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Kerr-Newman Black Holes and Negative Temperatures

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Abstract

The application of the Unruh procedure to the Rindler approximation of the Kerr-Newman metric in the neighborhood of the event and Cauchy horizons leads, unambiguously, to the well-known positive Hawking (black hole) temperature at the outer horizon, but to a negative (white hole) temperature at the inner horizon. Some consequences for the heat capacities and the status of the third law of thermodynamics are also discussed.

Keywords

Kerr-Newman Metric, Cauchy Horizon, Negative Temperature

1. Introduction

The Rindler [1] approximation to the Kerr-Newman metric [2] in the neighborhoods of the outer (event) and inner (Cauchy) horizons, and the application to it of the Unruh [3] procedure using the equivalence of freely falling observers with inertial systems, and rest observers with accelerated systems, respectively in curved and flat spacetimes, leads, without any ambiguity, to thermal blackbody radiation temperatures which are, respectively, positive and negative at the outer and inner horizons, in correspondence with the black hole and white hole radiations. The Rindler approximations hold outside the event horizon and inside the Cauchy horizon [4]. In particular, the result for the inner emission temperature is in agreement with those in references [5] [6] [7] [8]. These facts are discussed in Section 2 (We remind the reader that negative absolute temperatures have been predicted long ago by Ramsey [9] and Klein [10] and found experimentally by Purcell and Pound [11] and Abragam and Procter [12] in nuclear spin systems. For a recent review see Baldovin *et al.* [13]).

In Section 3, we show how, in the Kerr case, the heat capacity at the event horizon passes from a negative to a positive value at a critical value of the angular momentum *J*, while at the inner horizon it remains negative for all values of *J* in the interval $\left(0, M^2\right)$.

In Section 4, the status of the validity of the entropic and unattainability versions of the third law of thermodynamics is discussed in the present context. At both horizons, and in particular at that with $T<0$, the Planck-Nernst version is clearly violated, while the second version is fulfilled due to the impossibility of adding an infinite amount of energy (mass) in a finite time: at the event, horizon $T_+=0$ is unattainable, while at the Cauchy horizon what is unattainable is a zero value of the entropy S_-.

Finally, in Section 5, we comment on the possibility that the Cauchy horizons could be considered to lie beyond the physical regions of the solutions.

2. Rindler Approximation and Negative Temperature at Inner Horizon

The Kerr-Newman metric (KN) is given by

$$\mathrm{d}s^2=\frac{\Delta}{\Sigma}\left(\mathrm{d}t-a\sin^2\theta\mathrm{d}\varphi\right)^2-\frac{\Sigma}{\Delta}\mathrm{d}r^2-\Sigma\mathrm{d}\theta^2-\frac{r^2+a^2}{\Sigma}\sin^2\theta\left(\mathrm{d}\varphi-\frac{a}{r^2+a^2}\mathrm{d}t\right)^2 \quad (1)$$

where $t,r\in(-\infty,+\infty)$, $\varphi\in(0,2\pi)$, $\theta\in[0,\pi]$, $\Delta=r^2+a^2-2Mr+Q^2$, $\Sigma=r^2+a^2\cos^2\theta$, M is the gravitational mass (energy), $a=J/M$ the angular momentum/unit mass, and $Q^2=q^2+p^2$, with q the electric charge and p the Dirac magnetic charge. For $Q^2=0$, the KN metric reduces to the Kerr (K) metric. We consider the non-extremal case $M^2-\left(a^2+Q^2\right)>0$. The roots of Δ give the outer (event) and inner (Cauchy) horizons $h_\pm$:

$$r_\pm=M\pm\sqrt{M^2-\left(a^2+Q^2\right)}, \quad (2)$$

with areas

$$A_\pm=8\pi Mr_\pm\left(1-\frac{Q^2}{2Mr_\pm}\right). \quad (3)$$

In the neighborhood of $h_\pm$, one defines the radial coordinate ρ through

$$r:=r_\pm+\frac{\alpha_\pm}{r_\pm}\rho^2 \quad (4)$$

with $[\rho]=[L]$ and $\alpha_\pm$ numerical functions of θ to be later conveniently chosen. A detailed analysis of the metric (1) shows that, up to terms of order $O\left(\rho^2\right)$ (which cancels its 4th term), at fixed θ (which cancels its 3rd term), and with $\tilde{\varphi}_\pm=\varphi-\omega_\pm t$: co-rotating azimutal angles, where $\omega_\pm=a/(r_\pm^2+a^2)$ is the dragging angular velocity of spacetime at the horizons, Equation (1) becomes [4]

$$\left.\mathrm{d}s^2\right|_{r\sim r_\pm}\cong\frac{4\Sigma_\pm\alpha_\pm}{r_\pm\left(r_\pm-r_\mp\right)}\left(\left(\kappa_\pm\rho\right)^2\mathrm{d}t^2-\mathrm{d}\rho^2\right) \quad (5)$$

where $\Sigma_\pm=r_\pm^2+a^2\cos^2\theta$ and

$$\kappa_\pm = \frac{r_\pm (r_\pm - r_\mp)}{2(r_\pm^2 + a^2)} \tag{6}$$

are the surface gravities at $r_\pm$, with $\kappa_+ > 0$ and $\kappa_- < 0$. Choosing

$$\alpha_\pm \equiv \tilde{\alpha}_\pm := \frac{r_\pm (r_\pm - r_\mp)}{4\Sigma_\pm} \tag{7}$$

with $\tilde{\alpha}_+ > 0$ at h_+ and $\tilde{\alpha}_- < 0$ at h_-, Equation (5) is nothing but the Rindler metric

$$\mathrm{d}s^2\Big|_{r\sim r_\pm,\alpha=\tilde{\alpha}_\pm} = \mathrm{d}s^2_{Rindler} = (\kappa_\pm \rho)^2 \mathrm{d}t^2 - \mathrm{d}\rho^2, \tag{8}$$

with κ_+ near r_+ and κ_- near r_- playing the role of the corresponding proper accelerations. Note that the coordinate change (4) leads to $r = r_\pm + \frac{\tilde{\alpha}_\pm(\theta)}{r_\pm}$

$$= r_+ + \frac{r_+ - r_-}{4\Sigma_+(\theta)}\rho^2 > r_+, \text{ near } h_+, \text{ and } = r_- - \frac{r_+ - r_-}{4\Sigma_+(\theta)}\rho^2 < r_+, \text{ near } h_-. \tag{9}$$

This means that the Rindler approximation to the KN metric only holds outside h_+ and inside h_-. We exhibit this fact in **Figure 1**, the Penrose-Carter (PC) diagram of the basic cell of the KN spacetime.

If one assumes the existence of a free real massless quantum scalar field in the spacetime of the KN solution, the application of the Unruh procedure in the regions where the Rindler approximation holds, one obtains a black body Bose-Einstein distribution which, for the frequancy ω_k is given by $\frac{1}{\mathrm{e}^{\omega_k/T_\pm} - 1}$ [14], with positive temperature at the event horizon [15]:

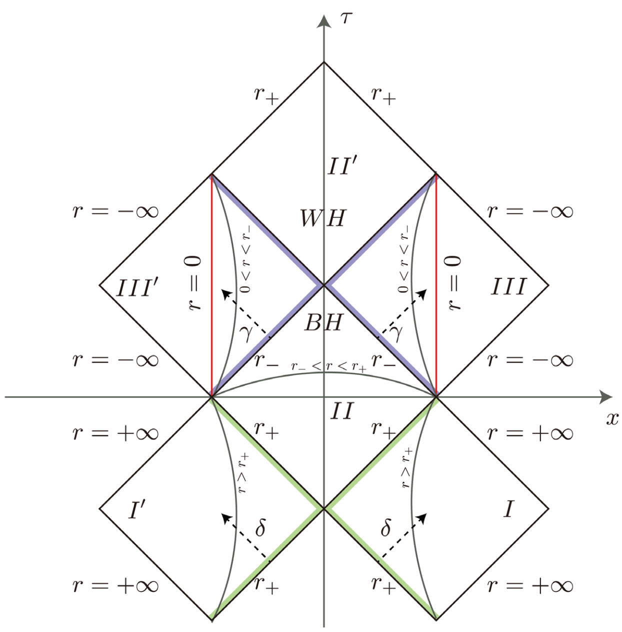

Figure 1. Region where rindler approximation holds: Blue: Inside r_-. Green: outside r_+, (τ, x): Penrose-carter time/radial coordinates.

$$T_+ = \frac{\kappa_+}{2\pi} > 0 \quad (10)$$

and negative temperature at the Cauchy horizon ([5] [6] [7] [8]):

$$T_- = \frac{\kappa_-}{2\pi} < 0. \quad (11)$$

(In Ref. [4], it was defined $T_- := \frac{|\kappa_-|}{2\pi} > 0$ in order to preserve a non negative value of the absolute temperature.) In the PC diagram, particles γ from r_- at $T_- < 0$ which are not swallowed by the singularity ring at $r = 0$, $\theta = \pi/2$, escape to the future asymptotically flat region with $r = -\infty$, while particles δ from r_+ at $T_+ > 0$ escape to the future null infinity at $r = +\infty$.

From (6),

$$\frac{T_+}{|T_-|} = \frac{r_-^2 + a^2}{r_+^2 + a^2} < 1 \quad (12)$$

i.e. $T_+ < |T_-|$. It can be easily verified that

$$A_+ T_+ + A_- T_- = 0 \quad (13)$$

and therefore $A_- = \frac{T_+}{|T_-|} A_+$. At the horizons the entropies are

$$S_\pm = \frac{A_\pm}{4} = 2\pi M r_\pm \left(1 - \frac{Q^2}{2Mr_\pm}\right), \quad (14)$$

and so

$$S_- = \frac{2Mr_- - Q^2}{2Mr_+ - Q^2} S_+ < S_+. \quad (15)$$

In particular, for the K case ($Q^2 = 0$),

$$T_\pm = \frac{r_\pm - r_\mp}{8\pi M r_\pm}, \; S_\pm = 2\pi M r_\pm \quad (16)$$

i.e.

$$T_\pm = \pm \frac{\sqrt{M^2 - a^2}}{4\pi M \left(M \pm \sqrt{M^2 - a^2}\right)}, \; S_\pm = 2\pi M \left(M \pm \sqrt{M^2 - a^2}\right). \quad (17)$$

3. Heat Capacities

As is well-known, in the Schwarzschild case ($a = 0$, $Q^2 = 0$) there is only one horizon at $r = 2M$ (the Cauchy horizon disappears), the temperature is positive:

$$T = \frac{1}{8\pi M}, \quad (18)$$

and the resulting heat capacity becomes negative:

$$C = \frac{\partial M}{\partial T} = -\frac{1}{8\pi T^2}. \quad (19)$$

In the K and KN cases, the situation is more complicated. For simplicity, we shall consider here the K case [16].

At the event horizon, the condition $C_+^{-1} = \frac{\partial T_+}{\partial M} < 0$ leads to the inequality

$$1 + \left(\frac{J}{M^2}\right)^2 < 2\left(\frac{\surd}{M}\right)^2 + \left(\frac{\surd}{M}\right)^3 \tag{20}$$

where $\surd = \sqrt{M^2 - J^2/M^2}$. It is then clear that there exists a value of the angular momentum, $\bar{J}$, such that

$$C_+ < 0 \text{ for } J < \bar{J} \tag{21}$$

and

$$C_+ > 0 \text{ for } \bar{J} < J < M^2. \tag{22}$$

That is, for slow rotation of the spacetime the sign of C_+ is the same as that of $C_{Schw.}$, while the opposite occurs for a quick rotation. $\bar{J}$ is the solution of the equation

$$1 + \left(\frac{\bar{J}}{M^2}\right)^2 = 2\left(\frac{\sqrt{M^2 - \bar{J}/M^2}}{M}\right)^2 + \left(\frac{\sqrt{M^2 - \bar{J}/M^2}}{M}\right)^3. \tag{23}$$

At the Cauchy horizon, the situation is different: the condition $C_-^{-1} = \frac{\partial T_-}{\partial M} < 0$ leads to the inequality

$$1 < 3\left(\frac{J}{M^2}\right)^2 + \left(\frac{\surd}{M}\right)^3, \tag{24}$$

which can be seen to hold for all angular momenta J in the interval $(0, M^2)$. This behavior is due to the negative value of the temperature: since $T_- = -|T_-|$, $|C_-^{-1}| = \frac{\partial |T_-|}{\partial M} > 0$, which, at least in absolute values, matches with the fact that the spacetime dragging angular velocity $\omega_- > \omega_+$, making the inner horizon "further away" from the Schwarzschild case which does not rotate at all.

4. Third Law of Thermodynamics

It is well-known that both the K and KN solutions violate the "entropic" or Planck-Nernst formulation of the third law of thermodynamics ([16] [17] [18] [19]). To begin with, this violation is extremely easy to verify in the Schwarzschild case: from (18), $T_{Schw.} \to 0_+$ as $M \to \infty$, while the entropy $S_{Schw.} = 4\pi M^2 \to +\infty$ in the same limit.

From Equations (2), (3), (6), (10), (11) and (14), in the KN case one obtains:

$$S_\pm = \frac{A_\pm}{4} = 2\pi M^2 \left(1 \pm \sqrt{1 - \left(J^2/M^4 + Q^2/M^2\right)}\right)\left(1 - \frac{Q^2}{2M^2\left(1 \pm \sqrt{1 - \left(J^2/M^4 + Q^2/M^2\right)}\right)}\right) \tag{25}$$

and

$$T_{\pm} = \left(\left(\frac{\partial S_{\pm}}{\partial M} \right)_{J,Q} \right)^{-1} = \pm \frac{1}{4\pi M} \times \frac{\sqrt{1-\left(J^2/M^4 + Q^2/M^2\right)}}{1 \pm \sqrt{1-\left(J^2/M^4 + Q^2/M^2\right)} - Q^2/2M^2}. \quad (26)$$

Then, as $M \to +\infty$,

$$T_{+} \to 0_{+}, S_{+} \to +\infty, \quad (27)$$

and

$$T_{-} \to -\infty, S_{-} \to 0_{+}. \quad (28)$$

For the K case, with $T_{\pm} = \left(\left(\frac{\partial S_{\pm}}{\partial M} \right)_{J} \right)^{-1}$, one obtains, as $M \to +\infty$, the same limiting behavior as in KN, *i.e.* $T_{+} \to 0_{+}$, $S_{+} \to +\infty$ and $T_{-} \to -\infty$ and $S_{-} \to 0_{+}$.

Note that both in KN and K, at the event horizon the disagreement with the Planck-Nernst third law is the same as in the Schwarzschild case, but in the Cauchy horizon at negative temperature, the opposite disagreement (except for a sign) between T and S occurs.

The previous analysis agrees, however, with the "unattainability" version of the third law; in the present case, the impossibility to achieve zero temperature (zero surface gravity) at the event horizons of the black holes in a finite time interval or, equivalently, after a finite number of steps, since an infinite amount of mass M is required [20]. At the Cauchy horizon, the impossibility is to obtain a vanishing entropy.

5. Final Comment

The Cauchy horizons are the boundaries of the closure of the domains of dependence of the Cauchy surfaces for both the K and KN solutions. So, they lie outside the corresponding globally hyperbolic regions. If these regions were the only final product of a collapse and therefore the only physical ones [21], then the distinct thermodynamic behavior of the inner horizons would be irrelevant.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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