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Ruin Probabilities in Finite Time

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Abstract

This paper considers the solution of the equations for ruin probabilities in finite time. Using the Fourier Transform and certain results from the theory of complex functions, these solutions are obtained as complex integrals in a form which may be evaluated numerically by means of the inverse Fourier Transform.

Keywords

Reserves, Fourier Transform, Inverse Fourier Transform, Analytic Functions, Cauchy's Theorem

1. Introduction

The ruin probability in infinite time has been extensively studied. Indeed it can be formulated in several cases. The same does not hold in finite time. [1] was the first to consider this problem; [2] considers the general case, which is difficult to compute, and [3] which is slightly less so. An extensive survey is found in [4].

This paper is focused on exact solutions, or general solutions which do not involve series computations. The ruin problem can be examined under a variety of conditions: with investment income, dividends, or time dependent claims distributions. For example, [5] discusses the situations where the reserve before ruin is different after ruin (but restricted to Erlang and Lindley distributions, say, due to investment income).

An integrated, general and computable approach would be useful, without the use of series expansions. This paper follows the philosophy of [6].

2. Basic Equations

Consider a risk business involving the following parameters:

- P is the rate of premium received per unit time;
- ζ is the stochastic variable measuring the amount of claim (given that a

claim has occurred) with probability density function $p(\zeta)$;
- u is the reserve held at any time;
- $\psi(u,t)$ is the probability of ruin of the business within time t, where the initial reserve was u at time $t=0$;
- $\varphi(u,t)$ is the corresponding probability of survival, with $\psi+\varphi=1$.

It will be seen that, since claim amount ζ is part of the change in reserve u, the symbols ζ and u will be used interchangeably. The framework is that of [6].

We define $L^1(R)$ as the space of Lebesgue integrable functions with finite norm

$$\|f\|_1=\int|f(\zeta)|\mathrm{d}\zeta<\infty.$$

Throughout this paper all integrals are taken to be defined in the sense of Lebesgue unless otherwise specified. We also consider later (in connection with the inverse Fourier Transform) the space of square integrable functions $L^2(R)$ with norm.

If $f\in L^1(R)$ and f is bounded, then it is clear that $f\in L^2(R)$.

In general we require that the claim amount density $p(\zeta)$ satisfy the conditions $p\geq 0$ for $\zeta\geq 0$, and $p=0$ for $\zeta<0$. In addition we require that the claim amount density satisfy

$$p(\zeta),\zeta p(\zeta),\zeta^2 p(\zeta)\in L^1(R).$$

These conditions are to ensure that the probability density of claims is sensible, and that it has a finite mean and variance. Additional restrictions on $p(\zeta)$ will be imposed as required.

Without loss of generality we may scale the claim amount ζ so that the mean claim is 1 and $\|\zeta f\|_1=1$. In practice this means that we take always the gross premium rate $P>1$.

We now consider how the method of [6] for deriving the survival probability ψ in infinite time may be extended to the finite time case, which satisfies:

$$\varphi_t=P\varphi_u-\varphi+p*\varphi.$$

This may be expressed in terms of the ruin probability ψ as follows:

$$\psi_t=P\psi_u-\psi-(p*1-1)+p*\psi \tag{1}$$

with

$$p*1-1=\int_u^\infty p(v)\mathrm{d}v=-g(u)\text{ say,}$$

and the FT of ψ_u by integration by parts is

$$\begin{aligned}\hat{\psi}_u&=\int \mathrm{e}^{iuz}\psi_u\mathrm{d}u=\mathrm{e}^{iuz}\psi\Big|_0^\infty-iz\hat{\psi}\\&=-\Lambda(t)-iz\hat{\psi},\text{ say}\end{aligned}$$

where $\Lambda(t)=\psi(t,0)$ introduces the role of time.

There are two approaches to the finite time problem. One is the classical of [3], the other is that of differential equations, relying on the FT wrt u the other on t.

Taking the FT of 1 wrt u:

$$\hat{\psi}_t = \eta\hat{\psi} + \hat{g} - P\Lambda \tag{2}$$

where

$$\eta = \hat{p} - 1 - izP \tag{3}$$

This may be written as

$$\frac{\partial}{\partial t}\left(\mathrm{e}^{-\eta t}\hat{\psi}\right) = \mathrm{e}^{-\eta t}\left[\hat{g}(z) - P\Lambda(t)\right] \tag{4}$$

Integrating over (t,∞) we get

$$\hat{\psi} = \frac{\hat{g}(z)}{\eta} - izP\mathrm{e}^{\eta t}\int_t^\infty \mathrm{e}^{-\eta s}\Lambda(s)\mathrm{d}s \tag{5}$$

Now we can express and take the inverse FT:

$$\psi(u,t) = \int \mathrm{e}^{-iuz}\left[\frac{\hat{g}(z)}{\eta} - izP\mathrm{e}^{\eta t}\int_t^\infty \mathrm{e}^{-\eta s}\Lambda(s)\mathrm{d}s\right]\mathrm{dz} \tag{ift}$$

We show that $\psi(u,t)$ is Hermitian in u:

$$\begin{aligned}\overline{\psi(u,t)} &= \int \mathrm{e}^{iu\bar{z}}\left[\frac{\hat{g}(-\bar{z})}{\eta(-\bar{z})} + izP\mathrm{e}^{\bar{\eta}t}\int_t^\infty \mathrm{e}^{\eta s}\Lambda(s)\mathrm{d}s\right]\mathrm{dz} \\ = \overline{\psi(u,t)} &= \int \mathrm{e}^{iu\bar{z}}\left[\frac{\hat{g}(-\bar{z})}{\eta(-\bar{z})} + izP\mathrm{e}^{\bar{\eta}t}\int_t^\infty \mathrm{e}^{\eta s}\Lambda(s)\mathrm{d}s\right]\mathrm{dz}\end{aligned} \tag{6}$$

The first term in is easily dealt with:

$$\begin{aligned}\int \mathrm{e}^{-iuz}\frac{\hat{g}(z)}{\eta}\mathrm{dz} &= \int \mathrm{e}^{-iuz}\frac{\frac{\hat{p}-1}{iz}}{\hat{p}-1-iPz}\mathrm{dz} \\ &= \int \frac{\mathrm{e}^{-iuz}}{iz}\frac{\hat{p}-1-iPz+iPz}{\hat{p}-1-Pz}\mathrm{dz} \\ &= \int \frac{\mathrm{e}^{-iuz}}{iz}\frac{\hat{p}-1-iPz+iPz}{\hat{p}-1-iPz}\mathrm{dz} \\ &= \int \frac{\mathrm{e}^{-iuz}}{iz}\left[1 + \frac{1}{\frac{\hat{p}-1}{iPz} - 1}\right]\mathrm{dz} \\ &= \int \frac{\mathrm{e}^{-iuz}}{iz}\left[1 + \frac{iz - P + 1 + P - 1}{P - 1 - iz}\right]\mathrm{dz} \\ &= \int \frac{\mathrm{e}^{-iuz}}{iz}\left[\frac{iP-1}{P-1-iz}\right]\mathrm{dz} \\ &= \delta(z)/i\end{aligned} \tag{7}$$

where $\delta(z)$ is the Dirac delta function. Thus the finite time probability rests on computing:

$$\int_t^\infty \mathrm{e}^{-iuz}\mathrm{e}^{-\eta(-\bar{z})s}\Lambda(s)\mathrm{dsdz} \tag{8}$$

Since ψ is bounded we know that $\hat{\psi}$ exists for $z \in C^{+}$, but not necessarily for $z \in \mathbb{R}$. The order in which the time derivative $(\hat{\psi})_t = \hat{\psi}_t$ is written is immaterial, as is seen by taking the time derivative as a limit and applying the dominated convergence theorem.

3. An Exponential Example

We have

$$\psi(u,t) = \int \mathrm{e}^{-iuz} \left[\frac{\hat{g}(z)}{\eta} - izP\mathrm{e}^{\eta t} \int_t^{\infty} \mathrm{e}^{-\eta s} \Lambda(s)\,\mathrm{d}s \right] \mathrm{d}z \tag{9}$$

with

$$\eta = \hat{p} - 1 - izP\Lambda(t).$$

In this case the FT wrt u becomes $\frac{\hat{g}(z)}{\eta} = \frac{P-1}{P} \mathrm{e}^{-\left(1-\frac{1}{P}\right)u}$. We need to compute $\int \mathrm{e}^{-iuz} \left[\frac{\hat{g}(z)}{\eta} \right] \mathrm{d}z$. This can be accomplished, with $\hat{g}(z) = \frac{\hat{p}(z)-1}{iz} = \frac{\frac{i}{i+z}-1}{iz} = i$.

Further,

$$\begin{aligned} \frac{1}{\eta} &= \frac{1}{\frac{i}{i+z} - 1 - iPz} \\ &= \frac{i+z}{-z + Pz - iPz^z} \\ &= \frac{i+z}{z(1 - Pz + iPz)} \\ &= \frac{A}{z} + \frac{B}{1 - P + iPz} \\ &= \frac{A(1-P+iPz) + Bz}{z(1-P+iPz)} \\ &= \frac{A(1-P+iPz) + Bz}{z(1-P+iPz)} \end{aligned} \tag{10}$$

so that

$$A = \frac{i}{P}$$

$$\begin{aligned} B &= 1 - \frac{i}{P\Lambda} i\Lambda P \\ &= 1 + \Lambda P \end{aligned}$$

so that

$$\frac{1}{\eta} = \frac{i}{zP} + \frac{2P}{1-iz}$$

and we have the following general FTs:

$$f = \frac{1}{2\pi} \int \frac{\mathrm{e}^{izu}}{\alpha + iz} \mathrm{d}z \tag{11}$$

$$\hat{f}(u) = \frac{1}{2\pi}\int \frac{ize^{izu}}{\alpha + iz}\mathrm{dz} \tag{12}$$

Hence

$$\frac{\partial}{\partial u}\left(\mathrm{e}^{au}\alpha f\right) = \frac{1}{2\pi}\int \mathrm{e}^{izu} = \mathrm{e}^{au}\delta(u) \tag{13}$$

so that since

$$\left|\begin{matrix} f & \hat{f} \\ \frac{1}{z} & \frac{1}{i\pi}sign(u) \\ \frac{1}{a+iz} & H(u)\mathrm{e}^{-au} \end{matrix}\right| \tag{14}$$

$$\int \mathrm{e}^{-iuz}\left[\frac{\hat{g}(z)}{\eta}\right]\mathrm{dz} = \int(1+i) + \left[2P\mathrm{e}^{-\mathrm{e}^{\Lambda}}\right]\mathrm{dz} \tag{15}$$

$$\mathrm{e}^{au}\alpha f = h(u)\mathrm{e}^{au} \tag{16}$$

$$\psi(u,t) = \int \mathrm{e}^{-iuz}\left[\frac{\hat{g}(z)}{\eta} - izP\mathrm{e}^{\eta t}\int_t^{\infty}\mathrm{e}^{-\eta s}\Lambda(s)\mathrm{ds}\right]\mathrm{dz} \tag{17}$$

Remark 1 *Generalized functions are also known as functionals, or distributions in the sense of Schwartz are readily discussed* [7] [8].

4. Conclusion

In this paper we have attempted to demonstrate how complex function theory enables an integrated approach to the solution of ruin probability problems. This has involved a heavy application of the Cauchy theorem for analytic functions. It might be noted that the solutions obtained are complex, but computable. If desired, Appendices A and B provide proof of $\Lambda(t)$ as in [1], but this may be omitted on first reading.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Beard, R.E. Pentikäinen, T. and Pesonen, E. (1969) Risk Theory. Chapman and Hall, London.

[2] Michna, Z. (2022) Ruin Probability on the Finite Time Horizon. Wrocław University of Economics. https://www.researchgate.net/publication/274359880_Ruin_probability_on_a_finite_time_horizon

[3] Seal, H.L. (1979) Stochastic Theory of a Risk Business. Wiley and Sons, Hoboken.

[4] Asmussen, S. (2000) Ruin Probabilities (Vol. 2). World Scientific, Singapore. https://doi.org/10.1142/2779

[5] Rebello, J.J. and Thampi, K.K. (2017) The Distribution of the Time of Ruin, the Surplus Immediately before Ruin and Deficit at Ruin under Two Sided Risk Renewal Process. *Journal of Mathematical Finance*, **7**, 624-632. https://doi.org/10.4236/jmf.2017.73032

[6] Leung, A.P. (2022) Ruin Probability and Complex Analysis. *Journal of Mathematical Finance*, **12**, 214-237. https://doi.org/10.4236/jmf.2022.121013

[7] Arsac, J. (1966) Fourier Transforms and the Theory of Distributions. Prentice-Hall, Hoboken.

[8] Zemanian, A.H. (1965) Distribution Theory and Transform Analysis. Dover, New York.

[9] Prabhu, N.U. (2012) Stochastic Storage Processes: Queues, Insurance Risk, Dams, and Data Communications (Vol. 15). Springer Science and Business Media, Berlin.

[10] Goodstein, R.L. (1965) Complex Functions. McGraw-Hill, New York.

[11] Bowers, N.L., Gerber, H.U., Hickman, J.C., Jones, D.A. and Nesbitt, C.J. (1986) Risk Theory. Education and Examination Committee of the Society of Actuaries, Itaxa, Illinois.

Appendix A

Equation (17) is a PDE of the first order in two dimensions, *u* and *t*, so should be solvable under boundary suitable conditions. The natural condition for *u* is that $\psi(t,0)=0$.

Letting $\theta \downarrow 0$ in (17) and using the boundary condition $\psi = \hat{\psi}(\theta) = \Lambda = 0$ for $t=0$, we immediately get $P\hat{\lambda} = \chi(0,z) = \hat{g}$. Hence it may be written as

$$\hat{\chi}(\theta,z) = \frac{P\hat{\lambda} - \hat{g}}{\eta(z) + i\theta}. \tag{18}$$

The function $\hat{\chi}$ must be finite at least for all $\theta, z \in C^{+}$. This follows from the fact that ψ must be bounded by 1, for it to have any physical significance. From Appendix B we know that for all $\theta \in C^{+}$ there exists a unique $\omega \in C^{+}$ such that $\theta = i(\omega)$.

From the property of FTs we deduce that

$$\hat{\lambda} = \frac{\hat{g}(\omega)}{P} \tag{19}$$

whenever $\theta = i\eta(\omega) \in C^{+}$.

This last result immediately gives us $\psi(\infty,u) \in L^{1}(R)$, as was required in the solution of the infinite time case. For $\hat{\lambda}(0) = \frac{1}{P}$ is implied by 19, whence

$$\hat{\chi}(0,z) = \frac{1-\hat{g}(z)}{\eta(z)} = \hat{\psi}(\infty,z) = \hat{\psi}(z)$$

This is a bounded function at $z=0$, so that $\|\hat{\psi}(\infty,z)\| = \|\hat{\psi}(0,z)\| < \infty$.

Remark 2 *Equations* (18) *and* (19), *expressed as Laplace Transforms, are attributed to* [3]. *The difficulty with these equations is that they depend implicitly on the relation* $\theta = i\eta(z)$, *which is required to be solved in order for the inverse FT to be applied. However it will be seen that Cauchy's theorem permits us to write these equations in a form which is more amenable to the inverse FT.*

In Figure 1 the contour of η passes through the origin since $\eta(0)=0$. As the parameter $\theta \to \infty$ we have $|z| \to \infty$, since $|\hat{p}|$ is bounded for $z \in C^{+}$. For large *z* we have approximately:

$$\theta = i(\hat{p} - 1 - iPz)\psi \approx -i + Pz$$

since $|\hat{p}| \to 0$ as $|z| \to \infty$ by the property of FTs. Thus $z = \frac{i+\theta}{P}$ for large θ. This gives $\Im(z) = \frac{1}{P}$ as an asymptote to Γ, as depicted above.

As a consequence of the asymptotic property of *r* above, we have $\hat{\lambda}(\theta) \in L^{2}(R)$, since $\hat{g}(z) \in L^{2}(R)$. This ensures that we can employ inverse FTs, at least for $L^{2}(R)$, in what follows. The solution for ψ in finite time is now discussed for two distinct cases of interest. The first case corresponds to using the inverse FT to obtain $\psi(t,0)$ from (19); the second case to obtain $\psi(t,u)$ for $u>0$. These cases need to be handled separately because of the discontinuity of ψ at $u=0$.

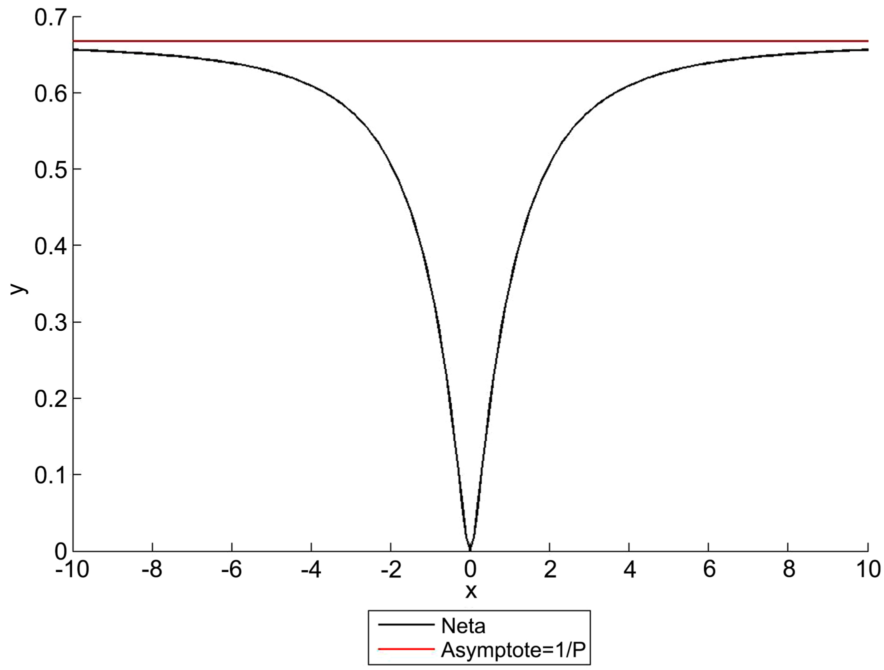

Figure 1. η for values of *z*.

Taking the inverse FT we then get:

$$\begin{aligned}\Lambda(t) &= \frac{1}{P} - \frac{1}{2\pi}\int \mathrm{e}^{-i\theta t}\frac{\hat{\lambda}(\theta)-\hat{\lambda}(0)}{i\theta}\mathrm{d}\theta \\ &= \frac{1}{P} + \frac{1}{2\pi P}\int_{\Gamma} \mathrm{e}^{i\eta}\frac{\eta'(z)}{z}\mathrm{dz} - \frac{P-1}{2\pi P}\int_{\Gamma}\frac{\mathrm{e}^{-i\theta t}}{i\theta}\mathrm{d}\theta\end{aligned} \tag{20}$$

where we have used the result in (20) and made the transformation $\theta = i\eta(z)$. Both integrals above exist as improper integrals; the second is evaluated along the real axis, whilst the first is evaluated along the contour $\Gamma \subset C^{+}$, parameterized by $\theta = i\eta(z) \in R$.

We show that the Γ integral appearing in (20) may be replaced by an integral along the real axis *R*. For this purpose we consider integration of the function $k(z) = \mathrm{e}^{i\eta}\frac{\eta'(z)}{z}$ along the closed contour bounded by the contour Γ, the vertical lines $z = \pm X$, and the horizontal line $R_{\varepsilon} = \{z : \Im(z) = \varepsilon\}$, for small $\varepsilon > 0$.

Now apply Cauchy's theorem to the integrand $k(z)$, which is analytic for $z \in C^{+}$. Let Γ_{ε} denote that section of Γ cut off by R_{ε}, at the point with real part *x*. As $|X| \to \infty$ the contribution along the vertical lines vanishes, as may be seen from the bounds $|\hat{p}| \le 1$, $|\hat{p}'| \le 1$ for all $z \in C^{+} \cup R$. Hence we get:

$$\int_{\Gamma_{\varepsilon}} k(z)\mathrm{dz} = \int_{R_{\varepsilon}} k(z)\mathrm{dz}$$

where the integral on R_{ε} is taken over $|\Re(z)| > x$.

As $\varepsilon \downarrow 0$ the left hand side of the above equality approaches the improper integral for Γ. It is also easy to show that the right hand side approaches the real line improper integral $\int k(z)\mathrm{d}z$ by means of the dominated convergence theorem.

This implies that $\Lambda(t)$ may be written solely in terms of real line integrals as:

$$\Lambda(t)=\frac{1}{P}+\frac{1}{2\pi P}\int_{\Gamma}\mathrm{e}^{i\eta}\frac{\eta'(z)}{z}\mathrm{d}z-\frac{P-1}{2\pi P}\int_{\Gamma}\frac{\mathrm{e}^{-iPtz}}{iz}\mathrm{d}z$$
$$=\frac{1}{P}+\frac{1}{2\pi P}\int_{\Gamma}\frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}z}\left[\mathrm{e}^{i\eta}-\mathrm{e}^{-iPtz}\right]\mathrm{d}z+\frac{1}{2\pi P}\int_{\Gamma}\frac{\mathrm{e}^{-iPzt}}{iz}\mathrm{d}z$$

Then integrating by parts, and noting that $\dfrac{\mathrm{e}^{i\eta}-\mathrm{e}^{-iPtz}}{z}$ is bounded at $z=0$, we get finally

$$\Lambda(t)=\frac{1}{P}+\frac{1}{2\pi Pt}\int\mathrm{e}^{-iPzt}\left[\frac{\mathrm{e}^{i(t\hat{p}-1)}-1}{z^2}+\frac{t}{iz}\right]\mathrm{d}z.$$

This last expression leads to the formula for ruin, with zero reserve, attributed to [9] by [3]:

Proposition 3 *The finite time probability for zero reserve is equal to*:

$$\psi(t,0)=\frac{1}{Pt}\int_{Pt}^{\infty}\int_{x}^{\infty}f(t,\zeta)\mathrm{d}\zeta\mathrm{d}x\quad \textit{for}\ \ t>0 \tag{21}$$

where $f(t,\zeta)$ is the probability density of total claim amount ζ in a finite time interval $(0,t)$.

Proof. To prove the equivalence of the equality for $\Lambda(t)$ in (20) and the expression above, we show that the appropriate FTs are the same. Since two functions having the same FT must be equal (almost everywhere) this would then prove that the expressions are equal if they are both continuous. Using a well known result for generating functions [1], the FT of $f(t,\zeta)$ is given as $\mathrm{e}^{t(\hat{p}-1)}$ for Poisson distributed claim frequency with parameter 1. The FT, with respect to *u*, of the function

$$f_1(t,u)=\int_{x}^{\infty}f(t,\zeta)\mathrm{d}\zeta$$

is thus $\dfrac{\mathrm{e}^{t(\hat{p}-1)}-1}{iz}$, which has the value t at $z=0$. The FT, with respect to *u*, of the function

$$f_2(t,u)=\int_{x}^{\infty}f_1(t,\zeta)\mathrm{d}\zeta$$

is thus given as $-\dfrac{\mathrm{e}^{t(\hat{p}-1)}-1}{z^2}-\dfrac{t}{iz}$, which is precisely the integrand appearing in the expression for $\Lambda(t)$ in §5.13, after putting $u=Pt$.

Appendix B

Proposition 4 (*a*) *For any* $\theta\in C^{+}\cup R$ *the equation* $\theta=i\eta(z)$ *has a unique* *solution* $z \in C^+$.

(*b*) *If* $\hat{p}$ *can be analytically continued to a neighborhood of* $z = 0$, *then there exists a root of* $\eta(z)$ *with* $z \in C^-$ *and* $\Re(z) = 0$. *In addition this root has the smallest modulus of all roots in* C^-.

Proof. (a) We first demonstrate the proposition for $\theta = 0$. The function η clearly has a root at $z = 0$. To show that it is unique in C^+ define the function

$$g(z) = \int_u^\infty p(v)\mathrm{d}v \in L^1(R) \tag{22}$$

We have $\|g\|_1 = 1$ and $\hat{g} =$ so that:

$$|g| = \left|\frac{\hat{p}-1}{z}\right| \le 1 < P \quad \text{for} \quad z \in C^+ \cup R,$$

from which the result follows.

If $\theta \in C^+$ then the circle γ lies completely within C^+, whereas if $\theta \in R$ then it touches the real axis at $x = re\dfrac{\theta}{P}$. In either case, $|\hat{p}| \le 1$ for $z \in C^+ \cup R$ the property of FTs, so that $\eta(z)$ cannot have a zero outside γ.

In the case of $\theta \in C^+$ it is clear that a closed curve $\Gamma \subseteq C^+$ may be constructed surrounding γ, on which holds the inequality:

$$|\hat{p}| \le 1 < |1 + iPz - i\theta|.$$

Hence by Rouché's theorem ([10], §8,2), the function $1 + iPz - i\theta$ has precisely the same number of zeros within Γ as $1 + iPz - i\theta - \hat{p}$. But it is easily shown that the former function has precisely one such zero, from which the result follows for $\theta \in C^+$. (Note that this also gives the proof where $\theta \in R$, but only if $\hat{p}\left(\dfrac{\theta}{P}\right) < 1$.)

In the case of $\theta \in R$ we use a continuity argument to establish the existence of a root of $\eta(z) = i\theta$. Let $\{i\theta_n \in C^+, n = 1, 2, 3, \cdots\}$ be a sequence such that $\theta_n \to \theta$. Then from the previous case, there exist $\{z_n\}$ such that $\eta(z_n) = i\theta_n$. Now the sequence $\{z_n\}$ is bounded and hence must have a limit point *z* with $z \in C^+ \cup R$. If necessary we can construct a convergent subsequence so that $z_n \to z$ say. Since the function η is continuous, we have $\eta(z) = i\theta$, which proves existence of a root. To show uniqueness, let ω be another zero, so that we have:

$$\hat{p}(z) - \hat{p}(\omega) = iP(z - \omega).$$

Using the same argument as for the proof of part (a), we consider in place of $p(u)$ the function $\mathrm{e}^{iu\omega} p(u) \in L^1(R)$ and the related function

$$g(u) = \int_u^\infty \mathrm{e}^{iv\omega} p(v)\mathrm{d}v \in L^1(R).$$

We have $|\hat{g}|_1 \le 1$, which yields the inequality:

$$|\hat{p}(z) - \hat{p}(\omega)| \le |z - \omega| < P|z - \omega|.$$

This implies that $z=\omega$ and thus uniqueness of the zero in the case $\theta \in R$.

(b) It is important to note that not all functions p satisfy the condition stated, for example the Pareto distribution $p(u)=\frac{9}{8}u^{-4}$ for $u>3/2$ does not[1]

The FT $\hat{p}(z)$ for $re(z)=0$, $\theta \in C^{+}$ corresponds to the moment generating function of p; it may be shown by considering the derivative of $\eta(z)$ at $z=0$ [11] that an appropriate root $z=i\xi$ for $\xi<0$ exists. Part of this proof demonstrates that the inequality

$$\hat{p}(iy)<1-Py$$

holds for $\xi<y<0$. It is clear the same inequality applies to $z=x+iy$ since

$$\left|\hat{p}(z)\right| \le \hat{p}(iy)<1-Py \le \left|1+iPz\right|.$$

Thus η has no roots in the region $\xi \le \Im(z) \le 0$ apart from 0 and $i\xi$.

[1]In fact a necessary and sufficient condition for $\hat{p}$ to be analytically continued at the origin $z=0$ is that the moments $M_n=\int u^n p(u)\mathrm{d}u<\infty$, and that $\frac{M_n^{1/n}}{n}$ is uniformly bounded for $n \ge 0$. In this sense p is short-tailed, as it must converge to 0 sufficiently fast $u \to \infty$.