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Perpetual American Call Option under Fractional Brownian Motion Model

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Abstract

In this paper, we consider perpetual American options under a fractional Brownian motion and give the closed-form solution for their value function. We discuss the pricing model when the underlying asset pays dividends continuously and derive the value functions. In order to get an analytical solution, we use the quadratic approximation method. By this approximation, we have Black-Scholes ordinary differential equation. Solving this equation with the boundary conditions, we get the value function and its optimal boundary.

Keywords

Option Pricing, Fractional Brownian Motion, Optimal Stopping Boundary

1. Introduction

In the Black-Scholes model, we suppose that the log return of the stock price is driven by geometric Brownian motion. However, there is a fat tail problem in this assumption. In order to solve this problem, Merton [1] considered that the stock price process has discontinuous points and derived a closed-form solution for perpetual American options. Kou and Wang [2] proposed that jump sizes have a double exponential distribution and Kou and Wang [3] presented the value function of perpetual American put options without a dividend for double exponential jump-diffusion processes. Rogers [4] pointed out that there is an arbitrage in the fractional Brownian motion model using pathwise integration. On the other hand, Hu and Øksendal [5] showed that there is no arbitrage under this model by the Wick product and derived the pricing formula of a European call option. Elliott and Chan [6] gave the closed-form solution for perpetual American put option without dividend. Xiao *et al.* [7] presented the pricing formula of currency options under a fractional Brownian motion with jumps. Barone-Adesi and Whaley [8] derived the value of American options by quadratic approximation and Barone-Adesi and Elliott [9] revised this approximation by the estimate for the error. In a fractional Brownian motion model literature, Funahashi and Kijima [10] examine an approximation for option pricing when the volatility follows a fractional Brownian motion. Guasoni, Nika and Rásonyi [11] studied the maximization of expected terminal wealth and Czichowsky *et al.* [12] showed the existence of a shadow price process.

In this paper, we derive the value function of the perpetual American put option divided by quadratic approximation and optimal stopping boundary under the fractional Brownian motion model. This paper is organized as follows. In Section 2, we set up a pricing model under fractional Brownian motion. Section 3 presents the value function of perpetual American call options with dividends and Section 4 gives numerical examples to verify analytical results.

2. Fractional Brownian Motion Model

Throughout this paper, let be $\left(\Omega,\mathcal{F},P,\{\mathcal{F}_t\}_{0\le t\le T}\right)$ a filtered probability space. In the following we consider the model for pricing option.

Definition 2.1. Gaussian process $B_H(t)$ on the probability space is called fractional Brownian motion with Hurst index $0<H<1$ if

1) $B_H(0)=0$;
2) $E\left[B_H(t)\right]=0, t\ge 0$;
3) $E\left[B_H(t)B_H(s)\right]=\frac{1}{2}\left\{t^{2H}+s^{2H}-|t-s|^{2H}\right\}, s,t\ge 0$.

Remark 2.1. If $H=\frac{1}{2}$, fractional Brownian motion coincides with Brownian motion.

Fractional Brownian motion has stationary increments and not independent ones. When $H>\frac{1}{2}$, $B_H(t)$ has the long distance memory. This means that covariance of $B_H(1)$ and $B_H(n+1)-B_H(n)$ satisfies

$$\sum_{n=1}^{\infty}Cov\left(B_H(1),B_H(n+1)-B_H(n)\right)=\infty.$$

We describe the pricing model. The risk-free asset price $S_0(t)$ at time t is determined by

$$\mathrm{d}S_0(t)=rS_0(t)\mathrm{d}t,\ S_0(0)=1,\ r>0,$$

where r is risk-free interest rate and a positive constant. The risky asset price $S(t)$ at time t satisfies the stochastic differential equation

$$\mathrm{d}S(t)=\mu S(t)\mathrm{d}t+\sigma S(t)\lozenge\mathrm{d}B_H(t),\ S(0)=x>0, \tag{1}$$

where μ and σ are positive constants, the mean of return and the volatility, respectively. $S(t)\lozenge\mathrm{d}B_H(t)$ represents wick product(See Hu and Øksendal [5]).

Remark 2.2. Since fractional Brownian motion is not semi-martingale if the Hurst index $H\ne\frac{1}{2}$, Therefore we cannot define the Ito-stochastic integral. See

Hu and Øksendal [5].

By Girsanov theorem for fractional Brownian motion,

$$\tilde{B}_H(t) = B_H(t) + \frac{\mu - r + \delta}{\sigma} \tag{2}$$

is a fractional Brownian motion, where $\delta > 0$ is dividend rate. By (2), we can rewrite (1) as

$$dS(t) = (r - \delta)S(t)\mathrm{d}t + \sigma S(t) \lozenge \mathrm{d}\tilde{B}_H(t). \tag{3}$$

Solving (3), we have

$$S(t) = x\exp\left\{(r-\delta)t - \frac{1}{2}\sigma^2 t^{2H} + \sigma\tilde{B}_H(t)\right\}.$$

From Hu and Øksendal [5], it follows that the value function $c(t, S(t))$ of European call option with the exercise price $K > 0$ and the maturity T is given by

$$c(t, S(t)) = S(t)\mathrm{e}^{-\delta(T-t)}N(d_1) - K\mathrm{e}^{-r(T-t)}N(d_2), \tag{4}$$

where $N(\cdot)$ is the cumulative standard normal distribution and

$$d_1 = \frac{\log\frac{S(t)}{K} + (r-\delta)(T-t) + \frac{1}{2}\sigma^2\left(T^{2H} - t^{2H}\right)}{\sigma\sqrt{T^{2H} - t^{2H}}},$$

$$d_2 = d_1 - \sigma\sqrt{T^{2H} - t^{2H}}.$$

If $H = \frac{1}{2}$, the value function $c(t, S(t))$ is equal to the Black-Scholes formula.

3. Perpetual American Options

Let $V_A(t,x)$ and $V_E(t,x)$ be the value function of American and European options respectively. Since it holds $V_A(t,x) \geq V_E(t,x)$, we can see that

$$V_A(t,x) = V_E(t,x) + \varepsilon(t,x),$$

where $\varepsilon(t,x) > 0$ is so-called early exercise premium. If the buyer does not exercise it, $V_A(t,x)$ and $V_E(t,x)$ satisfy the Black-Scholes partial differential equation. From this, Barone-Adesi and Whaley [8] obtained Black-Scholes ordinary differential equation by the quadratic approximation and gave the approximation of American options value. On the other hand, Elliott and Chan [6] derived the value function of perpetual American put option under a fractional Brownian motion model. Therefore, we give the value function of perpetual American call option with the dividend in this model.

Theorem 3.1. The value function $C(t,S)$ of perpetual American call option under fractional Brownian motion with the exercise price K is given by

$$C(t,S) = \begin{cases} \dfrac{S_C}{q_2}\left(\dfrac{S}{S_C}\right)^{q_2}, & 0 < S < S_C, \\ S - K, & S \geq S_C, \end{cases} \tag{5}$$

where S_C is the optimal boundary for the buyer and is represented by

$$S_C = \frac{K}{1 - \frac{1}{q_2}}, \tag{6}$$

and q_2 is the following.

$$q_2 = \frac{1}{2t^{2H-1}} \left\{ -\left(M_1 - t^{2H-1}\right) + \sqrt{\left(M_1 - t^{2H-1}\right)^2 + 4t^{2H-1} M_2} \right\}. \tag{7}$$

Proof. The early exercise premium $\varepsilon(t,x)$ satisfies the Black-Scholes partial differential equation

$$\frac{\partial \varepsilon}{\partial t} + (r - \delta) S \frac{\partial \varepsilon}{\partial S} + H\sigma^2 S^2 t^{2H-1} \frac{\partial^2 \varepsilon}{\partial S^2} - r\varepsilon = 0. \tag{8}$$

We assume that the early exercise premium $\varepsilon(\tau, S) = A(\tau) f(S, A)$, then we have the following.

$$S^2 (T - \tau)^{2H-1} \frac{\mathrm{d}^2 f}{\mathrm{d}S^2} + M_1 S \frac{\mathrm{d}f}{\mathrm{d}S} - M_2 f \left(1 + \frac{1}{rA} \frac{\mathrm{d}A}{\mathrm{d}\tau} + \frac{1}{rf} \frac{\mathrm{d}f}{\mathrm{d}A} \frac{\mathrm{d}A}{\mathrm{d}\tau} \right) = 0, \tag{9}$$

where $M_1 = \frac{r - \delta}{H\sigma^2}, M_2 = \frac{r}{H\sigma^2}, \tau = T - t$. If we have $A(\tau) = 1 - \mathrm{e}^{-r\tau}$, we can write (9) as

$$S^2 (T - \tau)^{2H-1} \frac{\mathrm{d}^2 f}{\mathrm{d}S^2} + M_1 S \frac{\mathrm{d}f}{\mathrm{d}S} - \frac{M_2 f}{A} - M_2 \frac{\mathrm{d}f}{\mathrm{d}A} (1 - A) = 0. \tag{10}$$

We approximate the partial differential Equation (10) by the differential equation

$$S^2 (T - \tau)^{2H-1} \frac{\mathrm{d}^2 f}{\mathrm{d}S^2} + M_1 S \frac{\mathrm{d}f}{\mathrm{d}S} - \frac{M_2 f}{A} = 0. \tag{11}$$

The general solution of (11) is expressed as

$$f(S) = a_1 S^{q_1} + a_2 S^{q_2},$$

where q_1 and q_2 are two solutions of the quadratic equation

$$(T - \tau)^{2H-1} q^2 + \left(M_1 - (T - \tau)^{2H-1} \right) q - \frac{M_2}{A} = 0$$

and

$$q_{1,2} = \frac{-\left(M_1 - (T - \tau)^{2H-1} \right) \pm \sqrt{\left(M_1 - (T - \tau)^{2H-1} \right)^2 + \frac{4(T - \tau)^{2H-1} M_2}{A}}}{2(T - \tau)^{2H-1}}.$$

Since we consider call option, a_1 is equal to 0. Therefore, the value function of American call option $C(\tau, S)$ is the following.

$$C(\tau, S) = c(\tau, S) + A a_2 S^{q_2}. \tag{12}$$

Let S_C be the optimal boundary. From (12), it follows that

$$\left. \frac{\partial C}{\partial S} \right|_{S = S_C} = \left. \frac{\partial c}{\partial S} \right|_{S = S_C} + A a_2 q_2 S_C^{q_2 - 1}.$$

Moreover, by smooth-pasting condition

$$1 = \frac{\partial c}{\partial S} + A a_2 q_2 S_C^{q_2 - 1}, \tag{13}$$

and

$$\left.\frac{\partial c}{\partial S}\right|_{S=S_C} = \mathrm{e}^{-\delta\tau} N\left(d_1\left(S_C\right)\right),$$

a_2 is given by

$$a_2 = \frac{1 - \mathrm{e}^{-\delta T} N\left(d_1\right)}{A q_2 S_C^{q_2 - 1}}. \tag{14}$$

Substituting (14) for value matching condition

$$S_C - K = c\left(\tau, S_C\right) + A a_2 S^{q_2}, \tag{15}$$

we have

$$S_C - K = c\left(\tau, S_C\right) + \frac{S_C}{q_2}\left\{1 - \mathrm{e}^{-\delta\tau} N\left(d_1\right)\right\}. \tag{16}$$

Therefore, we can approximate the value function of America call option with the finite maturity by

$$C\left(\tau, S\right) = \begin{cases} c\left(\tau, S\right) + \dfrac{S_C}{q_2}\left\{1 - \mathrm{e}^{-\delta\tau} N\left(d_1\right)\right\}\left(\dfrac{S}{S_C}\right)^{q_2}, & 0 < S < S_C, \\ S - K, & S \ge S_C. \end{cases} \tag{17}$$

When it goes to τ as $\to \infty$ in (16) and (17), we get (5) and (6). □

Similarly, we can obtain the value function of perpetual American put option with dividend.

Theorem 3.2. The value function $P\left(t, S\right)$ of perpetual American call option with the exercise price K is given by

$$P\left(t, S\right) = \begin{cases} K - S, & 0 \le S \le S_P, \\ -\dfrac{S_P}{q_1}\left(\dfrac{S}{S_P}\right)^{q_1}, & S > S_P, \end{cases} \tag{18}$$

where S_P is the optimal boundary for the buyer and is represented by

$$S_P = \frac{K}{1 - \dfrac{1}{q_1}}, \tag{19}$$

and q_1 is

$$q_1 = \frac{1}{2t^{2H-1}}\left\{-\left(M_1 - t^{2H-1}\right) - \sqrt{\left(M_1 - t^{2H-1}\right)^2 + 4t^{2H-1} M_2}\right\}. \tag{20}$$

In the following, we give numerical examples. We set parameters as $t = 0.5$, $H = 0.3$, $r = 0.2$, $\delta = 0.1$, $K = 100$, $S = 100$. Figure 1 and Figure 2 show that the optimal boundary from these figures, we can see that the optimal boundary is decreasing in the time t and concave function in Hurst index H. Figure 3 and Figure 4 demonstrate the value function of perpetual American option with the dividend. From these figures, we can recognize that $C\left(t, s\right)$ is convex and increasing function in stock price S and is decreasing in the time t.

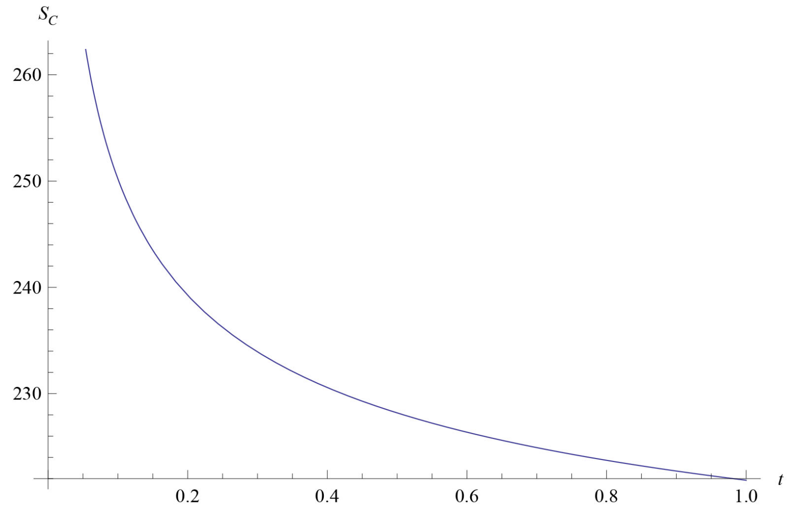

Figure 1. Optimal boundary S_C in t.

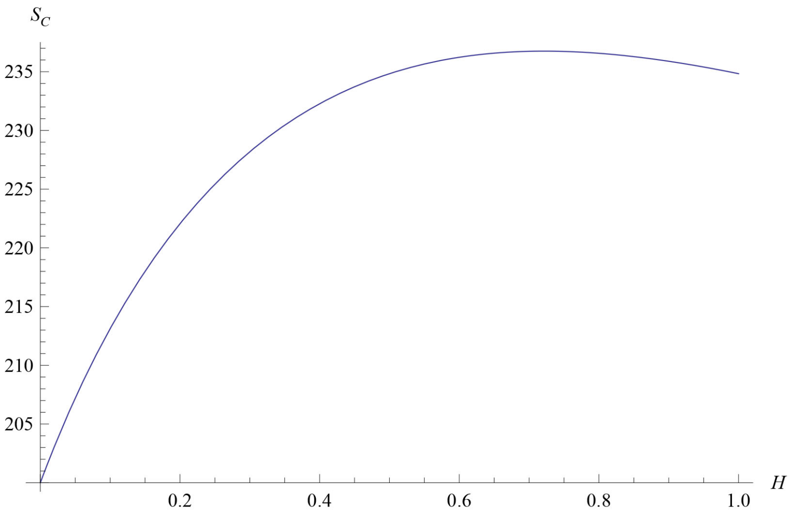

Figure 2. Optimal boundary S_C in H.

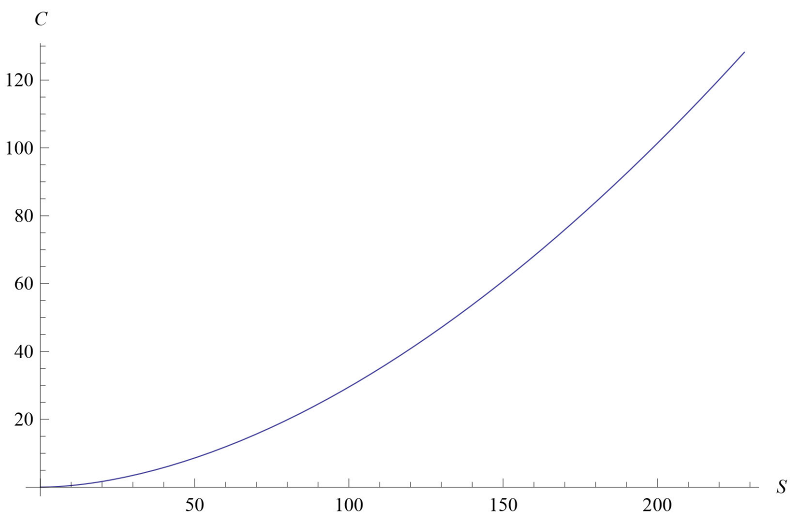

Figure 3. The value function C in S.

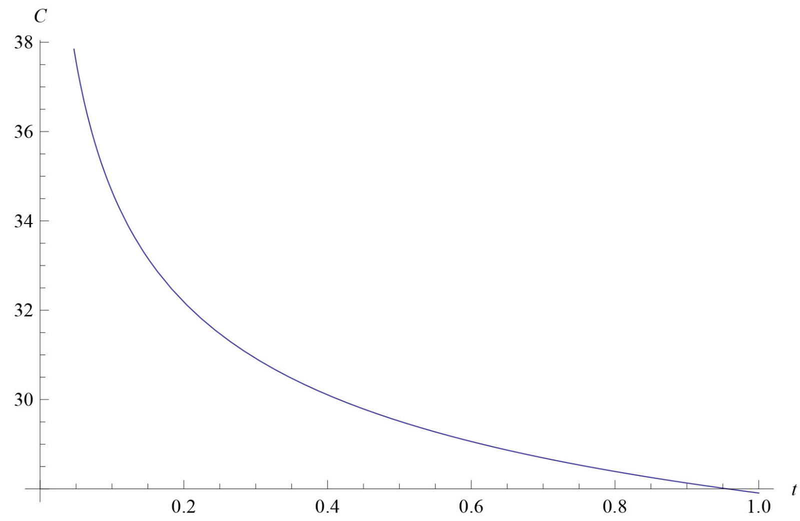

Figure 4. The value function C in t.

4. Concluding Remarks

In this paper, we discussed perpetual American options with dividends under the fractional Brownian motion model. Moreover, we obtained the value function of them and also explored some analytical properties of the value function and the optimal boundaries, which are useful to provide an approximation of the finite-lived American option. We apply the results to the valuation of convertible bonds. A convertible bond is a hybrid security that permits the investor to convert at any time until maturity. After the conversion, this bond has American call option features. We shall leave it as future work.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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