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Spherical and Circular Non-Equatorial Photon Orbits around Kerr Black Holes

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Abstract

By analytically solving the equation of azimuthal null geodesics for spherical photon trajectories, a parametric representation of the corresponding segment of the orbit is obtained. The solution parameter is the latitude coordinate. The dependences of the orbital radius on the black hole spinning parameter and the angle of inclination of its plane with respect to the rotation axis are calculated for flat circular non-equatorial orbits. It is proved that all spherical photon trajectories in the Kerr spacetime are unstable, as well as equatorial ones, and the critical photon orbits in the Schwarzschild metric.

Keywords

Black Holes, Kerr Metric, Photon Trajectories, Null Geodesics, Orbits Instability

1. Introduction

Due to the commissioning of the LIGO and VIRGO gravitational waves observatories, the scientific community has received a powerful new tool for obtaining information about events occurring in the vicinity of black holes and neutron stars. However, as before, the main means of obtaining this information remains electromagnetic, in other words, photon radiation. A recent major success in this direction is the photography of a supermassive object in the heart of the M87 galaxy in the constellation Virgo [1].

A recent analysis of the results obtained by the NuSTAR space observatory showed that all 14 discovered black holes had their own spinning close to the theoretical limit [2].

In this regard, the problem of lightlike (null) and timelike geodesics of photons and massive particles in the spacetime of spinning Karr black holes remains invariably attractive.

The first null geodesics in the Kerr metric were discovered by the authors of Ref. [3]. These were 2 groups of geodesic with prograde and retrograde rotation directions lying in the equatorial plane. The former occupy the region of radii from $1M$ to $3M$, depending on the spinning of the black hole, while the latter occupy the region from $3M$ to $4M$ (M is the mass of the black hole). Hence it is clear that in the Schwarzschild metric in the absence of central body's spinning, the radius of rotation is $3M$. The essential difference, however, is that in the spherically symmetric Schwarzschild metric, any plane can be considered equatorial. In the Kerr metric, due to the lower symmetry, the situation is more complicated. In addition to trajectories that lying in one plane, volumetric trajectories are also possible, and their diversity is large enough. Therefore, in recent years, a lot of works have appeared devoted to geodesics of various types [4]-[9].

Substantial progress has been made in the work [10]. The author managed to find out a combination of trajectory parameters, photons and a black hole characteristics, a combination of which allows specific spherical orbits—those that are characterized by a radius, some constant along the orbit. By numerically solving the system of differential equations of motion obtained earlier, the author calculated a number of spherical trajectories in the Kerr metric. In addition, he found that the direction of rotation of photons around a black hole is entirely determined by the sign of the projection of the angular momentum on the spinning axis. In a later work by this author [11], light-like spherical orbits were considered in detail. In addition, for null geodesics, analytical solutions were obtained using the Mino parameter [12].

An interesting special case of spherical orbits are planar circular orbits similar to equatorial ones, but located at an angle to the rotation axis. The effective orbital inclination angle i had been proposed and used [13] [14] to describe such trajectories. However, only very approximate results have been obtained so far.

This work is structured as follows: in Section II, based on the system of null geodesic equations the motion of a photon along the latitudinal part of the orbit is analytically considered; in Section III the planar circular non-equatorial orbits are considered and their radii are calculated; and finally, in conclusion, the results are summarized.

2. Latitudinal Motion

Geodesics around the Kerr spinning black holes are described by the set of 4 1st order differential equations [3] [15] in spherical coordinates (r,t,φ,θ)

$$\Sigma^2\dot{r}^2 = R(r) = \left[E\left(r^2+a^2\right)-aL_z\right]^2 - \Delta\left[Q+\left(L_z-aE\right)^2\right] \tag{1a}$$

$$\Sigma^2\dot{\theta}^2 = Q-\cos^2\theta\left[L_z^2\sin^{-2}\theta - a^2E^2\right] \tag{1b}$$

$$\Sigma\Delta\dot{\varphi} = L_z\sin^{-2}\theta\left(\Sigma-2Mr\right)+2MarE \tag{1c}$$

$$\Sigma\Delta\dot{t} = a\Delta\left(L_z-aE\sin^2\theta\right)+\left(r^2+a^2\right)\left[E\left(r^2+a^2\right)-aL_z\right] \tag{1d}$$

where a is the spinning parameter meaning angular momentum of the black hole per unit of its mass M. E and Lz are the integrals of motion determining the total energy and component of angular momentum with respect to the axis of rotation, respectively. The third constant Q is the Carter's constant which affects the motion in the latitudinal direction. Upper points mean, as usual, differentiation with respect to the affine parameter. In addition, we have introduced the notation

$$\Sigma \equiv r^2 + a^2\cos^2\theta \quad \text{and} \quad \Delta \equiv r^2 - 2Mr + a^2 \tag{2}$$

In the case of a spherical trajectory, its radius is a constant for given one, which makes it possible to integrate the Equations (1b)-(1d).

As a result of dividing Equation (1b) by the square of Equation (1c) and the corresponding transformations, we obtain

$$\frac{\mathrm{d}\varphi}{\mathrm{d}u} = \frac{\dfrac{a(2Mr - al_z)}{\Delta} + \dfrac{l_z}{1-u^2}}{\sqrt{q - \left(l_z^2 + q - a^2\right)u^2 - a^2u^4}}, \tag{3}$$

where a new variable $u = \cos\theta$ is introduced and denoted $l_z = \dfrac{L_z}{E}$ and $q = \dfrac{Q}{E^2}$.

As a result of integrating this equation using the known formulas for integrals [16] (Eqs. 3.152.3 and 3.157.3), we get the following expression

$$\varphi(u) = \left[\frac{2Mr - al_z}{\Delta} + \frac{l_z}{a(1+w)}\right]\frac{F(\gamma,m)}{\sqrt{v+w}} + \frac{l_z\Pi(c,\gamma,m)}{(1+w)\sqrt{v+w}}, \tag{4}$$

where $F(\gamma,m)$ and $\Pi(c,\gamma,m)$ are incomplete elliptic integrals of the 1st and 3rd kind, respectively, and their parameters are

$$\gamma = \arcsin\left(\frac{u}{\sqrt{m\left(w+u^2\right)}}\right), \quad m = \frac{v}{v+w}{}^{1}, \text{ and } \quad c = m(1+w), \tag{5}$$

v and $-w$ are the roots of biquadrate polynomial under the square root of the Equation (3).

Similarly, from Equations (1b) and (1d), using again [16] (Eq. 3.153.2), one can obtain the following expression

$$\begin{aligned} t(u) = {} & \left[\frac{\left(r^2+a^2\right)^2 - 2Marl_z}{a\Delta} - a\left(1 + \frac{v}{\sqrt{v+w}}\right)\right]F(\gamma,m) \\ & + a\sqrt{v+w}E(\gamma,m) - au\sqrt{\frac{v-u^2}{w+u^2}} \end{aligned} \tag{6}$$

where $E(\gamma,m)$ is an incomplete elliptic integral of the 2nd kind.

Expressions (4) and (6) are in fact a parametric description of the azimuthal motion along a spherical trajectory. **Figure 1(a)** and **Figure 1(b)** show the corresponding part of the trajectory in the cases of the photon rotation in the prograde and retrograde directions, respectively. Trajectory parameters for the first case are (in fact a/M and so on) a = 0.8, r = 2, l_z = 2.525, q = 7 and for the

[1]We use m instead of the usual for elliptic integrals k^2.

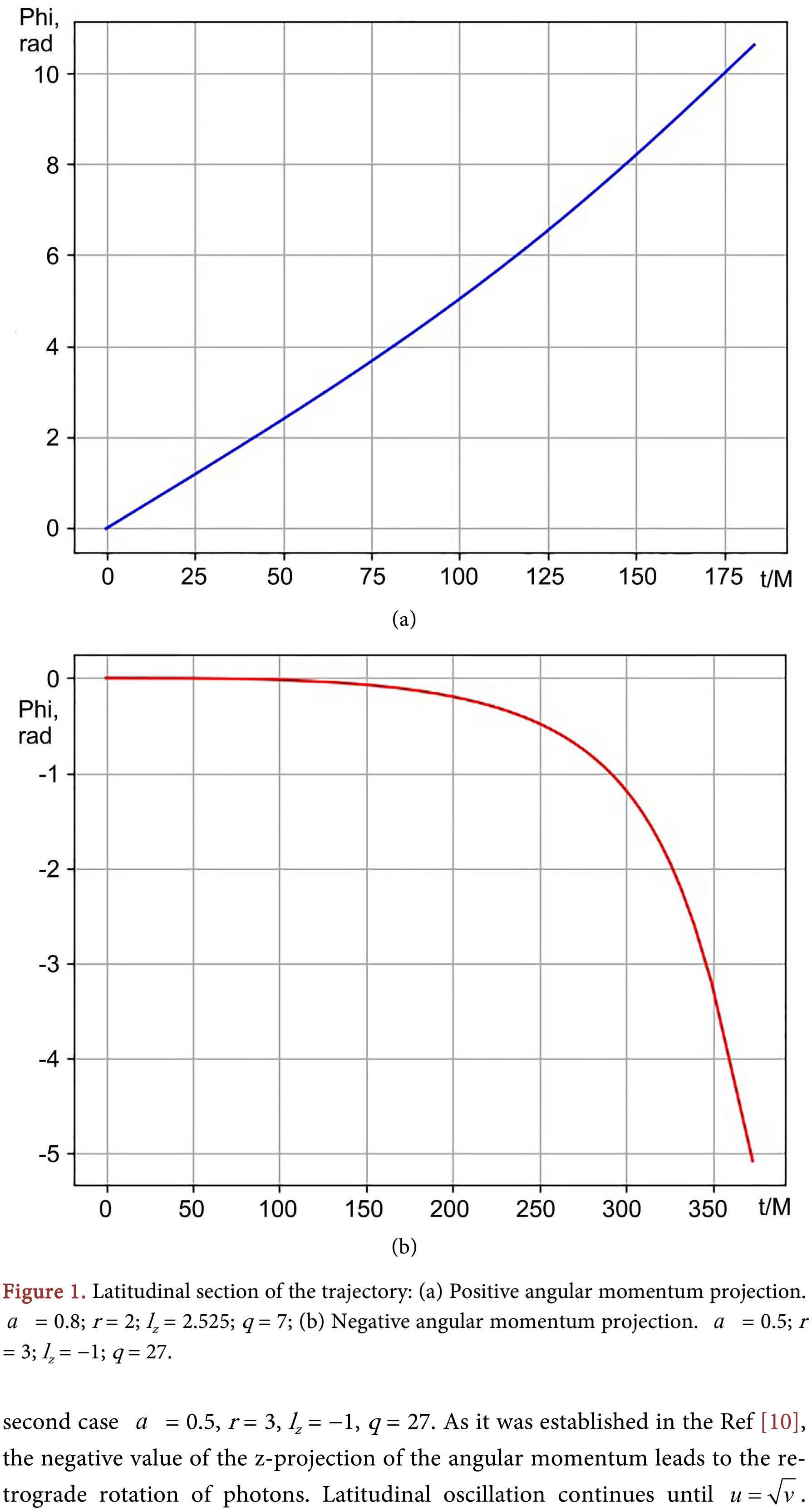

Figure 1. Latitudinal section of the trajectory: (a) Positive angular momentum projection. a = 0.8; r = 2; l_z = 2.525; q = 7; (b) Negative angular momentum projection. a = 0.5; r = 3; $l_z = -1$; q = 27.

second case a = 0.5, r = 3, $l_z = -1$, q = 27. As it was established in the Ref [10], the negative value of the z-projection of the angular momentum leads to the retrograde rotation of photons. Latitudinal oscillation continues until $u = \sqrt{v}$. Wherein, $\gamma = \pi/2$ and the elliptic integrals turn into complete ones. This gives an increment in the azimuthal coordinate $\Delta\varphi$ and in the flight time Δt

$$\Delta\varphi=\frac{4}{\sqrt{v+w}}\left[\left(\frac{2Mr-al_z}{\Delta}+\frac{l_z}{a(1+w)}\right)K(m)+\frac{wl_z\Pi(c,m)}{a(1+w)}\right], \tag{7}$$

$$\Delta t=4\left[\frac{\left(r^2+a^2\right)^2-2Mral_z}{a\Delta}-a\left(1+\frac{v}{\sqrt{v+w}}\right)\right]K(m)+4a\sqrt{v+w}E(m), \tag{8}$$

where K, E and Π are the complete elliptic integrals of the 1st, 2nd and 3rd kind, respectively.

3. Planar Circular Non-Equatorial Orbits

Spherical trajectories must satisfy two conditions simultaneously [Equation (1a)] $R(r)=\frac{\mathrm{d}R}{\mathrm{d}r}=0$ [17]. These two conditions give the fourth and third degrees algebraic equations. In Ref. [8], an attempt was made to solve the problem directly by finding the roots of these equations separately by the usual algebraic solution. However, it turned out that to find the common root of both equations in this way was practically impossible due to the extreme cumbersomeness of these solutions. More successful was the attempt by the author [14] to use the previously introduced [13] an effective angle of inclination of the trajectory. A polynomial equation of the 5th degree was obtained, which also contained a square root which included two terms. The only possible approximate analysis did not give convincing results. However the idea of using this angle did not go unnoticed.

By resolving Equations (1a) together with its derivative with respect to the parameters l_z and q (here and beyond we use values per unit mass, *i.e.* we write r instead of r/M, etc.), the author [10] obtained

$$l_z=-\frac{r^2(r-3)+a^2(r+1)}{a(r-1)} \quad\text{and}\quad q=-\frac{r^4(r-3)^2-4a^2r^3}{a^2(r-1)^2}. \tag{9}$$

It is not difficult to prove that for 2 possible groups of circular equatorial trajectories (prograde and retrograde) with radii [3]

$$r=2\left(1+\cos\left(\frac{2}{3}\arccos\left(\mp|a|\right)\right)\right) \tag{10}$$

$q=0$. At the same time, for polar trajectories with radii [10]

$$r=1+2\sqrt{1-\frac{a^2}{3}}\cos\left(\frac{1}{3}\arccos\frac{1-a^2}{\left(1-\frac{a^2}{3}\right)^{3/2}}\right) \tag{11}$$

$$l_z=0.$$

The question of the stability of spherical null geodesics in the Kerr metric deserves separate consideration. It is known that the critical photon orbit in the Schwarzschild metric is unstable [18]. Stability is determined by the sign of the second derivative of the Equation (1a)

$$R'' = 12r - 2\left(q + l_z^2 - a^2\right) \tag{12}$$

Substituting here the expressions (9) after simplifications, we get

$$R'' = 8r\left(r-1\right) > 0 \tag{13}$$

Hence, it follows that all the corresponding orbits are also unstable with respect to radial perturbations.

It makes sense, to distinguish between equatorial and similar but inclined trajectories. This purpose is served by the previously introduced an effective inclination angle, defined as [13] [14]

$$\cos^2 i = \frac{l_z^2}{l_z^2 + q}. \tag{14}$$

Due to this definition for the equatorial trajectories (q = 0) cos$i$ = ±1 for prograde and retrograde orbits, respectively, whereas i = 0. At the same time i = π/2 for the polar orbits.

After substituting expressions (9) into the formula (14) and the corresponding transformations, we get the 6th degree polynomial expression for determining the radii of this type of spherical orbits [19]

$$r^6 - 6r^5 + \left(9 + 2a^2\lambda\right)r^4 - 4a^2r^3 + \left(a^2 - 6\right)a^2\lambda r^2 + 2a^4\lambda r + a^4\lambda = 0, \tag{15}$$

where $\lambda = \sin^2 i$.

From the Equation (15) it is not difficult to obtain well-known special cases. So for a = 0 we have

$$r^3\left(r-3\right)^2 = 0, \tag{16}$$

from which follows r = 3M, the well-known result for the Schwarzschild metric.

In the case $\lambda = 0$, *i.e.* for the Kerr equatorial orbits Equation (15) turns into the numerator of the equality q = 0 (9) and hence the Equation (10). Finally, when $\lambda = 1$ for polar orbits, Equation (15) takes the form

$$\left(r^3 - 3r^2 + a^2r + a^2\right)^2 = 0, \tag{17}$$

whence follows (11).

Equation (15) implies that the radii of photon orbits generally depend on a^2. This in turn means that they are not affected by the spinning direction of the central body. Therefore, the radii of orbits with direct and reverse rotation are the same, and the statement of the author [14] about their difference is erroneous.

As is well known (Ruffini-Abel theorem), the solution of a polynomial equation with a degree higher than 4 in radicals is impossible. Therefore, Equation (15) was solved numerically. The calculation results are presented in **Figure 2** and **Figure 3**. The first of them shows the dependence of the radius of a circular orbit on the rotation parameter of a black hole for various angles of inclination of the rotation plane between 15 and 60 degrees. The orbit radius increases monotonically with increasing of the spin-parameter a, and increment is especially

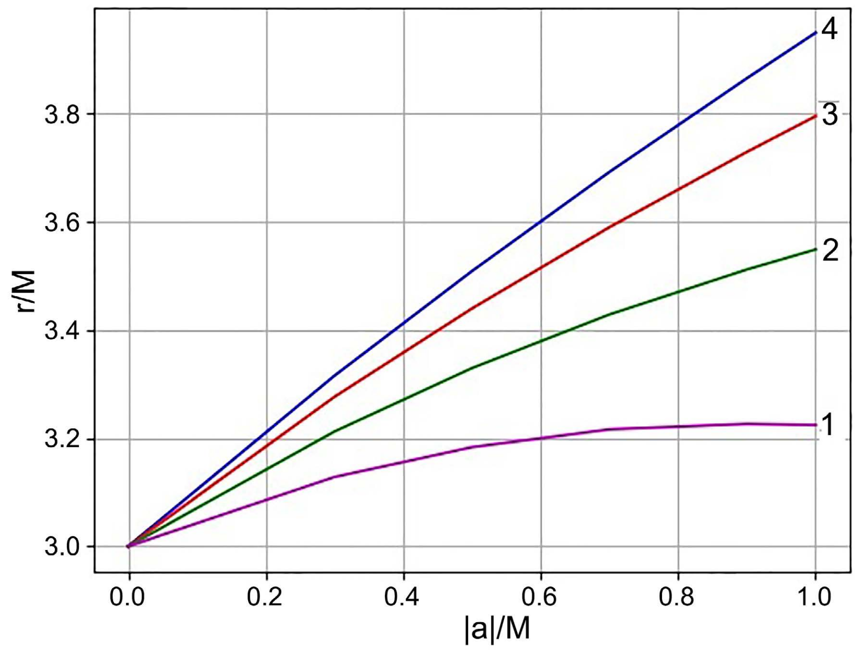

Figure 2. Dependence of the circular radius on spinning parameter for different inclination angles of the orbital plane. 1—60; 2—45; 3—30 and 4—15 degrees.

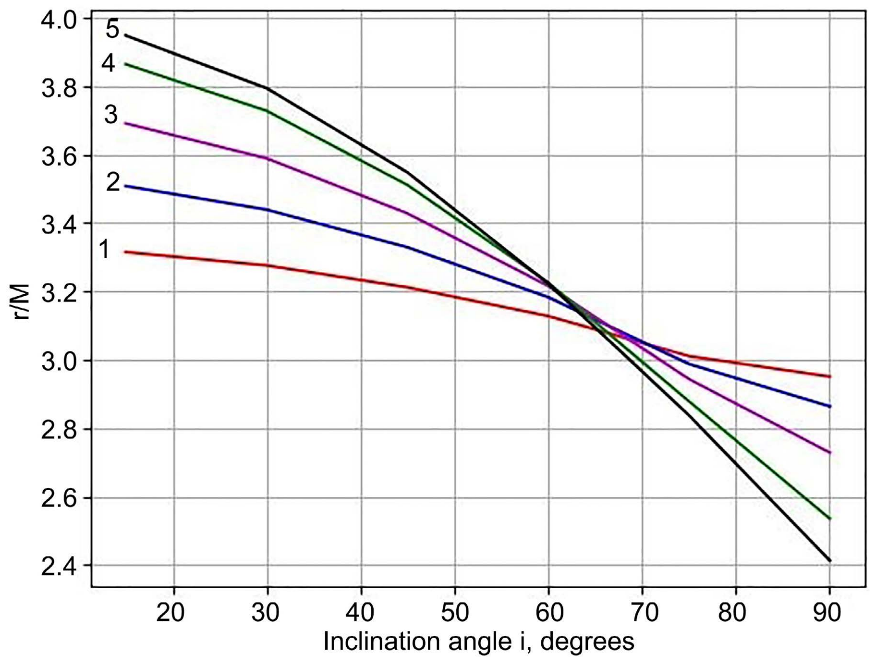

Figure 3. Dependence of circular radius on inclination angle for different spinning parameters. 1— a = 0.3; 2— a = 0.5; 3— a = 0.7; 4— a = 0.9; 5— a = 1.0.

steep for small inclination angles close to the equator. For angles close to the pole, the increment is relatively small. Figure 3 shows the dependence of the radius of the orbit on the tilt of the orbital plane. It is also monotonic and much stronger for spinning parameters a/M close to 1. As a result, the radii of orbits at an orbital inclination of about 65 degrees are approximately the same for all black hole rotation rates.

The range of possible radii is from $2.4M$ to $4.0M$. Our results contradict the conclusions [14] about the presence of non-monotonic sector in the dependences. Dependence $r(a)$ is monotonic, exactly as in the case of equatorial orbits [3].

4. Conclusions

1) By solving the equations of azimuthal motion, parametric solution is found for a section of a spherical photon trajectory around a rotating Kerr black hole without using the Mino parameter. The parameter is the latitudinal coordinate.

2) It has been proved that the motion of photons along spherical trajectories in a Kerr spacetime is unstable with respect to radial perturbations, just like equatorial ones and critical photon trajectory in the Schwarzschild metric.

3) The dependences of the radius of the circular non-equatorial motion of photons on the spinning parameter of the black hole and the angle of inclination of the rotation plane with respect to the rotation axis are obtained. Both dependences are monotonic, which refutes the result of Ref. [14].

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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