponential rate. The volatility ν is annual volatility. The exponential rate θ estimated from one year of daily prices is a daily exponential rate each year. The estimated volatility ν is daily volatility.

The empirical research is to find out if the exponential rate θ and the volatility ν are statistically significant at the 5% level of significance. The mean stock price μ is tested to see if it is a significant statistic.

Table 1 shows the point estimates of the annual parameters of an AR (1) process with autoregressive coefficient $e^{-\theta}$ and the half-life of an AR (1) process. The data consist of the end of December closing S&P 500 adjusted for stock splits from 1990 to 2019. The annual exponential rate θ and the annual volatility ν are restricted from being negative. The critical value of the t-statistic for θ and ν is 1.70 at the 5% level of significance. The parameter μ is the mean S&P 500. The t-statistics for point estimates of the parameters are given within parentheses below the point estimates. The number of observations is n. The half-life of an AR (1) process is h years.

The annual exponential rate for the lagged demeaned S&P 500 as shown in **Table 1** was 43.14% from 1990 to 2019.

An application of the annual exponential rate is the estimation of an annual cost of equity for a firm. The current annual cost of equity, c, which is a *discount*

Table 1. Estimates of an annual exponential rate and volatility.

Security	Period	n	θ	ν	μ	h
S&P 500	1990-2019	30	0.4314 (2.01)	0.2009 (6.31)	1316 (4.75)	1.60

rate, can be estimated from $\mu + \mathrm{e}^{-\theta}\left(S_{t-1} - \mu\right) = D/c$, where D is the current annual dividend per share paid by a firm. The equation is the equity/present-value model. The above equation represents the equity value and the present value of a firm's common stock. The present value of a firm's common stock is equal to the present value of dividends on a firm's common stock. The annual cost of equity for a firm, which does not pay a dividend, is zero. A firm is a legal entity and incurs financing costs when it has equity and debt instruments in its capital structure.

The annual volatility ν of the S&P 500 was 20.09% from 1990 to 2019.

The half-life of the current demeaned S&P 500 with annual prices from 1990 to 2019 was 1.6 years.

Table 2 shows the annual speculation risk premium for the S&P 500 stock portfolio from 1990 to 2019, and a first-order autoregressive demeaned price movement measured as volatility $(1-\lambda)\nu$. The parameter ν is the annual volatility. The parameter λ is a function of θ the annual exponential rate. The annual volatility $\lambda\nu$ is the residual volatility of the continuous-time AR (1) process. The annual speculation risk premium for the S&P 500 stock portfolio is $\gamma(\lambda\nu)^2$, and γ is a risk parameter. The level of annual speculation on the S&P 500 is inferred from the speculation risk premium. Observations are the end of December closing S&P 500 adjusted for stock splits. The number of observations is n.

From Table 2, the annual volatility of the S&P 500 of 3.65% was due to changes in the valuation of the S&P 500 stock portfolio. The annual volatility of the S&P 500 of 16.44% was due to speculation on the S&P 500.

The stock price was more volatile (20.09%) than the equity value (3.65%). The extra volatility (16.44%) was due to speculation.

The annual speculation risk premium for the S&P 500 stock portfolio was γ 0.1644^2. The S&P 500 was speculative from 1990 to 2019.

Table 3 shows the point estimates of the daily parameters of an AR (1) process with autoregressive coefficient $\mathrm{e}^{-\theta}$ and the half-life of an AR (1) process. The time series is one year, and three months for the COVID-19 pandemic sample. The daily exponential rate θ and the daily volatility ν are restricted from being negative. The critical value of the t-statistic for θ and ν is 1.64 at the 5% level of significance for the samples of one year of the S&P 500 and 1.67 for the COVID-19 pandemic sample. The parameter μ is the mean S&P 500. The t-statistics for point estimates of the parameters are given within parentheses below the point estimates. Observations are daily closing S&P 500 adjusted for stock splits. The number of observations is n. The half-life of an AR (1) process is h days. Period: January-December 2018, January-December 2019, and March 2, 2020-May 29, 2020 for the COVID-19 pandemic sample.

Table 2. Annual speculation risk premium.

Security	Period	n	$\lambda\nu$	$\gamma(\lambda\nu)^2$	$(1-\lambda)\nu$
S&P 500	1990-2019	30	0.1644	$\gamma 0.1644^2$	0.0365

Table 3. Estimates of a daily exponential rate and volatility.

Security	Period	n	θ	ν	μ	h
S&P 500	2018	251	0.0491 (2.42)	0.0125 (21.88)	2746 (68.09)	14
S&P 500	2019	252	0.0583 (2.63)	0.0159 (21.83)	2913 (52.54)	11
S&P 500	COVID-19 Pandemic	63	0.1248 (1.86)	0.0338 (10.55)	2773 (26.67)	5

From Table 3, the daily exponential rate θ for the lagged demeaned S&P 500 was 4.91% in 2018 and 5.83% in 2019. The daily exponential rates, not reported in Table 3, were 4.93% in 2012, 3.72% in 2013, 3.13% in 2014, 6.66% in 2015, 2.35% in 2016, and 3.39% in 2017, and were significant at 5% level.

The exponential rate θ is estimated as a boundary-condition rate and is stable. The standard error for θ is small in each of the eight years. The time-series parameter ϕ is not estimated. The volatility ν shown in Table 3 is estimated as a boundary-condition parameter and is stable. The standard error for ν is small. The volatility ν is not estimated as a time-series parameter.

The exponential rate θ and the volatility ν are estimated as boundary-condition parameters over a thirty-year period and over a three-month COVID-19 pandemic. Standard errors for θ are small and standard errors for ν are small.

A daily exponential factor, $e^{-\theta}$*, serves the purpose of discounting a day's lagged demeaned S&P* 500 *and values the security*. For example, the equity value of the S&P 500 stock portfolio on December 31, 2019, is given by $\mu + e^{-0.0583}(S_{t-1} - \mu) = 2913 + 0.9433(3221 - 2913) = 3203$. The current S&P 500 on December 31, 2019 was 3230.

The daily volatility ν of the S&P 500 shown in Table 3 was 1.25% in 2018 and 1.59% in 2019.

An investment in the S&P 500 stock portfolio carries volatility risk. An increase in the exponential rate reduces the S&P 500 due to volatility risk. During the COVID-19 pandemic, the exponential rate for the lagged demeaned S&P 500 rose to a high rate (Table 3). The S&P 500 fell. The mean S&P 500 for the COVID-19 sample was 140 lower than the mean S&P 500 for the 2019 sample.

Table 3 shows the half-lives for the current demeaned S&P 500 over a sample period of one year. The half-lives for the current demeaned S&P 500, which ranged from 14 days in 2018 to 11 days in 2019, were short. The daily volatility of the S&P 500 due to changes in valuation was low, at about 0.05% (Table 4). The daily volatility of the S&P 500 due to speculation on the S&P 500 ranged

Table 4. Daily speculation risk premium.

Security	Period	n	$\lambda\nu$	$\gamma(\lambda\nu)^2$	$(1-\lambda)\nu$
S&P 500	2018	251	0.0122	$\gamma 0.0122^2$	0.0003
S&P 500	2019	252	0.0154	$\gamma 0.0154^2$	0.0005
S&P 500	COVID-19 Pandemic	63	0.0317	$\gamma 0.0317^2$	0.0021

from 1.22% in 2018 to 1.54% in 2019 (Table 4). The daily volatility of the S&P 500 was due to speculation on the S&P 500 in 2018 and 2019.

Table 4 shows the daily speculation risk premium for the S&P 500 stock portfolio, and a first-order autoregressive demeaned price movement measured as volatility $(1-\lambda)\nu$. The time series is one year, and three months for the COVID-19 pandemic sample. The parameter ν is the daily volatility estimated over one year. The parameter λ is a function of θ the daily exponential rate estimated over one year. The COVID-19 pandemic sample; λ is a function of the daily exponential rate estimated over three months, and ν is also estimated over three months. The daily volatility $\lambda\nu$ is the residual volatility of the continuous-time AR (1) process. The speculation risk premium for the S&P 500 stock portfolio is $\gamma(\lambda\nu)^2$, and γ is a risk parameter. The level of daily speculation on the S&P 500 is inferred from the speculation risk premium. Observations are daily closing S&P 500 adjusted for stock splits. The number of observations is n. Period: January-December 2018, January-December 2019, and March 2, 2020-May 29, 2020, for the COVID-19 pandemic sample.

As shown in Table 4, the speculation risk premiums for the S&P 500 stock portfolio from one year of data ranged from $\gamma 0.0122^2$ in 2018 to $\gamma 0.0154^2$ in 2019. The S&P 500 was speculative from 2018 to 2019.

An investment in the S&P 500 stock portfolio carries speculation risk. During the COVID-19 pandemic, speculation on the S&P 500 rose and residual volatility was higher than usual (Table 4). Traders were pessimistic ($\mathrm{d}z=-1$). The current S&P 500 was less than the equity value of the S&P 500 stock portfolio due to speculation risk.

Very major shocks occurred one after another during the sample period from March 2 to May 29, 2020. The World Health Organization declared the outbreak of COVID-19 a pandemic on March 11, 2020. The period, March-May 2020, is an exceptional period to study the market activity of the S&P 500.

The daily exponential rate θ for the lagged demeaned S&P 500 of 12.48% at the start of the COVID-19 pandemic is estimated over three months (Table 3). The daily volatility ν of the S&P 500 was 3.38% at the start of the COVID-19 pandemic. Both the exponential rate and the daily volatility were high at the start of the COVID-19 pandemic.

The half-life of 5 days was short. The daily volatility of the S&P 500 due to changes in the valuation of the S&P 500 stock portfolio was 0.21%. The daily volatility of the S&P 500 due to speculation on the S&P 500 was 3.17%. The daily volatility of the S&P 500 during the early stage of the COVID-19 pandemic was due to speculation on the S&P 500.

The daily speculation risk premium for the S&P 500 stock portfolio in the early COVID-19 pandemic market was $\gamma 0.0317^2$. The S&P 500 was over-speculative in trading at the start of the COVID-19 pandemic.

As shown in Table 1 and Table 3, the mean S&P 500 is a significant statistic of the current demeaned S&P 500. The mean S&P 500 is equity value. The daily and annual discount rates are positive. The discounted lagged demeaned S&P 500 is added equity value. The equity value and the residual volatility are components of a current stock price which is the principal finding.

6. Conclusions

I derive a continuous-time first-order autoregressive process for a current demeaned stock price. A current demeaned stock price shows the composition of the current stock price.

A current stock price is the sum of the equity value and the residual volatility of the current stock price. The residual volatility is a fraction of the volatility of the current stock price. The remaining volatility of the current stock price is due to changes in valuation. The mean stock price is equity value. A current stock price has an added value, which forms part of the equity value. The added value to a firm's common stock is computed by discounting a lagged demeaned stock price.

Equity value is created from the valuation of corporate and economic events. Valuation starts from the process of mean reversion, where corporate and economic events move the current demeaned stock price. Valuation ends at the process of autoregression, where the lagged demeaned stock price is discounted. Discounting is valuation.

The valuation is carried out using a continuous-time AR (1) process for a current demeaned stock price. The continuous-time AR (1) process is derived from mean reversion of a current demeaned stock price.

A continuous-time AR (1) process was tested on the S&P 500. The empirical results show that the mean S&P 500 and the discounted lagged demeaned S&P 500 are equity values. The autoregressive speed is a discount rate.

The inefficiency of a stock market is measured by the daily residual volatility of the current stock price. The S&P 500 market was 1.54% inefficient in 2019. The inefficiency was small.

Growth and value stocks are known to perform differently. A growth stock is identified by a high discount rate and high volatility. A value stock is identified by a low discount rate and low volatility.

Applications of determining the equity value of a current stock price are estimating an annual cost of equity for a firm, estimating the gains in synergies of mergers, any gain in the equity value of stock splits, and the gains in the equity value of the breakup of conglomerates.

Conflicts of Interest

The author declares no conflicts of interest.

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