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Maximum Principle for Rotating Banack-Hausdorff Stocks

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Abstract

Suppose the loss-gain path of a stock say m is extended to enter the gain state of another stock n through its loss state. The path for $m \times n$ could explain the dynamics of bi-perfect production centers within the Hausdorff topological frame with completeness. We identify the rotation path for two independent stock investments and state its finance characteristics and implications.

Keywords

Rotation, Knot, Banack-Hausdorff Space, Stationary

1. Introduction

In business finance, managers are challenged by the n-investment problem in both solution techniques, analyses, and applications. The case of investor A with stocks in two distinct investments valued at $V_s(t)$ and $V_b(t)$ respectively suffices. Suppose the said investor fails to rotate his investments by their time-dependent performance behaviours within the limit of exercise $t_i : t \le T; 1 \le i \le N$ so that

$$V_s(t_2) + V_b(t_2) < V_s(t_1) + V_b(t_1). \tag{1}$$

In continuous trading times, the sequence $\{V\}$ for his joint value distribution is a decreasing sequence in $\mathbb{R}$ a.s. The loss-in-value magnitude becomes more pronounced as $V_s(t)$ and $V_b(t)$ decrease simultaneously in time leading to business discontinuities over $t \le T$; making matters worst for investors, managers, and owners of business enterprises with such stock holdings. For this portfolio problem, continuous loss streaks posit the need for care in understanding the true impact of macroeconomic variables like rotating stock holdings for proper investing and high-level operational management.

The investor described above may choose to rotate his stocks using poor models out of an obsession to rotate. In this case, conflict of operational time for purposeful management may be set into statutory operations. Here, an investor may fail to rotate at an optimal time $t \le T$ when rotating the said stocks is at its best. The poor model adopted predicts wrong τ when it is non-profitable for better business practice. For clarity purposes, suppose an investor rotates at τ^- such that $V\left(t=\tau^-\right)$ decreases, given the initial value $V\left(t=\tau\right)$. It is clear that

$$V\left(\tau\right)-V\left(\tau^-\right)=\varepsilon\left(\tau^-\right)>0 \tag{2}$$

The greater the $\varepsilon\left(T\right)>0$, the greater the loss-in-value of his joint investment. Consequently, it is imperative to save investment resources by constructing rotation models that minimize loss-in-value of investment resources for the sake of operations research. In this respect, this article designs a rotation strategy for 2 stocks trading within a Hausdorff topological frame with complete rotation paths for business benefits and management.

The 2-stock problem studied in this work is key in both framework and application because of the number of likelihoods an investor faces relative to decision-making for profit maximization and cost minimization. The serious case is when one stock is making a profit at t and the other stock is making a loss at t. This investment dynamic is overall, the larger expectation over $\mathbb{R}$ in view of the economic notion of where to invest.

The goal of this article is to assist investor A arrives at the best possible decision through cost and loss minimisations and analysis principally due to the art of rotating traded stocks under completeness known for Banack algebras in Mathematical analysis. The literature covering these considerations and designs is not extensively covered especially under the path-breakage and distribution designed and analyzed in this work.

2. Literature Review

Vannestal [1] rotated two assets in trade and obtained the maximal value function given that one asset price X changes according to the SDE.

$$\mathrm{d}X_t=\alpha X_t\mathrm{d}t+\sigma X_t\mathrm{d}W_t. \tag{3}$$

Here, $\mathrm{d}B_t=rB_t\mathrm{d}t$ and α is a measure of the growth rate of the stock and σ is the volatility parameter. X_t and B_t are stock and bond prices at time t. The analysis showed that, if α is not higher enough compared to the interest rate r, one should rotate to sell at a given threshold implying that rotating stock prices grow at a continuously compounded rate of α at t.

Merton [2] rotated stocks and t-bills assets to investigate the maximum path to the investor's utility. The generalized Wiener process was considered in capturing key aspects of the stock price. The biggest shortfall is the inability of the model to capture the percentage return required by investors when the stock asset is rotated independently of the stock's price. Bjork [3] analyzed price movements of two rotated stocks and obtained the density functions of the associated Wiener process. Using the central limit theorem, it was shown that market noises can conveniently be modeled as Poisson processes. Bjork [3] also looked at the stationary rotation timing of derivative purchases in incomplete markets.

Here, an investor attempts to maximize the spread between assets offered at market price through stationary timing. Both the investor and the market value of the options are assumed to have a risk-neutral probability and expectations under different equivalent martingale measures representing varying market views. The structure of the resulting stationary stopping problem depends on the interaction between the respective market price of risk and the option pay-off. In particular, a crucial role is played by the delayed purchase premium related to the stochastic bracket between the market price and the buyer's risk premia.

Explicit characterization of the rotation timing is presented for two representative classes of Markovian models which are equity models with local intensity and diffusion stochastic volatility models. In this respect, Shiryaev [4] posited that the geometric Brownian motion is a good model for conducting stock rotation and analysis. In line with both Bjork [3] and Shiryaev [4], Winkel [5] posited that maximum strategies could generate significant gains when used as rotation models.

Bensoussan *et al.* [6] compared different rotation timings; see also Thomakos [7] and Rishel [8]. In the former case, the rotation is based not only on a risk-free and risky asset, but also between pairs of risky assets. Additionally, asymmetric response terms for the relative returns on the pair of assets being rotated are calculated. Dai and Zhong [9] also examined the predictability and profitability of a similar market timing approach and obtained similar results to those listed above.

Moalosi-Court [10] rotated two company stocks with additional stochastic income and derived the rewarding process for stock numbered $\alpha_i, i = 1,2$ units at the rate $n\mathcal{L}(t, Y_t) \geq 0$. The analysis showed that stochastic posing is good for the business of rotating stocks and investments and provides results that are analytically tractable.

Zhang [11] provided a variational inequality sufficient condition for optimal stopping problems during asset rotation. The result was illustrated by computing solutions to the optimal stock rotating problem generally. The decision on when to rotate stocks that have rapid growth in value and then rapidly decline in value is considered one important assumption leading to the construction of an idealized model with both positive and negative growth rates.

Feller [12] studied the stationary behavior of two assets rotating in two distinct markets using the pairs rotation strategy. The observed process for each asset is assumed to be a sequence of the random walk $Y_t : t = 0, \pm 1, \pm 2, \cdots$ leading to continuous time analysis. A simulation of the random walk with 100 observations was used. The work concluded that the relationships between the two assets are more likely to hold over a shorter period of time than over a long period. In addition, the pairs rotation strategy was most effective when the market showed a higher trend and higher volatility in pricing.

Olofsson [13] studied the optimal rotation problem using backward stochastic differential equations and variational inequalities. A proof for comparison principle for sub and super solutions is provided. Again, the existence of limiting solution to iteratively defined interconnected obstacle problems is provided. Again, Olofsson [13] constructed certain backward SDEs in which the solutions are reflected to stay inside a time-dependent domain. The driving process is of Wiener-Poisson type allowing finite jumps.

By a penalization technique, the existence of a solution when the bounding domain has convex and non-increasing time slices was proved. Uniqueness is proved by an argument based on Ito's formula that proves the convergence of a scheme when the underlying processes fit into the framework of the Kalman-Bucy filter. The work proves that the value of information is positive and that the value function under incomplete information converges to that under full information when the noise in the observation vanishes.

From related literature, it is clear that the rotation of stocks and investments has been studied extensively. The idea of rotation time densities and expectations is important in deciding when to rotate for selling stocks and even when to buy for higher future profits. The concerns on when to rotate and when not to rotate under variance performance are at the heart of investors; Lamba [14] and missing in most research works reviewed in this area of operational management. This article fills the gap by providing path analysis that incorporates the expected rotation time idea in the analysis of rotating stocks for better yields. The methodology used is unique and easy to understand for the benefits of trade and investment.

3. Methodology

Suppose $q_1 = 1,2,3,\cdots,\mathbb{N}$ and $q_2 = 1,2,3,\cdots,\mathbb{N}$ are stock sizes carried by an investor in markets m and n in the presence of other $N \in \mathbb{N}$ investors such that, the said investor enters m from state N and leaves m from state $N+1$ as shown in **Figure 1** below.

Consider a rotation process $(Y_t, \phi_t)_{t \ge 0}$ where Y_t denotes the number of rotations at time t and ϕ_t is the remaining time to complete the last rotation at m from n. Let $j+1$ denote the present rotation state at the boundary of market m. Clearly, state $(j+1)(n)$ drops to state $j(n)$ upon departures from n and state $j(m)$ adds a single rotation $(j+1)(m)$ upon successive arrivals. This transition continues in time and space until $(Y_t, \phi_t)_{t \ge 0}$ is stationary, time-independent process (Y, ϕ). By the ergodic theorem for stationary Markov processes, there exists an invariant measure P such that

$$P(Y_{t+1} = j+1 \mid Y_t) = P(j+1, j). \tag{4}$$

Let P_j denote the transition probability from state N of market m when there are j rotations to state $N+1$ of market n when there are $j+1$ rotations.

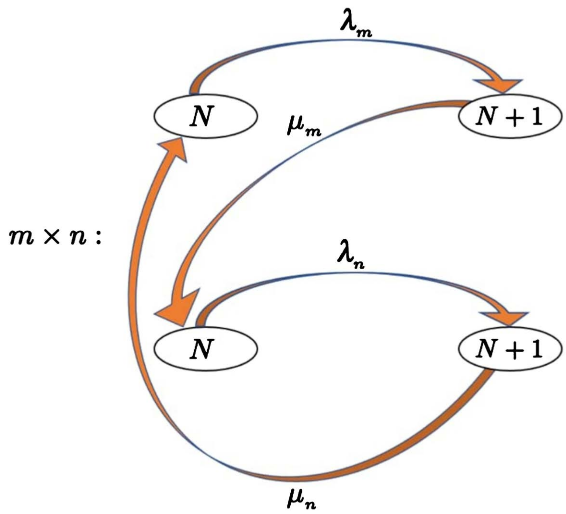

Figure 1. Transition path for $m \times n$.

From **Figure 1**, the associated Kolmogorov difference equations are

$$f(\alpha)\lambda_j P_0 = \mu_j P_1 \quad j = 0. \tag{5}$$

$$\left(f(\alpha)\lambda_j + \mu_j\right)P_j = f(\alpha)\lambda_j P_{j-1} + \mu_j P_{j+1} \quad j > 0. \tag{6}$$

where $f(\alpha)$ is a Haursdoff function equipped with complete rotation topology $\mathbb{X}$ and α is a fixed point of *f*. We make the following proposition.

Remark 3.1. *For* $f : \mathbb{H} \times \mathbb{X} \to \mathbb{R}$ *and* $\alpha \in \mathbb{R}$ *is a fixed point of f such that*

$$\rho_{m\times n} = \alpha\rho_{M/M/1}, \tag{7}$$

then (5) and (6) have a unique solution given by

$$P_j = \left(\alpha\rho_{M/M/1}\right)^j \left(1 - \alpha\rho_{M/M/1}\right) \quad j \geq 0. \tag{8}$$

Remark 3.1 implies that, one can use the joint setting in **Figure 1** by selecting the knot function α representing rotation lengths within constructed Hausdorff topology innerly regulated by α with some compact support within $m \times n$.

4. Analytic Results

Lemma 1. *The Banack-Haursdoff topology* $\mathbb{H}^{\mathbb{X}}$ *consists of investor rotation moments and paths such that*

$$M^{nth} = n!\left(\frac{\alpha\rho_{M/M/1}}{1 - \alpha\rho_{M/M/1}}\right)^n. \tag{9}$$

Proof. Define for Equation (8) a probability generating function (PGF) $M(z)$ such that

$$M(z) = \sum_{j=0}^{\infty} P_j z^j \quad z \leq 1. \tag{10}$$

Expanding Equation (10) in the light of Equation (8) gives

$$M(z) = P_0\left[1 + \alpha\rho_{M/M/1} z + \left(\alpha\rho_{M/M/1} z\right)^2 + \cdots\right], \tag{11}$$

which simplifies to

$$M(z)=\frac{P_0}{1-\alpha\rho_{M/M/1}z}. \tag{12}$$

Differentiating Equation (12) twice w.r.t. to z at 1 gives

$$M'(z=1)=1!\left(\frac{\alpha\rho_{M/M/1}}{1-\alpha\rho_{M/M/1}}\right)^1, \tag{13}$$

and

$$M''(z=1)=2!\left(\frac{\alpha\rho_{M/M/1}}{1-\alpha\rho_{M/M/1}}\right)^2. \tag{14}$$

The lemma follows upon finding higher-order derivatives of (12) and applying the principle of induction of Knuth [15] on successive moments.☐

Corollary 2. *Suppose two stocks are rotated in the Banack-Hausdorff space described in Lemma* 1 *above. Then the* $E[R_T]$ *is given by*

$$E[R_T]=\frac{\rho_{m/m/1}}{\lambda_j(1-\alpha\rho_{m/m/1})}. \tag{15}$$

Proof. This follows consequence of Little's theorem on (13) as in remark 3.1.☐

Remark 4.1. *The maximum rotation time* R_T *for* 2*-stocks in* $\mathbb{H}^{\mathbb{X}}$ *is given by*

$$R_T(\text{stationary})=\tau E[R_T]. \tag{16}$$

Here, τ is the length of investment time, Medhi [16].

Lemma 3. *Suppose* $j\le N\in\mathbb{R}$. *Then*

$$E[R_T]=\frac{\rho_{M/M/1}\left[(1-\mathbb{C}^N)-N(1-\mathbb{C})\mathbb{C}^{N-1}\right]}{\lambda_j(1-\mathbb{C}^N)(1-\mathbb{C})}. \tag{17}$$

Proof. Expanding Equation (8) in view[1] of the normalization condition for $j\le N$. Then

$$P_0+\alpha\rho_{M/M/1}P_0+(\alpha\rho_{M/M/1})^2P_0+\cdots+(\alpha\rho_{M/M/1})^NP_0=1. \tag{18}$$

Factorizing Equation (18) and applying certain properties of geometric series, one obtains that

$$P_0=\frac{1-\alpha\rho_{M/M/1}}{1-(\alpha\rho_{M/M/1})^N}. \tag{19}$$

By the same process as in Equation (18) for $j\le N$, we have

$$P_j=\frac{(\alpha\rho_{M/M/1})^j(1-\alpha\rho_{M/M/1})}{1-(\alpha\rho_{m/m/1})^N}. \tag{20}$$

Let $T(z)$ be a PGF for the number of complete rotations for Equation (19) such that

$$T(z)=\sum_{j=0}^{N}P_jz^j \qquad z\le 1. \tag{21}$$

[1] $\mathbb{C}=\alpha\rho_{M/M/1}$.

Expanding Equation (21) in the light of Equation (18) and Equation (19), one obtains that

$$T(z) = P_0 \left[\frac{1 - \left(\alpha \rho_{M/M/1} z\right)^N}{1 - \alpha \rho_{M/M/1} z} \right]. \tag{22}$$

Finally, the lemma follows upon differentiating Equation (22) w.r.t. z at $z = 1$.□

5. Experiments and Discussions

Simulated results of Equation (15) as in Table 1 indicate that if $\rho < 1$, then

Table 1. Values of $E[R_T]$ for selected α.

ρ $0<\rho<1$	$E[R_T]$ $\alpha = 1$	$E[R_T]$ $\alpha = 1.5$	$E[R_T]$ $\alpha = 1.9$
0.1	0.1110	0.1176	0.1235
0.2	0.1250	0.1429	0.1613
0.3	0.1429	0.1818	0.2326
0.4	0.1667	0.2500	0.4167
0.5	0.200	0.400	2.0000
0.6	0.2500	1.000	−0.7143
0.7	0.3333	−2.0000	−0.3030
0.8	0.5000	−0.5000	−0.1923
0.9	1.0000	−0.2857	−0.1408

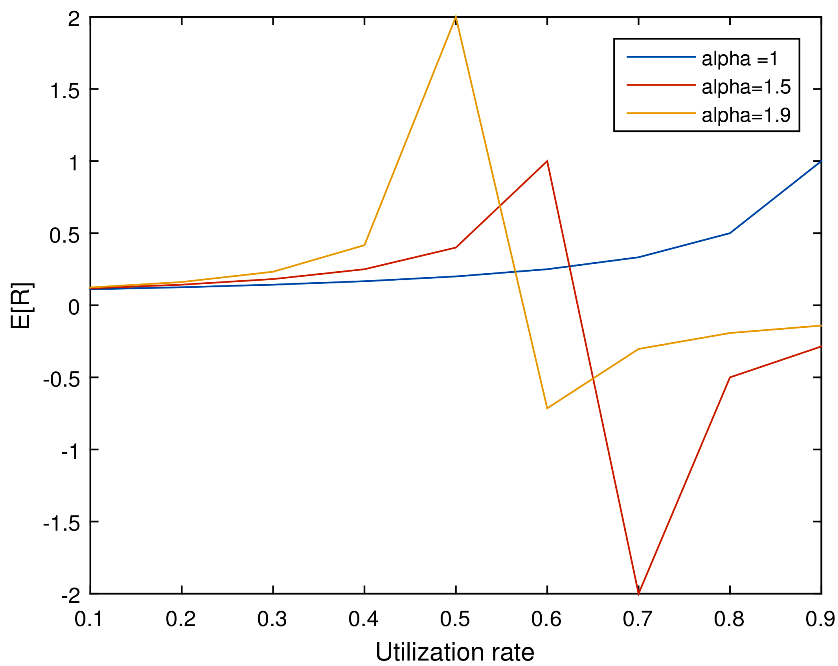

Figure 2. $E[R_T]$ against ρ.

maximizing $E[R_T]$ depends principally on ρ (see **Figure 2**). This means that rotating stocks at specified time provides added benefits as profits to the investor. Again, there is a time maximization knot α that increases profit margins to the right. Rotation stability goes left as in **Table 1** implying that, the shorter the knot, the more stable the system but unfortunately, the lesser the profits. This holds good since, if the random rotation walk is 50% more than the $M/M/1$ rotation system, the system stabilizer coefficient ρ cannot exceed 60%. After this value, the system loses control with maximum loss tendencies.

In the light of Equation (16) when $\tau = 100$ years, **Table 1** shows that at 20% utilization, R_T is at the 13$^{\text{th}}$ year provided that, one stock value rises. Again, at a 50% rotation rate more than that of the $M/M/1$ rotation model when $\rho = 20\%$, the R_T is at the 14$^{\text{th}}$ year under the stock condition described above. Finally, at 90% walk, the R_T is at the 16$^{\text{th}}$ year.

For a complete discussion and analysis, we simulated (17) to uncover the effects of the number of rotations N. The results are tabulated in **Tables 2-4** and **Figures 3-5** respectively. Clearly, the case of $N = 10$ rotations implies more stationarity compared to other N sizes as in **Table 3** and **Table 4**. Thus, in accordance with (17), the lesser the N, the greater the stability of the trading system and the more profit maximization tendency for the investor within the constructed Banack-Hausdorff space.

Table 2. $E[R_T]$ for $N = 10$.

ρ $0<\rho<1$	$E[R_T]$ $\alpha = 1$	$E[R_T]$ $\alpha = 1.5$	$E[R_T]$ $\alpha = 1.9$
0.1	0.1111	0.1176	0.1235
0.2	0.2500	0.2857	0.3223
0.3	0.4285	0.5432	0.6786
0.4	0.6656	0.9594	1.3051
0.5	0.9902	1.6022	2.1467
0.6	1.4392	2.4311	2.9216
0.7	2.0426	3.2673	3.4645
0.8	2.7971	3.9508	3.8059
0.9	3.6466	4.4442	4.0203

Table 3. $E[R_T]$ for $N = 100$.

ρ $0<\rho<1$	$E[R_T]$ $\alpha = 1$	$E[R_T]$ $\alpha = 1.5$	$E[R_T]$ $\alpha = 1.9$
0.1	0.1111	0.1176	0.1235
0.2	0.2500	0.2857	0.3226
0.3	0.4286	0.5455	0.6977
0.4	0.6667	0.1000	1.6667
0.5	1.0000	2.0000	9.6865

Continued

0.6	1.5000	5.9982	48.3460
0.7	2.3333	53.1775	50.5104
0.8	4.0000	62.6667	51.0931
0.9	8.9973	64.0952	51.3640

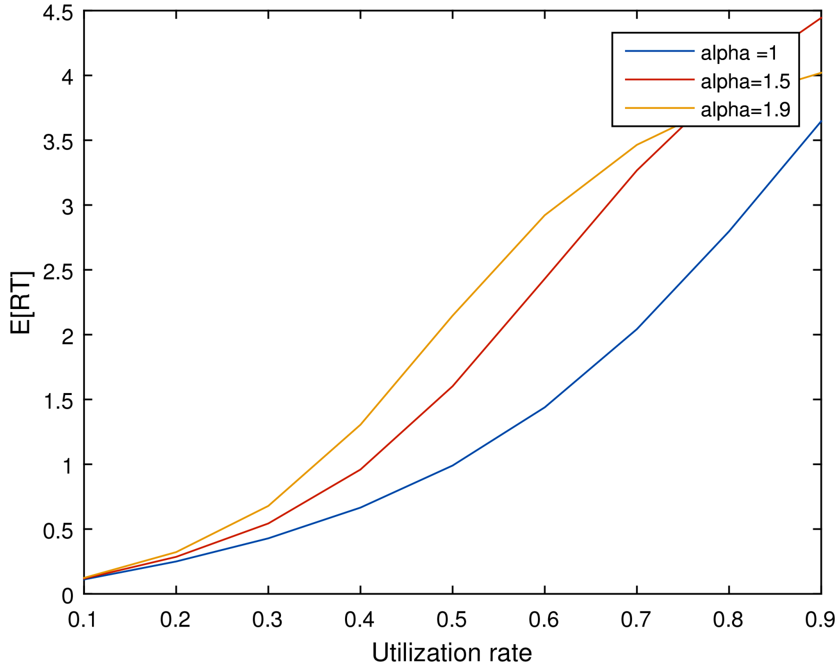

Figure 3. $N = 10$.

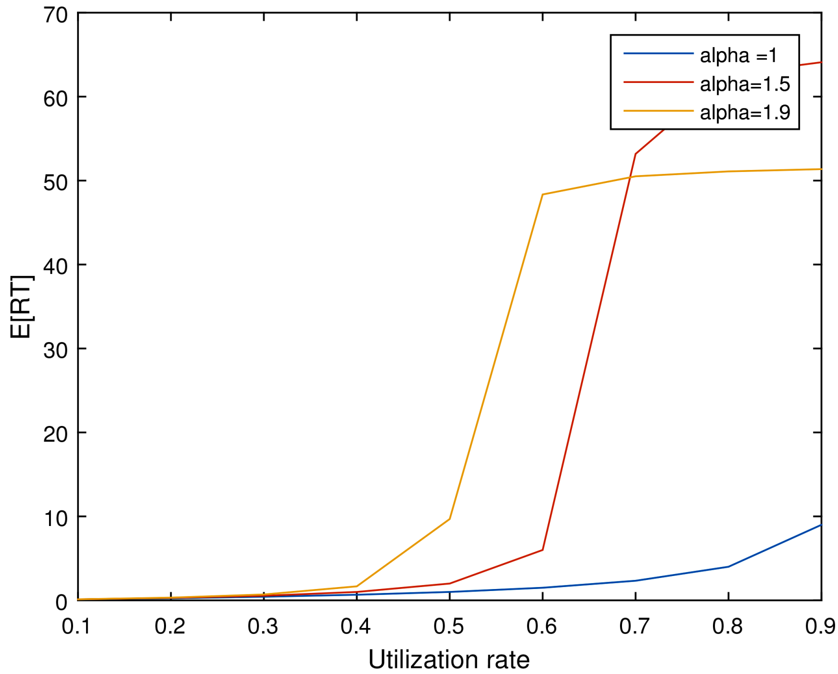

Figure 4. $N = 100$.

Table 4. $E[R_T]$ for $N = 1000$.

ρ $0<\rho<1$	$E[R_T]$ $\alpha = 1$	$E[R_T]$ $\alpha = 1.5$	$E[R_T]$ $\alpha = 1.9$
0.1	0.1111	0.1176	0.1235
0.2	0.2500	0.2857	0.3226
0.3	0.4286	0.5455	0.6977
0.4	0.6667	1.0000	1.6667
0.5	1.0000	2.0000	10.0000
0.6	1.5000	6.0000	522.0301
0.7	2.3333	652.6667	524.1946
0.8	4.0000	662.6667	524.7773
0.9	9.0000	664.0952	525.0482

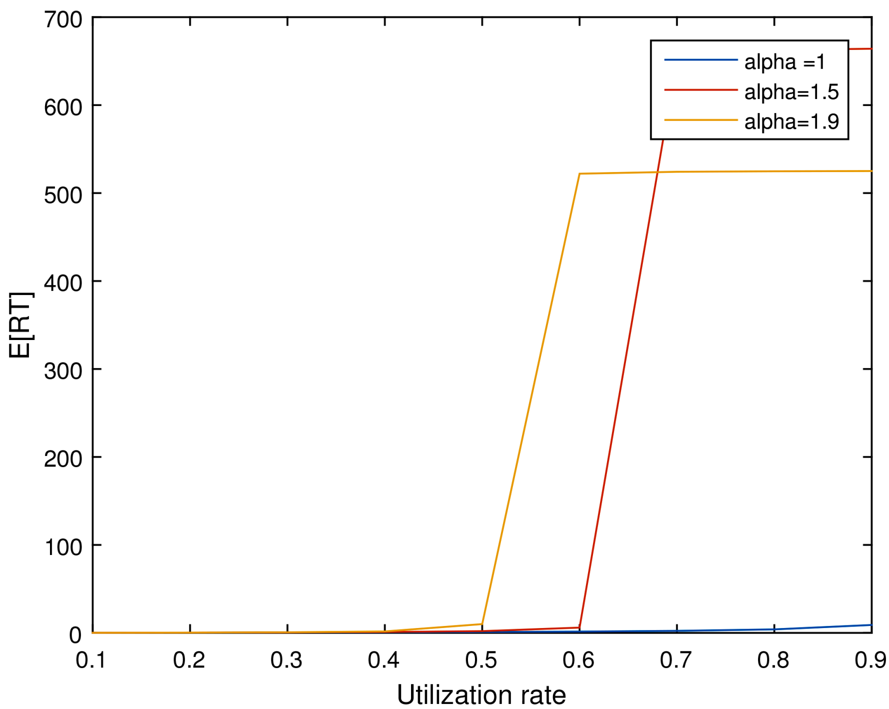

Figure 5. $N = 1000$.

6. Conclusions

This article presents the maximum principle for 2 stocks trading within the Banack-Haursdoff frame. The goal is to provide the methodology for rotating 2 independent stocks with complete rotational paths for profits. We have shown that the optimal gain path is the path closed compact to the path of the M/M/1 queueing system. We have also shown that the size of the knot is the most important parameter especially when the rotation system is stable.

The authors are grateful for all the sources of literature used in this article.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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