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Ruin Probability for Risk Model with Random Premiums

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Abstract

Based on Invariance Principle for Brownian Motion, we obtained a closed-form expression of the ruin probability for the Discrete-Time Risk Model with Random Premiums that was recently introduced by Korzeniowski [1]. We show that in this model, given two strategies that have the same probability of ultimate ruin, the strategy with larger initial capital and smaller loading factor is less risky than the strategy with smaller initial capital and larger loading factor in that it lowers the probability of ruin on the finite time horizon.

Keywords

Discrete-Time Risk Process, Random Walk with Drift, Invariance Principle, Probability of Ruin

1. Introduction

Considerations about uncertainties arising in the area of insurance and finance are often modeled by the following Risk Process:

$$U(t) = u + ct - \sum_{i=1}^{N_t} X_i$$

where $U(t)$ represents the capital available at time $t > 0$, given the initial capital $U(0) = u \geq 0$, after paying claims X_i which occurred at random times during the interval (0, t] according to a Poisson process N_t. The premium income stream ct is deterministic with premium rate c per unit of time. $U(t)$ is known as the Crámer-Lundberg model and represents the risk reserve of a company at time t. The main objective is to calculate the odds that the company reserve will ever become negative, referred to as the probability of ultimate ruin.

Except for a few special cases with closed-form solutions, the analysis of this process is usually carried out by numerical inversion of the associated Laplace

Transform to solve a renewal equation involving the probability of ruin.

Continuous time generalizations of the Cramer-Lundberg Model, where the deterministic premium stream ct is replaced by the Compound Poisson Premium Process [2] [3] provided estimates for the ultimate ruin probability. Furthermore, the Markov chains-based Discrete-Time Risk Model [4] considered expected dividend income until ruin without addressing ruin probabilities. On the other hand, in this paper, we establish explicit formulas for the ruin probability for both finite and infinite time horizons.

Following [1], we consider the Risk Process defined by

$$U_n = u + \sum_{i=1}^{n} Y_i - \sum_{i=1}^{n} X_i = u + \sum_{i=1}^{n} \left(Y_i - X_i \right), \quad Y_i \in [0,\infty), \; X_i \in [0,\infty), \tag{1.1}$$

where $\{Y_i\},\{X_i\}$ are i.i.d. of Random Premiums and Random Claims respectively. Risk Process U_n can be viewed as a random walk started at initial capital u at time 0.

Recall that the safety loading requires the expected value of the Risk Process gain $= \sum_{i=1}^{n} \left(Y_i - X_i \right)$ to be positive, for otherwise, the probability of eventual ruin is one. Therefore,

$$E\left[\sum_{i=1}^{n} \left(Y_i - X_i \right) \right] = n\left(EY - EX \right) = n\theta\mu, \text{ where } EY = (1+\theta)EX, \; EX = \mu, \tag{1.2}$$

where some positive θ is a safety loading factor.

U_n representation below will play a key role in establishing probability of ruin based on Invariance Principle. Namely, thanks to (1.1)-(1.2), we have

$$U_n = u + \theta\mu n - \sum_{i=1}^{n} Z_i, \text{ where } Z_i = X_i - Y_i + \theta\mu, \text{ with } EZ_i = 0, \; n = 1,2,\cdots, \tag{1.3}$$

which is a zero-mean random walk $\sum_{i=1}^{n} \left(-Z_i \right)$ with linear drift $\theta\mu n$ started at u.

2. Methodology

In this chapter, we will apply the celebrated Donsker Invariance Principle [5] for Brownian motion to derive the probability of ruin for the Risk Process on finite and infinite time horizons under the mild natural assumption of finite second moments for the claims X, and the premiums Y.

Typically, the Invariance Principle is stated on the Space of Continuous Functions $C[0,1]$ [6], however, our considerations require $C[0,\infty)$ and therefore the formulation here follows [7] (pp. 66-71). To this end consider $\{\xi_j\}_{j=1}^{\infty}$ i.i.d. with $E(\xi_j) = 0$, $Var(\xi_j) < \infty$, and $S_k = \sum_{j=1}^{k} \xi_j$, $S_0 = 0$. Define a continuous process $\{Y_t, t \ge 0\}$ obtained by linear interpolation

$$Y_t = S_{\lfloor t \rfloor} + t - \lfloor t \rfloor \xi_{\lfloor t \rfloor + 1}, t \ge 0, \tag{2.1}$$

where $\lfloor t \rfloor$ is the greatest integer less than or equal to t. Scaling appropriately both time and space consider a sequence of processes, $n \in \mathbb{N}$

$$X_t^n = \frac{1}{\sigma\sqrt{n}} Y_{nt},\ t \ge 0. \tag{2.2}$$

Donsker Invariance Principle (Theorem 9.20, [4]). *Let* $(\Omega, \mathcal{F}, P)$ *be the probability space on which* $\{\xi_j\}_{j=1}^{\infty}$ *is defined. Let* P_n *be the probability induced by* $\{X_t^n\}_{t\ge 0}$ *given by* (2.2) *on* $C[0,\infty), \mathcal{B}(C[0,\infty))$. *Then* P_n *converges weakly to the Wiener measure.*

The Wiener measure is the probability measure on $C[0,\infty)$ under which the coordinate mapping process $W_t(\circ) \equiv w(t)$ is the standard Brownian motion $B(t)$.

Equivalently,

$$X_t^n \Rightarrow B(t) \text{ as } n \to \infty, \tag{2.3}$$

where "$\Rightarrow$" stands for convergence in distribution.

Remark 2.4. Since $(t - \lfloor t \rfloor)\xi_{\lfloor t \rfloor + 1} \to 0$ in probability, (2.3) can be stated, due to (2.1) and (2.2), as

$$\frac{S_{\lfloor nt \rfloor}}{\sigma\sqrt{n}} \Rightarrow B(t) \text{ as } n \to \infty, \tag{2.5}$$

where, again, $\Rightarrow$ is convergence in distribution.

Turning to the Risk Process

$$U_n = u + \theta\mu n - S_n,\ n \in \mathbb{N},$$

where $S_n = \sum_{i=1}^n Z_i$, $Z_i = X_i - Y_i + \theta\mu$, $E(Z_i) = 0$, $E(X_i) = \mu$, $E(Y_i) = (1+\theta)\mu$, X_i, Y_i are i.i.d. and X_i independent of Y_i. Denote the variance of Z_i by

$$\begin{aligned}\sigma^2 &= Var(X_i - Y_i + \theta\mu)\\ &= Var(X_i) + Var(Y_i)\\ &= \sigma_X^2 + \sigma_Y^2,\end{aligned}$$

and set $\sigma = \sqrt{\sigma_X^2 + \sigma_Y^2}$.

First, we rename n as t and write

$$U(t) = u + \theta\mu t - S_t,\ t \in \mathbb{N}.$$

Then, we extend the above to

$$U(t) = u + \theta\mu t - S_t,\ t \ge 0, \tag{2.6}$$

where S_t is obtained by linear interpolation

$$S_t = S_{\lfloor t \rfloor} + (t - \lfloor t \rfloor) Z_{\lfloor t \rfloor + 1}.$$

Now, we rescale the time by a factor $\frac{1}{n}$, and consider a family of processes for $n \in \mathbb{N}$ corresponding to (2.6).

$$U^n(t) = \sqrt{n}u + \frac{1}{\sqrt{n}}\theta\mu t n - S_{\lfloor nt \rfloor} \tag{2.7}$$

$$= \sqrt{n}\left[u + \theta\mu t - \frac{S_{\lfloor nt \rfloor}}{\sqrt{n}}\right]. \tag{2.8}$$

Lemma 2.9 *Given the above notation and assumptions,*

$$\frac{U^n(t)}{\sqrt{n}} \Rightarrow u + \theta\mu t - \sigma B(t).$$

Proof. Apply Donsker Invariance Principle to

$$\sigma \frac{S_{\lfloor nt \rfloor}}{\sigma\sqrt{n}} \Rightarrow \sigma B(t).$$

Now, let

$$T_n = \inf\{t > 0 \mid U^n(t) \le 0\}$$

and observe that by (2.7) for all $n \in \mathbb{N}$

$$T_n = \inf\left\{t > 0 \mid u + \theta\mu t - \frac{S_{\lfloor nt \rfloor}}{\sqrt{n}} \le 0\right\}. \quad (2.10)$$

Define

$$T = \inf\{t > 0 \mid u + \theta\mu t - \sigma B(t) \le 0\}. \quad (2.11)$$

Before stating our main result we need the following fact concerning Brownian motion:

Fact 2.12 ([7], p. 196). *Given Brownian motion with drift* $W(t) = ct + B(t)$. *Let* τ *be the hitting time of the barrier* $a \ne 0$ *given* $c \ne 0$. *Then, the density*

$$f_\tau(t) = \frac{a}{\sqrt{2\pi t^3}} \mathrm{e}^{-\frac{a-ct}{2t}}, \; t \ge 0 \quad (2.13)$$

is Inverse Gaussian and

$$P(\tau \le t) = \int_0^t f_\tau(s)\,\mathrm{d}s,$$
$$P(\tau < \infty) = \mathrm{e}^{ca-|ca|},$$

where

$$\mathrm{e}^{ca-|ca|} = \begin{cases} 1 & \text{if } a \text{ and } c \text{ have the same sign,} \\ \mathrm{e}^{2ca} < 1 & \text{otherwise.} \end{cases}$$

The case $P(\tau < \infty) = 1$ corresponds to a drift c pointing toward the barrier, whereas $P(\tau < \infty) < 1$ corresponds to a drift c pointing in the direction opposite to the barrier, and signifies the fact that the density $f_\tau(t)$ is defective, *i.e.* $P(\tau = \infty) = 1 - \mathrm{e}^{2ca}$.

Theorem 2.14

$$\lim_{n\to\infty} P(T_n \le t) = P(T \le t) \quad (2.15)$$

$$= \int_0^t \frac{u}{\sqrt{2\pi s^3}} \mathrm{e}^{-\frac{(u+\theta\mu s)^2}{2s\sigma^2}} \mathrm{d}s. \quad (2.16)$$

Proof. The main benefit of the functional Central Limit Theorem, such as Donsker In-Variance Principle, is that continuous functionals of processes converging in distribution (here $U^n \Rightarrow u + \theta\mu t + \sigma B(t)$) also converge in distribution. The random variables T_n, T, defined by (2.10)-(2.11) and respectively, both satisfy $T_n \Rightarrow T$ because both are derived from a continuous map on the space of trajectories on $C[0,\infty)$. Namely,

$$\tau : C[0,\infty) \mapsto \mathbb{R}$$

defined by $\tau(x) = \inf\{t > 0 \mid X(t) \le 0\}$ if non-empty, and $+\infty$ otherwise, is measurable, and almost surely continuous with respect to $u + \theta\mu t + \sigma B(t)$. This establishes the equality (2.15). To show the second equality (2.16) notice that

$$\begin{aligned} T &= \inf\{t > 0 \mid u + \theta\mu t - \sigma B(t) \le 0\} \\ &= \inf\left\{t > 0 \mid \frac{u}{\sigma} \le -\frac{\theta\mu}{\sigma} + B(t)\right\}, \end{aligned}$$

whence by Fact 2.12 with $a = \frac{u}{\sigma}$, $c = -\frac{\theta\mu}{\sigma}$ the proof is complete.

Corollary 2.17

$$P(\text{ultimate ruin}) = P(T < \infty) = \mathrm{e}^{-\frac{2\theta\mu}{\sigma^2}u}. \tag{2.18}$$

Proof. Observe that the limiting distribution T of T_n corresponds to T for the original process $U(t)$ defined by 2.6).

We note that a similar analysis for the Cramer-Lundberg model $U(t) = u + ct - \sum_{i=1}^{N(t)} X_i$, with Poisson Process $N(t)$ was carried out by Iglehart [8] on the space $D[0,\infty)$ of right continuous left limit functions.

3. Ruin on Finite vs. Infinite Time Horizon

In this section, we will elaborate on some interesting consequences stemming from the ultimate ruin probability formula (2.18). In practice, risk models are typically considered equivalent whenever their respective ruin probabilities coincide. Turns out, however, as will be explained in an example below, that such notion of equivalence is quite misleading, when ruin probabilities on finite time horizon are not being taking into the account.

Example. Let the expected value of the claim size distribution be 1, that is, $\mu = 1$ and the standard deviation of the claims plus premium be 1, that is, $\sigma = 1$. Suppose we have two different Models A and B with the initial capital for A being $u = 24$ and the safety loading $\theta = 0.11$, whereas for B we have initial capital 48 and safety loading $\theta = 0.055$.

Then by (2.18) of the above corollary, we have

$$P(\text{ultimate ruin for } A) = \mathrm{e}^{-2(0.11)24} = 0.00509 = \mathrm{e}^{-2(0.055)48} = P(\text{ultimate ruin for } B).$$

Therefore, Model A and Model B both satisfy the same standard of having approximately $\frac{1}{2}$ of 1% probability of ultimate ruin. On the other hand, the probability of ruin on a finite time horizon for Model A and Model B correspond to very different Inverse Gaussian distributions. For Model A, the probability of ruin by time t corresponds to the following integral of the defective density of ruin probability shown in **Figure 1**.

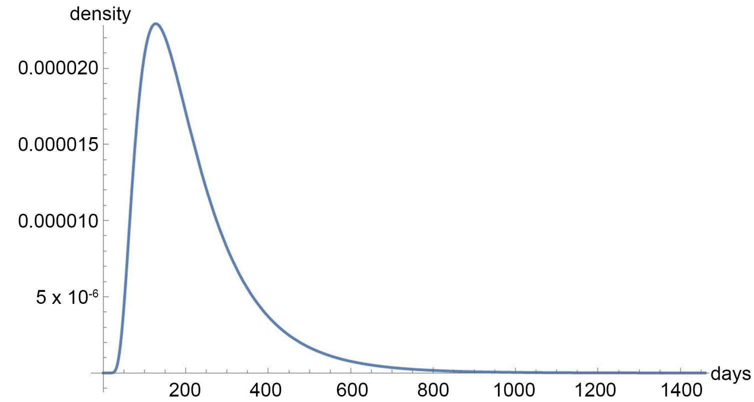

Figure 1. Defective density of ruin $f_\tau(t)=\frac{24}{\sqrt{2\pi t^3}}\mathrm{e}^{-\frac{(24+0.11t)^2}{2t}}$ for Model *A*.

$$\int_0^t \frac{24}{\sqrt{2\pi s^3}}\mathrm{e}^{-\frac{(24+0.11s)^2}{2s}}\mathrm{d}s.$$

We then see that after 4 years, or 1460 days, the probability of ruin in the tail for Model *A* is negligible. That is, we have already equaled the ultimate ruin probability since

$$\int_0^{1460} \frac{24}{\sqrt{2\pi s^3}}\mathrm{e}^{-\frac{(24+0.11s)^2}{2s}}\mathrm{d}s = 0.00509.$$

On the other hand, for Model *B*, it takes 4 times as long for the tail ruin probability to become negligible. The probability of ruin by time t corresponds to the following integral

$$\int_0^t \frac{48}{\sqrt{2\pi s^3}}\mathrm{e}^{-\frac{(48+0.055s)^2}{2s}}\mathrm{d}s.$$

After 4 years, we have

$$\int_0^{1460} \frac{48}{\sqrt{2\pi s^3}}\mathrm{e}^{-\frac{(48+0.055s)^2}{2s}}\mathrm{d}s = 0.00447,$$

and by year 20

$$\int_0^{5840} \frac{48}{\sqrt{2\pi s^3}}\mathrm{e}^{-\frac{(48+0.055s)^2}{2s}}\mathrm{d}s = 0.00509.$$

It follows that by year 20 the ruin probability on the finite time horizon for Model *B* equals its ultimate ruin probability. As a result, Model *A* is more risky than Model *B* in the first 4 years, despite the standard of regarding *A* and *B* as equivalent while failing to highlight this distinction. Figure 2 shows the difference.

The above example illustrates a general principle which we summarize as

Proposition 3.1 Let (u_A,θ_A) and (u_B,θ_B) be the initial capital and the safety loading factor for Models *A* and *B* respectively. If $u_A\theta_A = u_B\theta_B$, then Models *A* and *B* have the same probability of ultimate ruin, whereas for $u_B > u_A$

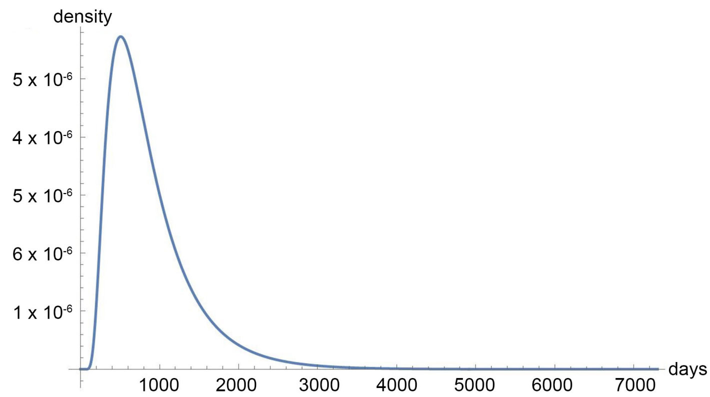

Figure 2. Defective density of ruin $f_\tau(t) = \frac{48}{\sqrt{2\pi t^3}} e^{-\frac{(48+0.055t)^2}{2t}}$ for Model *B*.

$$P(T_B \le t_A) < P(T_A \le t_A) \approx P(\text{ultimate ruin for } A) = e^{-\frac{2\theta\mu}{\sigma^2}u}, \tag{3.2}$$

where T_B, T_A are the respective times of ruin and t_A is chosen large enough to approximate

$$P(T_A < \infty) = P(\text{ultimate ruin for } A) = P(T_B < \infty) = P(\text{ultimate ruin for } B).$$

Proof. Thanks to formula (2.18) it remains to verify inequality (3.2). To that end we note that

$$\text{defective density} \quad f_\tau(t) = \frac{a}{\sqrt{2\pi t^3}} e^{-\frac{a-ct}{2t}}$$

achieves its maximum at

$$t = \frac{\sqrt{9+4a^2c^2} - 3}{2c^2}.$$

Since $a = \frac{u}{\sigma}$, $c = -\frac{\theta\mu}{\sigma}$, we have

$$t = \frac{\sqrt{9 + 4\frac{\mu^2}{\sigma^4}(u\theta)^2} - 3}{2\mu^2\theta^2}\sigma^2. \tag{3.3}$$

By assumption $u_A\theta_A = u_B\theta_B$ and $\frac{\theta_A}{\theta_B} = \frac{u_B}{u_A} > 1$.

This means $\theta_A > \theta_B$ and therefore the peak for $f_\tau(t)$ corresponding to Model *A* is attained at the smaller value of *t* than the value of *t* for the peak in Model *B*, thanks to (3.3) for $\theta = \theta_A$.

We have

$$f_\tau(t) = \frac{\frac{u}{\sigma}}{\sqrt{2\pi t^3}} e^{-\frac{\left(\frac{u}{\sigma}+\frac{\theta\mu}{\sigma}t\right)^2}{2t}},$$

so for Model *A*, with $\theta_A > \theta_B$, $f_\tau(t)$ falls-off faster than for Model *B*. Consequently, as illustrated in **Figure 3**, the tail of Model *B* is dominant and has some of its mass located to the right of t_A, which is the region of negligible probability for Model *A*.

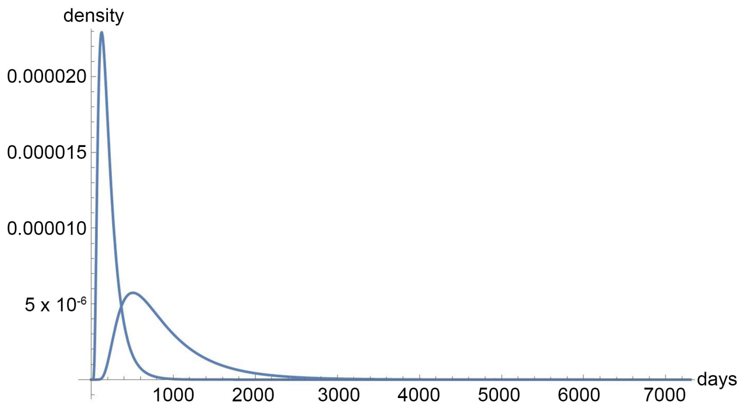

Figure 3. Combined plot of defective densities for Models *A* and *B*.

4. Conclusion

By deriving a closed-form expression for the probability of ruin in both the finite time horizon [0, t] and the infinite time horizon [0, ∞), we demonstrated that among strategies with the same ultimate ruin probability, which are characterized by having the same product of the respective initial capital and the loading factor, the strategy with the largest initial capital is least risky on the finite time horizon. Another key benefit of our Risk Process model is simplicity manifested by its discrete nature, which can be interpreted as an "end of the day model" for keeping track of daily premium income Y_i and liability payout X_i over any given period while adjusting the initial capital u and loading factor θ to minimize the probability of ruin based on formulas given.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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