Scientific Research Publishing

Journal of High Energy Physics, Gravitation and Cosmology, 2023, 9, 7-26
https://www.scirp.org/journal/jhepgc
ISSN Online: 2380-4335
ISSN Print: 2380-4327

Second Approximation of the Generalized Planetary Equation Based upon Golden Metric Tensors

Nura Yakubu[1*], Hayatu Abbba Ibrahim[1], Musa Hashimu Umar[1], Bappah Alkali[2]

[1]Department of Physics, University of Maiduguri, Borno State, Nigeria
[2]Aminu Saleh College of Education Azare, Bauchi State, Nigeria
Email: *nuraphy@gmail.com

How to cite this paper: Yakubu, N., Ibrahim, H.A., Umar, M.H. and Alkali, B. (2023) Second Approximation of the Generalized Planetary Equation Based upon Golden Metric Tensors. *Journal of High Energy Physics, Gravitation and Cosmology*, **9**, 7-26.
https://doi.org/10.4236/jhepgc.2023.91002

Received: August 7, 2022
Accepted: November 18, 2022
Published: November 21, 2022

Abstract

In this paper, we consider the Post Einstein Planetary equation of motion. We succeeded in offering a solution using second approximation method, in which we obtained eight exact mathematical solutions that rebel amazing theoretical results. To the order of C^{-2}, two of these exact solutions are reduced to the approximate solutions from the method of successive approximations.

Keywords

Golden Matric Tensors, Einstein's Planetary Equation

1. Introduction

Einstein planetary equation can be solved using the method of successive approximation which gives rise to two linearly independent solutions. While solving Post Einstein Planetary Equation, we employed Taylor's series expansion which gives rise to eight linearly independent mathematical solutions.

2. Mathematical Model

Using the expression:

$$\frac{\mathrm{d}^2 u(\varphi)}{\mathrm{d}\varphi^2}+\left(1+\frac{k^2}{c^2 l^2}\right)u(\varphi)-\frac{k}{l^2}=\frac{3ku^2(\varphi)}{c^2} \tag{1}$$

Equation (1) is called Einstein Planetary Equation while the left hand side of the equation is General Relativistic Contribution [1].

Now consider Newton's part of Equation (1) above given by

$$\frac{\mathrm{d}^2u(\varphi)}{\mathrm{d}\varphi^2}+u=\frac{k}{l^2} \tag{2}$$

the complementary solution of Equation (2) is

$$u_c(\varphi)=A\cos(\varphi+\alpha) \text{ or } -B\sin(\varphi+\alpha)$$

particular solution

$$u_p(\varphi)=\frac{k}{l^2}$$

the general solution $=u_c+u_p$

$$u(\varphi)=\frac{k}{l^2}+A\cos(\varphi+\alpha)$$

$$r(\varphi)=\frac{1}{\frac{k}{l^2}+A\cos(\varphi+\alpha)}=\frac{1}{\frac{k}{l^2}\left[1+\varepsilon\cos(\varphi+\alpha)\right]}$$

where

$$\varepsilon=\frac{Al^2}{k}$$

or

$$r(\varphi)=\frac{\frac{l^2}{k}}{1+\varepsilon\cos(\varphi+\alpha)}. \tag{3}$$

Equation (3) is Newton's solution of the planetary equation.

Equation (3) physically corresponds to the polar equation of a conic section. The conic section is characterised by the parameter ε (eccentricity) as follows:

$$\varepsilon>1\Rightarrow \text{hyperbola}$$

$$\varepsilon<1\Rightarrow \text{ellipse}$$

$$\varepsilon=1\Rightarrow \text{parabola}$$

$$\varepsilon=0\Rightarrow \text{circle}.$$

According to these equations, a planet traces the same path throughout its orbit.

To solve the relativistic planetary equation, we employ the method of successive approximation.

In this method, we start with the first iterate which is the solution of Newton's planetary equation

$$r(\varphi)=\frac{l^2}{k}=\frac{1}{1+\varepsilon\cos(\varphi+\alpha)} \tag{4}$$

$$\frac{k}{l^2}\left(1+\varepsilon\cos(\varphi+\alpha)\right) \tag{5}$$

putting this expression for $u(\varphi)$ into the right hand side of Equation (1)

$$\frac{\mathrm{d}^2u(\varphi)}{\mathrm{d}\varphi^2}+u-\frac{k}{l^2}=\frac{3k}{c^2}\left\{\frac{k}{l^2}\left(1+\varepsilon\cos(\varphi+\alpha)\right)\right\}^2 \tag{6}$$

now let perihelion be attained first at $\varphi=0$, then the next perihelion will be attained at an angle φ such that

$$\left(1-\frac{3k^2}{c^2l^2}\right)\varphi=2\pi \quad (\text{phase angle } \alpha=0) \tag{7}$$

or

$$\varphi=2\pi\left(1-\frac{3k^2}{c^2l^2}\right)^{-1} \tag{8}$$

for small $\frac{3k}{c^2l^2}$ compared to (1) and expanding gives

$$\varphi=2\pi\left(1+\frac{3k^2}{c^2l^2}\cdots\right)=2\pi+\frac{6\pi k^2}{c^2l^2}. \tag{9}$$

Consequently, the perihelion of the orbit has advanced beyond that of the first orbit.

The displacement of the planetary orbit from revolution to revolution is called PRECESSION.

Let Δ be the angle through which the planetary orbit is displaced in each revolution, then

$$\Delta=\frac{6\pi G^2M^2}{c^2l^2} \tag{10}$$

where $k=GM$.

The precession of planets in the solar system was discovered as far back as 1845 by a French man Leverier.

While in this paper we employ second approximation method to solve Post Einstein Planetary Equation to obtain exact analytical solutions given as

$$\frac{\mathrm{d}^2u(\varphi)}{\mathrm{d}\varphi^2}+\left(1+\frac{k^2}{c^2l^2}\right)u=\frac{k}{l^2}+\frac{3ku^2(\varphi)}{c^2}.$$

3. Theory

Different methods have been employed to obtain Einstein's planetary equation, one of such is method of successive approximations to obtain two linearly independent approximate solutions.

The aim of this work is to show how post Einstein's planetary equations may be solved analytically. The result is altogether eight linearly independent mathematical solutions altogether. To the order of C^{-2}, two of these exact solutions reduced to the approximate solutions from the method of successive approximations.

Let us seek the exact analytical solution of post Einstein Planetary equation

$$\frac{d^2u(\varphi)}{d\varphi^2}+\left(1+\frac{k^2}{c^2l^2}\right)u(\varphi)-\frac{k}{l^2}=\frac{3ku^2(\varphi)}{c^2} \tag{1}$$

in the form of a successive approximation

$$u(\varphi)=\sum_{n=0}^{\infty}A_n\exp^{\left[n_i(\omega\varphi+\alpha)\right]} \tag{2}$$

where A_n, ω and α are constants. Putting Equation (2) into Equation (1) and applying the linearly independence of the exponential functions, we equate corresponding coefficients on both sides to obtain the following system of equations, from Equation (2) in

$$u(\varphi)=\sum_{n=0}^{\infty}A_n\exp^{\left[n_i(\omega\varphi+\alpha)\right]}$$

$$u(\varphi)=A_0+A_1\exp^{\left[i(\omega\varphi+\alpha)\right]}+A_2\exp^{\left[2i(\omega\varphi+\alpha)\right]}+A_3\exp^{\left[3i(\omega\varphi+\alpha)\right]} \tag{3}$$

$$u'(\varphi)=i\omega A_1\exp^{\left[i(\omega\varphi+\alpha)\right]}+2i\omega A_2\exp^{\left[2i(\omega\varphi+\alpha)\right]}+3i\omega A_3\exp^{\left[3i(\omega\varphi+\alpha)\right]} \tag{4}$$

$$u''(\varphi)=-\omega^2A_1\exp^{\left[i(\omega\varphi+\alpha)\right]}-4\omega^2A_2\exp^{\left[2i(\omega\varphi+\alpha)\right]}-9\omega^2A_3\exp^{\left[3i(\omega\varphi+\alpha)\right]} \tag{5}$$

$$\begin{aligned}u^2(\varphi)=&\left\{A_0+A_1\exp^{\left[i(\omega\varphi+\alpha)\right]}+A_2\exp^{\left[2i(\omega\varphi+\alpha)\right]}+A_3\exp^{\left[3i(\omega\varphi+\alpha)\right]}\right\}\\&\times\left\{A_0+A_1\exp^{\left[i(\omega\varphi+\alpha)\right]}+A_2\exp^{\left[2i(\omega\varphi+\alpha)\right]}+A_3\exp^{\left[3i(\omega\varphi+\alpha)\right]}\right\}\end{aligned} \tag{6}$$

$$\begin{aligned}u^2(\varphi)=&A_0^2+A_0A_1\exp^{\left[i(\omega\varphi+\alpha)\right]}+A_0A_2\exp^{\left[2i(\omega\varphi+\alpha)\right]}+A_0A_3\exp^{\left[3i(\omega\varphi+\alpha)\right]}\\&+A_0A_1\exp^{\left[i(\omega\varphi+\alpha)\right]}+A_1^2\exp^{\left[2i(\omega\varphi+\alpha)\right]}+A_1A_2\exp^{\left[3i(\omega\varphi+\alpha)\right]}\\&+A_1A_3\exp^{\left[4i(\omega\varphi+\alpha)\right]}+A_0A_2\exp^{\left[2i(\omega\varphi+\alpha)\right]}+A_1A_2\exp^{\left[3i(\omega\varphi+\alpha)\right]}\\&+A_2^2\exp^{\left[4i(\omega\varphi+\alpha)\right]}+A_2A_3\exp^{\left[5i(\omega\varphi+\alpha)\right]}+A_0A_3\exp^{\left[3i(\omega\varphi+\alpha)\right]}\\&+A_1A_3\exp^{\left[4i(\omega\varphi+\alpha)\right]}+A_2A_3\exp^{\left[5i(\omega\varphi+\alpha)\right]}+A_3^2\exp^{\left[6i(\omega\varphi+\alpha)\right]}\end{aligned} \tag{7}$$

$$\begin{aligned}u^2(\varphi)=&A_0^2+2A_0A_1\exp^{\left[i(\omega\varphi+\alpha)\right]}+\left(A_1^2+2A_0A_2\right)\exp^{\left[2i(\omega\varphi+\alpha)\right]}\\&+\left(2A_0A_3+2A_1A_2\right)\exp^{\left[3i(\omega\varphi+\alpha)\right]}+\left(A_2^2+2A_1A_3\right)\exp^{\left[4i(\omega\varphi+\alpha)\right]}\\&+2A_2A_3\exp^{\left[5i(\omega\varphi+\alpha)\right]}+A_3^2\exp^{\left[6i(\omega\varphi+\alpha)\right]}+\cdots\end{aligned} \tag{8}$$

putting u,u^2 and u'' Equation (1) we get

$$\begin{aligned}&-\omega^2A_1\exp^{\left[i(\omega\varphi+\alpha)\right]}-4\omega^2A_2\exp^{\left[2i(\omega\varphi+\alpha)\right]}-9\omega^2A_3\exp^{\left[3i(\omega\varphi+\alpha)\right]}\\&+\left(1+\frac{k^2}{c^2l^2}\right)\left[A_0+A_1\exp^{\left[i(\omega\varphi+\alpha)\right]}+A_2\exp^{\left[2i(\omega\varphi+\alpha)\right]}+A_3\exp^{\left[3i(\omega\varphi+\alpha)\right]}\right]-\frac{k}{l^2}\\&=\frac{3k}{c^2}\left[A_0^2+2A_0A_1\exp^{\left[i(\omega\varphi+\alpha)\right]}+\left(A_1^2+2A_0A_2\right)\exp^{\left[2i(\omega\varphi+\alpha)\right]}\right.\\&\left.+\left(2A_0A_3+2A_1A_2\right)\exp^{\left[3i(\omega\varphi+\alpha)\right]}+\cdots\right]\end{aligned} \tag{9}$$

for $n=0$ in above equation, we get

$$0+\left(1+\frac{k^2}{c^2l^2}\right)A_0-\frac{k}{l^2}=\frac{3k}{c^2}A_0^2$$

$$\Rightarrow\frac{3k}{c^2}A_0^2-\left(1+\frac{k^2}{c^2l^2}\right)A_0+\frac{k}{l^2}=0 \tag{10}$$

Equation (10) above is a quadratic equation in A_0, compare it with general equation rule

$$ax^2 + bx + c = 0$$

$$\Rightarrow a = \frac{3k}{c^2}, b = -\left(1 + \frac{k^2}{c^2 l^2}\right), c = \frac{k}{l^2}$$

$$\therefore A_0 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-\left(1 + \frac{k^2}{c^2 l^2}\right) \pm \sqrt{\left(1 + \frac{k^2}{c^2 l^2}\right)^2 - \frac{12k^2}{c^2 l^2}}}{2\left(\frac{3k}{c^2}\right)}$$

$$= \frac{-\left(1 + \frac{k^2}{c^2 l^2}\right) \pm \left(1 + \frac{2k^2}{c^2 l^2} + \frac{k^4}{c^4 l^4} - \frac{12k^2}{c^2 l^2}\right)^{1/2}}{6\left(\frac{k}{c^2}\right)}$$

$$= \frac{\left(1 + \frac{k^2}{c^2 l^2}\right) \pm \left(1 - \frac{10k^2}{c^2 l^2} + 0c^{-4}\right)^{1/2}}{6\left(\frac{k}{c^2}\right)}$$

$$A_0 = A_{0^-} = \frac{c^2}{6k}\left[\left(1 + \frac{k^2}{c^2 l^2}\right) - \left(1 - \frac{10k^2}{c^2 l^2}\right)^{1/2}\right] \tag{11a}$$

and

$$A_0 = A_{0^+} = \frac{c^2}{6k}\left[\left(1 + \frac{k^2}{c^2 l^2}\right) + \left(1 - \frac{10k^2}{c^2 l^2}\right)^{1/2}\right] \tag{11b}$$

for $n = 1$, from Equation (9) we get

$$-\omega^2 A_1 \exp^{[i(\omega\varphi+\alpha)]} + \left(1 + \frac{k^2}{c^2 l^2}\right) A_1 \exp^{[i(\omega\varphi+\alpha)]} = \frac{6k}{c^2} A_0 A_1 \exp^{[i(\omega\varphi+\alpha)]}$$

or

$$\left[-\omega^2 + \left(1 + \frac{k^2}{c^2 l^2}\right) - \frac{6k}{c^2} A_0\right] A_1 \exp^{[i(\omega\varphi+\alpha)]} = 0$$

either

$$A_1 \exp^{[i(\omega\varphi+\alpha)]} = 0$$

$$A_1 \cdot 0 = 0$$

$\Rightarrow A_1$ is an arbitrary constant.

$$-\omega^2 + \left(1 + \frac{k^2}{c^2 l^2}\right) - \frac{6k}{c^2} A_0 = 0 \tag{12a}$$

or

$$\omega^2 - \left(1 + \frac{k^2}{c^2 l^2}\right) + \frac{6k}{c^2} A_0 = 0 \tag{12b}$$

or

$$\omega^2 = \left(1+\frac{k^2}{c^2l^2}\right) - \frac{6k}{c^2}A_0 \tag{12c}$$

for $n = 2$ in Equation (9) we obtain

$$\left[-4\omega^2 + \left(1+\frac{k^2}{c^2l^2}\right)\right]A_2 \exp^{[2i(\omega\varphi+\alpha)]} = \frac{3k}{c^2}\left(A_1^2 + 2A_0A_2\right)\exp^{[2i(\omega\varphi+\alpha)]}$$

$$\left[-4\omega^2 + \left(1+\frac{k^2}{c^2l^2}\right)\right]A_2 = \frac{3k}{c^2}A_1^2 + \frac{6k}{c^2}A_0A_2$$

or

$$\left[\left(1+\frac{k^2}{c^2l^2}\right) - 2^2\omega^2 - \frac{6k}{c^2}A_0\right]A_2 = \frac{3k}{c^2}A_1^2$$

$$A_2 = \frac{3k}{c^2}\left[\left(1+\frac{k^2}{c^2l^2}\right) - 2^2\omega^2 - \frac{6k}{c^2}A_0\right]^{-1}A_1^2 \tag{13}$$

next for $n = 3$ in Equation (9) we get

$$-9\omega^2 A_3 \exp^{[3i(\omega\varphi+\alpha)]} + \left(1+\frac{k^2}{c^2l^2}\right)A_3 \exp^{[3i(\omega\varphi+\alpha)]}$$
$$= \frac{3k}{c^2}\left(2A_0A_3 + 2A_1A_2\right)\exp^{[3i(\omega\varphi+\alpha)]}$$

or

$$-9\omega^2 A_3 + \left(1+\frac{k^2}{c^2l^2}\right)A_3 = \frac{3k}{c^2}\left(2A_0A_3 + 2A_1A_2\right)$$

or

$$A_3 = \left[\left(1+\frac{k^2}{c^2l^2}\right) - 3^2\omega^2 - \frac{6k}{c^2}A_0\right]^{-2}A_1A_2 \tag{14}$$

putting the rule of A_2 from Equation (13) into Equation (14)

$$\begin{aligned} A_3 = \frac{(3)(6)k^2}{c^2}&\left[\left(1+\frac{k^2}{c^2l^2}\right) - 3^2\omega^2 - \frac{6k}{c^2}A_0\right]^{-1} \\ &\times\left[\left(1+\frac{k^2}{c^2l^2}\right) - 2^2\omega^2 - \frac{6k}{c^2}A_0\right]^{-1}A_1^3 \end{aligned} \tag{15}$$

putting Equation (11a) into Equation (12) we get

$$\omega^2 = \left(1+\frac{k^2}{c^2l^2}\right) - \frac{6k}{c^2}A_0 \tag{16}$$

$$\omega^2 = \left(1+\frac{k^2}{c^2l^2}\right) - \frac{6k}{c^2}\times\frac{c^2}{6k}\left[\left(1+\frac{k^2}{c^2l^2}\right) - \left(1-\frac{10k^2}{c^2l^2}\right)^{1/2}\right] \tag{17}$$

$$\omega^2 = \left(1+\frac{k^2}{c^2l^2}\right) - \left[\left(1+\frac{k^2}{c^2l^2}\right) - \left(1-\frac{10k^2}{c^2l^2}\right)^{1/2}\right] \tag{18}$$

$$\omega=\pm\left(1+\frac{k^2}{c^2l^2}\right)-\left[\left(1+\frac{k^2}{c^2l^2}\right)-\left(1-\frac{10k^2}{c^2l^2}\right)^{1/2}\right]^{1/2} \tag{19}$$

$$\omega=\omega\alpha=\left(1+\frac{k^2}{c^2l^2}\right)-\left[\left(1+\frac{k^2}{c^2l^2}\right)-\left(1-\frac{10k^2}{c^2l^2}\right)^{1/2}\right]^{1/2} \tag{20}$$

$$\omega=\omega\beta=\left(1+\frac{k^2}{c^2l^2}\right)-\left[\left(1+\frac{k^2}{c^2l^2}\right)-\left(1-\frac{10k^2}{c^2l^2}\right)^{1/2}\right]^{1/2} \tag{21}$$

similarly, putting Equation (11b) into Equation (12) we get

$$\omega^2=\left(1+\frac{k^2}{c^2l^2}\right)-\frac{6k}{c^2}A_{0^+} \tag{22}$$

$$\omega^2=\left(1+\frac{k^2}{c^2l^2}\right)-\frac{6k}{c^2}\times\frac{c^2}{6k}\left[\left(1+\frac{k^2}{c^2l^2}\right)+\left(1-\frac{10k^2}{c^2l^2}\right)^{1/2}\right] \tag{23}$$

$$\omega=\pm\left(1+\frac{k^2}{c^2l^2}\right)-\left[\left(1+\frac{k^2}{c^2l^2}\right)+\left(1-\frac{10k^2}{c^2l^2}\right)^{1/2}\right]^{1/2} \tag{24}$$

$$\omega=\omega_\gamma=\left(1+\frac{k^2}{c^2l^2}\right)-\left[\left(1+\frac{k^2}{c^2l^2}\right)+\left(1-\frac{10k^2}{c^2l^2}\right)^{1/2}\right]^{1/2} \tag{25}$$

$$\omega=\omega_\gamma=-\left(1+\frac{k^2}{c^2l^2}\right)-\left[\left(1+\frac{k^2}{c^2l^2}\right)+\left(1-\frac{10k^2}{c^2l^2}\right)^{1/2}\right]^{1/2} \tag{26}$$

it now follows from Equation (13) that A_2 has eight possible values given by

$$A_2=A_{2\alpha^-}=\frac{3k}{c^2}\left[\left(1+\frac{k^2}{c^2l^2}\right)-2^2\omega_\alpha^2-\frac{6k}{c^2}A_{0^-}\right]^{-1}A_1^2 \tag{27}$$

$$A_2=A_{2\alpha^+}=\frac{3k}{c^2}\left[\left(1+\frac{k^2}{c^2l^2}\right)-2^2\omega_\alpha^2-\frac{6k}{c^2}A_{0^+}\right]^{-1}A_1^2 \tag{28}$$

$$A_2=A_{2\beta^-}=\frac{3k}{c^2}\left[\left(1+\frac{k^2}{c^2l^2}\right)-2^2\omega_\beta^2-\frac{6k}{c^2}A_{0^-}\right]^{-1}A_1^2 \tag{29}$$

$$A_2=A_{2\beta^+}=\frac{3k}{c^2}\left[\left(1+\frac{k^2}{c^2l^2}\right)-2^2\omega_\beta^2-\frac{6k}{c^2}A_{0^-}\right]^{-1}A_1^2 \tag{30}$$

$$A_2=A_{2\gamma^-}=\frac{3k}{c^2}\left[\left(1+\frac{k^2}{c^2l^2}\right)-2^2\omega_\gamma^2-\frac{6k}{c^2}A_{0^-}\right]^{-1}A_1^2 \tag{31}$$

$$A_2=A_{2\gamma^+}=\frac{3k}{c^2}\left[\left(1+\frac{k^2}{c^2l^2}\right)-2^2\omega_\gamma^2-\frac{6k}{c^2}A_{0^-}\right]^{-1}A_1^2 \tag{32}$$

$$A_2=A_{2\delta^-}=\frac{3k}{c^2}\left[\left(1+\frac{k^2}{c^2l^2}\right)-2^2\omega_\delta^2-\frac{6k}{c^2}A_{0^-}\right]^{-1}A_1^2 \tag{33}$$

$$A_2 = A_{2\delta^+} = \frac{3k}{c^2}\left[\left(1+\frac{k^2}{c^2l^2}\right)-2^2\omega_\delta^2-\frac{6k}{c^2}A_{0^-}\right]^{-1}A_1^2. \quad (34)$$

Similarly it follows from Equation (15) that A_3 has eight possible corresponding values which may be labelled as $A_{3\alpha^-}, A_{3\alpha^+}, A_{3\beta^-}, A_{3\beta^+}, A_{3\gamma^-}, A_{3\gamma^+}, A_{3\delta^-}, A_{3\delta^+}$, respectively. It is now obvious how the above sequence may be continued to derive the eight possible corresponding values for each of the constants $A_4, A_5, \cdots$ in terms of the arbitrary constants A_i.

This sequence implies eight mathematically possible except analytical solutions of Post Einstein's Planetary equation of the form

$$u(\varphi) = A_0 + A_1\exp^{[i(\omega\varphi+\alpha)]} + A_2\exp^{[2i(\omega\varphi+\alpha)]} + \cdots \quad (35)$$

where α and A_1 are arbitrary constants. Consequently, eight exacts and analytical solutions are given by inserting the combinations of Equations (9) in (22) and (24) and

$$A_n = f_{n\alpha^-},\ (n=3,4,5),(9,22,26)$$
$$A_n = f_{n\alpha^+},\ (n=3,4,5),(9,23,27)$$
$$A_n = f_{n\beta^-},\ (n=3,4,5),(9,23,28)$$
$$A_n = f_{n\beta^+},\ (n=3,4,5),(9,23,29)$$
$$A_n = f_{n\gamma^-},\ (n=3,4,5),(9,24,30)$$
$$A_n = f_{n\gamma^+},\ (n=3,4,5),(9,24,31)$$
$$A_n = f_{n\delta^-},\ (n=3,4,5),(9,24,32)$$
$$A_n = f_{n\delta^+},\ (n=3,4,5),(9,24,33)$$

into Equation (14) consecutively.

4. Results

4.1. The First Exact Analytical Solution

Let us consider the first analytical solution corresponding to Equation (21) and Equation (24).

In this case, it follows from Equation (10) that

$$A_n = f_{2\alpha^-}(A_1) = \frac{3k}{c^2}\left[\left(1+\frac{k^2}{c^2l^2}\right)-2^2\omega^2-\frac{6k}{c^2}A_0\right]^{-1}A_1^2 \quad (36)$$

putting Equation (12) into Equation (36) above we get

$$A_2 = \frac{3k}{c^2}\left[-3\left(1+\frac{k^2}{c^2l^2}\right)-2^2\omega^2-\frac{18k}{c^2}A_0\right]^{-1}A_1^2 \quad (37)$$

and

$$A_n = f_{3\alpha^-}(A_1) = 3,4,5. \quad (38)$$

In the case the exact analytical solution of Post Einstein's Planetary equation is a complex function of φ which may be written in Cartesian form as

$$u(\varphi)=x(\varphi)+y(\varphi) \tag{39}$$

where

$$\begin{aligned} x(\varphi)=A_{0^-}+A_1(wx\varphi+\alpha)+f_{2\alpha^-}(A_1)\cos\left[2(wx\varphi+\alpha)\right] \\ +f_{3\alpha^-}(A_1)\sin\left[3(wx\varphi+\alpha)\right] \end{aligned} \tag{40}$$

and

$$\begin{aligned} y(\varphi)=A_1+\sin(wx\varphi+\alpha)+f_{2\alpha^-}(A_1)\sin\left[2(w\varphi+\alpha)\right] \\ +f_{3\alpha^-}(A_1)\sin\left[3(w\varphi+\alpha)\right] \end{aligned} \tag{41}$$

therefore it may be expressed in Euler form as

$$u(\varphi)=R(\varphi)\mathrm{e}^{i\phi(\varphi)} \tag{42}$$

where R is the magnitude given by

$$u(\varphi)=\left[x^2(\varphi)+y^2(\varphi)\right] \tag{43}$$

and φ is the argument given by

$$\phi(\varphi)=\tan^{-1}\left\{\frac{y(\varphi)}{x(\varphi)}\right\} \tag{44}$$

Hence by definition, the instantaneous radial coordinate of the planet from sun, r is given as

$$r(\varphi)=R^{-1}(\varphi)\mathrm{e}^{-i\phi(\varphi)} \tag{45}$$

and it is a complex function.

From Equation (45) and definition, the instantaneous complex radial displacement r of the planet from the sun is given in terms of the angular displacement φ as

$$r(\varphi)=R^{-1}(\varphi)\mathrm{e}^{-i\phi(\varphi)}$$

we shall therefore interpret the magnitude of the instantaneous radial displacement of the planet from the sun as the real physical measurable instantaneous radial displacement r_p given as

$$r_p(\varphi)=R^{-1}(\varphi)=\left[x^2(\varphi)+y^2(\varphi)\right]^{1/2}. \tag{46}$$

It may be noted from Equation (37) that for $n>1$, $f_n(A_1)$ is of order e^{-2n}.

Therefore as the first approximation, let us neglect all terms in $f_n(A_1)$ for $n>1$. Then it follows from Equation (27) and (45) that

$$r(\varphi)=\frac{A}{1+e_1\cos(\omega_\alpha\varphi+\alpha)}. \tag{47}$$

Let

$$x(\varphi)=A_{0^-}+A_1(\omega_\alpha\varphi+\alpha)$$

$$\begin{aligned} x^2(\varphi)&=\left[A_{0^-}+A_1(\omega_\alpha\varphi+\alpha)\right]\times\left[A_{0^-}+A_1(\omega_\alpha\varphi+\alpha)\right] \\ &=A_0^2+2\left[A_{0^-}A_1\cos(\omega_\alpha\varphi+\alpha)\right]+A_1^2\cos^2(\omega_\alpha\varphi+\alpha) \end{aligned}$$

$$r_p(\varphi) = \left[x^2(\varphi) + y^2(\varphi)\right]^{1/2}$$

$$r_p(\varphi) = \left[A_0^2 + 2A_{0^-}A_1\cos(\omega_\alpha\varphi+\alpha) + A_1^2\cos^2(\omega_\alpha\varphi+\alpha) + A_1^2\sin^2(\omega_\alpha\varphi+\alpha)\right]^{1/2}$$

$$= A_{0^-}\left[\left(1+\frac{A_1^2}{A_0^2}\right) + \frac{2A_{0^-}A_1}{A_{0^-}^2}\cos(\omega_\alpha\varphi+\alpha)\right]^{1/2}$$

$$= A_{0^-}\left[\left(1+\frac{A_1^2}{A_0^2}\right)^{1/2} + \left[1+\frac{2A_1}{A_{0^-}}\left(1+\frac{A_1^2}{A_0^2}\right)^{-1}\cos(\omega_\alpha\varphi+\alpha)\right]\right]^{1/2}$$

$$= A_{0^-}\left[\left(1+\frac{A_1^2}{A_0^2}\right)^{1/2} + \left[1+\frac{A_1}{A_{0^-}}\left(1+\frac{A_1^2}{A_0^2}\right)^{-1}\cos(\omega_\alpha\varphi+\alpha)\right]\right]$$

$$r(\varphi) = \frac{1}{u(\varphi)} \tag{48}$$

$$= \frac{\frac{1}{A_{0^-}}\left(1+\frac{A_1^2}{A_0^2}\right)^{-1/2}}{\left[1+\frac{A_1}{A_{0^-}}\left(1+\frac{A_1^2}{A_0^2}\right)^{-1}\cos(\omega_\alpha\varphi+\alpha)\right]} \tag{49}$$

Compare Equation (49) with Equation (24) below

$$r(\varphi) = \frac{A}{\left[1+e_1\cos(\omega_\alpha\varphi+\alpha)\right]} \tag{50}$$

where

$$A = \frac{1}{A_{0^-}}\left(1+\frac{A_1^2}{A_0^2}\right)^{-1/2} \tag{51}$$

and

$$E_1 = \frac{A_1}{A_{0^-}}\left(1+\frac{A_1^2}{A_0^2}\right)^{-1}. \tag{52}$$

Consequently, the orbit in a processing conic section with eccentricity E_1 and hence semi major axis is given by

$$a = \frac{A}{1-E_1^2} \tag{53}$$

and partition displacement angle Δ given by

$$\Delta = 2\pi\left(\omega_\alpha^{-1} - 1\right) \tag{54}$$

In the first place, it follows from (54) and (21) that the partition displacement angle from analytical solution is given explicitly as

$$\Delta = \frac{4\pi k^2}{c^2 l^2} + \frac{8\pi k^2}{c^4 l^4} \tag{55}$$

where

$$k = C\mu. \tag{56}$$

This is an advance precisely as obtained from well known method of successive approximation. And this leading to term (55) is identically the same as leading term of the corresponding advance from the method of successive approximation [2]. But over and the above this term, our analytical solution reveals the corrections of all order of C^{-2n} to the leading term in (55). In the second place it follows (52) and (22) that the original eccentricity E_1 from our analytical solution is given explicitly as

$$E_1 = \frac{l^2 A_1}{k}\left(1+\frac{2k^2}{c^2 l^2}+\cdots\right)^{-1}\left[1+\frac{l^2 A_1^2}{k^2}\left(1+\frac{2k^2}{c^2 l^2}+\cdots\right)^{-2}\right]^{-1}. \quad (57)$$

Thus from the result (57) an experimental measurement of the orbital eccentricity E_1 is enough to determine the parameter A_1 that occurs in the exact analytical solution. It also follows from this result that the our analytical solution in this work reveals post Newtonian correction of all order of C^{-2n} to the planetary orbital eccentricity which have so far not been found from the method of successive approximations.

In the third place, it follows from (53) and (21) that the orbital semi major axis from our analytical solution are given explicitly as

$$a = \frac{A^2}{\left(1-E_1^2\right)k}\left(1+\frac{2k^2}{c^2 l^2}+\cdots\right)^{-1}\left[1+\frac{l^2 A_1^2}{k^2}\left(1+\frac{2k^2}{c^2 l^2}+\cdots\right)^{-2}\right]^{-1}. \quad (58)$$

Thus, our analytical solution that reveals post Einstein's correlations of all other of C^{-2n} to the planetary semi major axis which here is so far not been found from the method of successive approximations.

4.2 The Second Exact Analytical Solution

The second exact analytical solution is given by substituting the combination of (16), (13), (26) and $A_n = f_{n\alpha^+}, n = 3,4,5,\cdots$ into Equation (15). In this case, the solution is

$$u_2(\varphi) = A_{0^+} + A_1 \exp^{\left[i(\omega_\alpha \varphi+\alpha)\right]} + A_{2\alpha^+} \exp^{\left[2i(\omega_\alpha \varphi+\alpha)\right]} + \sum_i^\infty f_{n\alpha^+}(A_1)\exp^{\left[ni(\omega_\alpha \varphi+\alpha)\right]} \quad (59)$$

where ω_α is given by (20) as

$$\begin{aligned}\omega_\alpha &= \left\{\left(1+\frac{k^2}{c^2 l^2}\right)-\left(1+\frac{k^2}{c^2 l^2}\right)-\left(1-\frac{10k^2}{c^2 l^2}\right)\right\}^{1/2}\\ &= \left\{\left(1+\frac{k^2}{c^2 l^2}\right)-\left(1+\frac{k^2}{c^2 l^2}\right)-\left(1-\frac{5k^2}{c^2 l^2}\right)\right\}^{1/2}\\ &\approx \left\{\left(1+\frac{k^2}{c^2 l^2}\right)-\left[1+\frac{k^2}{c^2 l^2}-1+\frac{5k^2}{c^2 l^2}\right]\right\}^{1/2}\\ &\approx \left\{\left(1+\frac{k^2}{c^2 l^2}\right)-\left[\frac{6k^2}{c^2 l^2}\right]\right\}^{1/2}\\ &\approx \left\{1+\frac{k^2}{c^2 l^2}-\frac{3k^2}{c^2 l^2}\right\}\end{aligned}$$

$$\omega_\alpha = \left(1 - \frac{2k^2}{c^2 l^2} + \cdots\right). \tag{60}$$

Now consider post Einstein's equation of motion *i.e.*

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\varphi^2} + \left(1 + \frac{k^2}{c^2 l^2}\right)u - \frac{k}{l^2} = \frac{3ku^2}{c^2}$$

follow by substituting

$$u(\varphi) = \frac{k}{l^2}\left[\left(1 + \frac{k^2}{c^2 l^2}\right) + \varepsilon \cos\left(\varphi - \alpha - \frac{3k^2}{c^2 l^2}\varphi\right)\right].$$

This relativistic equation is solved by the particular and complementary solution to obtain

$$u(\varphi) = \frac{k}{l^2}\left[\left(1 + \frac{k^2}{c^2 l^2}\right) + \varepsilon \cos\left(\varphi - \alpha - \frac{3k^2}{c^2 l^2}\varphi\right)\right]. \tag{61}$$

Now let perihelion be obtain at $\varphi = 0$, then the next perihelion will be obtain at an angle φ such that

$$\left(1 - \frac{k^2}{c^2 l^2}\right)\varphi = \omega_\alpha \varphi = 2\pi \tag{62}$$

as obtained in Equation (60).

Or

$$\varphi = 2\pi\left(1 - \frac{k^2}{c^2 l^2}\right)^{-1} \tag{63}$$

for small $\frac{2k^2}{c^2 l^2}$ as compared to (1)

Expanding gives

$$\begin{aligned} \varphi &= 2\pi\left(1 - \frac{2k^2}{c^2 l^2} + \cdots\right) \\ &= 2\pi + \frac{4\pi k^2}{c^2 l^2} + \cdots \end{aligned}. \tag{64}$$

Equation (42) is real and consequently, the second exact mathematical solution is given by

$$\begin{aligned} u_2(\varphi) = \Big\{ & A_{0^+} + A_1 \cos\left[(\omega_\alpha \varphi + \alpha)\right] + A_{2\alpha^+} \cos\left[2(\omega_\alpha \varphi + \alpha)\right] \\ & + A_{3\alpha^+} \cos\left[3(\omega_\alpha \varphi + \alpha)\right]\Big\} + i\Big\{A_1 \sin\left[(\omega_\alpha \varphi + \alpha)\right] \\ & + A_{2\alpha^+} \sin\left[2(\omega_\alpha \varphi + \alpha)\right] + A_{3\alpha^+} \sin\left[3(\omega_\alpha \varphi + \alpha)\right] + \cdots\Big\} \end{aligned}. \tag{65}$$

If all the terms in $f_{n\alpha^+}(A_1)$ for $n > 1$ are neglected, the first approximation solution for the orbital equation is given by

$$r_2(\varphi) = \frac{1}{u_2(\varphi)} = \frac{A}{\left[1 + \varepsilon_2 \cos(\omega_\alpha \varphi + \alpha)\right]} \tag{66}$$

evaluation

$$r(\varphi)=\left\{x^2(\varphi)+y^2(\varphi)\right\}^{1/2}$$

Let

$$x(\varphi)=A_1\cos(\omega_\alpha\varphi+\alpha)+A_{0^+}$$

$$\begin{aligned}x^2(\varphi)&=\left[A_{0^+}+A_1\cos(\omega_\alpha\varphi+\alpha)\right]\times\left[A_{0^+}+A_1\cos(\omega_\alpha\varphi+\alpha)\right]\\&=A_{0^+}^2+2A_1\cos(\omega_\alpha\varphi+\alpha)+A_1^2\cos^2(\omega_\alpha\varphi+\alpha)\end{aligned}$$

$$y(\varphi)=A_1\sin(\omega_\alpha\varphi+\alpha)$$

$$y^2(\varphi)=A_1^2\sin^2(\omega_\alpha\varphi+\alpha)$$

$$\begin{aligned}\therefore r(\varphi)&=\left\{A_{0^+}^2+2A_1\cos(\omega_\alpha\varphi+\alpha)+A_1^2\cos^2(\omega_\alpha\varphi+\alpha)+A_1^2\sin^2(\omega_\alpha\varphi+\alpha)\right\}^{1/2}\\&=\left[A_{0^+}^2+A_1^2+A_{0^+}A_1\cos(\omega_\alpha\varphi+\alpha)\right]^{1/2}\\&=A_{0^+}^2\left[\left(1+\frac{A_1^2}{A_{0^+}^2}\right)+\frac{2A_{0^+}A_1}{A_{0^+}^2}\cos(\omega_\alpha\varphi+\alpha)\right]^{1/2}\end{aligned}$$

$$\begin{aligned}r(\varphi)&=A_{0^+}\left\{\left(1+\frac{A_1^2}{A_{0^+}^2}\right)^{1/2}+\left[1+\frac{2A_1}{A_{0^+}}\left(1+\frac{A_1^2}{A_{0^+}^2}\right)^{-1}\cos(\omega_\alpha\varphi+\alpha)\right]\right\}^{1/2}\\&=A_{0^+}\left\{\left(1+\frac{A_1^2}{A_{0^+}^2}\right)^{1/2}+\left[1+\frac{A_1}{A_{0^+}}\left(1+\frac{A_1^2}{A_{0^+}^2}\right)^{-1}\cos(\omega_\alpha\varphi+\alpha)\right]\right\}\end{aligned}.$$

Using Binomial theorem

$$(1+x)^{+n}\approx 1+nx$$

$$\Rightarrow r(\varphi)=\frac{1}{u(\varphi)}$$

$$r(\varphi)=\frac{\frac{1}{A_{0^+}}\left(1+\frac{A_1^2}{A_{0^+}^2}\right)^{-1/2}}{\left[1+\frac{A_1}{A_{0^+}}\left(1+\frac{A_1^2}{A_{0^+}^2}\right)^{-1}\cos(\omega_\alpha\varphi+\alpha)\right]}. \tag{67}$$

Comparing Equation (49) with Equation (50), we get

$$A=\frac{1}{A_{0^+}}\left(1+\frac{A_1^2}{A_{0^+}^2}\right)^{-1/2} \tag{68}$$

and

$$\varepsilon_2=\frac{A_1}{A_{0^+}}\left(1+\frac{A_1^2}{A_{0^+}^2}\right)^{-1}. \tag{69}$$

From Equation (22), A_{0^+} is given by

$$A_{0^+}=\frac{c^2}{3k}-\frac{k}{l^2}\left(1+\frac{2k^2}{c^2l^2}+\cdots\right) \tag{70}$$

therefore the orbital eccentricity ε_2 from Equation (69) is given explicitly as

$$\varepsilon_2 = A_1\left[\frac{c^2}{3k}-\frac{k}{l^2}\left(1+\frac{2k^2}{c^2l^2}+\cdots\right)\right]^{-1}\times\left\{1+A_1^2\left[\frac{c^2}{3k}-\frac{k}{l^2}\left(1+\frac{2k^2}{c^2l^2}+\cdots\right)\right]^{-2}\right\}^{-1} \quad (71)$$

hence the semi major axis of the processing orbit is given by

$$a=\frac{A}{1-\varepsilon_2^2} \quad (72)$$

and procession angle Δ is given by

$$\Delta = 2\pi\left(\omega_\alpha^{-1}-1\right) \quad (73)$$

in the first place it follows from Equations (21) and (54) that the partition displacement angle from analytical solution is given as

$$\Delta=\frac{4\pi k^2}{c^2l^2}+\frac{8\pi k^2}{c^4l^4}. \quad (74)$$

In the first place, the procession angle given in Equation (74) and Equation (55) are the same however, there are differences between the first exact mathematical solution given in Equation (65).

Secondly, it may be noted that the first and the second exact mathematical solutions can be reduced approximately to the two known linearly independent solutions from the method of successive approximation.

4.3. The Third and Fourth Exact Mathematical Solutions

Consider the third and the fourth exact mathematical solutions corresponding to the combinations of equations $A_n = f_{n\beta^-}; n=3,4,5,\cdots$ and $A_n = f_{n\beta^+}; n=3,4,5,\cdots$ being substituted into Equation (21) respectively. In this case, the third mathematical solution is given by

$$u_3(\varphi)=A_{0^-}+A_1\mathrm{e}^{\left[i(\omega_\beta\varphi+\alpha_0)\right]}+A_{2\beta^-}\mathrm{e}^{\left[2i(\omega_\beta\varphi+\alpha_0)\right]}+\sum_3^\infty f_{n\beta^-}(A_1)\mathrm{e}^{\left[ni(\omega_\beta\varphi+\alpha_0)\right]} \quad (75)$$

where

$$\begin{aligned}\omega_\beta &= -\left\{\left(1+\frac{k^2}{c^2l^2}\right)-\left[\left(1+\frac{k^2}{c^2l^2}\right)-\left(1-\frac{10k^2}{c^2l^2}\right)^{1/2}\right]\right\}^{1/2}\\ &= -\left\{\left(1+\frac{k^2}{c^2l^2}\right)-\left[\left(1+\frac{k^2}{c^2l^2}\right)-\left(1-\frac{5k^2}{c^2l^2}\right)^{1/2}\right]\right\}^{1/2}\\ &\approx -\left\{\left(1+\frac{k^2}{c^2l^2}\right)-\left[1+\frac{k^2}{c^2l^2}-1+\frac{5k^2}{c^2l^2}\right]\right\}^{1/2}\\ &\approx -\left\{1+\frac{k^2}{c^2l^2}-\frac{6k^2}{c^2l^2}\right\}^{1/2}\\ &\approx -\left\{1-\frac{5k^2}{2c^2l^2}\right\}\end{aligned}$$

$$\omega_\beta \approx i\left(1-\frac{2k^2}{c^2l^2}\right) \tag{76}$$

ω_β is real and negative and therefore Equation (75) becomes

$$\begin{aligned} u_3(\varphi) = \Big\{ & A_{0^-} + A_1 \cos\left[\left(\omega_\beta\varphi+\alpha_0\right)\right] + A_{2\beta^-}\cos\left[2\left(\omega_\beta\varphi+\alpha_0\right)\right] \\ & + A_{3\beta^-}\cos\left[3\left(\omega_\beta\varphi+\alpha_0\right)\right]+\cdots\Big\} - \Big\{A_{0^-} + A_1\sin\left[\left(\omega_\beta\varphi+\alpha_0\right)\right]. \\ & + A_{2\beta^-}\sin\left[2\left(\omega_\beta\varphi+\alpha_0\right)\right] + A_{3\beta^-}\sin\left[3\left(\omega_\beta\varphi+\alpha_0\right)\right]+\cdots\Big\} \end{aligned} \tag{77}$$

So that it follows directly from Equation (75) and (77) that

$$r_3(\varphi) = \frac{A}{1+\varepsilon_3\cos\left(\omega_\beta\varphi+\alpha_0\right)} \tag{78}$$

and

$$\varepsilon_3 = \frac{A_1}{A_{0^-}}\left(1+\frac{A_1^2}{A_{0^-}^2}\right)^{-1} \tag{80}$$

$$= \frac{l^2A_1}{A_{0^-}}\left(1+\frac{A_1^2}{A_{0^-}^2}\right)^{-1}. \tag{81}$$

And so the orbit described by $r_3(\varphi)$ in Equation (78) is also a precessing conic section with eccentricity ε and semi major axis a given by

$$a = \frac{A_1}{\left(1-a_3^2\right)k}\left(1+\frac{3k^2}{c^2l^2}\right)^{-1}\cdot\left\{\frac{l^2A^2}{k^2}\left(1+\frac{3k^2}{c^2l^2}+\cdots\right)^2\right\}^{-1}. \tag{82}$$

The precession angle Δ is given by

$$\begin{aligned} \Delta &= 2\pi\left(\omega_\beta^- -1\right) \\ &= -\frac{5\pi k^2}{c^2l^2} - \frac{12.5\pi k^4}{c^4l^4}+\cdots \end{aligned} \tag{83}$$

or

$$\cong -\left(\frac{5\pi k^2}{c^2l^2}+\frac{12.5\pi k^4}{c^4l^4}+\cdots\right). \tag{84}$$

Consequently, the magnitude for procession angle form the third mathematical solution given in Equation (55) and (84) exact mathematical solution except for the negative sign (-).

The major implication of this negative sign, is that for every anticlockwise revolutionary movement of the planet around the sun, clockwise revolutionary movement is also a possibility mathematically.

Following the same mathematical steps for analysing the third exact mathematical solution is the same as that of the fourth exact mathematical solution except that the parameter A_{0^-} in the third exact mathematical solution is replaced by the parameter A_{0^+} in the fourth exact mathematical solution. Therefore the two solutions remain linearly independent of each other.

4.4. The Fifth and Sixth Exact Mathematical Solutions

Consider the fifth and sixth exact mathematical solutions corresponding to the combinations of equations $A_n = f_{n\gamma^-}; n = 3,4,5,\cdots$ and $A_n = f_{n\gamma^+}; n = 3,4,5,\cdots$ Being substituted into Equation (31) respectively.

In this case, the fifth and sixth mathematical solutions is given by

$$u_5(\varphi) = A_{0^-} + A_1 e^{\left[i(\omega_\gamma\varphi+\alpha_0)\right]} + A_{2\gamma^-} e^{\left[2i(\omega_\gamma\varphi+\alpha_0)\right]} + \sum_3^\infty f_{n\gamma^-}(A_1) e^{\left[ni(\omega_\gamma\varphi+\alpha_0)\right]} \tag{85}$$

$$u_6(\varphi) = A_{0^+} + A_1 e^{\left[i(\omega_\gamma\varphi+\alpha_0)\right]} + A_{2\gamma^+} e^{\left[2i(\omega_\gamma\varphi+\alpha_0)\right]} + \sum_3^\infty f_{n\gamma^+}(A_1) e^{\left[ni(\omega_\gamma\varphi+\alpha_0)\right]} \tag{86}$$

where A_{0^-} and A_{0^+} are given by Equations (11a) and (11b) respectively and ω_α si given by Equation (25)

$$\omega_\gamma = \left(1+\frac{k^2}{c^2l^2}\right) - \left[\left(1+\frac{k^2}{c^2l^2}\right) - \left(1-\frac{10k^2}{c^2l^2}\right)^{1/2}\right]^{1/2}$$

$$\omega_\gamma \approx -\left\{1-\frac{5k^2}{2c^2l^2}\right\}$$

$$\omega_\gamma \approx i\left(1-\frac{5k^2}{2l^2c^2}+\cdots\right). \tag{87}$$

It follows from Equation (87) that ω_γ is an imaginary number and therefore Equation (85) and (86) becomes

$$u_5(\varphi) = A_{0^-} + A_1 e^{\left[(-\omega_\gamma\varphi+i\alpha_0)\right]} + A_{2\gamma^-} e^{\left[(-2\omega_\gamma\varphi+2i\alpha_0)\right]} + \sum_3^\infty f_{n\gamma^-}(A_1) e^{\left[(n\omega_\gamma\varphi+in\alpha_0)\right]}$$

$$u_5(\varphi) = A_{0^-} + A_1 e^{\left\{\left(1-\frac{5k^2}{2l^2c^2}+\cdots\right)\varphi\right\}} + \cdots \tag{88}$$

$$u_6(\varphi) = A_{0^+} + A_1 e^{\left[(-\omega_\gamma\varphi+i\alpha_0)\right]} + A_{2\gamma^+} e^{\left[(-2\omega_\gamma\varphi+2i\alpha_0)\right]} + \sum_3^\infty f_{n\gamma^+}(A_1) e^{\left[(-n\omega_\gamma\varphi+in\alpha_0)\right]}$$

$$u_6(\varphi) = A_{0^+} + A_1 e^{\left\{-\left(1-\frac{5k^2}{2l^2c^2}+\cdots\right)\varphi\right\}} + \cdots. \tag{89}$$

As a first approximation, let the terms in $f_{n\gamma}(A_1)$ for $n > 1$ be neglected. Then it follows from (68) and (69) that

$$r_5(\varphi) = \frac{1}{A_{0^-} + A_1 e^{\left\{-\left(1-\frac{5}{2}\frac{k^2}{l^2c^2}\right)\varphi\right\}}} \tag{90}$$

$$u_6(\varphi) = \frac{1}{A_{0^+} + A_1 e^{\left\{-\left(1-\frac{5}{2}\frac{k^2}{l^2c^2}\right)\varphi\right\}}}. \tag{91}$$

It follows directly from equation xxx that in the limit $\varphi \to 0$, $r_5(\varphi) \to \frac{1}{A_{0^-} + A_1}$ and as $\varphi \to \infty$, $r_5(\varphi) \to \frac{1}{A_{0^-}}$ and $\frac{1}{A_{0^-} + A_1} < \frac{1}{A_{0^-}}$ it means that $r_5(\varphi)$ is a spiral orbit. Consequently, a planet in a stable orbit at a distance $\frac{1}{A_{0^-}+A_1}$ from the sun can be ejected from this orbit and spiral outwardly until after several turns it settles in an orbit with a finite distance $r_5(\varphi)\to\frac{1}{A_{0^-}}$ from the sun.

In the first place, this finding corroborate adequately the "ejection hypothesis for planetary formation" which asserts that the planets were formed by the ejection of matter from the sun which then came together under gravitational attraction and then settles into orbits as we know today [3].

In the second place, it was shown that the planets are receding from the sun due to mass loss by the sun [4]. The sun as a source of gravitational field is losing mass at the rate 4.5 million tons per second [5], which ultimately affects the planetary orbit of the solar system.

Therefore, it can be inferred directly from this work exact mathematical solution that the only way a planet can recede from the sun is by spiralling outwardly. And this clearly shows the distinctive consequences of exact mathematical solution compared to that obtained from method of successive approximation in view of this Equation (70) is hereafter referred to as the "recession model for planetary system"

4.5. The Seventh and Eighth Exact Mathematical Solutions

Consider the seventh and eighth exact mathematical solutions given by substituting $A_n=f_{n\delta^-};n=3,4,5,\cdots$ and $A_n=f_{n\delta^+};n=3,4,5,\cdots$ into Equation (16) respectively.

In this case, the seventh and eighth mathematical solutions is given by

$$u_7(\varphi)=A_{0^-}+A_1\mathrm{e}^{[i(\omega_\delta\varphi+\alpha_0)]}+A_{2\delta^-}\mathrm{e}^{[2i(\omega_\delta\varphi+\alpha_0)]}+\sum_3^\infty f_{n\delta^-}(A_1)\mathrm{e}^{[ni(\omega_\delta\varphi+\alpha_0)]}\tag{92}$$

$$u_8(\varphi)=A_{0^+}+A_1\mathrm{e}^{[i(\omega_\delta\varphi+\alpha_0)]}+A_{2\delta^+}\mathrm{e}^{[2i(\omega_\delta\varphi+\alpha_0)]}+\sum_3^\infty f_{n\delta^+}(A_1)\mathrm{e}^{[ni(\omega_\delta\varphi+\alpha_0)]}.\tag{93}$$

From Equation (26) that is

$$\omega_\delta=-\left\{\left(1+\frac{k^2}{c^2l^2}-\right)-\left[\left(1+\frac{k^2}{c^2l^2}\right)-\left(1-\frac{10k^2}{c^2l^2}\right)^{1/2}\right]\right\}^{1/2}$$

$$\omega_\delta\approx-\left\{1-\frac{5k^2}{c^2l^2}\right\}$$

$$\omega_\delta\approx i\left(1-\frac{5k^2}{2l^2c^2}+\cdots\right).\tag{94}$$

Assumes A_n are real then (3) becomes

$$u_7(\varphi)=A_{0^-}+A_1\mathrm{e}^{\left\{\left(1-\frac{5k^2}{2l^2c^2}+\cdots\right)\varphi\right\}}+\cdots\tag{95}$$

$$u_8(\varphi)=A_{0^+}+A_1\mathrm{e}^{\left\{-\left(1-\frac{5k^2}{2l^2c^2}+\cdots\right)\varphi\right\}}+\cdots\tag{96}$$

As a first approximation, let the terms in $f_{n\delta^{+}}(A_1)$ for $n>1$ be neglected. Then it follows from (51) and (52) that

$$r_7(\varphi)=\frac{1}{u_7(\varphi)}=\frac{1}{A_{0^-}+A_1 e^{\left\{\left(1-\frac{5}{2}\frac{k^2}{l^2c^2}\right)\varphi\right\}}} \tag{97}$$

$$r_8(\varphi)=\frac{1}{u_8(\varphi)}=\frac{1}{A_{0^+}+A_1 e^{\left\{-\left(1-\frac{5}{2}\frac{k^2}{l^2c^2}\right)\varphi\right\}}}. \tag{98}$$

It follows directly from equations that as the limit $\varphi\to 0$, $r(\varphi)\to\infty$ where A_0 cab be either A_{0^-} or A_{0^+}. Consequently, the orbit of the planet in this case could be spiral inwardly. In the other word, the planet could start from an orbit which is of finite distance $\frac{1}{A_{0^-}+A_1}$ and spiral inward until after several orbits crashes into the sun. There is an analogy between this mathematical method and the planetary model for H-atom. According to this model, the nucleus is at rest while the electron is going round it in a circular or elliptical orbit. This is like the planetary model of the solar system, with the nucleus, the sun, the electron, the planet and the coulomb force between electron and the nucleus like the gravitational force between the planet and the sun. According to classical electromagnetic theory, an accelerated electron charge must radiate electromagnetic waves. In this process, of radiation, the electron naturally loses energy and hence the binding energy E_b of the electron decreases. This is possible only if the distance of separation r decreases. Since the electron accelerates constantly, it radiates continuously and hence E_b and r decreases continuously. This means that the electron spiral inwards and caught eventually to collapse into the nucleus. But this will not happen.

The hydrogen atom does not itself radiates electromagnetic waves because it is stable. Hence this classical planetary model for hydrogen is not tenable (*i.e.* rejected). However the planetary model uncovered in this work may not be rejected because the planet—sun system is not stable as the H-atom but losses mass continuously in the form of elementary particles into the interstellar medium. Therefore this mathematical model may be tenable, especially for process containing accretion in binary stars system [6].

5. Conclusions

- Equation (1) is the generalised Einstein's planetary equation of motion. Solving this equation using the second approximation method, the results obtained are eight (8) exact linearly independent mathematical solutions. The physical interpretation of eight solutions was carried out and the results were compared with those obtained from the well-known method of successive approximations as well as those from other theories.
- The generalised Einstein's planetary parameters for the analytical method

have more advantage compared to the approximation method based on the experimental measured values in the case of each planet.

- Results obtained for the first analytical solution of the Einstein's planetary equation to the order of C^{-2}, reveal some post Newtonian correction terms to the orbital eccentricity of the planet.
- Results obtained reveal that it is possible for a planet in a particular orbit to spiral either outwardly or inwardly, which is in agreement with the ejection hypothesis for planetary formation.
- Results obtained reveal the productions of perturbation terms to the well-known Einstein's planetary parameters, which can be applied in the motion of satellite, planets around the sun, comets and variations of weather and climatic conditions.
- Exact analytical solutions uncover eight mathematical solutions, each of which constitutes significant progress towards the solution of phenomenon of the anomalous orbital procession of the orbits of the planet.
- Exact analytical solution corresponds to the parameter A_{0^+} and ω_1 except that one is opposite in sign to the other. The implication of the results obtained, therefore, is that for every clockwise movement of the planet around the sun or satellite around the planet, clockwise movement is also possible mathematically.

The second approximation method which is also known as the method of successive approximations was applied to the generalised Einstein's planetary equation of motion to obtain eight exact linearly independent mathematical solutions [7]. The physical interpretation of the eight solutions was carried out and the results obtained were compared with the results obtained from the existing theories.

- It is interesting to note that the leading term in Equation (68) is identically the same as the leading term of the corresponding method successive approximation [8]. But over and above this term, the analytical method reveals the correction of all orders of C^{-2} to the leading term in Equation (69).
- It also follows that Equation (51) and (4.36) that the original eccentricity ε from our analytical method used reveals post Newtonian correction of all order of C^{-2} to the planetary orbit eccentricity which has so far not been from the method of successive approximation [9].
- From Equation (55), it follows that the orbital semi-major axis from the analytical method reveals post Einstein's correction term of all order of C^{-2} to the planetary semi-major axis, which has not been derived from the method of successive approximation.
- The analytical solution also reveals several terms contributing to the anomalous perihelion advance with the first term in Equation (53) corresponding to the result from the method of successive approximation. But the correction term to Equation (53) makes it greater than the corresponding expression from the method of successive approximation. Since perihelion advance from the method of successive approximations is less the experimen-

tally measured values for the planets, it follows that the exact analytical method brings us closer to the complete resolution of the phenomenon of the anomalous orbital precession in the solar system [10].

- It is most interesting to note that even to the order of C^{-2}, the analytical method reveals post Newtonian effect in the motion of the planet apart from the anomalous perihelion advance whereas the method of successive approximation does not. The analytical method reveals the seriousness of the well-known method of successive approximation in the solution of non-linear equations in General Relativity in particular and physics in general.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Nura, Y., Howusu, S.X.K., Lumbi, L.W., Nuhu, I. and Hayatu, A. (2017) The Generalised Planetary Equations Based Upon Riemannian Geometry and the Golden Metric Tensor. *International Journal of Theoretical and Mathematical Physics*, **7**, 25-35.

[2] Nura, Y. and Howusu, S.X.K. (2017) General Geodesic Equation of Motion based Upon the Riemannian Geometry. LAP LAMBERT Academic Publishing Group, Moldova, 29-31

[3] Anderson, J.L. (1967) Solution of the Einstein of the Einstein's Equations. In *Principles of Relativity Physics*, New York Academic Press, New York, 372-402.

[4] Weinberg, S. (1972) Principle and Application of the General Theory of Relativity. John Wiley & Sons, New York, 333-340.

[5] Marion, J.B. (1965) Classical Dynamics. Academic Press, New York, 400-402.

[6] Rosser, W.G.V. (1964) Introduction to the Theory of Relativity. Cambridge University Press, New York, 437-466.

[7] Ter, H. and Cameron, A. (1967) The Solar System Annual Review of Astronomy and Astrophysics. *Annual Review of Astronomy and Astrophysics*, **5**, 267-278. http://doi.org/10.1146/annurev.aa.05.090167.001411

[8] Bakwa, D. and Howusu, S. (2004) The Solution of Complete Schwarzchild's Planetary Equation, with Methid of Successive Approximations-Nigeria. *Journal of Physics*, **17**, 7-10. https://doi.org/10.4314/njphy.v17i1.37984

[9] Howusu, S.X.K. (2010) Exact Analytical Solutions of Einstein's Geometrical Gravitational Field Equation. Jos University Press Ltd., Jos, 40-70.

[10] Roman, S. (1983) The Solar System. Scientific American, New York, 32-33.