Journal of High Energy Physics, Gravitation and Cosmology, 2024, 10, 197-227
https://www.scirp.org/journal/jhepgc
ISSN Online: 2380-4335
ISSN Print: 2380-4327

Solving the Three-Neutrino Problem as One of the Three-Body Problems in Physics

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How to cite this paper: Todorovic, Z.B. (2024) Solving the Three-Neutrino Problem as One of the Three-Body Problems in Physics. *Journal of High Energy Physics, Gravitation and Cosmology*, **10**, 197-227.
https://doi.org/10.4236/jhepgc.2024.101018

Received: November 22, 2023
Accepted: January 22, 2024
Published: January 25, 2024

Abstract

By using the standard PMNS (Pontecorvo-Maki-Nakagawa-Sakata) mixing matrix and applying the rule for the sum of the oscillation probabilities of three neutrinos, the equations of motion were derived in which the Dirac CP violating phase appeared as an unknown quantity. The equations of motion were separately derived for each of the three possible transitions for flavor-neutrino oscillations. Two roots of those equations were obtained in the form of two formulas for the Dirac CP violating phase with opposite signs. In the mathematical sense, the connection between those formulas was established in order to maintain the continuous process of oscillation of three neutrinos. This made it possible to calculate the numerical value for the Dirac CP violating phase, the Jarlskog invariant and to write the general form of the PMNS mixing matrix in the final form in which all its elements are defined with explicit numerical values.

Keywords

Ordinary Neutrino, PMNS Matrix, Dirac CPV Phase, Jarlskog Invariant

1. Introduction

In the process of theoretical investigation of possible physical properties and parameters for three neutrinos, our intention was to show that the obtained results could be consistent with the results published in Refs. [1] [2] [3] [4] in which the hypothesis of the possible existence of the fourth-sterile neutrino is challenged. That is why we devoted ourselves to the research of three neutrinos and the results that we obtained at the end of this paper could be considered as a confirmation of agreement with the STEREO experiments [2] that rejected the possibility of the existence of a sterile neutrino in nature.

In previous papers [5] [6] [7] an explicit formula for the Dirac CPV phase was derived. It can be seen from the form of the formula that it does not depend on the mixing angles embedded in the PMNS mixing matrix, but that it directly depends on the ratio between the corresponding differences of the squares of the neutrino masses.

In this paper, special attention is devoted to researching the application of rules for maintaining the sum of the probability of oscillation of three neutrinos, which is equal to one, for all three possible transitions: $\nu_e \to \nu_\mu$, $\nu_e \to \nu_\tau$, $\nu_e \to \nu_e$; $\nu_\mu \to \nu_e, \nu_\mu \to \nu_\tau, \nu_\mu \to \nu_\mu$ and $\nu_\tau \to \nu_e; \nu_\tau \to \nu_\mu, \nu_\tau \to \nu_\tau$.

In this sense, the PMNS (Pontecorvo-Maki-Nakagawa-Sakata) mixing matrix was applied, and the final results of the introduced procedure are in the form of formulas for the Dirac CP violation phase.

However, as will be seen in the following chapters, the resulting formulas for the transitions in question differ in sign and this is a kind of unexpected phenomenon that could be characterized as an anomalous phenomenon.

Thus, in the mathematical sense, two formulas for Dirac's CP phase violation appeared, which differ from each other only in sign, and how this problem was solved can be seen in the following chapters.

2. Equation of Motion for Three Neutrinos and Final Solutions for Dirac's CP Violation Phase

The procedure for deriving the equation of motion for three neutrinos and its final form is given in papers [5] [6] [7], which reads:

$$\left(2W\cos\delta - V\sin\delta\right)\times 0 = 0 \tag{1}$$

where W and V are represented by the corresponding ratio between the differences of the squares of the eigenstates of the neutrino masses:

$$W = \sin^2 \pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2} = \sin^2 \pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}, V = \sin 2\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2} = \sin 2\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}. \tag{2}$$

And δ is the Dirac CP violation phase as an unknown quantity of this equation.

By inserting explicit values for parameters (2) into Equation (1), it is reduced to the following two identical forms. The first form looks like this:

$$\begin{gathered}\left(2\sin^2 \pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\cos\delta - \sin 2\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\sin\delta\right)\times 0 = 0 \to \\ 2\sin\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\left(\sin\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\cos\delta - \cos\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\sin\delta\right)\times 0 = 0 \to \\ 2\sin\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\left[\sin\left(\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2} - \delta\right)\right]\times 0 = 0;\end{gathered} \tag{3}$$

And the second is:

$$\begin{gathered}\left(2\sin^2 \pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}\cos\delta - \sin 2\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}\sin\delta\right)\times 0 = 0 \to \\ 2\sin\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}\left(\sin\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}\cos\delta - \cos\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}\sin\delta\right)\times 0 = 0 \to \\ 2\sin\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}\left[\sin\left(\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2} - \delta\right)\right]\times 0 = 0\end{gathered} \tag{4}$$

Mathematically, we see that both Equation (3) and Equation (4) consist of a general solution and a particular one. The general solution satisfies all the values from the set $\delta \in [0, 2\pi)$, which are countless and such solutions have no physical meaning.

But from the particular equation

$$
\begin{aligned}
& 2\sin\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\left[\sin\left(\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}-\delta\right)\right]=0\to\sin\left(\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}-\delta\right)=0\to \\
& \left(\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}-\delta\right)=0,\pm\pi\to \\
& A:\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}-\delta=0\to\delta=\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}; \\
& B:\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}-\delta=+\pi\to\delta=\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}-\pi=\pi\left(\frac{\Delta m_{32}^2}{\Delta m_{21}^2}+1\right)-\pi=\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}; \\
& C:\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}-\delta=-\pi\to\delta=\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}+\pi=\pi\left(\frac{\Delta m_{31}^2}{\Delta m_{21}^2}+1\right)+\pi=\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}+2\pi=\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}; \\
& 2\sin\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}\left[\sin\left(\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}-\delta\right)\right]=0\to\sin\left(\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}-\delta\right)=0\to \\
& \left(\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}-\delta\right)=0,\pm\pi\to \\
& A:\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}-\delta=0\to\delta=\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}; \\
& B:\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}-\delta=+\pi\to\delta=\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}-\pi=\pi\left(\frac{\Delta m_{31}^2}{\Delta m_{21}^2}-1\right)-\pi=\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}-2\pi=\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}; \\
& C:\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}-\delta=-\pi\to\delta=\pi\left(\frac{\Delta m_{31}^2}{\Delta m_{21}^2}-1\right)+\pi=\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}-\pi+\pi=\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}.
\end{aligned}
\tag{5}
$$

we extract only two solutions that satisfy a particular equation from countless possible ones and they make physical sense:

$$
\begin{aligned}
\delta &= \pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}; \\
\delta &= \pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}.
\end{aligned}
\tag{6}
$$

Everything we said about Equation (3) also applies to Equation (4). Namely, we can also explain it in the following way:

$$
\begin{aligned}
& \left(1-\tan^{-1}\left(\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)\times\tan\delta\right)=0\to\tan\left(\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)-\tan\delta=0\to \\
& \frac{\sin\left(\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)}{\cos\left(\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)}-\frac{\sin\delta}{\cos\delta}=0\to\sin\left(\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)\cos\delta-\cos\left(\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)\sin\delta=0
\end{aligned}
$$

$$\rightarrow \sin\left(\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}-\delta\right)=0 \rightarrow \left(1-\tan^{-1}\left(\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}\right)\times\tan\delta\right)=0$$

$$\rightarrow \tan\left(\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}\right)-\tan\delta=0 \rightarrow \frac{\sin\left(\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}\right)}{\cos\left(\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}\right)}-\frac{\sin\delta}{\cos\delta}=0\rightarrow \tag{7}$$

$$\sin\left(\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}\right)\cos\delta-\cos\left(\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}\right)\sin\delta=0\rightarrow\sin\left(\left(\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}\right)-\delta\right)=0.$$

Which gives solutions that are identical to solutions (6).

3. Sum Rule for Three Neutrino Oscillation Probabilities

In general, we can write sum rules for oscillation probabilities for three neutrinos in the form of the following relations:

$$P(\nu_e\rightarrow\nu_\mu)+P(\nu_e\rightarrow\nu_\tau)+P(\nu_e\rightarrow\nu_e)=1 \tag{8}$$

$$P(\nu_\mu\rightarrow\nu_e)+P(\nu_\mu\rightarrow\nu_\tau)+P(\nu_\mu\rightarrow\nu_\mu)=1 \tag{9}$$

$$P(\nu_\tau\rightarrow\nu_e)+P(\nu_\tau\rightarrow\nu_\mu)+P(\nu_\tau\rightarrow\nu_\tau)=1 \tag{10}$$

We are in Refs. [5] [6] [7] already used one of the sum rules for the oscillation probabilities for three neutrinos, which is related to the transition $\nu_e\rightarrow\nu_\mu,\nu_e\rightarrow\nu_\tau,\nu_e\rightarrow\nu_e$ and then we derived an equation of the form (1).

Appearance of Factor $\Delta+\delta=\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}+\delta$ in Oscillation Probabilities for Three Neutrinos

In Ref. [8] are derived relations for the probability of oscillation for three neutrinos under the conditions when the neutrino beam moves through the medium with a constant density of matter and corresponding mathematical developments and approximations (See details in the mentioned paper Ref. [8]) and each of them written individually looks like this:

$$P_{\mu e}=\alpha^2\sin^2 2\theta_{12}C_{23}^2\frac{\sin^2 A\Delta}{A^2}+4S_{13}^2S_{23}^2\frac{\sin^2(A-1)\Delta}{(A-1)^2}$$
$$+2\alpha S_{13}\sin 2\theta_{12}\sin 2\theta_{23}\left[\cos(\Delta+\delta)\right]\frac{\sin A\Delta}{A}\frac{\sin(A-1)\Delta}{(A-1)},$$

$$P_{\mu\tau}=\sin^2 2\theta_{23}\sin^2\Delta-\alpha C_{12}^2\sin^2 2\theta_{23}\Delta\sin 2\Delta+\alpha^2C_{12}^4\sin^2 2\theta_{23}\Delta^2\cos 2\Delta$$
$$-\frac{1}{2A}\alpha^2\sin^2 2\theta_{12}\sin^2 2\theta_{23}\left[\sin\Delta\frac{\sin A\Delta}{A}\cos(A-1)\Delta-\frac{1}{2}\sin 2\Delta\right]$$
$$+\frac{2}{A-1}S_{13}^2\sin^2 2\theta_{23}\left[\sin\Delta\cos A\Delta\frac{\sin(A-1)\Delta}{(A-1)}-\frac{A}{2}\Delta\sin 2\Delta\right]$$
$$+2\alpha S_{13}\sin 2\theta_{12}\sin 2\theta_{23}\left[\sin\Delta\sin\delta\right]\frac{\sin A\Delta}{A}\frac{\sin(A-1)\Delta}{(A-1)}$$
$$-\frac{2}{A-1}\alpha S_{13}\sin 2\theta_{12}\sin 2\theta_{23}\cos 2\theta_{23}\cos\delta\sin\Delta\left[A\sin\Delta-\frac{\sin A\Delta}{A}\cos(A-1)\Delta\right],$$

$$
\begin{aligned}
&P_{\mu\mu} = 1 - \sin^2 2\theta_{23} \sin^2 \Delta + \alpha C_{12}^2 \sin^2 2\theta_{23} \Delta \sin 2\Delta \\
&- \alpha^2 \sin^2 2\theta_{12} C_{23}^2 \frac{\sin^2 A\Delta}{A^2} - \alpha^2 C_{12}^4 \sin^2 2\theta_{23} \Delta^2 \cos 2\Delta \\
&+ \frac{1}{2A} \alpha^2 \sin^2 2\theta_{12} \sin^2 2\theta_{23} \left[\sin\Delta \frac{\sin A\Delta}{A} \cos(A-1)\Delta - \frac{1}{2} \sin 2\Delta \right] \\
&- 4 S_{13}^2 S_{23}^2 \frac{\sin^2 (A-1)\Delta}{(A-1)^2} - \frac{2}{A-1} S_{13}^2 \sin^2 2\theta_{23} \left[\sin\Delta \cos A\Delta \frac{\sin(A-1)\Delta}{(A-1)} - \frac{A}{2} \Delta \sin 2\Delta \right] \\
&- 2\alpha S_{13} \sin 2\theta_{12} \sin 2\theta_{23} [\cos\Delta \cos\delta] \frac{\sin A\Delta}{A} \frac{\sin(A-1)\Delta}{(A-1)} \\
&+ \frac{2}{A-1} \alpha S_{13} \sin 2\theta_{12} \sin 2\theta_{23} \cos 2\theta_{23} \cos\delta \sin\Delta \left[A \sin\Delta - \frac{\sin A\Delta}{A} \cos(A-1)\Delta \right]
\end{aligned}
\tag{11}
$$

We note that in Ref. [8] marked factor Δ defined with the following formula:

$$
\Delta = \pi \frac{\Delta m_{31}^2}{\Delta m_{21}^2} \tag{12}
$$

We will use these relations by applying the rule for the sum of the oscillation probabilities for three neutrinos (9), on the basis of which we obtain the following equation:

$$
\begin{aligned}
&2\alpha S_{13} \sin 2\theta_{12} \sin 2\theta_{23} [\cos(\Delta+\delta)] \frac{\sin A\Delta}{A} \frac{\sin(A-1)\Delta}{(A-1)} \\
&+ 2\alpha S_{13} \sin 2\theta_{12} \sin 2\theta_{23} [\sin\Delta \sin\delta] \frac{\sin A\Delta}{A} \frac{\sin(A-1)\Delta}{(A-1)} \\
&- 2\alpha S_{13} \sin 2\theta_{12} \sin 2\theta_{23} [\cos\Delta \cos\delta] \frac{\sin A\Delta}{A} \frac{\sin(A-1)\Delta}{(A-1)} = 0 \rightarrow \\
&[\cos(\Delta+\delta)] \times 2\alpha S_{13} \sin 2\theta_{12} \sin 2\theta_{23} \frac{\sin A\Delta}{A} \frac{\sin(A-1)\Delta}{(A-1)} \\
&- [\cos\Delta \cos\delta - \sin\Delta \sin\delta] \times 2\alpha S_{13} \sin 2\theta_{12} \sin 2\theta_{23} \frac{\sin A\Delta}{A} \frac{\sin(A-1)\Delta}{(A-1)} = 0 \rightarrow \\
&[\cos(\Delta+\delta)] \times 2\alpha S_{13} \sin 2\theta_{12} \sin 2\theta_{23} \frac{\sin A\Delta}{A} \frac{\sin(A-1)\Delta}{(A-1)} \\
&- [\cos(\Delta+\delta)] \times 2\alpha S_{13} \sin 2\theta_{12} \sin 2\theta_{23} \frac{\sin A\Delta}{A} \frac{\sin(A-1)\Delta}{(A-1)} = 0
\end{aligned}
\tag{13}
$$

We can write this equation in the form:

$$
\begin{aligned}
&[\cos(\Delta+\delta)] \left[2\alpha S_{13} \sin 2\theta_{12} \sin 2\theta_{23} \frac{\sin A\Delta}{A} \frac{\sin(A-1)\Delta}{(A-1)} \right. \\
&\left. - 2\alpha S_{13} \sin 2\theta_{12} \sin 2\theta_{23} \frac{\sin A\Delta}{A} \frac{\sin(A-1)\Delta}{(A-1)} \right] = 0 \rightarrow \\
&[\cos(\Delta+\delta)] \times 0 = 0
\end{aligned}
\tag{14}
$$

This equation has a general solution that is satisfied for every arbitrarily taken value from the set $(0,\mp 2\pi)$ and such solutions have no physical meaning. That's why we write a particular equation and the solutions that make physical sense are extracted from the set $(0,\mp 2\pi)$ and they read:

$$\begin{aligned}
&\cos(\Delta+\delta)=\pm 1,\rightarrow \\
&(\Delta+\delta)=0,\pm\pi\rightarrow \\
&A:\Delta+\delta=0\rightarrow\delta=-\Delta=-\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}; \\
&B:\Delta+\delta=\pi\rightarrow\delta=-\Delta+\pi=-\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}+\pi=-\pi\left(\frac{\Delta m_{32}^2}{\Delta m_{21}^2}+1\right)+\pi \\
&\quad=-\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}-\pi+\pi=-\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}; \\
&C:\Delta+\delta=-\pi\rightarrow\delta=-\Delta-\pi=-\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}-\pi=-\pi\left(\frac{\Delta m_{32}^2}{\Delta m_{21}^2}+1\right)-\pi \\
&\quad=-\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}-\pi-\pi=-\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}-2\pi=-\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}.
\end{aligned}\tag{15}$$

We obtained a formula for the Dirac CP phase with a negative sign compared to the original Formula (1). Where such a formula comes from will be explained in the next chapter.

4. Derivation of the Equation of Motion for Three Neutrinos for Transitions $\nu_\mu\rightarrow\nu_e,\nu_\mu\rightarrow\nu_\tau,\nu_\mu\rightarrow\nu_\mu$ and $\nu_\tau\rightarrow\nu_e,\nu_\tau\rightarrow\nu_\mu,\nu_\tau\rightarrow\nu_\tau$: The Occurrence of an Anomaly

4.1. Transition $\nu_\mu\rightarrow\nu_e,\nu_\mu\rightarrow\nu_\tau,\nu_\mu\rightarrow\nu_\mu$

Derivation of the equation without approximations for three neutrinos during the motion of the neutrino beam through the vacuum

In order to eliminate any ambiguities and for the sake of comparison between the case without approximations and the example with approximations, we will show again the way in which the final Equation (1) and Equation (2) was derived, which is shown in Refs. [6] [7].

In further research, we will see that we will get the same formula for the Dirac CP violating phase (15) when the neutrino beam moves through the physical vacuum.

We emphasize this especially because we have already derived the equation of motion (14) and found its roots (15) in the analysis of the case when the neutrino beam moves through a medium with a constant density of matter.

Based on the derived final formulas for the Dirac CP violation phase, it can be seen that they all coincide with each other, which shows that the Dirac CP violation phase does not depend on the medium through which the neutrino beam propagates.

And what is very important to point out is that the Dirac CP violating phase depends exclusively on the ratio between the differences of the squares of the neutrino masses and does not depend on the mixing angles.

In the processes known as neutrino flavor oscillations, the Dirac CP violation phase δ is singled out as the cause of those oscillations in the propagation of the neutrino beam through the physical vacuum. For that reason, there arises the question of writing the equation in which δ would appear as an unknown quantity. On the basis of that equation, it would be possible to determine that unknown quantity. So far, there appears to be only one way to derive that equations for a neutrino beam, and it is related to the use of the equations of the neutrino oscillations probabilities. The procedure for deriving those equations is given here.

To find the equation of motion for three neutrinos, we use the standard form of the mixing matrix [9] [10] [11] [12] [13] which reads:

$$
\begin{aligned}
U_{PMNS} &= \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \\
&= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}\mathrm{e}^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}\mathrm{e}^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}\mathrm{e}^{i\delta} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}\mathrm{e}^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}\mathrm{e}^{i\delta} & c_{13}c_{23} \end{pmatrix} \\
&= \begin{pmatrix} U_{e1} & U_{e2} & J\mathrm{e}^{-i\delta} \\ -A - B\mathrm{e}^{i\delta} & C - D\mathrm{e}^{i\delta} & U_{\mu 3} \\ E - F\mathrm{e}^{i\delta} & -G - H\mathrm{e}^{i\delta} & U_{\tau 3} \end{pmatrix}
\end{aligned}
\tag{16}
$$

where the mixing angles from the (16) are taken into consideration:

$$c_{ij} = \cos\theta_{ij}, s_{ij} = \sin\theta_{ij}; i \neq j = 1,2,3.$$

What is the reason for the introduction of this matrix and what is its role are given in the following considerations. For now, let us note that the introduction of this matrix is of essential importance due to the need to indicate by theoretical considerations that the standard PMNS matrix of the form 3 × 3 is unique in depicting the physical characteristics of neutrinos. And therefore it can be considered that the entire description of the functionalization of this matrix essentially represents a theoretical proof of the existence in nature of only three neutrinos, and that the idea related to the hypothesis of the existence of sterile neutrinos could not be found in the Standard Model.

In our considerations, we will use the general formula for neutrino oscillations given in [11] [14]:

$$
\begin{aligned}
&P(\nu_\alpha \to \nu_\beta) \\
&= \delta_{\alpha\beta} - 4\sum_{i<j} R_e\left(U_{\alpha i}U_{\beta i}^{*}U_{\alpha j}^{*}U_{\beta j}\right)\sin^2\left(\frac{\Delta m_{ji}^2 Lc^3}{4E\hbar}\right) \\
&\quad + 2\sum_{i<j}\mathrm{Im}\left(U_{\alpha i}U_{\beta i}^{*}U_{\alpha j}^{*}U_{\beta j}\right)\sin\left(\frac{\Delta m_{ji}^2 Lc^3}{2E\hbar}\right); i,j = 1,2,3; \alpha,\beta = e,\mu,\tau.
\end{aligned}
\tag{17}
$$

We will derive the equations of motion of the three neutrinos in such a way that we will use the property of the mixing matrix (16) which is expressed by the following relation:

$$P\left(\nu_\mu \to \nu_e\right) + P\left(\nu_\mu \to \nu_\tau\right) + P\left(\nu_\mu \to \nu_\mu\right) = 1 \tag{18}$$

The main goal of this work is to derive the equations of three neutrinos and then to determine their root from those equations, which represents the solution for Dirac's ČP violating phase. In this sense, we will separately analyze the case for the normal neutrino mass hierarchy. In further considerations, we use the original PMNS matrix (16), where by applying the rule for maintaining the sum of oscillation probabilities for three neutrinos (18), we find:

$$\begin{aligned}
&P\left(\nu_\mu \to \nu_e\right) + P\left(\nu_\mu \to \nu_\tau\right) + P\left(\nu_\mu \to \nu_\mu\right) \\
&= 1 - 4R\left\{U_{\mu 1}U_{e1}^{*}U_{\mu 2}^{*}U_{e2}\sin^2\left(\pi\frac{\Delta m_{21}^2}{\Delta m_{21}^2}\right)\right\} + 2\,\mathrm{Im}\left\{U_{\mu 1}U_{e1}^{*}U_{\mu 2}^{*}U_{e2}\sin\left(2\pi\frac{\Delta m_{21}^2}{\Delta m_{21}^2}\right)\right\} \\
&\quad - 4R\left\{U_{\mu 1}U_{e1}^{*}U_{\mu 3}^{*}U_{e3}\sin^2\left(\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)\right\} + 2\,\mathrm{Im}\left\{U_{\mu 1}U_{e1}^{*}U_{\mu 3}^{*}U_{e3}\sin\left(2\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)\right\} \\
&\quad - 4R\left\{U_{\mu 2}U_{e2}^{*}U_{\mu 3}^{*}U_{e3}\sin^2\left(\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}\right)\right\} + 2\,\mathrm{Im}\left\{U_{\mu 2}U_{e2}^{*}U_{\mu 3}^{*}U_{e3}\sin\left(2\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}\right)\right\} \\
&\quad - 4R\left\{U_{\mu 1}U_{\tau 1}^{*}U_{\mu 2}^{*}U_{\tau 2}\sin^2\left(\pi\frac{\Delta m_{21}^2}{\Delta m_{21}^2}\right)\right\} + 2\,\mathrm{Im}\left\{U_{\mu 1}U_{\tau 1}^{*}U_{\mu 2}^{*}U_{\tau 2}\sin\left(2\pi\frac{\Delta m_{21}^2}{\Delta m_{21}^2}\right)\right\} \\
&\quad - 4R\left\{U_{\mu 1}U_{\tau 1}^{*}U_{\mu 3}^{*}U_{\tau 3}\sin^2\left(\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)\right\} + 2\,\mathrm{Im}\left\{U_{\mu 1}U_{\tau 1}^{*}U_{\mu 3}^{*}U_{\tau 3}\sin\left(2\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)\right\} \\
&\quad - 4R\left\{U_{\mu 2}U_{\tau 2}^{*}U_{\mu 3}^{*}U_{\tau 3}\sin^2\left(\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}\right)\right\} + 2\,\mathrm{Im}\left\{U_{\mu 2}U_{\tau 2}^{*}U_{\mu 3}^{*}U_{\tau 3}\sin\left(2\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}\right)\right\} \\
&\quad - 4\left|U_{\mu 1}\right|^2\left|U_{\mu 2}\right|^2\sin^2\left(\pi\frac{\Delta m_{21}^2}{\Delta m_{21}^2}\right) - 4\left|U_{\mu 1}\right|^2\left|U_{\mu 3}\right|^2\sin^2\left(\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right) \\
&\quad - 4\left|U_{\mu 2}\right|^2\left|U_{\mu 3}\right|^2\sin^2\left(\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}\right) \\
&= 1
\end{aligned} \tag{19}$$

And, from the Equation (19), the equation of neutrino motion is formed with a condition that the travelled distance of the neutrino beam, moving through a vacuum from the source, equals the neutrino wavelength $L = L_{12}$. So, it can be written as:

$$\begin{aligned}
&-4R\left\{U_{\mu 1}U_{e1}^{*}U_{\mu 3}^{*}U_{e3}W\right\} + 2\,\mathrm{Im}\left\{U_{\mu 1}U_{e1}^{*}U_{\mu 3}^{*}U_{e3}V\right\} \\
&-4R\left\{U_{\mu 2}U_{e2}^{*}U_{\mu 3}^{*}U_{e3}W\right\} + 2\,\mathrm{Im}\left\{U_{\mu 2}U_{e2}^{*}U_{\mu 3}^{*}U_{e3}V\right\} \\
&-4R\left\{U_{\mu 1}U_{\tau 1}^{*}U_{\mu 3}^{*}U_{\tau 3}W\right\} + 2\,\mathrm{Im}\left\{U_{\mu 1}U_{\tau 1}^{*}U_{\mu 3}^{*}U_{\tau 3}V\right\} \\
&-4R\left\{U_{\mu 2}U_{\tau 2}^{*}U_{\mu 3}^{*}U_{\tau 3}W\right\} + 2\,\mathrm{Im}\left\{U_{\mu 2}U_{\tau 2}^{*}U_{\mu 3}^{*}U_{\tau 3}V\right\} \\
&-4\left|U_{\mu 1}\right|^2\left|U_{\mu 3}\right|^2 W - 4\left|U_{\mu 2}\right|^2\left|U_{\mu 3}\right|^2 W = 0
\end{aligned} \tag{20}$$

where

$$V_{(NO)}=\sin\left(2\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)=\sin\left(2\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}\right),$$
$$W_{(NO)}=\sin^2\left(\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)=\sin^2\left(\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}\right). \tag{21}$$

Detailed algebraic calculations in Equation (20) yield the explicit form of the Equation (it is understood that it is written for the normal hierarchy of neutrino masses):

$$\begin{aligned}&\left(4WU_{e1}U_{\mu3}AJ-4WU_{e2}U_{\mu3}CJ-4WU_{\mu3}U_{\tau3}AF+4WU_{\mu3}U_{\tau3}BE\right.\\&\left.+4WU_{\mu3}U_{\tau3}CH-4WU_{\mu3}U_{\tau3}DG\right)\cos\delta+\left(2VU_{e1}U_{\mu3}AJ-2VU_{e2}U_{\mu3}CJ\right.\\&\left.-2VU_{\mu3}U_{\tau3}AF-2VU_{\mu3}U_{\tau3}BE+2VU_{\mu3}U_{\tau3}CH+2VU_{\mu3}U_{\tau3}DG\right)\sin\delta\\&+\left(4WU_{e1}U_{\mu3}BJ+4WU_{e2}U_{\mu3}DJ+4WU_{\mu3}U_{\tau3}\left(AE-BF\right)\right.\\&+4WU_{\mu3}U_{\tau3}\left(CG-DH\right)-4WU_{\mu3}^2\left(A^2+B^2+2AB\cos\delta\right)\\&\left.-4WU_{\mu3}^2\left(C^2+D^2-2CD\cos\delta\right)\right)=0;\end{aligned}$$

$$\begin{aligned}&\left(4WU_{e1}U_{\mu3}BJ+4WU_{e2}U_{\mu3}DJ+4WU_{\mu3}U_{\tau3}\left(AE-BF\right)\right.\\&+4WU_{\mu3}U_{\tau3}\left(CG-DH\right)-4WU_{\mu3}^2\left(A^2+B^2+2AB\cos\delta\right)\\&\left.-4WU_{\mu3}^2\left(C^2+D^2-2CD\cos\delta\right)\right)\\&=C_{13}^2S_{23}^2\left(C_{23}^2+S_{23}^2S_{13}^2\right)-C_{13}^2S_{23}^2\left(S_{13}^2+C_{23}^2C_{13}^2\right)=0,\\&AB=CD,BE=DG,U_{e1}A=U_{e2}C,AF=CH\end{aligned}\tag{22}$$

The sum of all free members is equal to zero, so they are omitted in this equation.

The first thing that can be noticed in this equation is that the algebraic expressions with $\cos\delta$ and $\sin\delta$ are not identical.

The second thing we can calculate is that each of these expressions are equal to zero, so in that case we can write:

$$0\times\cos\delta+0\times\sin\delta=0 \tag{23}$$

In the mathematical sense, this equation is satisfied for any arbitrarily taken value from the set $\delta\in\left(0,2\pi\right)$, so such solutions do not make physical sense.

However, if we separate the members with the coefficients BE and DG we see that they repel each other because $BE=DG$, then we can write the Equation (22) in the form:

$$\begin{aligned}&\left(4WU_{e1}U_{\mu3}AJ-4WU_{e2}U_{\mu3}CJ-4WU_{\mu3}U_{\tau3}AF+4WU_{\mu3}U_{\tau3}CH\right)\cos\delta\\&+\left(2VU_{e1}U_{\mu3}AJ-2VU_{e2}U_{\mu3}CJ-2VU_{\mu3}U_{\tau3}AF+2VU_{\mu3}U_{\tau3}CH\right)\sin\delta=0\end{aligned}\tag{24}$$

Now we see that the algebraic expressions with $\cos\delta$ and $\sin\delta$ are mutually identical, so we can extract them as a common factor:

$$\begin{aligned}&\left(U_{e1}U_{\mu3}AJ-4WU_{e2}U_{\mu3}CJ-4WU_{\mu3}U_{\tau3}AF+4WU_{\mu3}U_{\tau3}CH\right)\\&\times\left(4W\cos\delta+2V\sin\delta\right)=0\end{aligned}\tag{25}$$

Without going into the numerical value for the common factor, we find the solution of this equation:

$$4W\cos\delta+2V\sin\delta=0\to 2\sin^2\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\cos\delta+\sin 2\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\sin\delta=0\to$$

$$2\sin^2\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\cos\delta+2\sin\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\cos\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\sin\delta\to$$

$$2\sin\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\left(\sin\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\cos\delta+\cos\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\sin\delta\right)=0\to$$

$$2\sin\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\left[\sin\left(\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}+\delta\right)\right]=0\to\sin\left(\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}+\delta\right)=0\to$$

$$\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}+\delta=0,\pm\pi;\to$$

$$\begin{aligned}
&A:\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}+\delta=0\to\delta=-\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2};\\
&B:\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}+\delta=\pi\to\delta=-\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}+\pi=-\pi\left(\frac{\Delta m_{32}^2}{\Delta m_{21}^2}+1\right)+\pi\\
&\quad=-\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}-\pi+\pi=-\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2};\\
&C:\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}+\delta=-\pi\to\delta=-\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}-\pi=-\pi\left(\frac{\Delta m_{32}^2}{\Delta m_{21}^2}+1\right)-\pi\\
&\quad=-\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}-\pi-\pi=-\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}-2\pi=-\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}.
\end{aligned}\tag{26}$$

We subsequently calculate the numerical value for the joint expression with $\cos\delta$ and $\sin\delta$ and find

$$\begin{aligned}&\left(U_{e1}U_{\mu3}AJ-4WU_{e2}U_{\mu3}CJ-4WU_{\mu3}U_{\tau3}AF+4WU_{\mu3}U_{\tau3}CH\right)\\&\times\left(4W\cos\delta+2V\sin\delta\right)=0\end{aligned}\tag{27}$$

So the Equation (25) takes a definite form:

$$0\times\left(4W\cos\delta+2V\sin\delta\right)=0\tag{28}$$

With this form of the equation, the solution (26) represents a particular solution and only it has physical meaning, while the general solution is satisfied for every arbitrarily taken value from the set $\delta\in(0,2\pi)$ and they have no physical meaning.

Therefore, the essence of solving the Equation (25) is to find a solution that makes physical sense. In the first step of solving the Equation (22), we saw that due to the inequality of the expressions with $\cos\delta$ and $\sin\delta$, there could not be a solution that makes physical sense because it would be satisfied for any arbitrarily taken value from the set $\delta\in(0,2\pi)$.

However, since we see that in these apparently unequal expressions there are terms that cancel each other out, then we understand that in essence the algebraic expressions with $\cos\delta$ and $\sin\delta$ are identical to each other.

And for that reason, we can write the Equation (27) without entering the numerical value of that common factor.

4.2. Transition $\nu_\tau \to \nu_e, \nu_\tau \to \nu_\mu, \nu_\tau \to \nu_\tau$

And in further considerations, we use the original PMNS matrix (16), where we apply the rules for maintaining the sum of oscillations for three neutrinos for the next transition:

$$P(\nu_\tau \to \nu_e) + P(\nu_\tau \to \nu_\tau) + P(\nu_\tau \to \nu_\mu) = 1 \tag{29}$$

The main goal of this work is to derive the equations of three neutrinos and then to determine their root from those equations, which represents the solution for Dirac's ČP violating phase. Using relations (16) and (17) we can write:

$$\begin{aligned}
&P(\nu_\tau \to \nu_e) + P(\nu_\tau \to \nu_\mu) + P(\nu_\tau \to \nu_\tau) \\
&= 1 - 4R\left\{U_{\tau 1}U_{e1}^{*}U_{\tau 2}^{*}U_{e2}\sin^2\left(\pi\frac{\Delta m_{21}^2}{\Delta m_{21}^2}\right)\right\} + 2\,\mathrm{Im}\left\{U_{\tau 1}U_{e1}^{*}U_{\tau 2}^{*}U_{e2}\sin\left(2\pi\frac{\Delta m_{21}^2}{\Delta m_{21}^2}\right)\right\} \\
&\quad - 4R\left\{U_{\tau 1}U_{e1}^{*}U_{\tau 3}^{*}U_{e3}\sin^2\left(\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)\right\} + 2\,\mathrm{Im}\left\{U_{\tau 1}U_{e1}^{*}U_{\tau 3}^{*}U_{e3}\sin\left(2\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)\right\} \\
&\quad - 4R\left\{U_{\tau 2}U_{e2}^{*}U_{\tau 3}^{*}U_{e3}\sin^2\left(\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}\right)\right\} + 2\,\mathrm{Im}\left\{U_{\tau 2}U_{e2}^{*}U_{\tau 3}^{*}U_{e3}\sin\left(2\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}\right)\right\} \\
&\quad - 4R\left\{U_{\tau 1}U_{\mu 1}^{*}U_{\tau 2}^{*}U_{\mu 2}\sin^2\left(\pi\frac{\Delta m_{21}^2}{\Delta m_{21}^2}\right)\right\} + 2\,\mathrm{Im}\left\{U_{\tau 1}U_{\mu 1}^{*}U_{\tau 2}^{*}U_{\mu 2}\sin\left(2\pi\frac{\Delta m_{21}^2}{\Delta m_{21}^2}\right)\right\}
\end{aligned}$$

$$\begin{aligned}
&P(\nu_\tau \to \nu_e) + P(\nu_\tau \to \nu_\mu) + P(\nu_\tau \to \nu_\tau) \\
&= 1 - 4R\left\{U_{\tau 1}U_{e1}^{*}U_{\tau 2}^{*}U_{e2}\sin^2\left(\pi\frac{\Delta m_{21}^2}{\Delta m_{21}^2}\right)\right\} + 2\,\mathrm{Im}\left\{U_{\tau 1}U_{e1}^{*}U_{\tau 2}^{*}U_{e2}\sin\left(2\pi\frac{\Delta m_{21}^2}{\Delta m_{21}^2}\right)\right\} \\
&\quad - 4R\left\{U_{\tau 1}U_{e1}^{*}U_{\tau 3}^{*}U_{e3}\sin^2\left(\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)\right\} + 2\,\mathrm{Im}\left\{U_{\tau 1}U_{e1}^{*}U_{\tau 3}^{*}U_{e3}\sin\left(2\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)\right\} \\
&\quad - 4R\left\{U_{\tau 2}U_{e2}^{*}U_{\tau 3}^{*}U_{e3}\sin^2\left(\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}\right)\right\} + 2\,\mathrm{Im}\left\{U_{\tau 2}U_{e2}^{*}U_{\tau 3}^{*}U_{e3}\sin\left(2\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}\right)\right\} \\
&\quad - 4R\left\{U_{\tau 1}U_{\mu 1}^{*}U_{\tau 2}^{*}U_{\mu 2}\sin^2\left(\pi\frac{\Delta m_{21}^2}{\Delta m_{21}^2}\right)\right\} + 2\,\mathrm{Im}\left\{U_{\tau 1}U_{\mu 1}^{*}U_{\tau 2}^{*}U_{\mu 2}\sin\left(2\pi\frac{\Delta m_{21}^2}{\Delta m_{21}^2}\right)\right\}
\end{aligned} \tag{30}$$

So, it can be written as

$$\begin{aligned}
&-4WR\left\{U_{\tau 1}U_{e1}^{*}U_{\tau 3}^{*}U_{e3}\right\} + 2V\,\mathrm{Im}\left\{U_{\tau 1}U_{e1}^{*}U_{\tau 3}^{*}U_{e3}\right\} \\
&-4WR\left\{U_{\tau 2}U_{e2}^{*}U_{\tau 3}^{*}U_{e3}\right\} + 2V\,\mathrm{Im}\left\{U_{\tau 2}U_{e2}^{*}U_{\tau 3}^{*}U_{e3}\right\} \\
&-4WR\left\{U_{\tau 1}U_{\mu 1}^{*}U_{\tau 3}^{*}U_{\mu 3}\right\} + 2V\,\mathrm{Im}\left\{U_{\tau 1}U_{\mu 1}^{*}U_{\tau 3}^{*}U_{\mu 3}\right\} \\
&-4WR\left\{U_{\tau 2}U_{\mu 2}^{*}U_{\tau 3}^{*}U_{\mu 3}\right\} + 2V\,\mathrm{Im}\left\{U_{\tau 2}U_{\mu 2}^{*}U_{\tau 3}^{*}U_{\mu 3}\right\} \\
&-4\left|WU_{\tau 1}\right|^2\left|U_{\tau 3}\right|^2 - 4W\left|U_{\tau 2}\right|^2\left|U_{\tau 3}\right|^2 = 0
\end{aligned} \tag{31}$$

where

$$V=\sin\left(2\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)=\sin\left(2\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}\right),$$
$$W=\sin^2\left(\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)=\sin^2\left(\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}\right). \quad (32)$$

Detailed algebraic calculations in Equation (31) yield:

$$\begin{aligned}&\left(-4WU_{e1}U_{\tau3}EJ+4WU_{e2}U_{\tau3}GJ+4WU_{\mu3}U_{\tau3}EB-4WU_{\mu3}U_{\tau3}FA\right.\\&\left.-4WU_{\mu3}U_{\tau3}GD+4WU_{\mu3}U_{\tau3}HC\right)\cos\delta+\left(-2VU_{e1}U_{\tau3}EJ+2VU_{e2}U_{\tau3}GJ\right.\\&\left.+2VU_{\mu3}U_{\tau3}EB+2VU_{\mu3}U_{\tau3}FA-2VU_{\mu3}U_{\tau3}GD-2VU_{\mu3}U_{\tau3}HC\right)\sin\delta\\&+4WU_{e1}U_{\tau3}FJ+4WU_{e2}U_{\tau3}HJ+4WU_{\mu3}U_{\tau3}\left(EA-FB\right)\\&+4WU_{\mu3}U_{\tau3}\left(GC-HD\right)-4WU_{\tau3}^2\left(E^2+F^2-2EF\cos\delta\right)\\&-4WU_{\tau3}^2\left(G^2+H^2+2GH\cos\delta\right)=0;\end{aligned}$$

$$\begin{aligned}&4WU_{e1}U_{\tau3}FJ+4WU_{e2}U_{\tau3}HJ+4WU_{\mu3}U_{\tau3}\left(EA-FB\right)\\&+4WU_{\mu3}U_{\tau3}\left(GC-HD\right)=C_{13}^2C_{23}^2\times\left(S_{13}^2+S_{23}^2C_{13}^2\right),\\&4WU_{\tau3}^2\left(E^2+F^2-2EF\cos\delta\right)+4WU_{\tau3}^2\left(G^2+H^2+2GH\cos\delta\right)\\&=C_{13}^2C_{23}^2\times\left(S_{23}^2+C_{23}^2S_{13}^2\right);\to\left(S_{13}^2+S_{23}^2C_{13}^2\right)=\left(S_{23}^2+C_{23}^2S_{13}^2\right),\\&EF=GH,FA=HC,U_{e1}E=U_{e2}G,EB=GD.\end{aligned} \quad (33)$$

As can be seen in the expressions shown in (33), the free terms are equal to zero, but it seems that the algebraic expressions with $\cos\delta$ and $\sin\delta$ are not equal to each other, so we write the following equation in the form:

$$0\times\cos\delta+0\times\sin\delta=0 \quad (34)$$

Mathematically, this equation is satisfied for any arbitrarily taken value from the set $\delta\in(0,2\pi)$, so such solutions do not make physical sense.

The sum of all free members is equal to zero, so they are omitted in the Equation (33) and we write the final form of this equation:

$$\begin{aligned}&\left(-4WU_{e1}U_{\tau3}EJ+4WU_{e2}U_{\tau3}GJ+4WU_{\mu3}U_{\tau3}EB-4WU_{\mu3}U_{\tau3}GD\right)\cos\delta\\&+\left(-2VU_{e1}U_{\tau3}EJ+2VU_{e2}U_{\tau3}GJ+2VU_{\mu3}U_{\tau3}EB-2VU_{\mu3}U_{\tau3}GD\right)\sin\delta=0\end{aligned} \quad (35)$$

Now we see that the algebraic expressions with $\cos\delta$ and $\sin\delta$ are mutually identical, so we can extract them as a common factor:

$$\left(-U_{e1}U_{\tau3}EJ+U_{e2}U_{\tau3}GJ+U_{\mu3}U_{\tau3}EB-U_{\mu3}U_{\tau3}GD\right)\left(4W\cos\delta+2V\sin\delta\right)=0 \quad (36)$$

Without going into the numerical value for the common factor, we find the solution of this equation:

$$\begin{aligned}&4W\cos\delta+2V\sin\delta=0\to\\&2\sin^2\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\cos\delta+2\sin\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\cos\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\sin\delta=0\to\\&2\sin\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\left(\sin\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\cos\delta+\cos\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\sin\delta\right)=0\\&\to2\sin\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\left(\sin\left(\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}+\delta\right)\right)=0\to\sin\left(\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}+\delta\right)=0\to\\&\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}+\delta=0,\pm\pi\to\end{aligned}$$

$$4W\cos\delta + 2V\sin\delta = 0 \to$$
$$2\sin^2\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\cos\delta + 2\sin\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\cos\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\sin\delta = 0 \to$$
$$2\sin\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\left(\sin\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\cos\delta + \cos\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\sin\delta\right) = 0 \quad (37)$$
$$\to 2\sin\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\left(\sin\left(\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2} + \delta\right)\right) = 0 \to \sin\left(\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2} + \delta\right) = 0 \to$$
$$\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2} + \delta = 0, \pm\pi \to$$

We subsequently calculate the numerical value for the joint expression with $\cos\delta$ and $\sin\delta$ and find:

$$-U_{e1}U_{\tau3}EJ + U_{e2}U_{\tau3}GJ + U_{\mu3}U_{\tau3}EB - U_{\mu3}U_{\tau3}GD = 0,$$
$$U_{e1}E = U_{e2}G, EB = GD. \quad (38)$$

so the equation takes a definite form:

$$0 \times \left(4W\cos\delta + 2V\sin\delta\right) = 0 \quad (39)$$

With this form of the equation, the solution (37) represents a particular solution and only it has physical meaning, while the general solution is satisfied for every arbitrarily taken value from the set $\delta \in (0, 2\pi)$ and they have no physical meaning.

Note. The entire procedure that was done for the normal hierarchy of neutrino masses is also applied for the inverted hierarchy of neutrino masses, as long as the corresponding labeling should be adjusted accordingly.

5. Unified Equation of Motion for Three Neutrinos

The derived equations of motion for three neutrinos for transitions $\nu_\mu \to \nu_e$, $\nu_\mu \to \nu_\tau, \nu_\mu \to \nu_\mu$ and $\nu_\tau \to \nu_e; \nu_\tau \to \nu_\mu, \nu_\tau \to \nu_\tau$ it gave solutions for the Dirac CP phase in the form of Formula (37), which in comparison with (5) has the opposite sign to the solution which is related to the transition $\nu_e \to \nu_\mu, \nu_e \to \nu_\tau$, $\nu_e \to \nu_e$.

For transition $\nu_e \to \nu_\mu, \nu_e \to \nu_\tau, \nu_e \to \nu_e$ the sign in front of the formula is a plus:

$$\delta = \pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2};$$
$$\delta = \pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}; \quad (40)$$

which represent the roots of the equation:

$$2W\cos\delta - V\sin\delta = 0 \quad (41)$$

While for transition $\nu_\mu \to \nu_e, \nu_\mu \to \nu_\tau, \nu_\mu \to \nu_\mu$ and $\nu_\tau \to \nu_e; \nu_\tau \to \nu_\mu$, $\nu_\tau \to \nu_\tau$ the sign in front of the formula is minus:

$$\delta = -\pi \frac{\Delta m_{31}^2}{\Delta m_{21}^2}; \quad \delta = -\pi \frac{\Delta m_{32}^2}{\Delta m_{21}^2}; \tag{42}$$

which represent the roots of the equation:

$$2W\cos\delta + V\sin\delta = 0 \tag{43}$$

The process of three-flavor neutrino oscillations is continuous in the neutrino beam, and as we can see from the solution of the particular Equations (40) and (42), two types of Dirac CP phases could participate in these processes: one phase is with a plus sign (40) and the other with a minus sign (42).

It is obvious that the appearance of Dirac CP phases with different signs at first glance could not indicate any continuity of the process.

However, in further considerations, we will see that this phenomenon is justified in a mathematical sense, and we will first show this by forming a union equation composed of two particular equations:

$$\begin{aligned} 2W\cos\delta - V\sin\delta = 0, \\ 2W\cos\delta + V\sin\delta = 0. \end{aligned} \tag{44}$$

These two equations give a common solution for the Dirac CP violating phase and it will be valid for the neutrino beam regardless of its distance from the source.

By adding the left and right sides of the Equation (44), we get the general equation:

$$4W\cos\delta = 0. \tag{45}$$

In this equation, we can make a substitution for W given in (32). We can even take both signs $\pm W$, and the equation will not lose its generality.

That's why we can write:

$$4\sin^2\left(\pm\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)\cos\delta = 0 \to \sin^2\left(\pm\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}\right)\cos\delta = 0 \to \sin^2\delta\cos\delta = 0 \tag{46}$$

So, we got the equation of union

$$\sin^2\delta\cos\delta = 0 \tag{47}$$

which contains two equations: the Equation (41) with the root (40) and the Equation (43) with the root (42).

The unification Equation (47) should also include every particular solution shown in (40) and (42). We will show what that procedure looks like in further considerations.

Another way of deriving the unification equation

$$\begin{aligned}
&2W\cos\delta - V\sin\delta = 0 \to 2\sin^2\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\cos\delta - \sin 2\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\sin\delta = 0 \to \\
&2\sin\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\left(\sin\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\cos\delta - \cos\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\sin\delta\right) = 0 \to \\
&\sin\left(\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2} - \delta\right) = 0 \to \delta = \pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}; \\
&2W\cos\delta + V\sin\delta = 0 \to \sin\left(\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2} + \delta\right) = 0 \to \delta = -\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}.
\end{aligned} \tag{48}$$

By adding the equations

$$\sin\left(\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}-\delta\right)+\sin\left(\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}+\delta\right)=0\rightarrow$$
$$\sin\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\cos\delta-\cos\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\sin\delta+\sin\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\cos\delta+\cos\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\sin\delta \tag{49}$$

we get the unification equation:

$$2\sin\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\cos\delta=0 \tag{50}$$

Based on the roots of particular Equations (40) and (42) we have:

$$2\sin(\pm\delta)\cos\delta=0 \tag{51}$$

As we have already mentioned $\sin(\pm\delta)$ is not allowed to be equal to zero, so the only root of this equation is determined with the following formula:

$$\cos\delta=0\rightarrow\delta=-\frac{\pi}{2}=3\frac{\pi}{2}=270^{\circ}. \tag{52}$$

for two key reasons:

1) $\sin\delta$ cannot be equal to zero because in that case the Jarlskog invariant would be equal to zero and thus there would be no neutrino oscillations.

2) The measured values of the Dirac CP violating phase are mostly found in III. Quadrant of the trigonometric circle [14] [15] [16].

After all, from this solution (52) we see that the sign has no effect.

Analysis of solutions for the unification equation

According to the structure of the Equation (44), the following solutions are mathematically possible:

$$\begin{aligned}
&1)\quad \delta=0,\sin 0=0.\\
&2)\quad \delta=\pi/2,\sin\pi/2=1.\\
&3)\quad \delta=\pi,\sin\pi=0.\\
&4)\quad \delta=3\pi/2,\sin 3\pi/2=-1.\\
&5)\quad \delta=0,\cos 0=1.\\
&6)\quad \delta=\pi/2,\cos\pi/2=0.\\
&7)\quad \delta=\pi,\cos\pi=-1.\\
&8)\quad \delta=3\pi/2,\cos 3\pi/2=0.
\end{aligned} \tag{53}$$

Selection of solutions:

The solutions under 1) and 5) fall away because the Jarlskog invariant would be equal to zero, which physically denies the flavor of neutrino oscillations.

Solutions under 2) and 6) fall away because that value is out of range in the data in Ref. [14] [15] and Ref. [16].

Solutions under 3) and 7) fall away because the Jarlskog invariant would be equal to zero.

Solutions under 4) and 8) are acceptable.

Therefore, we adopt the value for the Dirac CP violating phase for the solution of the unification equation

This value for Dirac's CP violating phase means that in the nature of three-flavor neutrino oscillations, the highest possible value for the Jarlskog invariant is present.

Therefore, we adopt the value for the Dirac CP violating phase for the solution of the unification equation

$$\delta = 3\pi/2 = 270^\circ \tag{54}$$

This value for Dirac's CP violating phase means that in the nature of three-flavor neutrino oscillations, the highest possible value for the Jarlskog invariant is present.

6. Coupling between Dirac's CP Violating Phases

In this chapter, we will show how the coupling between two formulas for Dirac's CP violating phases with different signs is realized. For this purpose, we will use two sources of data from experimental measurements of neutrino parameters.

The calculations that we release will indicate that the published data from experimental measurements must be corrected in some cases.

The basic idea is to equate Dirac's CP violating phases with different signs, which is in agreement with the results that come from solving the unification equation.

Reference [14] **[15]**

In the following tables, the explicit numerous values that we will use in our procedure are given.

Based on the data from **Table 1**, numerous values are calculated that are shown in the Formula (55).

$$\begin{aligned}
\delta_{BF32+} &= 180^\circ \times \frac{\Delta m_{32}^2}{\Delta m_{21}^2} = \left[180^\circ \times (24490/739)/360^\circ - 16\right] \times 360^\circ = 205.0879^\circ \\
\delta_{BF32-} &= -180^\circ \times \frac{\Delta m_{32}^2}{\Delta m_{21}^2} = \left[-180^\circ \times (24490/739)/360^\circ + 17\right] \times 360^\circ = 154.912^\circ \\
\delta_{BF31+} &= 180^\circ \times \frac{\Delta m_{31}^2}{\Delta m_{21}^2} = \left[180^\circ \times (25229/739)/360^\circ - 17\right] \times 360^\circ = 25.0879^\circ \\
\delta_{BF31-} &= -180^\circ \times \frac{\Delta m_{31}^2}{\Delta m_{21}^2} = \left[-180^\circ \times (25229/739)/360^\circ + 18\right] \times 360^\circ = 334.912^\circ
\end{aligned} \tag{55}$$

Ensuring agreement with the unification equation is achieved by applying the roots obtained from particular equations by establishing equality between the corresponding formulas for Dirac's CP violating phase that have different signs.

We show how it looks in the next section.

The first thing we notice is that different values are obtained for the Dirac CP violating phase (55). This is due to the fact that there are some deviations during measurements from the true natural values for the ratio between the difference of the square of the neutrino masses.

Table 1. Measured neutrino parameters.

	Range of measured parameters	BF	-1σ	$+1\sigma$
$\frac{\Delta m_{21}^2}{10^{-5}\text{ eV}^2}$	$7.39_{-0.20}^{+0.21}$	7.39	7.19	7.60
$\frac{\Delta m_{32}^2}{10^{-3}\text{ eV}^2}$	$2.449_{-0.030}^{+0.032}$	2.4490	2.4190	2.4810
$\frac{\Delta m_{31}^2}{10^{-3}\text{ eV}^2}$		2.5229	2.4909	2.5570
$\delta_{CP}/^\circ$	222_{-28}^{+38}	222	194	260
$\frac{\sin^2\theta_{12}}{10^{-1}}$	$3.10_{-0.11.}^{+0.12}$	3.10	2.990	3.220
$\frac{\sin^2\theta_{23}}{10^{-1}}$	$5.58_{-0.23}^{+0.20}$	5.580	5.350	5.780
$\frac{\sin^2\theta_{13}}{10^{-2}}$	$2.241_{-0.065}^{+0.065}$	2.2410	2.1760	2.3070

Therefore, we will apply the rule that provides us with the same numerical value for Dirac's CP violating phase regardless of the sign in front of the formula.

In other words, it means that we need to equalize these formulas and mark which unknown in that case we need to correct, and it looks like this:

$$
\begin{aligned}
&\delta_{BF32+} = \delta_{BF31-} \rightarrow \\
&\left[180^\circ\left(\Delta m_{32}^2/7.39\times10^{-5}\text{ eV}^2\right)/360^\circ - 16\right]360^\circ \\
&= \left[-180^\circ\left(\Delta m_{31}^2/7.39\times10^{-5}\text{ eV}^2\right)/360^\circ + 18\right]360^\circ \rightarrow \\
&\left(\Delta m_{32}^2/7.39\times10^{-5}\text{ eV}^2\right)\frac{1}{2} - 16 = -\left(\Delta m_{32}^2/7.39\times10^{-5}\text{ eV}^2 + 1\right)\frac{1}{2} + 18 \rightarrow \\
&\Delta m_{32}^2 = 7.39\times10^{-5}\times33.5 = 2.47565\times10^{-3}\text{ eV}^2; \\
&\delta_{BF32-} = \delta_{BF31+} \rightarrow \\
&\left[-180^\circ\left(\Delta m_{31}^2/7.39\times10^{-5}\text{ eV}^2 - 1\right)/360^\circ + 17\right]360^\circ \\
&= \left[180^\circ\left(\Delta m_{31}^2/7.39\times10^{-5}\text{ eV}^2\right)/360^\circ - 17\right]360^\circ \rightarrow \\
&-\frac{1}{2}\left(\Delta m_{31}^2/7.39\times10^{-5}\text{ eV}^2 - 1\right) + 17 = \frac{1}{2}\left(\Delta m_{31}^2/7.39\times10^{-5}\text{ eV}^2\right) - 17 \rightarrow \\
&\Delta m_{31}^2 = 7.39\times10^{-5}\times34.5 = 2.54955\times10^{-3}\text{ eV}^2.
\end{aligned}
\tag{56}
$$

Here we introduced the assumption that the measured value for Δm_{21}^2 is the correct value during the measurement, and we marked with Δm_{32}^2 the unknown value that we will calculate. In this way, we declare the calculated value to be the correct value and it should appear in the measurements.

Based on the calculated values (56), we form the corrected **Table 2** which looks like this:

Table 2. Corrected values from measured neutrino parameters.

	Range of measured parameters	BF	-1σ	$+1\sigma$
$\frac{\Delta m_{21}^2}{10^{-5}\ \text{eV}^2}$	7.39	7.39		
$\frac{\Delta m_{32}^2}{10^{-3}\ \text{eV}^2}$	$(2.4490)_{exp}^{+0.02665}$	$2.4490^{+0.02665}=2.47565$		
$\frac{\Delta m_{31}^2}{10^{-3}\ \text{eV}^2}$		2.54955		
$(\delta_{CP}/^{\circ})_{exp}$	222_{-28}^{+38}	222	194	260
$(\delta_{CP}/^{\circ})_{th}$	270_{-28}^{+38}	270	242	308

In the next step, we use the values in the corrected Table 2. And we calculate the Dirac CP violating phases:

$$\begin{aligned}
\delta_{BF32+} &= 180^{\circ}\times\frac{\Delta m_{32}^2}{\Delta m_{21}^2}=\left[180^{\circ}\times(24756.5/739)/360^{\circ}-16\right]\times 360^{\circ}=270^{\circ}\\
\delta_{BF32-} &= -180^{\circ}\times\frac{\Delta m_{32}^2}{\Delta m_{21}^2}=\left[-180^{\circ}\times(24756.5/739)/360^{\circ}+17\right]\times 360^{\circ}=90^{\circ}\\
\delta_{BF31+} &= 180^{\circ}\times\frac{\Delta m_{31}^2}{\Delta m_{21}^2}=\left[180^{\circ}\times(25495.5/739)/360^{\circ}-17\right]\times 360^{\circ}=90^{\circ}\\
\delta_{BF31-} &= -180^{\circ}\times\frac{\Delta m_{31}^2}{\Delta m_{21}^2}=\left[-180^{\circ}\times(25495.5/739)/360^{\circ}+18\right]\times 360^{\circ}=270^{\circ}
\end{aligned} \tag{57}$$

From all these values (57), we choose those that are consistent with the root of the unification Equation (52), so we write:

$$\begin{aligned}
&\delta_{BF32+}=\delta_{BF31-}\rightarrow\\
&\left[180^{\circ}\times(24756.5/739)/360^{\circ}-16\right]\times 360^{\circ}\\
&=\left[-180^{\circ}\times(25495.5/739)/360^{\circ}+18\right]\times 360^{\circ}=270^{\circ}\\
&\sin\left[180^{\circ}\times(24756.5/739)\right]=-1,\\
&\sin\left[-180^{\circ}\times(25495.5/739)\right]=-1.
\end{aligned} \tag{58}$$

Therefore, on the basis of Formula (56), it can be seen that the achievement of the Dirac CP violation phase, which represents the root of the unification Equation (51), is only possible with the condition that the formulas for the Dirac CP violating phase with different signs produce mutually equal values.

Equating Dirac's CP violating phases with different signs leads to a direct transformation of numerous values shown in Formulas (55) into numerous values shown in Formulas (57) and this can be shown in this way:

$$\begin{aligned}
&\delta_{BF32+}:205.0879^{\circ}\rightarrow 270^{\circ},\\
&\delta_{BF32-}:154.912^{\circ}\rightarrow 90^{\circ},\\
&\delta_{BF31+}:25.0879^{\circ}\rightarrow 90^{\circ},\\
&\delta_{BF31-}:334.912^{\circ}\rightarrow 270^{\circ}.
\end{aligned} \tag{59}$$

Note: Neutrino oscillation processes take place over the calculated value in the best fit for Dirac's CP violating phase, which is 270 degrees. Deviation from that value is related to the precision of the measurements, which have nothing to do with the oscillation process. That is why measurements are adopted in the range $\pm 1\sigma$ which is shown in Table 2.

Calculation task:

Establish a connection between the experimental and theoretical values shown in Table 1 and Table 2. Taking into account $\left(\Delta m_{21}^2\right)_{BF} = 7.39 \times 10^{-5}\ \text{eV}^2$ as the exact value when measuring, and then giving an estimate for $\left[\left(\Delta m_{32}^2\right)_{BF}\right]_{exp}$ in BF range that the experiments should measure for the Dirac CP violating phase to be exactly 270 degrees.

Solution:

The difference in BF between the theoretical and experimental values is:

$$\left[\left(\Delta m_{32}^2\right)_{BF}\right]_{th} - \left[\left(\Delta m_{32}^2\right)_{BF}\right]_{exp} = (2.47565 - 2.4490) \times 10^{-3}\ \text{eV}^2. \quad (60)$$

This difference (60) is positive, therefore the new value in BF moves towards the area defined in $+1\sigma$ range in the original measurement, so we could establish a connection between the new value in BF and the original one:

$$\begin{aligned} &\left[\left(\Delta m_{32}^2\right)_{BF}\right]_{th} - \left[\left(\Delta m_{32}^2\right)_{BF}\right]_{exp} \\ &= (2.47565 - 2.4490) \times 10^{-3}\ \text{eV}^2 = 0.02665 \times 10^{-3}\ \text{eV}^2 \to \\ &\frac{\left[\left(\Delta m_{32}^2\right)_{BF}\right]_{exp}}{10^{-3}\ \text{eV}^2} = 2.4490^{+0.02665}. \end{aligned} \quad (61)$$

Based on the appearance of two forms for Dirac's CP violating phase with different signs and their unification, the corresponding transformations were formed and gave a unique numerical value of 270 degrees for the Dirac CP violating phase.

And since ideal values cannot be obtained in the measurements, due to the appearance of systematic errors and others, then it is recommended to pay attention to more precise measurements of the differences in the squared masses of neutrinos, bearing in mind that the theoretical value for the Dirac CP violating phase is equal to 270 degrees..

Because of the solution for the value of the Dirac CP violating phase obtained in this way, the following questions could be asked: Why exactly was the value of 270 degrees obtained for the Dirac CP violating phase? Could it not be some other value?

The answer is as follows: For an angle of 270 degrees, the highest possible value for the Jarlskog invariant is obtained, which is exactly:

$$J_{CP} = s_{12} c_{12} s_{23} c_{23} s_{13} c_{13}^2 \sin\delta = J_{CP}^{mas} \sin\delta = J_{CP}^{mas} \sin 270^\circ = -J_{CP}^{mas} \quad (62)$$

And it seems that this would be the possible behavior of three neutrinos in nature.

In a similar way as we obtained the results in the previous section, we will process the results given in the next section in Ref. [16]. In the following tables, the explicit numerous values that we will use in our procedure are given.

Ref. [16]

Based on the data from **Table 3**, numerous values are calculated that are shown in the Formula (63):

$$\begin{aligned}
\delta_{BF31+} &= 180^\circ \times (25110/741) = \left(180^\circ \times (25110/741)/360^\circ - 16\right) \times 360^\circ = 339.595^\circ \\
\delta_{BF31-} &= -180^\circ \times (25110/741) = \left(-180^\circ \times (25110/741)/360^\circ + 17\right) \times 360^\circ = 20.404^\circ \\
\delta_{BF32+} &= 180^\circ \times (24369/741) = \left(180^\circ \times (24369/741)/360^\circ - 16\right) \times 360^\circ = 159.595^\circ \\
\delta_{BF32-} &= -180^\circ \times (24369/741) = \left(-180^\circ \times (24369/741)/360^\circ + 17\right) \times 360^\circ = 200.404^\circ
\end{aligned} \quad (63)$$

Ensuring agreement with the unification equation is achieved by applying the roots obtained from particular equations by establishing equality between the corresponding formulas for Dirac's CP violating phase that have different signs.

We show how it looks in the next section.

The first thing we notice is that different values are obtained for the Dirac CP violating phase (63). This is due to the fact that there are some deviations during measurements from the true natural values for the ratio between the differences of the square of the neutrino mass eigenstates.

Therefore, we will apply the rule that provides us with the same numerical value for Dirac's CP violating phase regardless of the sign in front of the formula.

In other words, it means that we need to equalize these formulas and mark which unknown in that case we need to correct, and it looks like this:

Table 3. Measured neutrino parameters.

	Range of measured parameters	BF	-1σ	$+1\sigma$
$\frac{\Delta m_{21}^2}{10^{-5}\ \text{eV}^2}$	$7.41_{-0.20}^{+0.21}$	7.41	7.21	7.62
$\frac{\Delta m_{32}^2}{10^{-3}\ \text{eV}^2}$		2.4369	2.4119	2.4628
$\frac{\Delta m_{31}^2}{10^{-3}\ \text{eV}^2}$	$2.5110_{-0.027}^{+0.028}$	2.5110	2.4840	2.5390
$\delta_{CP}/^\circ$	197_{-25}^{+42}	197	172	239
$\frac{\sin^2\theta_{12}}{10^{-1}}$	$3.03_{-0.11}^{+0.12}$	3.03	2.920	3.150
$\frac{\sin^2\theta_{23}}{10^{-1}}$	$5.72_{-0.20}^{+0.28}$	5.720	5.520	6.00
$\frac{\sin^2\theta_{13}}{10^{-2}}$	$2.203_{-0.053}^{+0.056}$	2.203	2.150	2.261

$$\begin{aligned}
&\delta_{BF32+}=\delta_{BF31-}\to\\
&\left[180^\circ\left(\Delta m_{32}^2/741\times10^{-5}\,\text{eV}^2\right)/360^\circ-16\right]360^\circ\\
&=\left[-180^\circ\left(\Delta m_{32}^2/741\times10^{-5}\,\text{eV}^2+1\right)/360^\circ+17\right]360^\circ\to\\
&\left(\Delta m_{32}^2/741\times10^{-5}\,\text{eV}^2\right)\frac{1}{2}-16=-\left(\Delta m_{32}^2/741\times10^{-5}\,\text{eV}^2+1\right)\frac{1}{2}+17\to\\
&\Delta m_{32}^2=7.41\times10^{-5}\,\text{eV}^2\times32.5=2.40825\times10^{-3}\,\text{eV}^2;\\
&\delta_{BF32-}=\delta_{BF31+}\to\\
&\left[-180^\circ\left(\Delta m_{31}^2/741\times10^{-5}\,\text{eV}^2-1\right)/360^\circ+17\right]360^\circ\\
&=\left[180^\circ\left(\Delta m_{31}^2/741\times10^{-5}\,\text{eV}^2\right)/360^\circ-16\right]360^\circ\to\\
&-\frac{1}{2}\left(\Delta m_{31}^2/741\times10^{-5}\,\text{eV}^2-1\right)+17=\frac{1}{2}\left(\Delta m_{31}^2/741\times10^{-5}\,\text{eV}^2\right)-16\to\\
&\Delta m_{31}^2=741\times10^{-5}\,\text{eV}^2\times33.5=2.48235\times10^{-3}\,\text{eV}^2.
\end{aligned}\tag{64}$$

Based on the calculated values (64), we form the corrected **Table 4**, which looks like this:

In the next step, we use the values in the corrected **Table 4** and we calculate the Dirac CP violating phases:

$$\begin{aligned}
&\delta_{BF32+}=180^\circ\times\frac{\Delta m_{32}^2}{\Delta m_{21}^2}=\left[180^\circ\times(24082.5/741)/360^\circ-16\right]\times360^\circ=90^\circ\\
&\delta_{BF32-}=-180^\circ\times\frac{\Delta m_{32}^2}{\Delta m_{21}^2}=\left[-180^\circ\times(24082.5/741)/360^\circ+17\right]\times360^\circ=270^\circ\\
&\delta_{BF31+}=180^\circ\times\frac{\Delta m_{31}^2}{\Delta m_{21}^2}=\left[180^\circ\times(24823.5/741)/360^\circ-16\right]\times360^\circ=270^\circ\\
&\delta_{BF31-}=-180^\circ\times\frac{\Delta m_{31}^2}{\Delta m_{21}^2}=\left[-180^\circ\times(24823.5/741)/360^\circ+17\right]\times360^\circ=90^\circ
\end{aligned}\tag{65}$$

And with this source of information, based on the results (63) and (65), we see that mutual equating the formulas for Dirac's CP violating phases with different signs transforms one numerical value into another, and we can show it in the following way:

Table 4. Corrected values from measured neutrino parameters.

	Range of measured parameters	*BF*	-1σ	$+1\sigma$
$\frac{\Delta m_{21}^2}{10^{-5}\,\text{eV}^2}$	7.41	7.41		
$\frac{\Delta m_{32}^2}{10^{-3}\,\text{eV}^2}$		2.40825		
$\frac{\Delta m_{31}^2}{10^{-3}\,\text{eV}^2}$	$(2.5110)_{txp(-0.02865)}$	2.48235		
$\left(\delta_{CP}/^\circ\right)_{exp}$	197^{+42}_{-25}	197	172	239
$\left(\delta_{CP}/^\circ\right)_{th}$	270^{+42}_{-25}	270	245	312

$$\begin{aligned} &\delta_{BF31+} : 339.595^\circ \to 270^\circ, \\ &\delta_{BF31-} : 20.404^\circ \to 90^\circ, \\ &\delta_{BF32+} : 159.595^\circ \to 90^\circ, \\ &\delta_{BF32-} : 200.404^\circ \to 270^\circ. \end{aligned} \quad (66)$$

Note: Neutrino oscillation processes take place over the calculated value in the best fit for Dirac's CP violating phase, which is 270 degrees. Deviation from that value is related to the precision of the measurements, which have nothing to do with the oscillation process. That is why measurements are adopted in the range $\pm 1\sigma$ which is shown in Table 4.

From all these values (65), we choose those that are consistent with the root of the unification Equation (51), so we write:

$$\begin{aligned} &\delta_{BF32-} = \delta_{BF31+} \to \\ &\left[-180^\circ \times (24082.5/741)/360^\circ + 17\right] \times 360^\circ \\ &= \left[180^\circ \times (24823.5/741)/360^\circ - 16\right] \times 360^\circ = 270^\circ \\ &\sin\left[180^\circ \times (24823.5/741)\right] = -1, \\ &\sin\left[-180^\circ \times (24082.5/741)\right] = -1. \end{aligned}$$

Therefore, on the basis of Formula (66), it can be seen that the achievement of the Dirac CP violation phase, which represents the root of the unification Equation (47), is only possible with the condition that the formulas for the Dirac CP violating phase with different signs produce mutually equal values.

And this essentially represents the condition for spontaneous oscillation of three-flavor neutrinos during the movement of the neutrino beam from the source.

Because of the solution for the value of the Dirac CP violating phase obtained in this way, the following questions could be asked: Why exactly was the value of 270 degrees obtained for the Dirac CP violating phase? Could it not be some other value?

The answer is as follows: For an angle of 270 degrees, the highest possible value for the Jarlskog invariant is obtained, which is exactly:

$$J_{CP} = s_{12}c_{12}s_{23}c_{23}s_{13}c_{13}^2 \sin\delta = J_{CP}^{mas} \sin\delta = J_{CP}^{mas} \sin 270^\circ = -J_{CP}^{mas} \quad (67)$$

Calculation task:

Establish a connection between the experimental and theoretical values shown in Table 1 and Table 2. Taking into account $\left(\Delta m_{21}^2\right)_{BF} = 7.41 \times 10^{-5}\ \text{eV}^2$ as the exact value when measuring, and then giving an estimate for $\left[\left(\Delta m_{31}^2\right)_{BF}\right]_{exp}$ in BF range that the experiments should measure for the Dirac CP violating phase to be exactly 270 degrees.

Solution:

The difference in BF range between the theoretical and experimental values is:

$$\left[\left(\Delta m_{31}^2\right)_{BF}\right]_{exp} - \left[\left(\Delta m_{31}^2\right)_{BF}\right]_{th} = (2.5110 - 2.48235) \times 10^{-3} = 0.02865 \times 10^{-3}\ \text{eV}^2. \quad (68)$$

This difference (68), is positive, therefore the new value in BF range moves towards the area defined in -1σ range in the original measurement, so we could establish a connection between the new value in BF range and the original one:

$$\left[\left(\Delta m_{32}^2\right)_{BF}\right]_{th} = \left[\left(\Delta m_{32}^2\right)_{BF}\right]_{exp}$$
$$= (2.48235)\times 10^{-3}\ \text{eV}^2 \to \frac{\left[\left(\Delta m_{32}^2\right)_{BF}\right]_{exp}}{10^{-3}\ \text{eV}^2} = 2.5110_{-0.03865}. \tag{69}$$

By determining the numerical value for the Dirac CP violation phase in the amount of 270 degrees, the PMNS (Pontecorvo-Maki-Nakagawa-Sakata) matrix, which was written in the general form (16) so far, can now be written with all its elements that have numerical values as shown in the next section.

7. The Final Form of the PMNS (Pontecorvo-Maki-Nakagawa-Sakata) Mixing Matrix

With these theoretical investigations and the results we obtained indicate the possibility that the PMNS matrix, which until now in neutrino physics has always been written in a general form due to an undetermined value for Dirac's CP violating phase, is now given a final form [8]-[16]:

$$U_{PMNS} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix}$$
$$= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{13}c_{23} \end{pmatrix}$$
$$= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i270^\circ} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i270^\circ} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i270^\circ} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i270^\circ} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i270^\circ} & c_{13}c_{23} \end{pmatrix} \tag{70}$$
$$= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & +is_{13} \\ -s_{12}c_{23} + ic_{12}s_{23}s_{13} & c_{12}c_{23} + is_{12}s_{23}s_{13} & c_{13}s_{23} \\ s_{12}s_{23} + ic_{12}c_{23}s_{13} & -c_{12}s_{23} + is_{12}c_{23}s_{13} & c_{13}c_{23} \end{pmatrix}.$$

Using the data in the matrix (70), we find the values for the Jarlskog invariant [17]:

$$J_{CP} = \text{Im}\left(U_{\mu 2}^* U_{e2} U_{\mu 3} U_{e3}^*\right) = \text{Im}\left(U_{\mu 3}^* U_{e3} U_{\mu 1} U_{e1}^*\right) = -\left(U_{\mu 2}^* U_{e2} U_{\mu 1} U_{e1}^*\right) = -J_{CP}^{\max} \tag{71}$$

Leptonic unitariy triangle

Based on the parameters of the first and third columns of the mixing matrix (70), we write the well-known relation for the triangle from geometry that the vector sum of the sides written in complex form for the leptonic unitariy triangle is equal to zero:

$$U_{e1}U_{e3}^* + U_{\mu 1}U_{\mu 3}^* + U_{\tau 1}U_{\tau 3}^*$$
$$= -iC_{12}C_{13}S_{13} + \left(-S_{12}C_{23} + iC_{12}S_{23}S_{13}\right)\times C_{13}S_{23} + \left(S_{12}S_{23} + iC_{12}C_{23}S_{13}\right)\times C_{13}C_{23} \quad (72)$$
$$= 0.$$

Summary

In Ref. [18] we calculated the range in which values for the Dirac CP violating phase could be expected.

This could be done because we paid attention only to the transition $\nu_e \to \nu_\mu, \nu_e \to \nu_\tau, \nu_e \to \nu_e$ and it gave an equation whose root was the formula for the Dirac CP phase with a positive sign. We connected that formula with the rule for the sum of $\cos\delta$ [19] and that enabled us to form the procedure to determine the range for the numerical value for the Dirac CP violating phase.

However, when we derived the equations for the transitions $\nu_\mu \to \nu_e$, $\nu_\mu \to \nu_\tau, \nu_\mu \to \nu_\mu$ and $\nu_\tau \to \nu_e, \nu_\tau \to \nu_\mu, \nu_\tau \to \nu_\tau$, we obtained the roots of those equations for the formula for Dirac's CP violating phase with a negative sign.

This indicated the need to establish continuity in the appearance of Dirac's CP violating phase. This would mean that regardless of the sign in front of those formulas, they would have to give the same value for Dirac's CP violating phase in the calculation.

And that practically means that we should equalize those two formulas, which we did.

In the ideal case that the experimental measurements were made with the highest possible precision, it should be expected that the application of either the formula with a plus sign or the formula with a minus sign would yield an equal or approximately equal numerical value.

However, due to insufficient measurement accuracy, this did not happen, as can be seen from the calculation results (55) and (65)

This theoretical consideration clearly indicated that the published measurement results given in Refs. [14] [15] [16] must be corrected as shown in **Table 2** and **Table 4**.

Applying the formulas for Dirac's CP violating phase with both signs to the corrected results (T2 and T4), we came to a unique result for the numerical value for Dirac's CP violating phase for both sources of information. And it is exactly 270 degrees.

In the published results of NuFIT 5.2 (2022) [16], all neutrino parameters are shown in diagrams using computer simulations in which numerical values of those parameters are expected. Such diagrams also show areas for both the Dirac CP violating phase and the Jarlskog invariant.

And since ideal values cannot be obtained in the measurements, due to the appearance of systematic errors and others, then it is recommended to pay attention to more precise measurements of the differences in the squared masses of neutrinos, bearing in mind that the theoretical value for the Dirac CP violating phase is equal to 270 degrees.

8. Project Assignment

Definition of the task: Combine the data from the measurement results from the two mentioned sources of information [14] [15] [16] so that it will be adopted $\Delta m_{21}^2 = 7.43\times10^{-5}\,\text{eV}^2$ as an exact value. And then, by applying the procedure applied in the previous chapters, determine: Δm_{32}^2, Δm_{31}^2, and the Dirac CP violating phase which should be expected to be measured in laboratories..

8.1. Task Solution

Ref. [14] [15]

$$\begin{aligned}
\delta_{BF32+} &= 180^\circ\times\frac{\Delta m_{32}^2}{\Delta m_{21}^2} = \left[180^\circ\times\left(24490/743\right)/360^\circ - 16\right]\times 360^\circ = 172.964^\circ,\\
\delta_{BF32-} &= -180^\circ\times\frac{\Delta m_{32}^2}{\Delta m_{21}^2} = \left[-180^\circ\times\left(24490/743\right)/360^\circ + 17\right]\times 360^\circ = 187.0255^\circ,\\
\delta_{BF31+} &= 180^\circ\times\frac{\Delta m_{31}^2}{\Delta m_{21}^2} = \left[180^\circ\times\left(25229/743\right)/360^\circ - 16\right]\times 360^\circ = 352.005^\circ,\\
\delta_{BF31-} &= -180^\circ\times\frac{\Delta m_{31}^2}{\Delta m_{21}^2} = \left[-180^\circ\times\left(25229/743\right)/360^\circ + 17\right]\times 360^\circ = 7.994^\circ.
\end{aligned} \tag{73}$$

Ref. [16]

$$\begin{aligned}
\delta_{BF31+} &= 180^\circ\times\left(25110/743\right) = \left(180^\circ\times\left(25110/743\right)/360^\circ - 16\right)\times 360^\circ = 323.176^\circ\\
\delta_{BF31-} &= -180^\circ\times\left(25110/743\right) = \left(-180^\circ\times\left(25110/743\right)/360^\circ + 17\right)\times 360^\circ = 36.823^\circ\\
\delta_{BF32+} &= 180^\circ\times\left(24369/743\right) = \left(180^\circ\times\left(24369/743\right)/360^\circ - 16\right)\times 360^\circ = 143.660^\circ\\
\delta_{BF32-} &= -180^\circ\times\left(24369/743\right) = \left(-180^\circ\times\left(24369/743\right)/360^\circ + 17\right)\times 360^\circ = 216.339^\circ
\end{aligned} \tag{74}$$

We adopted $\Delta m_{21}^2 = 7.43\times10^{-5}\,\text{eV}^2$ as the exact value and Δm_{32}^2, Δm_{31}^2 represent unknown values that are related to the adopted exact value and we determine them from the following equations:

$$\begin{aligned}
&\delta_{BF32+} = \delta_{BF31-} \to 180^\circ\times\frac{\Delta m_{32}^2}{\Delta m_{21}^2} = -180^\circ\times\frac{\Delta m_{31}^2}{\Delta m_{21}^2} \to\\
&\left[\Delta m_{32}^2/\left(7.43\times10^{-5}\,\text{eV}^2\right)\right]\frac{1}{2} - 16 = -\left[\Delta m_{32}^2/\left(7.43\times10^{-5}\,\text{eV}^2\right) + 1\right]\frac{1}{2} + 17 \to\\
&\Delta m_{32}^2 = 32.5\times7.43\times10^{-5}\,\text{eV}^2 = 2.41475\times10^{-3}\,\text{eV}^2;\\
&\delta_{BF31+} = \delta_{BF32-} \to 180^\circ\times\frac{\Delta m_{31}^2}{\Delta m_{21}^2} = -180^\circ\times\frac{\Delta m_{32}^2}{\Delta m_{21}^2} \to\\
&\left[\Delta m_{31}^2/\left(7.43\times10^{-5}\,\text{eV}^2\right)\right]\frac{1}{2} - 16 = -\left[\Delta m_{31}^2/\left(7.43\times10^{-5}\,\text{eV}^2\right) - 1\right]\frac{1}{2} + 17 \to\\
&\Delta m_{31}^2 = 33.5\times7.43\times10^{-5}\,\text{eV}^2 = 2.48905\times10^{-3}\,\text{eV}^2;\\
&\frac{\Delta m_{31}^2}{\Delta m_{21}^2} + \frac{\Delta m_{32}^2}{\Delta m_{21}^2} = 66.
\end{aligned} \tag{75}$$

Now we use the values (75) and the calculated values for the Dirac CP violating phases are:

$$\begin{aligned}
&\delta_{BF31+} = 180^\circ \times (24890.5/743) = (180^\circ \times (24890.5/743)/360^\circ - 16) \times 360^\circ = 270^\circ \\
&\delta_{BF31-} = -180^\circ \times (24890.5/743) = (-180^\circ \times (24890.5/743)/360^\circ + 17) \times 360^\circ = 90^\circ \\
&\delta_{BF32+} = 180^\circ \times (24147.5/743) = (180^\circ \times (24147.5/743)/360^\circ - 16) \times 360^\circ = 90^\circ \\
&\delta_{BF32-} = -180^\circ \times (24147.5/743) = (-180^\circ \times (24147.5/743)/360^\circ + 17) \times 360^\circ = 270^\circ \\
&\delta_{BF31+} - \delta_{BF32-} = 0.
\end{aligned} \tag{76}$$

Therefore, by looking for a common solution for both sources of information Refs. [14] [15] [16] we can project the following parameters for three neutrinos, which should be aimed for during measurements:

$$\begin{aligned}
&(\Delta m_{21}^2)_{BF} = 7.43 \times 10^{-5}\ \mathrm{eV}^2, \\
&(\Delta m_{32}^2)_{BF} = 2.41475 \times 10^{-3}\ \mathrm{eV}^2, \\
&(\Delta m_{31}^2)_{BF} = 2.48905 \times 10^{-3}\ \mathrm{eV}^2; \\
&\delta_{BF31+} = \delta_{BF32-} = 270^\circ.
\end{aligned} \tag{77}$$

8.2. Projected Values to Aim for in Experimental Measurements

Ref. [14] [15]

$$\begin{aligned}
&\left[(\Delta m_{21}^2)_{BF}\right]_{th} - \left[(\Delta m_{21}^2)_{BF}\right]_{exp} = (7.43 - 7.39) \times 10^{-5}\ \mathrm{eV}^2 = 0.04 \times 10^{-5}\ \mathrm{eV}^2 \to \\
&\frac{\left[(\Delta m_{21}^2)_{BF}\right]_{exp}}{10^{-5}\ \mathrm{eV}^2} = 7.39^{+0.04} \\
&\left[(\Delta m_{32}^2)_{BF}\right]_{exp} - \left[(\Delta m_{32}^2)_{BF}\right]_{th} = (2.4490 - 2.41475) \times 10^{-3}\ \mathrm{eV}^2 \to \\
&\frac{\left[(\Delta m_{32}^2)_{BF}\right]_{exp}}{10^{-3}\ \mathrm{eV}^2} = 2.4490_{-0.03425} \\
&\left[(\Delta m_{31}^2)_{BF}\right]_{exp} - \left[(\Delta m_{31}^2)_{BF}\right]_{th} = (2.5229 - 2.48905) \times 10^{-3}\ \mathrm{eV}^2 \to \\
&\frac{\left[(\Delta m_{31}^2)_{BF}\right]_{exp}}{10^{-3}\ \mathrm{eV}^2} = 2.5229_{-0.03385} \\
&\delta_{BF31+} = \delta_{BF32-} = 270^\circ \to \delta_{\exp} = 222^{+38}_{-28} \to \delta_{th} = 270^\circ = \left[(222)^{+48}\right]_{exp}.
\end{aligned} \tag{78}$$

Ref. [16]

$$\begin{aligned}
&\left[(\Delta m_{21}^2)_{BF}\right]_{th} - \left[(\Delta m_{21}^2)_{BF}\right]_{exp} = (7.43 - 7.41) \times 10^{-5}\ \mathrm{eV}^2 = 0.02 \times 10^{-5}\ \mathrm{eV}^2 \to \\
&\frac{\left[(\Delta m_{21}^2)_{BF}\right]_{exp}}{10^{-5}\ \mathrm{eV}^2} = 7.41^{+0.02} \\
&\left[(\Delta m_{32}^2)_{BF}\right]_{\exp} - \left[(\Delta m_{32}^2)_{BF}\right]_{th} = (2.4369 - 2.41475) \times 10^{-3}\ \mathrm{eV}^2 \to \\
&\frac{\left[(\Delta m_{32}^2)_{BF}\right]_{exp}}{10^{-3}\ \mathrm{eV}^2} = 2.4490_{-0.02215}
\end{aligned}$$

$$\left[\left(\Delta m_{31}^2\right)_{BF}\right]_{exp} - \left[\left(\Delta m_{31}^2\right)_{BF}\right]_{th} = (2.5110 - 2.48905) \times 10^{-3}\ \text{eV}^2 \rightarrow$$

$$\frac{\left[\left(\Delta m_{31}^2\right)_{BF}\right]_{exp}}{10^{-3}\ \text{eV}^2} = 2.5110_{-0.02195} \tag{79}$$

$$\delta_{BF31+} = \delta_{BF32-} = \delta_{th} = 270^{\circ}$$

$$\delta_{exp} = \left(197_{-25}^{+42}\right)^0 \rightarrow \delta_{th} = \left(197^{+73}\right)_{exp}^0 .$$

Based on the calculations performed using the information sources [14] [15] [16] and the project assignment, we could conclude that the most optimal numerical value between the neutrino parameters could be represented by the following relationship:

$$\frac{\Delta m_{31}^2}{\Delta m_{21}^2} + \frac{\Delta m_{32}^2}{\Delta m_{21}^2} = 66. \tag{80}$$

9. Neutrino Mass Eigenstates and the Effective Value for the Majorana Neutrino Mass

By introducing into research two formulas for Dirac's CP violating phases that differ from each other in sign, it led to the formation of the unification equation and then to the unique root of that equation.

From a mathematical point of view, both formulas must satisfy both the unification equation and the particular equations.

Namely, by the simultaneous inclusion in the calculation of both formula, a unique result for the Dirac CP violating phase of 270 degrees was reached.

First of all, by including that value, there was a change in the numerical values that are displayed in **Table 2** and **Table 4**, and we will use them to calculate the numerical values for neutrino mass eigenstates. For calculations, we will use the formulas derived in Ref. [6].

And secondly, we emphasize that this value could have a deep physical meaning because the highest possible value for the Jarlskog invariant, which is $J_{CP} = J_{CP}^{nax} \sin 270^{\circ} = -J_{CP}^{nax}$, is associated with Dirac's CP violating phase of 270 degrees.

These formulas are derived by applying the rules for the sum of neutrino masses for Group A_4 Seesaw Type Weinberg Matrix M_{ν} [20].

9.1. Formulas for Calculating Numerical Values for Neutrino Mass Eigentates [6]

$$m_1 = \frac{2\Delta m_{32}^2 - 3/2\Delta m_{31}^2}{\sqrt{4\Delta m_{32}^2 - 2\Delta m_{31}^2}} = 0.016260\ \text{eV} \tag{81}$$

$$m_2 = -\frac{m_1}{2} + \frac{1}{2}\sqrt{m_1^2 + \Delta m_{31}^2} = 0.018393\ \text{eV} \tag{82}$$

$$m_3 = 2m_2 + m_1 = 0.0530467\ \text{eV} \tag{83}$$

$$\sum m_i = m_1 + m_2 + m_3 \approx 0.0877\ \text{eV}. \tag{84}$$

9.2. Effective Value for the Majorana Neutrino Mass

The formula for calculating the effective value for the Majorana neutrino mass reads:

$$\begin{aligned}|m_{ee}| &= \left|m_1 C_{12}^2 C_{13}^2 + m_2 S_{12}^2 C_{13}^2 \exp(i\alpha_{21}) + m_3 S_{13}^2 \exp\left[i(\alpha_{31} - 2\delta_{CP})\right]\right| \\ &= \left|m_1 C_{12}^2 C_{13}^2 + m_2 S_{12}^2 C_{13}^2 \exp(i\pi) + m_3 S_{13}^2 \exp\left[i\left(0 - 2\times 270^\circ\right)\right]\right| \\ &= \left|m_1 C_{12}^2 C_{13}^2 - m_2 S_{12}^2 C_{13}^2 - m_3 S_{13}^2\right| \approx 0.0042\ \text{eV}. \\ \alpha_{21} &= \pi, \alpha_{31} = 0, \delta_{CP} = 270^\circ.\end{aligned} \tag{85}$$

These formulas are derived by applying the rules for the sum of neutrino masses for Group A_4 Seesaw Type Weinberg Matrix M_ν.

9.3. Formulas for Calculating Numerical Values for Neutrino Mass Eigentates

$$m_1 = \frac{2\Delta m_{32}^2 - 3/2\Delta m_{31}^2}{\sqrt{4\Delta m_{32}^2 - 2\Delta m_{31}^2}} = 0.015996\ \text{eV} \tag{86}$$

$$m_2 = -\frac{m_1}{2} + \frac{1}{2}\sqrt{m_1^2 + \Delta m_{31}^2} = 0.0181657\ \text{eV} \tag{87}$$

$$m_3 = 2m_2 + m_1 = 0.052328\ \text{eV} \tag{88}$$

$$\sum m_i = m_1 + m_2 + m_3 \approx 0.0865\ \text{eV}. \tag{89}$$

9.4. Effective Value for the Majorana Neutrino Mass

The formula for calculating the effective value for the Majorana neutrino mass reads:

$$\begin{aligned}|m_{ee}| &= \left|m_1 C_{12}^2 C_{13}^2 + m_2 S_{12}^2 C_{13}^2 \exp(i\alpha_{21}) + m_3 S_{13}^2 \exp\left[i(\alpha_{31} - 2\delta_{CP})\right]\right| \\ &= \left|m_1 C_{12}^2 C_{13}^2 + m_2 S_{12}^2 C_{13}^2 \exp(i\pi) + m_3 S_{13}^2 \exp\left[i\left(0 - 2\times 270^\circ\right)\right]\right| \\ &= \left|m_1 C_{12}^2 C_{13}^2 - m_2 S_{12}^2 C_{13}^2 - m_3 S_{13}^2\right| \approx 0.0041\ \text{eV}. \\ \alpha_{21} &= \pi, \alpha_{31} = 0, \delta_{CP} = 270^\circ.\end{aligned} \tag{90}$$

9.5. Accounting Values According to the Project Assignment

9.5.1. Formulas for Calculating Numerical Values for Neutrino Mass Eigentates

$$m_1 = \frac{2\Delta m_{32}^2 - 3/2\Delta m_{31}^2}{\sqrt{4\Delta m_{32}^2 - 2\Delta m_{31}^2}} = 0.016018\ \text{eV} \tag{91}$$

$$m_2 = -\frac{m_1}{2} + \frac{1}{2}\sqrt{m_1^2 + \Delta m_{31}^2} = 0.018190\ \text{eV} \tag{92}$$

$$m_3 = 2m_2 + m_1 = 0.052398\ \text{eV} \tag{93}$$

$$\sum m_i = m_1 + m_2 + m_3 \approx 0.08660\ \text{eV}. \tag{94}$$

9.5.2. Effective Value for the Majorana Neutrino Mass

Approximate mean values for mixing angles that include both sources of information are also taken here.

The formula for calculating the effective value for the Majorana neutrino mass reads:

$$
\begin{aligned}
|m_{ee}| &= \left| m_1 \left\langle C_{12}^2 C_{13}^2 \right\rangle + m_2 \left\langle S_{12}^2 C_{13}^2 \right\rangle \exp(i\alpha_{21}) + m_3 \left\langle S_{13}^2 \right\rangle \exp\left[i\left(\alpha_{31} - 2\delta_{CP}\right)\right] \right| \\
&= \left| m_1 \left\langle C_{12}^2 C_{13}^2 \right\rangle + m_2 \left\langle S_{12}^2 C_{13}^2 \right\rangle \exp(i\pi) + m_3 \left\langle S_{13}^2 \right\rangle \exp\left[i\left(0 - 2\times 270^\circ\right)\right] \right| \\
&= \left| m_1 \left\langle C_{12}^2 C_{13}^2 \right\rangle - m_2 \left\langle S_{12}^2 C_{13}^2 \right\rangle - m_3 \left\langle S_{13}^2 \right\rangle \right| \approx 0.00424\ \text{eV};
\end{aligned}
\tag{95}
$$

$$\alpha_{21} = \pi, \alpha_{31} = 0, \delta_{CP} = 270^\circ.$$

10. Conclusions

To derive the equation of motion for three neutrinos, we use the rule for the sum of the probabilities of neutrino oscillation (8, 9, 10) for all three possible transitions: $\nu_e \to \nu_\mu, \nu_e \to \nu_\tau, \nu_e \to \nu_e$; $\nu_\mu \to \nu_e, \nu_\mu \to \nu_\tau, \nu_\mu \to \nu_\mu$ and $\nu_\tau \to \nu_e; \nu_\tau \to \nu_\mu, \nu_\tau \to \nu_\tau$.

By applying the standard PMNS matrix, the motion equations were obtained in which the Dirac CP violation phase appears as an unknown quantity.

One might expect that by solving those equations, completely identical formulas for Dirac's CP violation phase would be obtained.

However, this did not happen, formulas for Dirac's CP violation phase were obtained, which differ in sign, for the following transitions:

1) For the transition $\nu_e \to \nu_\mu, \nu_e \to \nu_\tau, \nu_e \to \nu_e$, the formula is obtained in the form:

$$\delta = \pi \frac{\Delta m_{31}^2}{\Delta m_{21}^2}, \delta = \pi \frac{\Delta m_{32}^2}{\Delta m_{21}^2}.$$

2) For transitions $\nu_\mu \to \nu_e, \nu_\mu \to \nu_\tau, \nu_\mu \to \nu_\mu$ and $\nu_\tau \to \nu_e, \nu_\tau \to \nu_\mu, \nu_\tau \to \nu_\tau$, the following mutually identical formulas were obtained, but which are opposite in sign compared to transition $\nu_e \to \nu_\mu, \nu_e \to \nu_\tau, \nu_e \to \nu_e$:

$$\delta = -\pi \frac{\Delta m_{31}^2}{\Delta m_{21}^2}, \delta = -\pi \frac{\Delta m_{32}^2}{\Delta m_{21}^2}.$$

Due to the fact that the oscillations of three neutrinos are reflected with the same value for the Dirac CP violating phase, it would mathematically mean that such transformations should be established that would give the same end result.

Such a mathematical transformation can only be achieved by equating both formulas with opposite signs.

For example, depending on the source of information [14] [15] [16] or the assigned design task, we had the following equations, respectively:

$\delta_{BF31-} - \delta_{BF32+} = 0$, $\delta_{BF31+} - \delta_{BF32-} = 0$, $\delta_{BF31+} - \delta_{BF32-} = 0$.

Applying such a procedure in all the above examples, we obtained a unique result for the Dirac CP violating phase in the amount of 270 degrees and the maximum value for the Jatlskog invariant: $J_{CP} = J_{CP}^{\max} \sin(3\pi/2) = -J_{CP}^{\max}$.

By equating formulas with opposite signs, it essentially describes continuity in neutrino oscillations, regardless of the type of transition.

By specifying a numerical value for the Dirac CP violating phase of 270 degrees, the PMNS (Pontecorvo-Maki-Nakagawa-Sakata) matrix, which until now was written in the general form (16), can now be written with all its elements having numerical values (70) and it looks like this:

$$U_{PMNS} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix}$$
$$= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{13}c_{23} \end{pmatrix}$$
$$= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & +is_{13} \\ -s_{12}c_{23} + ic_{12}s_{23}s_{13} & c_{12}c_{23} + is_{12}s_{23}s_{13} & c_{13}s_{23} \\ s_{12}s_{23} + ic_{12}c_{23}s_{13} & -c_{12}s_{23} + is_{12}c_{23}s_{13} & c_{13}c_{23} \end{pmatrix}.$$

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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