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Approximate and Exact GR-Solutions for the Two-Body Problem

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Abstract

A binary gravitational rotator, also called the two-body problem, is a pair of masses m_1, m_2 moving around their center-of-mass (com) in their own gravitational field. In Newtonian gravitation, the two-body problem can be described by a single reduced mass (gravitational rotator) $m_r = m_1 m_2 / (m_1 + m_2)$ orbiting around the total mass $m = m_1 + m_2$ situated in com in the distance r, which is the distance between the two original masses. In this paper, we discuss the rotator in Newtonian, Schwarzschild and Kerr spacetime context. We formulate the corresponding Kerr orbit equations, and adapt the Kerr rotational parameter to the Newtonian correction of the rotator potential. We present a vacuum solution of Einstein equations (Manko-Ruiz), which is a generalized Kerr spacetime with five parameters $g_{\mu\nu}(m_1, m_2, R, a_1, a_2)$, and adapt it to the Newtonian correction for observer orbits. We show that the Manko-Ruiz metric is the exact solution of the GR-two-body problem (*i.e.* GR-rotator) and express the orbit energy and angular momentum in terms of the 5 parameters. We calculate and discuss Manko-Ruiz rotator orbits in their own field, and present numerical results for two examples. Finally, we carry out numerical calculations of observer orbits in the rotator field for all involved models and compare them.

Keywords

General Relativity, Two-Body Problem, Gravitational Rotator, Kerr Metric, Generalized Kerr Metric

1. Introduction

A *binary gravitational rotator*, also called the *two-body problem*, is a pair of masses m_1, m_2 moving around their center-of-mass (com) in their own gravitational field.

The problem can be formulated as a *single rotator* under certain conditions.

In Newtonian gravitation, the two-body problem can be described by a single reduced mass (single rotator) $m_r = \frac{m_1 m_2}{m_1 + m_2}$ orbiting around the total mass $m = m_1 + m_2$ situated in com in the distance $r = |\boldsymbol{r}| = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$, which is the distance between the two original masses.

In Newtonian case, the com has zero acceleration, *i.e.* it is *a well-defined Lorentzian frame*.

In section 2, we discuss the two-body problem in Newtonian, Schwarzschild and Kerr spacetime context, and show that the (single) rotator is well-defined also in Schwarzschild spacetime, and in Kerr spacetime it is well-defined, if the rotational parameters of the two spinning Kerr-masses are equal.

We describe the general GR-ansatz for the two-body problem in section 3, and formulate the corresponding Kerr orbit equations in section 4.

In section 5, we adapt the Kerr rotational parameter to the Newtonian correction of the rotator-potential.

In section 6, we present the exact GR-solution (Manko-Ruiz) of the two-body problem, which is a generalized Kerr spacetime with four parameters, and adapt it to the Newtonian correction.

In section 7, we discuss extensions: the Einstein radiation power formula and complex rotation parameters in the generalized Kerr spacetime.

In subsection 8.1, we carry out numerical orbit calculations for observer orbit in rotator potential for all involved models and compare them.

In subsection 8.2, we derive the formula for the angular momentum of the Manko-Ruiz rotator in its own field, and calculate and discuss the exact rotator orbits in two examples.

The formulas for metrics, orbit equations, and Christoffel symbols are calculated symbolically from fundamental equations in [1] and [2], and are therefore errorless. In the text, they are typed by hand where needed, but are inserted additionally as images in the appendix to serve as protection against typos.

2. The Rotator in Newtonian, Schwarzschild and Kerr Spacetime

2.1. Schwarzschild and Kerr Spacetime

We start with exact solutions of Einstein equations in spherical coordinates for the non-rotating (Schwarzschild) and rotating (Kerr) black-hole.

The Kerr line element reads [3] [4] (with metric signature $\eta = (1, -1, -1, -1)$)

$$\begin{aligned} \mathrm{d}s^2 = &\left(1 - \frac{r r_s}{r^2 + \alpha^2 \cos^2\theta}\right)\mathrm{d}t^2 + \left(\frac{2 r r_s \alpha \sin^2\theta}{r^2 + \alpha^2 \cos^2\theta}\right)\mathrm{d}t\,\mathrm{d}\phi \\ &- \left(\frac{r^2 + \alpha^2\cos^2\theta}{r^2 - r r_s + \alpha^2}\right)\mathrm{d}r^2 - \left(r^2 + \alpha^2 + \frac{r r_s \alpha^2 \sin^2\theta}{r^2 + \alpha^2\cos^2\theta}\right)\sin^2\theta\,\mathrm{d}\phi^2 \\ &- \left(r^2 + \alpha^2\cos^2\theta\right)\mathrm{d}\theta^2 \end{aligned} \quad (1)$$

where $r_s = \frac{2GM}{c^2}$ is the Schwarzschild radius, and $\alpha = \frac{J}{Mc}$ is the angular momentum radius (amr), α has the dimension of a distance: $[\alpha] = [r]$, and *J* is the angular momentum.

In the limit $\alpha \to 0$ the Kerr line element becomes the standard Schwarzschild line element

$$\mathrm{d}s^2 = \left(1 - \frac{r_s}{r}\right)c^2\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{1 - \frac{r_s}{r}} - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi\right) \tag{2}$$

The total energy for a mass *m* in Newtonian gravitation field of a mass *M* is:

$$\frac{m\dot{r}^2}{2} + \frac{m\dot{\varphi}^2 r^2}{2} - \frac{GmM}{r} = E_t = \varepsilon_t mc^2 \tag{3}$$

where E_t is the *total energy* and ε_t the *relative total energy*. We use in the following the terminology of [4] for the GR energy and radial orbit equation:

$\varepsilon_t = \frac{F^2 - 1}{2}$, where $F^2 = 2\varepsilon_t + 1$ is the (dimensionless) *relativistic velocity factor*, and ε_t the (here negative) relative total energy, $\varepsilon = |\varepsilon_t|$ the absolute relative total energy.

Because of conservation of angular momentum *L* is

$l = \frac{L}{m} = \varphi'(\tau) r(\tau)^2 = const$, *l* = reduced angular momentum is a constant. Using this relation, (3) becomes the Newtonian orbit differential equation for the orbit radius *r*, with the parameters *l* and ε_t to be determined from the initial condition.

From the first Schwarzschild orbit equation (see below) we get

$t'(\tau)\left(1 - \frac{1}{r(\tau)}\right) = const = F$ [4], where *F* is the above *relativistic velocity factor* and has the dimensionality of velocity $F = v_F < c$.

In the general relativistic Schwarzschild case the Newtonian approximation (3) becomes the exact relativistic energy equation (radial orbit equation [5] [6] [7]):

$$\frac{\dot{r}^2}{2} + \frac{l^2}{2r^2}\left(1 - \frac{2GM}{c^2 r}\right) - \frac{GM}{r} = \varepsilon_t = \frac{F^2 - 1}{2} \tag{4}$$

or with Schwarzschild radius r_s.

$\frac{\dot{r}^2}{2} + \frac{l^2}{2r^2}\left(1 - \frac{r_s}{r}\right) - \frac{r_s c^2}{2r} = \varepsilon_t = \frac{F^2 - 1}{2}$, the relative total energy is negative for a bound state: $\varepsilon_t < 0$, we use in the following also the absolute relative energy $\varepsilon = |\varepsilon_t|$.

We can write the energy equation using the effective potential [5]

$$V_{eff} = \frac{l^2}{2r^2}\left(1 - \frac{r_s}{r}\right) - \frac{r_s c^2}{2r}$$

$$\frac{\dot{r}^2}{2} + V_{eff} = -\varepsilon \tag{4a}$$

or still simpler using the general effective potential

$$\tilde{V}_{eff} = \frac{l^2}{r^2}\left(1-\frac{r_s}{r}\right) - \frac{r_s c^2}{r} + 2\varepsilon = 2V_{eff} + 2\varepsilon$$

$$\dot{r}^2 + \tilde{V}_{eff} = 0 \tag{4b}$$

The second form is convenient for expansion of the energy equation in section 6.

The equation is solvable in integral form

$$\tau_S(r) = \int \frac{1}{\sqrt{2\varepsilon_t + \frac{r_s c^2}{r} - \frac{l^2}{r^2}\left(1-\frac{r_s}{r}\right)}} \mathrm{d}r$$

For the Newtonian case, the $\frac{l^2}{2r^2}\left(1-\frac{r_s}{r}\right)$ term is missing, and the integral can be calculated in closed form (see 5.1).

2.2. Newtonian, Schwarzschild, Kerr Rotator

Newtonian rotator

In Newtonian gravitation, the movement of two masses with location vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ in their own gravitational potential takes a very simple form, when formulated in their center-of-mass (com) reference frame [8] [9].

The location of com is $\boldsymbol{R} = \frac{m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2}{m_1 + m_2}$ and we get for its acceleration from the third Newtonian law

$m_1\boldsymbol{r}_1'' + m_2\boldsymbol{r}_2'' = (m_1+m_2)\boldsymbol{R}'' = \boldsymbol{F}_{12} + \boldsymbol{F}_{21}$, so $\boldsymbol{R}'' = 0$, com has constant velocity and is a well-defined reference frame, in which the com relation $m_2 r_2 = m_1 r_1$ is valid.

For the displacement vector $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$ we have then for a central force $\boldsymbol{F}$ then

$$\boldsymbol{r}'' = \boldsymbol{r}_1'' - \boldsymbol{r}_2'' = \left(\frac{\boldsymbol{F}_{12}}{m_1} - \frac{\boldsymbol{F}_{21}}{m_2}\right) = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\boldsymbol{F}_{12} = \frac{1}{m_r} F(r)\frac{\boldsymbol{r}}{r},$$

$$\text{or} \quad m_r \boldsymbol{r}'' = F(r)\frac{\boldsymbol{r}}{r} \quad \text{and} \quad m_2 r_2 = m_1 r_1 = m_r r$$

where $m_r = \frac{m_1 m_2}{m_1 + m_2}$ is the reduced mass: *i.e.* the movement of m_1 and m_2 is described by the movement of mass m_r with distance $\boldsymbol{r}$ in com reference frame, this orbit is the single rotator.

Schwarzschild rotator

We consider now the case of the Schwarzschild spacetime gravitational rotator, also called the Schwarzschild two-body problem: two point masses m_1 and m_2 with $m_2 \geq m_1$ rotating around their center-of-mass (com), in the com reference-frame with orbit radii r_1 and r_2 respectively.

We make the approximation, that the Schwarzschild potentials of the two masses add up to the total spacetime, which is approximate, since the Einstein equations are non-linear. Under this assumption, every mass "sees" the Schwarzschild potential of the other mass, like in Newtonian case.

The original Schwarzschild energy equation reads [5] [6]

$$\text{with} \quad \theta = \frac{\pi}{2} \quad \theta'(\tau) = 0 \quad r_s = \frac{2(m_1 + m_2)G}{c^2} \quad \varphi = \varphi_2 = \varphi_1 + \pi$$

$$\left(1 - \frac{r_s}{r(\tau)}\right) t'(\tau)^2 c^2 - \frac{r'(\tau)^2}{\left(1 - \frac{r_s}{r(\tau)}\right)} - r(\tau)^2 \varphi'(\tau)^2 = c^2 \tag{5}$$

and for the two masses

$$\left(1 - \frac{r_s}{r(\tau)}\right) t_1'(\tau)^2 c^2 - \frac{r_1'(\tau)^2}{\left(1 - \frac{r_s}{r(\tau)}\right)} - r_1(\tau)^2 \varphi'(\tau)^2 = c^2$$

$$\left(1 - \frac{r_s}{r(\tau)}\right) t_2'(\tau)^2 c^2 - \frac{r_2'(\tau)^2}{\left(1 - \frac{r_s}{r(\tau)}\right)} - r_2(\tau)^2 \varphi'(\tau)^2 = c^2$$

we use the two Schwarzschild orbit invariants l and F and get

$$r_1'(\tau)^2 + \frac{l_1^2}{r_1(\tau)^2}\left(1 - \frac{r_s}{r(\tau)}\right) - \frac{r_s}{r(\tau)} = -2\varepsilon_1 c^2$$

$$r_2'(\tau)^2 + \frac{l_2^2}{r_2(\tau)^2}\left(1 - \frac{r_s}{r(\tau)}\right) - \frac{r_s}{r(\tau)} = -2\varepsilon_2 c^2$$

Because of the com-condition $m_2 r_2 = m_1 r_1$, r_1 and r_2 can be calculated from the distance r_0 between m_1 and m_2

$r_1 = \frac{m_2}{m} r_0 = \frac{m_r}{m_1} r_0 = \frac{\mu}{1+\mu} r_0$ and $r_2 = \frac{m_1}{m} r_0 = \frac{m_r}{m_2} r_0 = \frac{1}{1+\mu} r_0$ where $m_r = \frac{m_1 m_2}{m_1 + m_2}$ is the *reduced mass*, and $m = m_1 + m_2$ is the total mass, furthermore we have $\varphi = \varphi_2 = \varphi_1 + \pi$ for the rotation angles, and the reduced angular momenta $l_1 = \varphi'(\tau) r_1(\tau)^2$, $l_2 = \varphi'(\tau) r_2(\tau)^2$, where $\frac{l_1}{r_1(\tau)^2} = \varphi'(\tau) = \frac{l_2}{r_2(\tau)^2} = \frac{l}{r(\tau)^2}$, so $l_1 (1+\mu)^2 = l_2 \frac{(1+\mu)^2}{\mu^2} = l$ is the reduced angular momentum of the rotator, with the relation $\sqrt{l} = \sqrt{l_1} + \sqrt{l_2}$.

From the above follows

$$r_1'(\tau)^2 + 2V_{eff}(r(\tau)) = -2\varepsilon_1 c^2, \quad V_{eff}(r) = \frac{l^2}{2r^2}\left(1 - \frac{r_s}{r}\right) - \frac{r_s}{2r}$$

$$r_2'(\tau)^2 + 2V_{eff}(r(\tau)) = -2\varepsilon_2 c^2, \text{ so } \frac{d}{d\tau} r_2'(\tau)^2 = \frac{d}{d\tau} r_1'(\tau)^2, \quad r_1''(\tau) = r_2''(\tau)$$

and the com acceleration vanishes: $\boldsymbol{R}''(\tau) = \boldsymbol{r}_1''(\tau) + \boldsymbol{r}_2''(\tau) = 0$, so the com frame has constant velocity relative to the observer, the Schwarzschild single rotator is well-defined.

We add the two orbit equations

$$m_1 r_1'(\tau)^2 + 2m_1 V_{eff}\left(r(\tau)\right) = -2\varepsilon_1 m_1 c^2$$

$$m_2 r_2'(\tau)^2 + 2m_2 V_{eff}\left(r(\tau)\right) = -2\varepsilon_2 m_2 c^2$$

and from $m_2 r_2 = m_1 r_1 = m_r r$ we get the well-known single rotator orbit equation

$$m_r^2\left(\frac{1}{m_1}+\frac{1}{m_2}\right)r'(\tau)^2 + 2(m_1+m_2)V_{eff}\left(r(\tau)\right) = -2\left(m_1\varepsilon_1 c^2 + m_2\varepsilon_2 c^2\right) \quad (6)$$

$$m_r r'(\tau)^2 + 2mV_{eff}\left(r(\tau)\right) = -2\left(m_1\varepsilon_1 c^2 + m_2\varepsilon_2 c^2\right) = -2m_r\varepsilon c^2$$
$$= 2\left(E_{1t}+E_{2t}\right) = 2E_t$$

The single rotator has the orbit of the reduced mass m_r in the potential $V_{eff}(r)$ with energy $E_t = E_{1t} + E_{2t}$ and reduced angular momentum $l = \left(\sqrt{l_1}+\sqrt{l_2}\right)^2$.

In order to calculate the individual relative energies $(\varepsilon_1, \varepsilon_2)$, we consider $\varepsilon_1 \simeq m_1$, since $\varepsilon_1 \to 0$ for $m_1 \to 0$. From $m_1\varepsilon_1 + m_2\varepsilon_2 = m_r\varepsilon$ follows

$$\varepsilon_1 = \frac{m_1 m_r \varepsilon}{m_1^2 + m_2^2}, \quad \varepsilon_2 = \frac{m_2 m_r \varepsilon}{m_1^2 + m_2^2}$$

Kerr rotator

In case of two rotating relativistic masses, the underlying individual spacetime is the Kerr spacetime.

We consider the $1/r$-expanded Kerr energy equation to third order and get (dimensionless $r_s = 1$, $c = 1$)

$$\frac{-l^2 - F^2\alpha^2 + 2Fl\alpha^3}{r(\tau)^3} + \frac{l^2 + \alpha^2 - F^2\alpha^2}{r(\tau)^2} - \frac{1-2Fl\alpha}{r(\tau)} + r'(\tau)^2 + 1 - F^2 = 0 \quad (7)$$

Like in the Schwarzschild case, we get (here with error $O\left(\frac{1}{r^4}\right)$) and setting $F \approx 1$

$$r_1'(\tau)^2 + 2V_{eff\,2}\left(r(\tau)\right) = -2\varepsilon_1 c^2,$$

$$V_{eff\,2}(r) = \frac{l^2+\alpha_2^2}{2r^2}\left(1-\frac{r_s}{r}\right) - \frac{r_s}{2r} + 2\frac{l\alpha_2}{r}\left(1+\frac{\alpha_2^2}{r^2}\right) \quad (8)$$

$$r_2'(\tau)^2 + 2V_{eff\,1}\left(r(\tau)\right) = -2\varepsilon_2 c^2, \quad V_{eff\,1}(r) = \frac{l^2+\alpha_1^2}{2r^2}\left(1-\frac{r_s}{r}\right) - \frac{r_s}{2r} + 2\frac{l\alpha_1}{r}\left(1+\frac{\alpha_1^2}{r^2}\right)$$

with rotation parameters α_1 and α_2.

Now the condition for a well-defined com reference frame is $V_{eff\,1} = V_{eff\,2}$, *i.e.* $\alpha_1 = \alpha_2$.

Under this condition, the movement of masses m_1 and m_2 is described by the binary Kerr rotator orbit $r(\tau)$ with the radial potential $V_{eff}(r) = \frac{l^2+\alpha^2}{2r^2}\left(1-\frac{r_s}{r}\right) - \frac{r_s}{2r} + 2\frac{l\alpha}{r}\left(1+\frac{\alpha^2}{r^2}\right)$, with reduced angular momentum l, rotation **parameter** α, and relative energy ε.

2.3. The GR Field and Orbit Equations

The Einstein field equations are [5] [6] [8]:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R_0 + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \tag{9}$$

where $R_{\mu\nu}$ is the Ricci tensor, R_0 the Ricci curvature, $\kappa = \frac{8\pi G}{c^4}$, $T_{\mu\nu}$ is the energy-momentum tensor, Λ is the cosmological constant. In the following Λ is neglected, *i.e.* set $\Lambda = 0$, because it is important only on cosmological scale, and here we consider a distance scale of star binaries, *i.e.* d = 10, …, 300 AU.

Further on, we use the Christoffel symbols (second kind)

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\kappa} \left(\frac{\partial g_{\kappa\mu}}{\partial x^{\nu}} + \frac{\partial g_{\kappa\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\kappa}} \right) \tag{10}$$

and the Ricci tensor

$$R_{\mu\nu} = \frac{\partial \Gamma^{\rho}_{\mu\rho}}{\partial x^{\nu}} - \frac{\partial \Gamma^{\rho}_{\mu\nu}}{\partial x^{\rho}} + \Gamma^{\sigma}_{\mu\rho}\Gamma^{\rho}_{\sigma\nu} - \Gamma^{\sigma}_{\mu\nu}\Gamma^{\rho}_{\sigma\rho} \tag{11}$$

The geodesic orbit equations O1…O4 in vacuum ($T_{\mu\nu} = 0$) are:

$$\frac{\mathrm{d}^2 x^{\kappa}}{\mathrm{d}\tau^2} + \Gamma^{\kappa}_{\mu\nu} \frac{\mathrm{d}x^{\mu}}{\mathrm{d}\tau} \frac{\mathrm{d}x^{\nu}}{\mathrm{d}\tau} = 0, \quad \kappa = 0, \cdots, 3 \tag{12}$$

where τ = proper time.

The four equations O1, O2, O3, O4 are the relativistic orbit equations for $x^{\kappa} = (ct, x, y, z)$ *i.e.* time t and the three spatial coordinates x, y, z in dependence on the relativistic proper time τ For τ we get for the line-element ds = dτ and therefore trivially:

$$g_{\mu\nu} \frac{\mathrm{d}x^{\mu}}{\mathrm{d}\tau} \frac{\mathrm{d}x^{\nu}}{\mathrm{d}\tau} - c^2 = 0 \tag{13}$$

This relation yields for the Kerr- and Schwarzschild-spacetimes the GR energy relation, we choose the denomination E1s for it, the orbit equations for the Kerr spacetime [1] [3] O1s, O2s, O3s, O4s are given in the appendix [9].

Setting [τ] = π/2 solves O3s = 0, and simplifies considerably the other equations.

The simplified expressions E1s1, O1s1, O2s1, O4s1 are given in the appendix [9].

In the following, we use the expression for the τ-derivative with dot or with prime: $t' = \dot{t} = \frac{\mathrm{d}t}{\mathrm{d}\tau}$.

3. The General Ansatz for the GR Two-Body Problem

The General Formulation of GR Two-Body Problem

The GR GR two-body problem is the GR-gravitationally bound binary system of two masses m_1 and m_2 moving around their center-of-mass (com) with respective distances (r_1, r_2) from com [10] [11].

The GR two-body problem has 4 independent parametersthe total mass $m = m_1 + m_2$ (resp. corresponding Schwarzschild radius $r_s = \frac{2Gm}{c^2}$), the mass ratio $\mu = \frac{m_2}{m_1}$, and two orbit parameters: the reduced angular momentum $l = \frac{L}{m}$ and the relative total energy $\varepsilon_t = \frac{E_t}{mc^2} < 0$ (resp. its absolute value $\varepsilon = |\varepsilon_t|$).

Equivalently we can use the two masses m_1 and m_2, $r_c = r_s \frac{1}{4\varepsilon}$ = mean distance, and l, which defines the eccentricity by the formula $e = \sqrt{1 - \frac{8\varepsilon l^2}{r_s^2 c^4}}$.

The corresponding spacetime has the form

$$M_{br} = \left(g_{\mu\nu}\left(m_1, m_2, r_c, l\right)\right)$$

In comparison, Schwarzschild spacetime has only one parameter r_s.

$M_S = \left(g_{\mu\nu}\left(r_s\right)\right)$ (see matrix form in the appendix) and Kerr spacetime has 2 parameters: r_s and $\alpha = \frac{J}{Mc}$ the angular momentum radius $M_K = \left(g_{\mu\nu}\left(r_s, \alpha\right)\right)$ (see matrix form in the appendix)

The **general formulation** of the GR two-body-problem is as follows [11] [12].

We formulate the problem in coordinates relative to center-of-mass, $x_1^\kappa = \left(ct_1, \boldsymbol{x}_1\right)$ and $x_2^\kappa = \left(ct_2, \boldsymbol{x}_2\right)$ for the two masses, and $g_{\mu\nu}\left(x_1^\kappa, x_2^\kappa\right)$ the metric of the GR two-body-system, with the com-equations $\boldsymbol{x}_1 m_1 = -\boldsymbol{x}_2 m_2$, $\mathrm{d}\tau^2 = \mathrm{d}t_1^2 - \sum_{k=1}^{3} \mathrm{d}x_{1k}^2 = \mathrm{d}t_2^2 - \sum_{k=1}^{3} \mathrm{d}x_{2k}^2$, and τ proper-time of the metric $g_{\mu\nu}$. We can eliminate x_2^κ using the com-equations and are left with $g_{\mu\nu}$ and x_1^κ as the 10 + 4 unknown variables of the problem.

Equivalently, we can express x_1^κ and x_2^κ by the rotator 4-vector $x^\kappa = \left(ct, r, \theta, \varphi\right)$ in center-of-mass (com) spatial spherical coordinates, with r being the distance between m_1 and m_2, $r_1 = \frac{m_2}{m} r_0 = \frac{m_r}{m_1} r_0 = \frac{\mu}{1+\mu} r_0$ and $r_2 = \frac{m_1}{m} r_0 = \frac{m_r}{m_2} r_0 = \frac{1}{1+\mu} r_0$ where $m_r = \frac{m_1 m_2}{m_1 + m_2}$ is the *reduced mass*, and $m = m_1 + m_2$ is the total mass. In Newtonian limit, this is a correct formulation of the two-body problem (see above), so it is also valid in the GR case.

We have then 10 Einstein-equations for the metric $g_{\mu\nu}$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R_0 = -\kappa T_{\mu\nu},$$

where the energy-mass tensor is [4]: $T_{\mu\nu} = \rho \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}$, and where for black-holes

$\rho = \frac{m}{4\pi r_s^3/3} = \frac{3c^2}{8\pi r_s^2 G}$ (or dimensionless $\rho = \frac{3m}{4\pi}$), and the 4 geodesic orbit equation for x^κ

$$\frac{d^2 x^\kappa}{d\tau^2} + \Gamma^\kappa_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0$$

So in principle, we can solve the 14 coupled partial deq's for the 14 unknowns imposing appropriate boundary conditions.

We show in section 6.3 that the Manko-Ruiz spacetime $M_{br} = \left(g_{\mu\nu}\left(m_1, m_2, R, a\right)\right)$ is the exact GR-solution of the relativistic two-body problem. However, the Manko-Ruiz solution is very complicated, since it has a complexity of $C\left(V_{eff}, M1\right) = 1.3 \times 10^5$ terms.

If the direct evaluation of the exact solution is too demanding, one is forced to use some iterative scheme.

The common approach is the so-called **post-Newtonian approximation** [10].

Here we make a series ansatz for the GR-Hamiltonian

$$H_{PN}(\boldsymbol{q}, \boldsymbol{p}) = H_0(\boldsymbol{q}, \boldsymbol{p}) + \frac{1}{c^2} H_{1PN}(\boldsymbol{q}, \boldsymbol{p}) + \frac{1}{c^4} H_{2PN}(\boldsymbol{q}, \boldsymbol{p}) + \cdots$$

in the reduced location 3-vector $\boldsymbol{q} = \frac{\boldsymbol{q}_1 - \boldsymbol{q}_2}{G(m_1 + m_2)}$ and reduced momentum 3-vector $\boldsymbol{p} = \frac{\boldsymbol{p}_1}{m_1 m_2/(m_1 + m_2)}$, where $\boldsymbol{p}_1 = -\boldsymbol{p}_2$ (from com-equations), unit vector $\boldsymbol{n} = \frac{\boldsymbol{q}}{|\boldsymbol{q}|}$, reduced mass ratio $\nu = \frac{m_1 m_2}{(m_1 + m_2)^2}$.

Note that here the reduced momentum has the dimensionality $[p] = \frac{\text{m}}{\text{s}}$ of velocity, and the reduced location has the dimensionality $[q] = \frac{\text{s}^2}{\text{m}^2}$ of 1/velocity^2, so the dimensionality of the Hamiltonian $[H] = \frac{\text{J}}{\text{kg}}$ is energy/mass.

The Newtonian Hamiltonian is then

$$H_0(\boldsymbol{q}, \boldsymbol{p}) = \frac{\boldsymbol{p}^2}{2} - \frac{1}{|\boldsymbol{q}|}$$

and the first post-Newtonian Hamiltonian is

$$H_{1PN}(\boldsymbol{q}, \boldsymbol{p}) = (3\nu - 1)\frac{\left(\boldsymbol{p}^2\right)^2}{8} - \frac{1}{2|\boldsymbol{q}|}\left((3+\nu)\boldsymbol{p}^2 + \nu(\boldsymbol{np})^2\right) + \frac{1}{2\boldsymbol{q}^2}$$

Now we can apply the Hamiltonian equations to get the approximate GR equation-of-movement $\frac{d\boldsymbol{p}}{dt} = -\frac{\partial H}{\partial \boldsymbol{q}}$.

4. The Kerr Orbit Equations

Of special importance is the solution of O1, which gives the derivative $t' = \frac{dt}{d\tau}$.

In the Newtonian approximation, is of course $t=\tau$ and $t'=1$.

In the Schwarzschild spacetime, O1 can be solved analytically, and the well-known solution is [5] [6]

$t'=\frac{F}{1-\frac{1}{r}}$, where $F^2=2\varepsilon_t+1$ the velocity factor, and reduced energy ε_t are constants of the orbit and result from initial conditions.

Furthermore in Schwarzschild spacetime, angular momentum L is conserved: $\varphi'(\tau)r(\tau)^2=l=\frac{L}{m}$, where l is the reduced angular momentum, and the azimuthal angle is constant, we can choose $\theta=\frac{\pi}{2}$.

In the Kerr spacetime, the solution cannot be given in analytical form, but it can be expressed as a series in r and α, it seems that it is derived here for the first time.

First, we bring O1 into a new form using $\theta=\pi/2$, $\theta'=0$ and $\varphi' r^2=l$, thus eliminating φ' and θ':

$$\frac{l\alpha\left(1+\frac{2r(\tau)^2}{\alpha^2+r(\tau)^2}\right)r'(\tau)}{r(\tau)^4\left(1-\frac{r(\tau)}{\alpha^2+r(\tau)^2}\right)}+\frac{r'(\tau)t'(\tau)}{r(\tau)^2\left(1-\frac{r(\tau)}{\alpha^2+r(\tau)^2}\right)}+t''(\tau)=0 \tag{14}$$

This has the general form $t''+t'r'f_2(r)-r'f_3(r)=0$ and after multiplication with a function $f_1(r)$ it can be made a total differential

$$t''f_1(r)+t'r'f_2(r)f_1(r)-r'f_3(r)f_1(r)=0 \quad \text{with} \quad \frac{\mathrm{d}f_1(r)}{\mathrm{d}\tau}=r'f_2(r)f_1(r)$$

And with this condition the formal solution can be derived immediately:

$f_1(r)=\exp\left(\int f_2(r)\mathrm{d}r\right)$ and $t'=\frac{\int f_3(r)f_1(r)+F}{f_1(r)}$ with an integration constant F.

In the Schwarzschild case with $\alpha=0$ and $f_3(r)=0$ this results immediately in $f_1(r)=1-\frac{1}{r}$ and $t'=\frac{F}{1-\frac{1}{r}}$.

In the Kerr case,

$$f_2(r)=\frac{1}{r^2\left(1-\frac{r}{r^2+\alpha^2}\right)},\quad f_3(r)=\frac{\alpha\left(1+\frac{2}{1+\frac{\alpha^2}{r^2}}\right)}{r^4\left(1-\frac{1}{r\left(1+\frac{\alpha^2}{r^2}\right)}\right)}$$

and after turning the integral in the numerator of t' into a series in α and $1/r$, t' becomes to first order in α

$$t' = t_s(r,\alpha) = \exp\left(-\frac{\alpha^2}{4r^4}\right)\left(\frac{rF}{r-1} + \frac{l\alpha\left(-4+\frac{1}{5r^2}+\alpha^2\left(\frac{1}{3r^3}-\frac{2}{7r^4}\right)\right)}{r^2(r-1)} + \cdots\right)$$

which for $\alpha = 0$ results again in $t' = \frac{F}{1-\frac{1}{r}}$.

So the first-order Kerr-correction to t' is of the order $F\frac{\alpha^2}{r^4}$ from the F-term (total energy) and of the order $l\frac{\alpha}{r^3}$ from the l-term (rotational energy).

Now we can eliminate $\varphi(\tau)$, $\theta(\tau)$, $t(\tau)$, using the above relations in the Kerr energy equation E1d, and we get the radial equation for $r(\tau)$ in the form $r'(\tau)^2 + V_{eff}(r) = const$, where $V_{eff}(r)$ is called the effective potential.

Inserting the Schwarzschild (dimensionless) invariants: $t'(\tau)\left(1-\frac{1}{r(\tau)}\right) = F$ and $\varphi'(\tau)r(\tau)^2 = l = \frac{L}{m}$ yields the simplified Kerr energy equation in α, l, and positive relative total energy $\varepsilon = |\varepsilon_t|$ (see also appendix [9]), with $1-F^2 = 2\varepsilon$

$$\begin{aligned} \text{E1dS0} &= r'(\tau)^2 - \frac{1}{r(\tau)} \\ &+1-\frac{(1-2\varepsilon)\left(1-\frac{1}{r(\tau)}+\frac{\alpha^2}{r(\tau)^2}\right)}{1-\frac{1}{r(\tau)}} + \frac{\alpha^2}{r(\tau)^2} \\ &-\frac{2l\alpha(1-\varepsilon)\left(1-\frac{1}{r(\tau)}+\frac{\alpha^2}{r(\tau)^2}\right)}{\left(1-\frac{1}{r(\tau)}\right)r(\tau)^3} \\ &+\frac{l^2\left(1-\frac{1}{r(\tau)}+\frac{\alpha^2}{r(\tau)^2}\right)\left(1+\frac{\alpha^2}{r(\tau)^2}+\frac{\alpha^2}{r(\tau)^3}\right)}{r(\tau)^2} \end{aligned} \tag{15}$$

and for pure Schwarzschild $\alpha = 0$ and we recover the Schwarzschild energy equation

$$\text{E1dS00} = \frac{l^2}{r(\tau)^2}\left(1-\frac{1}{r(\tau)}\right) - \frac{1}{r(\tau)} + r'(\tau)^2 + 2\varepsilon$$

E1 with Kerr-replacement (full Kerr)

$$\text{E1dA0} = \frac{r'(\tau)^2}{1 - \frac{1}{r(\tau)} + \frac{\alpha^2}{r(\tau)^2}} + 1 - \frac{\left(1 - \varepsilon - \frac{4l\alpha}{r(\tau)^3} + \frac{l\alpha}{5r(\tau)^5} + \frac{l\alpha^3}{3r(\tau)^6} - \frac{2l\alpha^3}{7r(\tau)^7}\right)^2}{1 - \frac{1}{r(\tau)}}$$
$$- \frac{2l\alpha\left(1 - \varepsilon - \frac{4l\alpha}{r(\tau)^3} + \frac{l\alpha}{5r(\tau)^5} + \frac{l\alpha^3}{3r(\tau)^6} - \frac{2l\alpha^3}{7r(\tau)^7}\right)}{\left(1 - \frac{1}{r(\tau)}\right) r(\tau)^3} \tag{16}$$
$$+ \frac{l^2\left(1 + \frac{\alpha^2}{r(\tau)^2} + \frac{\alpha^2}{r(\tau)^3}\right)}{r(\tau)^2}$$

The Kerr radial equation E1dS0 can be expanded in $1/r$ powers:

$$\text{K1N} = r'(\tau)^2 + 2\varepsilon - \frac{1}{r(\tau)} + \frac{l^2 + 2\alpha^2\varepsilon}{r(\tau)^2} - \frac{l^2 + 2l\alpha(1-\varepsilon) + \alpha^2(1-2\varepsilon)}{r(\tau)^3}$$
$$+ \frac{\alpha^2\left(2l^2 - 1 + 2\varepsilon\right)}{r(\tau)^4} \tag{17}$$

5. Adaptation of the Kerr-Parameter α to the Newtonian Rotator Correction

5.1. The Newtonian and Schwarzschild Orbit

The Newtonian Kepler orbit is a fundamental formula of classical celestial mechanics. We shall use it in the following sub-chapter and describe it here as a reminder.

The Kepler orbit results from the Newtonian gravitational potential $V_N = -\frac{MGm}{r} = -\frac{r_s mc^2}{2r}$ where r_s is the Schwarzschild radius of the central mass M.

The Kepler orbit is an ellipse with eccentricity e described by [9].
$\frac{1}{r} = \frac{MGm}{L^2}(1 + e\cos\varphi) = \frac{r_s c^2}{2l^2}(1 + e\cos\varphi)$, where the reduced angular momentum is $l = \frac{L}{m}$, the eccentricity is $e = \sqrt{1 + \frac{2E_t L^2}{mGM}} = \sqrt{1 + \frac{8\varepsilon_t l^2}{r_s^2 c^4}} = \sqrt{1 - \frac{8\varepsilon l^2}{r_s^2 c^4}} < 1$, and $\varepsilon_t = \frac{E_t}{m}$ is the relative total energy from the radial orbit equation, $\varepsilon_t < 0$ for bound states with elliptical orbits, $\varepsilon = |\varepsilon_t|$ is the absolute value of the (negative) relative total energy.

The orbit rotation angle $\varphi = \varphi(t)$ is time-dependent and obeys the relation $\frac{\mathrm{d}\varphi(t)}{\mathrm{d}t} r^2(t) = l$.

The Newtonian orbit Equation (4) is solvable in integral form

$$t_N(r,\varepsilon_t,l)=\int\frac{1}{\sqrt{2\varepsilon_t+\frac{r_sc^2}{r}-\frac{l^2}{r^2}}}\mathrm{d}r \tag{18}$$

The resulting formula is [13] (dimensionless, time in Schwarzschild-units $t_s=\frac{r_s}{c}$, radius in Schwarzschild-units r_s, $c=1$).

$$t_N(r,\varepsilon,l)=\frac{1}{8\varepsilon^{3/2}}\left(4\sqrt{\varepsilon l^2-\varepsilon r+2\varepsilon^2r^2}+\sqrt{2}\log\left(\varepsilon\left(-1+4\varepsilon r+2\sqrt{2}\sqrt{\varepsilon l^2-\varepsilon r+2\varepsilon^2r^2}\right)\right)\right) \tag{19}$$

The zeros of the root in the integral are the minimal and maximal radius, *i.e.* the semi-minor and the semi-major axis of the orbit ellipse [13].

$r_{\min}=\frac{1-\sqrt{1-8\varepsilon l^2}}{4\varepsilon}$, $r_{\max}=\frac{1+\sqrt{1-8\varepsilon l^2}}{4\varepsilon}$ dimensionless $r_{\min}=r_s\frac{1-\sqrt{1-\frac{8\varepsilon l^2}{r_s^2c^4}}}{4\varepsilon}$, $r_{\max}=r_s\frac{1+\sqrt{1-\frac{8\varepsilon l^2}{r_s^2c^4}}}{4\varepsilon}$ full dimensional, mean distance is $r_c=r_s\frac{1}{4\varepsilon}$.

For circular orbit, $r_{\min}=r_{\max}=r(t)$, $\omega=\sqrt{\frac{r_s}{2}}\frac{c}{r^{3/2}}$, $l=\omega r^2=\sqrt{r_s}c\sqrt{\frac{r}{2}}$.

The minimal and maximal radii $(r_{\min},r_{\max})$ are zeros of the Newtonian effective potential (4b) (dimensionless, *i.e.* $r_s=1,c=1$).

$\tilde{V}_{eff,N}=\frac{l^2}{r^2}-\frac{1}{r}+2\varepsilon$, which yields a quadratic equation for the two radii.

In case of the Schwarzschild metric with the corresponding effective potential $\tilde{V}_{eff,S}=\frac{l^2}{r^2}\left(1-\frac{1}{r}\right)-\frac{1}{r}+2\varepsilon$ the minimal and maximal radii $(r_{\min},r_{\max})$ are roots of a cubic equation in *r*.

The first root r_1 is the turning point of a parabolic orbit, the other two are $(r_{\min},r_{\max})$ of a closed Schwarzschild orbit [2], see appendix .

When $(r_{\min},r_{\max})$ become complex, there is no closed Schwarzschild orbit, this happens for small distances from the center. The mean radius $r_c(\varepsilon,l)=\frac{r_{\min}+r_{\max}}{2}$ has the form ([2])

$$r_c(\varepsilon,l)=\frac{6l^2\varepsilon-\left(-1+\left(1+9l^2\varepsilon(-1+6\varepsilon)+3\sqrt{3}l\varepsilon\sqrt{4+8l^4\varepsilon+l^2\left(-1-36\varepsilon+108\varepsilon^2\right)}\right)^{1/3}\right)^2}{12\varepsilon\left(1+9l^2\varepsilon(-1+6\varepsilon)+3\sqrt{3}l\varepsilon\sqrt{4+8l^4\varepsilon+l^2\left(-1-36\varepsilon+108\varepsilon^2\right)}\right)^{1/3}}$$

for small energies $\varepsilon\ll1$, it becomes $r_c=\frac{1}{6\varepsilon}$.

5.2. The Orbit Equations for the Remote Observer in Kerr Spacetime and in the Newtonian Limit

First, we calculate the two-body correction $O\left(\frac{1}{r^3}\right)$ to the Newtonian gravitational potential from the Newtonian Kepler orbits of the rotatoras shown in Figure 1.

Second, we take into account the Kerr correction $O\left(\frac{\alpha}{r^3}\right)$ in the Kerr energy E1dS0 and compare both expressions, from this we get a formula for α.

The z-axis of the observer is the z-axis of the rotator: perpendicular to the rotator plane, the origin is at rotator-com.

We make the assumptions:

- Observer distance from com is large compared to the rotator's orbit diameter, so φ_b = rotation angle of the rotator, changes much faster than φ_o = rotation angle of the observer orbit.

- The distance vector between observer and center-of-mass: $r_o \gg r_{0x}$, *i.e.* observer orbit movement is much slower than rotator movement

We have the following denominations:

$\boldsymbol{r}_o$ = vector(observer, com rotator)

$\boldsymbol{r}_o = (x_o, y_o, z_o)$ Cartesian observer coordinates $\boldsymbol{r}_o = (r_o, \theta_o, \varphi_o)$ spherical coordinates relative to com

$$x_o = r_o \sin\theta_o,\quad z_o = r_o \cos\theta_o$$

distance between masses $r_{0x} = d(m_1, m_2)$,

$$(x_{1phi}, x_{2phi}) = \text{projection}(r_o, \theta_o = \pi/2)$$

$$r_{01x} = d(\text{com}, m_1) = r_{0x}\frac{m_2}{m},\quad r_{02x} = d(\text{com}, m_2) = r_{0x}\frac{m_1}{m}$$

masses: m_1, m_2, $m = m_1 + m_2$, $m_r = \frac{m_1 m_2}{m}$, mass ratio $\mu = \frac{m_2}{m_1} \ge 1$ rotator orbit angle φ_b = angle(rotator-axis, observer x-projection) observer mass-distance projections:

$$x_{1phi} = \sqrt{x_o^2 + r_{01x}^2 - 2x_o r_{01x}\sin\varphi_b},\quad x_{2phi} = \sqrt{x_o^2 + r_{02x}^2 - 2x_o r_{02x}\sin\varphi_b}$$

distances between observer and masses $r_{1x} = d(\text{observer}, m_1)$, $r_{2x} = d(\text{observer}, m_2)$ are

$$r_{1x} = \sqrt{x_{1phi}^2 + z_o^2} = \sqrt{x_o^2 + r_{01x}^2 - 2x_o r_{01x}\sin\varphi_b + r_o^2\cos^2\theta_o}$$
$$= \sqrt{r_o^2 + r_{01x}^2 - 2r_o r_{01x}\sin\theta_o\sin\varphi_b}$$

$$r_{2x} = \sqrt{x_{2phi}^2 + z_o^2} = \sqrt{x_o^2 + r_{02x}^2 - 2x_o r_{02x}\sin\varphi_b + r_o^2\cos^2\theta_o}$$
$$= \sqrt{r_o^2 + r_{02x}^2 - 2r_o r_{02x}\sin\theta_o\sin\varphi_b}$$

We calculate now the correction of the Newtonian potential due to the two-body rotation of the rotator.

The Newtonian potential of the rotator mass $m = m_1 + m_2$ is

$$V_N(r_o) = \frac{r_s(m)c^2}{2r_o},\quad r_s(m) = \frac{2Gm}{c^2}$$

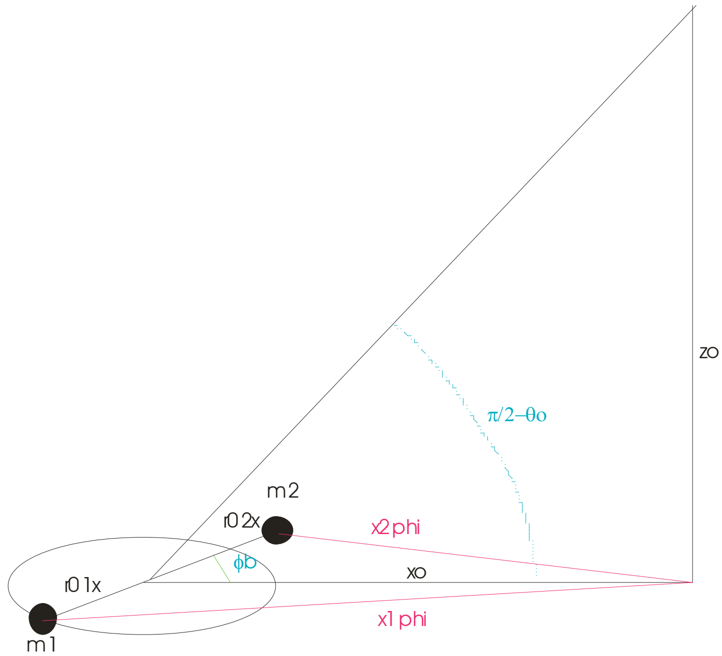

Figure 1. The Newtonian correction to the observer orbit in rotator potential.

We develop the distances r_{1x} and r_{2x}

$$r_{1x} = \sqrt{r_o^2 + r_{01x}^2 - 2r_o r_{01x} \sin\theta_o \sin\varphi_b} \approx r_o\left(1 + \frac{r_{01x}^2}{2r_o^2} - \frac{r_{01x}}{r_o}\sin\theta_o \sin\varphi_b\right)$$

$$r_{2x} \approx r_o\left(1 + \frac{r_{02x}^2}{2r_o^2} - \frac{r_{02x}}{r_o}\sin\theta_o \sin\varphi_b\right)$$

Now, we have to average over a rotation period over the angle φ_b. Theoretically, we have to insert $\varphi_b = \varphi(t)$ from the orbit equation $\frac{\mathrm{d}\varphi(t)}{\mathrm{d}t} r^2(t) = l$ and invert $t_N(r, \varepsilon, l)$ to get $r(t)$, and then integrate over a rotation period $\bar{r}_{01x} = \frac{1}{T}\int_{t=0}^{t=T} r_{01x}(t)\mathrm{d}t$.

It is hopeless to get an analytical expression for $\bar{r}_{01x}$, but we can assume that $\varphi(t)$ is approximately linear $\varphi(t) \approx \omega t$, which is exact for a circular orbit: $\varphi(t) = \omega t$. Then the term $\frac{r_{02x}}{r_o}\sin\theta_o \sin\varphi_b$ cancels by averaging over a rotation period, and we get the expression $r_{1x} \approx r_o\left(1 + \frac{\bar{r}_{01x}^2}{2r_o^2}\right)$, where the average $\bar{r}_{01x} \approx r_c\left(\frac{m_2}{m}\right)$, $r_c \approx \frac{r_{\min} + r_{\max}}{2}$ is the mean diameter of the rotator orbit ellipse,

$$r_{1x} \approx r_o\left(1 + \frac{r_c^2}{2r_o^2}\left(\frac{m_2}{m}\right)^2\right), \quad r_{2x} \approx r_o\left(1 + \frac{r_c^2}{2r_o^2}\left(\frac{m_1}{m}\right)^2\right)$$

With this approximation, we get the correction of the Newtonian potential

$$\Delta V_N = \frac{r_s c^2}{2r_o}\left(1-\left(\frac{\frac{m_1}{m}}{1+\frac{r_c^2}{2r_o^2}\left(\frac{m_2}{m}\right)^2}+\frac{\frac{m_2}{m}}{1+\frac{r_c^2}{2r_o^2}\left(\frac{m_1}{m}\right)^2}\right)\right)$$

$$=\frac{r_s r_c^2 c^2}{4r_o^3}\left(\frac{m_1 m_2^2}{m^3}+\frac{m_2 m_1^2}{m}\right)=\frac{r_s r_c^2 c^2}{4r_o^3}\frac{m_1 m_2}{m^2}=\frac{r_s r_c^2 c^2}{4r_o^3}\frac{\mu}{(1+\mu)^2}$$

or dimensionless

$$\Delta V_N = \frac{r_c^2}{4r_o^3}\frac{\mu}{(1+\mu)^2} \tag{20}$$

The term $O\left(\frac{1}{r(\tau)^4}\right)$ in E1dS0 is with $r(\tau)=r_o$ dimensionless

$$\frac{\Delta E_\alpha}{m}=\frac{\alpha^2}{r_o^4}\left(2l^2-(1-2\varepsilon)\right)\approx\frac{\alpha^2}{r_o^4}2l^2$$

Now, for circular orbit dimensionless $l=\sqrt{\frac{r_o}{2}}$, so $\frac{\Delta E_\alpha}{m}\approx\frac{\alpha^2}{r_o^3}$, and we get for $\alpha^2=\frac{r_c^2}{4}\frac{\mu}{(1+\mu)^2}$, and $\alpha=\frac{r_c}{2}\frac{\sqrt{\mu}}{(1+\mu)}$ and $\lim(\alpha(\mu),\mu\to\infty)=0$, as it should be, because for $m_1\to 0$ the rotator becomes a single mass, and the cylindrical symmetry becomes spherical, the Kerr-spacetime becomes Schwarzschild-spacetime with $\alpha=0$.

6. The Manko-Ruiz Solution as a Generalized Kerr Spacetime

6.1. Tomimatsu-Sato Solutions as Generalizations of Kerr Spacetime

The Kerr solution belongs to the class of Tomimatsu–Sato solutions describing exterior gravitational fields of stationary rotating axisymmetric sources introduced by Tomimatsu and Sato (1972, 1973).

In Weyl's canonical (cylindrical) coordinates (ρ, z), the axisymmetric Papapetrou space-time metric ([14] 19.17) is given by

$$\mathrm{d}s^2=\exp(-2U)\left(\exp(2k)\left(\mathrm{d}\rho^2+\mathrm{d}z^2\right)+\rho^2\mathrm{d}\varphi^2\right)-\exp(2U)\left(\mathrm{d}t+A\mathrm{d}\varphi\right)^2 \tag{21}$$

with functions U, k, A.

We get the Einstein field equations in the form

$$\Delta U=\rho^{-1}\sum_{i=1,2}\left(\rho U_{,i}\right)_{,i}=\sum_{i=1,2}\left(U_{,i}U_{,i}+k_{,ii}\right)$$

$$(\Delta U)^2=\left(2U_{,1}U_{,2}-\frac{k_{,2}}{\rho}\right)^2+\left(U_{,1}^2-U_{,2}^2-\frac{k_{,1}}{\rho}\right)^2 \quad \text{where } U_{,1}=\frac{\partial U}{\partial\rho},\ U_{,2}=\frac{\partial U}{\partial z}$$

For $\Gamma=\exp(2U)+i\omega$ we obtain in Weyl coordinates the Ernst equation ([14] 19.41)

$$\left(\Gamma+\bar{\Gamma}\right)\left(\Gamma_{,\rho\rho}+\rho^{-1}\Gamma_{,\rho}+\Gamma_{,zz}\right)=2\left(\Gamma_{,\rho}^{2}+\Gamma_{,z}^{2}\right) \quad (22)$$

from which we calculate the metric functions

$$k_{,\varsigma}=\sqrt{2}\rho\frac{\Gamma_{,\varsigma}\bar{\Gamma}_{,\varsigma}}{\left(\Gamma+\bar{\Gamma}\right)^{2}},\quad A_{,\varsigma}=2\rho\frac{\left(\Gamma-\bar{\Gamma}\right)_{,\varsigma}}{\left(\Gamma+\bar{\Gamma}\right)^{2}},\text{ where }\ \partial_{\varsigma}=\frac{\partial_{\rho}-i\partial_{z}}{\sqrt{2}}$$

The Ernst equation with Γ can be reformulated by introducing the new potential ξ by

$$\xi=\frac{1-\Gamma}{1+\Gamma}$$

We go over to the prolate spheroidal coordinates (x,y) connected with Weyl's canonical coordinates (ρ,z) by the relations

$$\rho=\sigma\sqrt{x^{2}-1}\sqrt{1-y^{2}},\quad z=\sigma xy,\quad \sigma=const$$

In these coordinates, the Ernst equation becomes ([14] 20.37)

$$\left(\xi\bar{\xi}-1\right)\left(\left(\left(x^{2}-1\right)\xi_{,x}\right)_{,x}+\left(\left(1-y^{2}\right)\xi_{,y}\right)_{,y}\right)=2\bar{\xi}\left(\left(x^{2}-1\right)\xi_{,x}^{2}+\left(1-y^{2}\right)\xi_{,y}^{2}\right)$$

Based on this ansatz, the Tomimatsu-Sato solution with integer δ can be obtained via a limiting process from the non-linear superposition of δ Kerr solutions with common symmetry axis (Tomimatsu and Sato, 1981).

The potential ξ of these solutions is a quotient $\xi=\frac{\beta}{\alpha}$, α and β being polynomials in the coordinates x and y, with parameters (p,q) where $1=p^{2}+q^{2}$. Here, the constant σ in the relation between the Weyl coordinates (ρ,z) and the prolate spheroidal coordinates (x,y), the angular momentum J and the quadrupole moment Q are given by

$$\sigma=\frac{mp}{\delta},\quad J=m^{2}q,\quad Q=m^{2}\left(q^{2}+\frac{\left(\delta^{2}-1\right)p^{2}}{3\delta^{2}}\right),\text{ where }\delta\text{ is an integer}$$

For $\delta=1$ we get the Kerr solution in prolate coordinates

$$\alpha=px-iqy,\quad \beta=1,\quad \xi=\frac{\beta}{\alpha}$$

The full Kerr metric can be obtained from ξ as ([14] 20.36)

$$\exp\left(2U\right)=\frac{p^{2}x^{2}+q^{2}y^{2}-1}{\left(px+1\right)^{2}+q^{2}y^{2}},$$

$$\exp\left(2k\right)=\frac{p^{2}x^{2}+q^{2}y^{2}-1}{p^{2}\left(x^{2}-y^{2}\right)},\quad A=\frac{2mq}{p^{2}x^{2}+q^{2}y^{2}-1}\left(1-y^{2}\right)\left(px+1\right)$$

where $mq=a$, $mp=\sigma$

$$q=\frac{a}{m},\quad p=\frac{\sigma}{m},\quad \sigma=m\sqrt{1-\frac{a^{2}}{m^{2}}},\quad x=\frac{r-m}{\sigma}=\frac{\frac{r}{m}-1}{\sqrt{1-\frac{a^{2}}{m^{2}}}}=\frac{\frac{2r}{r_{s}}-1}{\sqrt{1-\frac{4a^{2}}{r_{s}^{2}}}},\quad y=\cos\theta$$

For $\delta = 2$ we get the Kinnersley and Chitre (1978) solution

$$\alpha = p^2\left(x^4-1\right)-2ipqxy\left(x^2-y^2\right)+q^2\left(y^4-1\right),$$

$$\beta = 2px\left(x^2-1\right)-2iqy\left(1-y^2\right),\quad \xi=\frac{\beta}{\alpha}$$

6.2. Metric for Kerr-Kerr Binary Configuration

We consider the spacetime of two Kerr black-holes with masses m_1 and m_2, rotation parameters a_1 and a_2, in fixed distance R, and we are looking for the corresponding vacuum solution of Einstein equations, which is asymptotically flat, *i.e.* it becomes Minkowski metric at infinity (Manko & Ruiz [15]), it is also called the Kinnersley-Chitre five-parameter vacuum solution, Manko & Ruiz [16].

This spacetime generalizes the Tomimatsu-Sato solution for $\delta = 2$ [16].

Ernst complex potential Γ of the exact solution for two aligned Kerr sources is defined on the symmetry axis ($\rho = 0$) by the expression (Stephani [14] (34.94), Manko & Ruiz [15])

$$\Gamma(\rho=0,z)=\frac{z^2-(m+ia)z+s-\mu+i(\tau+\delta)}{z^2+(m-ia)z+s+\mu+i(\tau-\delta)}$$

where m is the total mass of the binary system, a is the rotational parameter, and the real quantities s, μ, τ, δ are related to the physical characteristics:

$$s=\frac{1}{4}\left(R^2+2\left(\sigma_1^2+\sigma_2^2-m^2+a^2\right)\right),\quad \delta=ma-m_1a_1-m_2a_2,$$

$$\tau=\frac{R}{2}(a_2-a_1)+\frac{(R+m)\left(m_2a_1(R+2m_1)-m_1a_2(R+2m_2)\right)}{(R+m)^2+a^2},$$

$$\mu=\frac{1}{2m}\left(R\left(\sigma_1^2-\sigma_2^2\right)-2a\tau\right)$$

We have the 5 real physical parameters $\{m_1, m_2, a_1, a_2, R\}$, with distance R, total mass m and total angular momentum J $m=m_1+m_2$, $J=m_1a_1+m_2a_2$, rotational parameters $a_1=\frac{j_1}{m_1}$, $a_2=\frac{j_2}{m_2}$, total rotational parameter a depends on J and is obtained from the cubic equation in a_1, a_2, R, m_1, m_2

$$J-ma+\frac{(a_1+a_2-a)\left(R^2-m^2+a^2\right)}{2(R+m)}=0$$

σ_1 and σ_2 are the half-horizon radii

$$\sigma_1=\sqrt{m_1^2-a_1^2+4m_2a_1d_1},\quad \sigma_2=\sqrt{m_2^2-a_2^2+4m_2a_2d_2}$$

where

$$d_1=\frac{\left(m_1(a_1-a_2+a)+Ra\right)\left((R+m)^2+a^2\right)+m_2a_1a^2}{\left((R+m)^2+a^2\right)^2}$$

$$d_2=\frac{\left(m_2(a_2-a_1+a)+Ra\right)\left((R+m)^2+a^2\right)+m_1a_2a^2}{\left((R+m)^2+a^2\right)^2}$$

Ernst complex potential Γ in the whole (ρ, z) space is then extended to

$$\Gamma = \frac{A-B}{A+B} \tag{23}$$

$$A = \left(R^2 + (\sigma_1 - \sigma_2)\right)(R_+ - R_-)(r_+ - r_-) + 4\sigma_1\sigma_2(R_+ - r_-)(R_- - r_+)$$

$$B = 2\sigma_1\left(R^2 - \sigma_1^2 + \sigma_2^2\right)(R_- - R_+) + 2\sigma_2\left(R^2 + \sigma_1^2 - \sigma_2^2\right)(r_- - r_+) + 4R\sigma_1\sigma_2(R_+ + R_- - r_+ - r_-)$$

with the radial functions

$$\tilde{r}_\pm = \sqrt{\rho^2 + \left(z - \frac{R}{2} \pm \sigma_1\right)^2}\,,\quad \tilde{R}_\pm = \sqrt{\rho^2 + \left(z - \frac{R}{2} \pm \sigma_2\right)^2}\,,\quad \mu_0 = \frac{R+m-ia}{R+m+ia}$$

$$r_\pm = \frac{\tilde{r}_\pm}{\mu_0}\frac{(\pm\sigma_1 - m_1 - ia_1)\left((R+m)^2 + a^2\right) + 2a_1\left(m_1 a + im(R+m)\right)}{(\pm\sigma_1 - m_1 + ia_1)\left((R+m)^2 + a^2\right) + 2a_1\left(m_1 a - im(R+m)\right)}$$

$$R_\pm = -\mu_0 \tilde{R}_\pm \frac{(\pm\sigma_2 + m_2 - ia_2)\left((R+m)^2 + a^2\right) - 2a_2\left(m_2 a - im(R+m)\right)}{(\pm\sigma_2 + m_2 + ia_2)\left((R+m)^2 + a^2\right) - 2a_2\left(m_2 a + im(R+m)\right)}$$

which satisfies the Ernst equation (Cabrera [13])

$$\left(\Gamma + \bar{\Gamma}\right)\Delta\Gamma = 2(\nabla\Gamma)^2 \tag{24}$$

$$\left(\Gamma + \bar{\Gamma}\right)\left(\Gamma_{,\rho\rho} + \rho^{-1}\Gamma_{,\rho} + \Gamma_{,zz}\right) = 2\left(\Gamma_{,\rho}^2 + \Gamma_{,z}^2\right)$$

The line element becomes now [13] in the Weyl-form from 6.1 with new denominations

$$f = \exp(2U),\quad \gamma = k,\quad \omega = -A \tag{24a}$$

$$\mathrm{d}s^2 = f^{-1}\left(\exp(2\gamma)\left(\mathrm{d}\rho^2 + \mathrm{d}z^2\right) + \rho^2\mathrm{d}\varphi^2\right) - f\left(\mathrm{d}t - \omega\mathrm{d}\varphi\right)^2$$

with the functions f, γ, ω of the Weyl coordinates ρ and z

$$f = \frac{A\bar{A} - B\bar{B}}{(A+B)\left(\bar{A}+\bar{B}\right)},\quad \exp(2\gamma) = \frac{A\bar{A} - B\bar{B}}{16|\sigma_1|^2|\sigma_2|^2 K_0^2 \tilde{R}_+\tilde{R}_-\tilde{r}_+\tilde{r}_-},$$

$$\omega = 2a - \frac{2Im\left[G\left(\bar{A}+\bar{B}\right)\right]}{A\bar{A} - B\bar{B}}, \tag{24b}$$

$$\begin{aligned} G = &-zB + \sigma_1\left(R^2 - \sigma_1^2 + \sigma_2^2\right)(R_- - R_+)(r_+ + r_- + R) \\ &+ \sigma_2\left(R^2 + \sigma_1^2 - \sigma_2^2\right)(r_- - r_+)(R_+ + R_- - R) \\ &- 2\sigma_1\sigma_2\left(2R\left(r_+r_- - R_+R_- - \sigma_1(r_- - r_+) + \sigma_2(R_- - R_+)\right)\right. \\ &\left.+ \left(\sigma_1^2 - \sigma_2^2\right)(r_+ + r_- - R_+ - R_-)\right) \end{aligned}$$

where

$$K_0 = \frac{\left((R+m)^2 + a^2\right)\left(R^2 - (m_1 - m_2)^2 + a^2\right) - 4a^2 m_1^2 m_2^2}{m_1 m_2\left((R+m)^2 + a^2\right)}$$

6.3. Transformation to Spherical Coordinates

It is advisable to transform the metric from the Weyl cylindrical coordinates (t, ρ, z, φ) to spherical coordinates (t, r, θ, φ).

The transformation reads: $\rho \to \sqrt{r^2 - mr}\sin\theta$, $z \to (r-m)\cos\theta$, where $m = m_1 + m_2$ is the total mass [14].

The differentials in the line element also have to be transformed correspondingly: $\mathrm{d}\rho \to \mathrm{d}r \frac{\mathrm{d}\rho}{\mathrm{d}r} + \mathrm{d}\theta \frac{\mathrm{d}\rho}{\mathrm{d}\theta}$ and $\mathrm{d}z \to \mathrm{d}r \frac{\mathrm{d}z}{\mathrm{d}r} + \mathrm{d}\theta \frac{\mathrm{d}z}{\mathrm{d}\theta}$. The transformed metric doubles in complexity [1] [2], $C(g_{\mu\nu}, (\rho, z)) = 127000$, $C(g_{\mu\nu}, (r, \theta)) = 478000$, but handling the Christoffel symbols and the Einstein equations in familiar spherical coordinates is easier.

The resulting Christoffel symbols have the complexity $C(\Gamma^{\kappa}_{\mu\nu}, (r, \theta)) = 34 \times 10^6$ terms, and the Einstein equations have the complexity $C(G_{\mu\nu}, (r, \theta)) = 240 \times 10^6$ terms, so symbolic verification of the Einstein equations is out of the question, but numeric verification by stochastic insertion works (although it is time-consuming), and the result is indeed zero within the limit of precision for this complexity (10^{-4}) [2].

In the following and in the program code, we use exclusively this transformed Manko-Ruiz metric.

6.4. Adaptation of the Manko-Ruiz Kerr-Kerrpotential to the Newtonian Correction for Observer Orbit

The Manko-Ruiz Kerr-Kerr-solution $M_{br} = (g_{\mu\nu}(m_1, m_2, R, a_1, a_2))$ has originally 5 parameters: two rotator masses m_1, m_2, the distance between the masses R, two orbit rotation parameter a_1, a_2. We interpret R as the mean rotator distance r_c, and a_1, a_2 as the two individual reduced angular momenta of the rotator masses. In this case, the rotation parameters have the ratio $\frac{a_2}{a_1} = \mu^2$, where $\mu = \frac{m_2}{m_1}$ (see 2.1), so we are left with 4 independent parameters:

$$M_{br} = (g_{\mu\nu}(m_1, m_2, R, a_0)), \text{ where } a_1 = \frac{a_0}{1+\mu^2}, \quad a_2 = \frac{a_0 \mu^2}{1+\mu^2}.$$

The 4 parameters have to be adapted to the Newtonian rotator correction: in order $O\left(\frac{1}{r^2}\right)$: $\Delta V_N \approx 0$, and in order $O\left(\frac{1}{r^3}\right)$: $\Delta V_N = \frac{r_s r_c^2 c^2}{4r^3} \frac{\mu}{(1+\mu)^2}$, preserving the mass ratio $\mu = \frac{m_2}{m_1}$; in order $O(1)$ and $O\left(\frac{1}{r}\right)$ the Manko-Ruiz metric is Schwarzschild.

The general effective potential of the Manko-Ruiz metric in the energy (or radial) equation is $\tilde{V}_{eff}(m_1, m_2, R, a_0, l, \varepsilon)$, it depends on the 4 metric parameters (m_1, m_2, R, a_0) and the two orbit parameters (l, ε) of the observer orbit.

The two free parameters (R, a_0) in $M_{br} = (g_{\mu\nu}(m_1, m_2, R, a_0))$ are in fact functions of the rotator orbit parameter r_c, *i.e.* $R = R(r_c)$, $a_0 = a_0(r_c)$. The coefficient of $O\left(\frac{1}{r^3}\right)$ in $\tilde{V}_{eff}$ is $c\left(m_1, m_2, R, a_0, l, \varepsilon, V_{eff}, \frac{1}{r^3}\right)$.

It is a quadratic polynomial in (l, ε), as evaluation shows [2]:

$c\left(m_1, m_2, R, a_0, l, \varepsilon, \tilde{V}_{eff}, \frac{1}{r^3}\right) = \sum_{i,j=0}^{2} k_{3ij} l^i \varepsilon^j$, the same is true for $O\left(\frac{1}{r^2}\right)$,

$c\left(m_1, m_2, R, a_0, l, \varepsilon, \tilde{V}_{eff}, \frac{1}{r^3}\right) = \sum_{i,j=0}^{2} k_{2ij} l^i \varepsilon^j$.

Adaptation means: in the effective orbit potential $\tilde{V}_{eff}(m_1, m_2, R, a_0, l, \varepsilon)$ the (l, ε) independent coefficient satisfies

$$k_{300}(m_1, m_2, R, a_0) = \Delta V_N = \frac{r_s r_c^2 c^2}{4} \frac{\mu}{(1+\mu)^2} \tag{25}$$

and the coefficient $\left|k_{200}(m_1, m_2, R, a_0)\right| = \min$.

This results in two equations for the two parameters (R, a_0), which are solvable, usually in real numbers, as the numerical example shows.

The adaptation cannot be done symbolically, because the radial equation potential $\tilde{V}_{eff}$ for Manko-Ruiz has a complexity of $C(\tilde{V}_{eff}, M1s) = 2.6 \times 10^5$ terms, and to calculate the expansion in $\frac{1}{r}$-powers takes several weeks on a desktop. In comparison, for Kerr-spacetime $C(\tilde{V}_{eff}, K1) = 200$ with the expansion computing time $t(\tilde{V}_{eff}, K1) = 0.015\,\text{s}$. The assessment for the computing time is $t \sim C^3$, so we get

$$t(\tilde{V}_{eff}, M1) = 3.3 \times 10^7\, s = 390\,\text{d}.$$

A numerical alternative is to fit the expansion in $\frac{1}{r}$-powers of the radial equation $\tilde{V}_{eff}(R, a, \mu)$ *with inserted parameters* (R, a, μ), for the two expansion coefficients: for $V_{E1,2}(R, a, \mu) = c\left(\tilde{V}_{eff}, \frac{1}{r^2}\right)$ coefficient of $O\left(\frac{1}{r^2}\right)$, and for $V_{E1,3}(R, a, \mu) = c\left(\tilde{V}_{eff}, \frac{1}{r^3}\right)$ coefficient of $O\left(\frac{1}{r^2}\right)$.

This is done in [2] in section "Interpolation Manko-Ruiz coeff-expansion ($1/r^2$), coeff-expansion ($1/r^3$), in *R*, *a*" as a fit in (R, a) with fixed $\mu = \mu_0 = 2$.

We find the solution of the relation (25) to be [2]:

$$(R = 22.6, a = 0.0021)$$

We use the approximate values $(R = 20, a = 0.05)$ for the calculation of Manko-Ruiz observer orbits with $\mu = \mu_0 = 2$ in subsection 8.1.

6.5. Manko-Ruiz Rotator Orbits in Its Own Potential

The Manko-Ruiz Kerr-Kerr-solution $M_{br} = (g_{\mu\nu}(m_1, m_2, R, a_1, a_2))$ has originally 5 parameters, we replace the two masses (m_1, m_2) by the total mass $m = m_1 + m_2$, and the ratio $\mu = \frac{m_2}{m_1}$. The rotational parameters (a_1, a_2) are proportional to the reduced orbit angular momentum of the two masses: $l_1 \simeq a_1$ and $l_2 \simeq a_2$, as in the case of the Kerr metric, where $l = \frac{a}{c}$. We have shown in 2.2 that the angular momentums of the rotator have the ratio $\frac{l_2}{l_1} = \mu^2$, so we can set $a_1 = a\frac{1}{1+\mu^2}$ and $a_2 = a\frac{\mu^2}{1+\mu^2}$, where a is the rotational parameter of the rotator.

Now the Manko-Ruiz metric becomes $M_{br} = \left(g_{\mu\nu}\left(m,R,a,\mu,r,\theta\right)\right)$ as a function in spherical coordinates r,θ with 3 parameters $\left(R,a,\mu\right)$ and the mass m. We make the metric dimensionless by setting, as usual, $2m = 1$ and $c = 1$, which is equivalent to calculation in Schwarzschild units $r_s = \frac{2mG}{c^2}$ and $t_s = \frac{r_s}{c}$.

In order to calculate Manko-Ruiz rotator orbits in its own potential, we have to derive a formula for the orbit parameters $\left(l,\varepsilon\right)$ in dependence of the three Manko-Ruiz parameters $\left(R,a,\mu\right)$: $l = l\left(R,a,\mu\right)$ and $\varepsilon = \varepsilon\left(R,\mu\right)$.

Relative energy $\varepsilon\left(R,\mu\right)$

Here $R = r_c$ is the mean radius of the orbit, and we know from 5.1 that the Kepler formula for ε is $\varepsilon = \frac{1}{4r_c}$.

As for ε, in the case of Manko-Ruiz rotator, we calculate $\varepsilon\left(R,\mu\right)$ at fixed μ by iteration with the ansatz $\varepsilon = \frac{c_1\left(\mu,R\right)}{4r_c}$: we start with $\varepsilon = \frac{1}{4R}$ and $l = l\left(R,a,\mu\right)$, calculate the orbit from the Manko-Ruiz orbit equations, calculate $r_c = \frac{r_{\min}+r_{\max}}{2}$, and adapt the factor c_1 in such a way to make $r_c = R$, The iteration normally converges after 2 - 3 steps with accuracy $\left|R - r_c\right| \le 0.5$, as demonstrated in [2].

In principle, we can calculate symbolically the roots $\left(r_{\min}, r_{\max}\right)$ of the Manko-Ruiz effective potential $\tilde{V}_{eff}\left(R,a,\mu,l,\varepsilon\right)$, calculate $r_c = \frac{r_{\min}+r_{\max}}{2}$, and solve the equation $r_c\left(R,a,\mu,l\left(R,a,\mu\right),\varepsilon\right) = R$ for ε.

In reality, the equation is of course hopelessly complicated.

However, we can solve the equation numerically for given discrete parameter values $\left(R,a,\mu\right) = \left(R_i,a_j,\mu_k\right)$ on lattice in a certain parameter region, and then fit the values $\varepsilon\left(R_i,a_j,\mu_k\right)$ in three dimensions. This has been done in [2] in subsection "fit epsc = c1(rc)/(4rc)" and the result is $\varepsilon\left(R\right) = \frac{c_1\left(R\right)}{4R}$, where the coefficient $c_1\left(R\right)$ is fitted by a rational quadratic function

$$c_1\left(R\right) = \frac{4.67 + 9.94R + 8.94R^2}{1 + 0.488R + 9.12R^2} \tag{26}$$

which depends only on *R*, and the fit-data plot is shown below **Figure 2**:

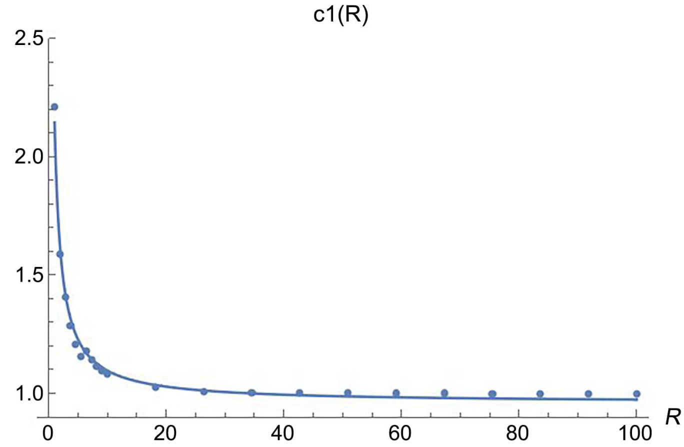

Figure 2. The ε-coefficient $c_1(R)$ as a function of R in Schwarzschild units.

For $R > 20$, $c_1(R) \approx 1$, and we have the Newtonian relation (19).

Reduced angular momentum $l = l(R, a, \mu)$.

In order to derive the formula $l = l(R, a, \mu)$, we start with the off-diagonal metric component $g_{14}(r, \theta) = \frac{r a \sin^2\theta}{r^2 + a^2\cos^2\theta}$ of the Kerr metric. In the orbit plane $\theta = \frac{\pi}{2}$ and we get $g_{14}\left(r, \theta = \frac{\pi}{2}\right) = \frac{a}{r}$, *i.e.* $l = a = c\left(\frac{1}{r}\right)$ is the coefficient of $O\left(\frac{1}{r}\right)$ in the expansion of $g_{14}\left(r, \theta = \frac{\pi}{2}\right)$ in $\frac{1}{r}$-powers.

We apply this to the Manko-Ruiz metric. Symbolic calculation is out of the question, but we can fit the expansion of $g_{14}\left(r, \theta = \frac{\pi}{2}\right)$ numerically in the parameters (R, a, μ), as we did in 6.4. in parameters (R, a).

This is done in [1] in subsections "interpolation M14" and "fit M14".

The result is the formula

$$l = l_{cf1n5c}(R, a, \mu) = \frac{aR}{2.99 + 4.718a + 1.553\mu + 1.137R} \tag{27}$$

which we use for the calculation of rotator orbits in 8.2.

7. Extended Interpretations of Rotator Energy Equation

7.1. The Gravitational Radiation Power

The Einstein-formula gravitational radiation power of a two-body rotator is as presented in [5]:

$$P_{gr} = \frac{32}{5} m_1^2 m_1^2 m \frac{G^4}{r_c^5 c^5} = \frac{32}{5} \frac{G^4}{r_c^5 c^5} m^5 \frac{\mu^2}{(1+\mu)^4} \tag{28}$$

where $r_c = \frac{r_{\min} + r_{\max}}{2}$ is the mean distance of the rotator. or expressed by the Schwarzschild radius $r_s(m) = \frac{2Gm}{c^2}$ $P_{gr} = \frac{2}{5}\left(\frac{r_s}{r_c}\right)^4 mc^2 \frac{c}{r_c} \frac{\mu^2}{(1+\mu)^4}$, which shows manifestly dimensionality energy/time, or dimensionless

$$P_{gr} = \frac{2}{5}\frac{1}{r_c^5}\frac{\mu^2}{(1+\mu)^4}$$

We compare it to the Kerr energy term $O\left(\frac{1}{r^3}\right)$ in E1dS0 with $\omega = \varphi' = \frac{l}{r_c^2}$, where we have to set $l = l_c$, since we consider now the rotator orbit in his own metric,

$$\frac{\Delta E_{\alpha,3}}{m} = -\frac{1}{r^3}\left(l_c^2 + 2l_c\alpha(1-\varepsilon) + \alpha^2(1-2\varepsilon)\right)$$

and calculate the corresponding power = energy/period dimensionless setting $\omega = \varphi'(\tau) = \frac{l_c}{r_c^2}$ and have

$P_{K,3} = \frac{\Delta E_\alpha}{T} = \Delta E_\alpha \frac{\omega}{2\pi} = \left(l_c^2 + 2l_c\alpha(1-\varepsilon) + \alpha^2(1-2\varepsilon)\right)\frac{ml_c}{2\pi r_c^5}$, or dimensionless $m = \frac{1}{2}$

$$P_{K,3} = \left(l_c^2 + 2l_c\alpha(1-\varepsilon) + \alpha^2(1-2\varepsilon)\right)\frac{l_c}{4\pi r_c^5}$$

Now, for circular Keplerian orbits, $l_c = \sqrt{\frac{r_c}{2}}$ (dimensionless), therefore $l_c \sim \sqrt{r_c}$, so the term is approximately

$$P_{K,3} \approx \left(\frac{r_c}{2} + 2\sqrt{\frac{r_c}{2}}\alpha(1-\varepsilon) + \alpha^2(1-2\varepsilon)\right)\frac{\sqrt{\frac{r_c}{2}}}{4\pi r_c^5}$$

We see, that none of the term has the correct order in $\frac{1}{r_c}$, and, since the term $O\left(\frac{1}{r^4}\right)$ in E1dS0 contains no l_c-term, there is no appropriate term in Kerr potential, the Kerr approximation cannot explain radiation loss.

In fact, there is another important precondition for radiation lass, this energy term is extracted from the gravitational system, therefore it must be imaginary in the potential $\tilde{V}_{eff}$. In optics and quantum mechanics, imaginary terms in energy are related to dissipation and wave damping.

What about the Manko-Ruiz metric?

One can see from numeric $\frac{1}{r}$-expansion of the Manko-Ruiz potential $\tilde{V}_{eff}$, that there is a l_c-term in the $O\left(\frac{1}{r^4}\right)$ coefficient, so the Manko-Ruiz potential yields an appropriate term for radiation energy. However, in order to be imaginary, the Manko-Ruiz metric itself has to be made complex and still satisfy the Einstein equations.

In fact, there is a natural way to do it.

In the basic equations of the Manko-Ruiz metric, in (24b) the line element coefficient of $\mathrm{d}\varphi$ is defined as

$$\omega = 2a - \frac{2Im\left[G\left(\bar{A}+\bar{B}\right)\right]}{A\bar{A} - B\bar{B}}$$

In fact, this equation is a little artificial, it is made so, to ensure that the metric is real.

The setting $\omega = 2a - \frac{2G\left(\bar{A}+\bar{B}\right)}{A\bar{A} - B\bar{B}}$ is natural, and of course it satisfies the Einstein equations as well.

The Manko-Ruiz metric thus becomes manifestly complex, and the orbits are calculated from complex orbit equations. With this ansatz, the solution $t(\tau), r(\tau), \varphi(\tau)$ becomes complex, and we simply take the absolute of the variables $|t(\tau)|, |r(\tau)|, |\varphi(\tau)|$, as all of them must be positive, as below in subsection 7.2.

So in Manko-Ruiz metric, the radiation power can be derived from the potential as the imaginary part of the term $O\left(\frac{l_c}{r^4}\right)$, but we will deal with it in a separate paper.

7.2. Taking into Account Self-Rotation of the Participating Masses m_1 and m_2

Self-rotation with the Kerr ansatz

The Kerr ansatz is valid as long as there is a θ-symmetry, *i.e.* the system is independent of φ.

If there is self-rotation (around z-axis) for the masses m_1 and m_2, the θ-symmetry is not disturbed and self-rotation (spin-) angular momentums L_1 and L_2 add up and contribute to the Kerr parameter α according to the formula $\alpha_s = \alpha_1 + \alpha_2$. For a rotating black hole $\alpha_x = \frac{L_x}{m_x c} = \frac{\kappa \omega_x r_s (m_x)^2 m_x}{m_x c}$, where $r_s(m_x) = \frac{2Gm_x}{c^2}$ is the Schwarzschild radius, κ is the inertia-factor ($\kappa = 2/3$ for a spherical shell) and ω_x the angular frequency.

Now we can make the ansatz that in the case of a binary rotator with self-rotation of the black-holes with masses m_1 and m_2, this contribution adds a part to the (real) α_b of the binary rotator, where from above $\alpha_b = \frac{r_c}{2}\frac{\sqrt{\mu}}{1+\mu}$.

For the masses (m_1, m_2) (dimensionless, *i.e.* $m = 1$) rotating with angular frequencies (ω_1, ω_2) we get then $\alpha = \alpha_b + \alpha_x = \frac{r_c}{2}\frac{\sqrt{\mu}}{1+\mu} + \frac{\kappa\left(\omega_1 + \omega_2 \mu^2\right)}{1+\mu^2}\frac{r_s^2}{c}$

For $\mu \to \infty$ the rotator becomes a single rotating Kerr- blackhole with $\alpha = \alpha_2$, as it should be. Also, the contribution α_b from gravitational rotation becomes zero for $\mu \to \infty$, and the spacetime becomes the normal real Kerr spacetime of a rotating black-hole.

We can generalize the self-rotation to arbitrary rotation axis: the z-component of the axis is yields the (real) Kerr parameter α_x as above, the perpendicular component, *i.e.* the projection on the rotator plane yields the imaginary Kerr parameter $i\alpha_y$, so that the total Kerr parameter becomes

$$\alpha = \alpha_b + \alpha_x + i\alpha_y$$

With this ansatz, the solution $t(\tau), r(\tau), \varphi(\tau)$ becomes complex, and we simply take the absolute of the variables $|t(\tau)|, |r(\tau)|, |\varphi(\tau)|$, as all of them must be positive. The numerical solution of the radial equation with complex coefficient, as well as that of the full orbit equations with complex coefficient presents no additional difficulties, so the ansatz is perfectly feasible.

The program code for the calculated examples can be found in [2].

Self-rotation with Manko-Ruiz rotator

In case of the Manko-Ruiz rotator, the metric parameter *a* is determined from the orbit reduced angular momentum l_c according to the formula

$$l_c = l(R, a, \mu) \quad \text{(see 6.5 (27))}.$$

If we take into account individual rotation of the two rotator masses, then the additional Kerr parameters (α_1, α_2) add up to the relative angular momentum l_c directly: $l_c = l(R, a, \mu) + \alpha_1 + \alpha_2$. If the individual rotation is inclined under angle $\theta_k, k = 1, 2$ to the rotator (*i.e.* orbit) z-axis, then the corresponding perpendicular components enter the formula as imaginary numbers, as described above:

$$l_c = l(R, a, \mu) + \alpha_1(\cos\theta_1 + i\sin\theta_1) + \alpha_2(\cos\theta_2 + i\sin\theta_2)$$

We get then, as above, orbit equations with complex coefficients, solve them in complex orbit radius $r(\tau)$, and the physical radius is then taken as the absolute value $r_p(\tau) = |r(\tau)|$.

8. Numerical Examples

In the following subsections 8.1 and 8.2, we present orbits around the binary rotator calculated numerically from different models.

In subsection 8.3, we present results of calculation of rotator orbits in its own metric for mean radius $r_c = R = 20$, and $r_c = R = 35$.

The details of the calculation, the results and graphics are accessible in the Mathematica source code [2].

8.1. Binary Rotator with Mass Ratio $\mu = 2$, Observer Orbit

We consider an example of a rotator consisting of a binary black-hole with mass ratio $\mu = \frac{m_2}{m_1} = 2$, in dimensionless spacetime, *i.e.* *r* in Schwarzschild-units r_s, and *t* in Schwarzschild-units $t_s = \frac{r_s}{c}$.

We consider an observer orbit around the rotator, with orbit parameters (l, F), or equivalently $\left(l, \varepsilon = \frac{1 - F^2}{2}\right)$.

For the Kerr radial equation, we have to specify the initial value for $r(0)=r_{c0}$, the orbit parameters $l=l_{cp0}$, $F=F_{c0}$, and the rotator parameters $\mu=\mu_0=2$ and $\alpha=\alpha_0$. Here we use the Kerr parameter value derived for the Newtonian correction in 5.2: $\alpha=\frac{r_c}{2}\frac{\sqrt{\mu}}{1+\mu}$, where $r_c=r_{c00}$ is the mean distance of the binary rotator. We have two scales here: r_{c0} determines the maximum orbit radius (orbit scale), r_{c00} is the mean rotator distance (rotator scale).

We use the following values:

$\mu_0=2$, $r_{c0}=100$, $r_{c00}=30$, $\omega_0=\frac{1}{\sqrt{2r_{c0}^3}}=0.000707$, $l_{cp0}=0.61\omega_0 r_{c0}^2=4.321$, $F_{c0}=1-\frac{2.5}{r_{c0}}=0.996$, relative absolute energy $\varepsilon_0=\frac{1-F_{c0}^2}{2}=0.00399$, eccentricity $e=\sqrt{1-8\varepsilon_{t0}l_{cp0}^2}=0.635$, $\alpha=\alpha_0=\frac{r_{c00}}{2}\frac{\sqrt{\mu_0}}{1+\mu_0}=7.071$ pure Newtonian correction, resp. complex α with additional self-rotation $\Delta\alpha=(1+i)$ for both black-holes 45˚ to rotator axis $\alpha_0=\frac{r_{c00}}{2}\frac{\sqrt{\mu_0}}{1+\mu_0}+(1+i)=8.071+i1.0$.

The reduced angular momentum l_{cp0} is set on purpose relatively low in order to make eccentricity large, to test the validity of the formula for α for non-circular orbits: the results show that the approximation of the Newtonian correction is still good in this case.

In the case of the Manko-Ruiz spacetime, we have made the adaptation of the orbit parameters by semi-manual gradient procedure.

In this way [2], we got the Manko-Ruiz parameter values with fixed mass ratio $\mu_0=2$, $\frac{m_2}{m_1}=\mu_0$, $\frac{a_2}{a_1}=\mu_0^2$.

The result values are: $m_1=0.5/3$, $m_2=0.5*2/3$, $R=35$, $a_1=0.05/5$, $a_2=0.05*4/5$.

We use the following calculation models without self-rotation: pure Newtonian N1, Newtonian with rotator correction N1s, Schwarzschild S1, Kerr approximation (E1dS0 Schwarzschild-Kerr) K1, Kerr approximation (E1dA0 full Kerr-approx.) L1, exact Kerr (full Kerr orbit equations) A1, the Manko-Ruiz 4-parameter Kerr-extension M1.

We use the following calculation models with self-rotation: Kerr approximation (E1dS0 Schwarzschild-Kerr) K1s, Kerr approximation (E1dA0 full Kerr) L1s, exact Kerr (full Kerr orbit equations) A1s.

8.2. Numerical Results

Orbit parameter table

The following table (**Table 1**) contains the principal orbit values for the models without self-rotation: minimal and maximal radius r_{min}, r_{max}, radius drift Δr, first rotation period T_1, period drift ΔT, all in Schwarzschild units.

Table 1. Orbit parameters of observer orbits in the rotator field in different models.

model	$r_{\min}$	$r_{\max}$	Δr	T_1	ΔT
N1 Newton	22.84	102.41		4403.7	
N1s Newton corr.	33.57	100.0		4469.4	
S1 Schwarzschild	21.49	102.56		4397.5	
K1 Kerr Schw-approx	20.33	102.6		4280.7	
K1s mit spin	19.9	102.6		4254.5	
L1 Kerr fK-approx	33.01	102.9		4545.7	
L1s mit spin	34.45	103.		4565.6	
A1 exact Kerr	22.6	102.76		4457.4	
A1s mit spin	22.34	102.76		4459.6	
M1s Manko-Ruiz	29.5	103.0	1.9×10^{-6}	4645.0	-1.7×10^{-4}

Orbit radius plots $r(\tau)$

Kerr exact, Kerr apS Schwarz-app, Schwarzschild, Newton

Here we present models, which approximate well the uncorrected original pure Newton orbit with values: $r_{\min}$: ~22, period T_1 ~4400. The main deviation in period is in the model Kerr apSas as shown in Figure 3.

Kerr apF full-Kerr-approx., Schwarzschild, Newton corrected, Manko-Ruiz

Here we present models as shown in Figure 4, which approximate well the rotator-corrected Newton orbit with values: $r_{\min}$: ~34, period T_1 ~4470, and for comparison Schwarzschild orbit with $r_{\min}$ = 21.5. The main deviation in period is in the Schwarzschild model.

Kerr exact, Kerr exact with spin

Here we show the influence of 2x spin (self-rotation) $\Delta\alpha = (1+i)$ added to the orbit angular momentumas shown in Figure 5, where the spin is inclined 45˚ to rotator axis. The exact Kerr solution shows only a small deviation $\Delta r_{\min} \approx -0.3$ and $\Delta T_1 \approx 2$.

Kerr apS Schwarzschild-approx, Kerr apS with spin

Here we show the influence of self-rotation on the Kerr solution with Schwarzschild-approximationas shown in Figure 6.

The Kerr solution shows a larger deviation $\Delta r_{\min} = -0.4$ and $\Delta T_1 \approx -26$.

Kerr apF full Kerr-approx, Kerr apF with spin

Here we show the influence of self-rotation on the Kerr solution with full Kerr-approximation.

The Kerr solution shows a larger deviation $\Delta r_{\min} = 1.5$ and $\Delta T_1 = -20$.

8.3. Manko-Ruiz Rotator Orbits in Its Own Field (Metric)

We consider the Manko-Ruiz rotator in its own field, with parameters (R, a, μ), relative orbit energy $\varepsilon = \frac{c_1(\mu, R)}{4R} \approx \frac{1.45}{4R}$ for $\mu = 2$, and reduced angular orbit momentum $l = l(R, a, \mu) = \frac{aR}{2.99 + 4.718a + 1.553\mu + 1.137R}$, where the two latter relations have been derived in subsection 6.5, we set the total mass $m = m_1 + m_2 = 0.5$ and calculate in Schwarzschild units, *i.e.* dimensionless.

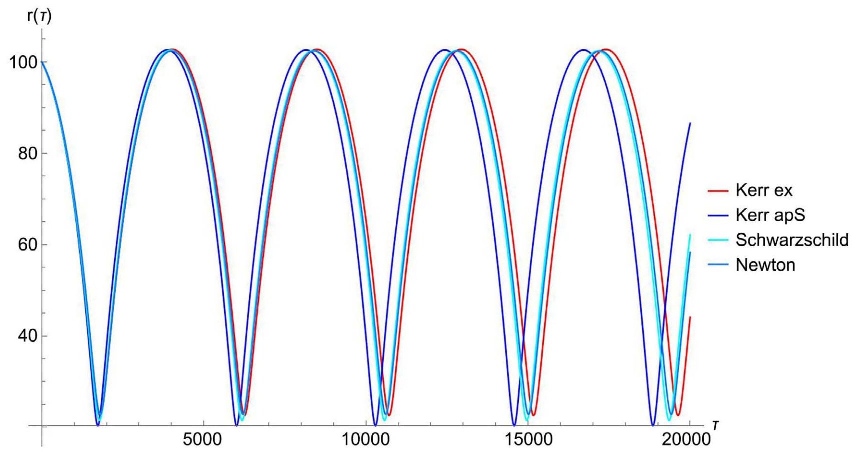

Figure 3. Orbit radius $r(\tau)$: Kerr exact, Kerr apS = Schwarz-app, Schwarzschild, Newton.

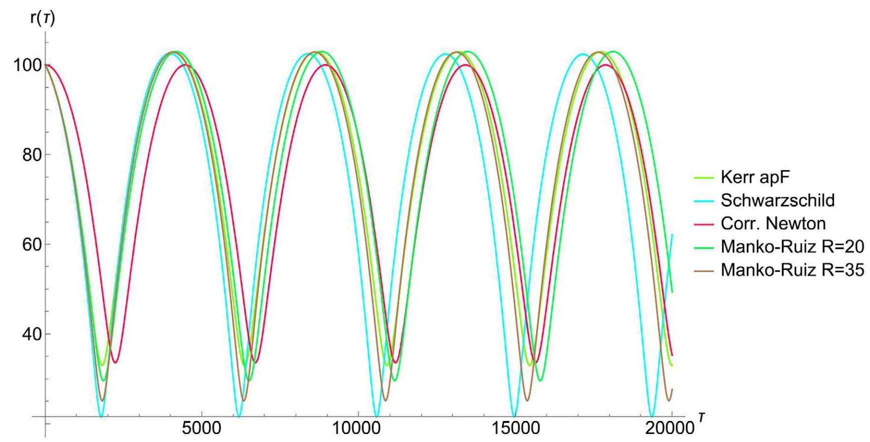

Figure 4. Orbit radius $r(\tau)$: Kerr apF full-Kerr-approx., Schwarzschild, Newton corrected, Manko-Ruiz.

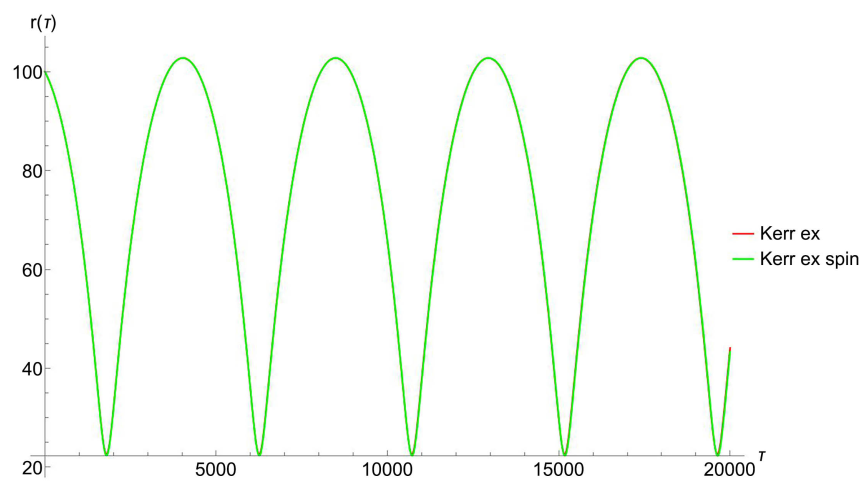

Figure 5. Orbit radius $r(\tau)$ without and with spin: Kerr exact.

A rotator orbit for a rotator with given mass ratio μ is determined by the mean radius r_c (which determines the relative orbit energy according to $\varepsilon_c = \frac{c_1(\mu, R)}{4R}$) and the reduced angular momentum l_c (which determines the eccentricity of the orbit). We extract the metric parameters from it: $R = r_c$, and a from the above formula $l_c = l(R, a, \mu)$.

We calculate for the rotator with $\mu = 2$ two orbits with mean radii $r_c = 20$ resp. $r_c = 35$ and $a = 0.2$.

We calculate also the corresponding Newton and Schwarzschild orbits with the same orbit parameters l_c, ε_c.

The result are presented in Schwarzschild units in the following **Table 2** and **Figure 7**, resp. **Table 3** and **Figure 8**.

Comparison of the two tables reveals strong differences in behavior of the Manko-Ruiz orbits and Kepler orbits as shown in **Figure 9**.

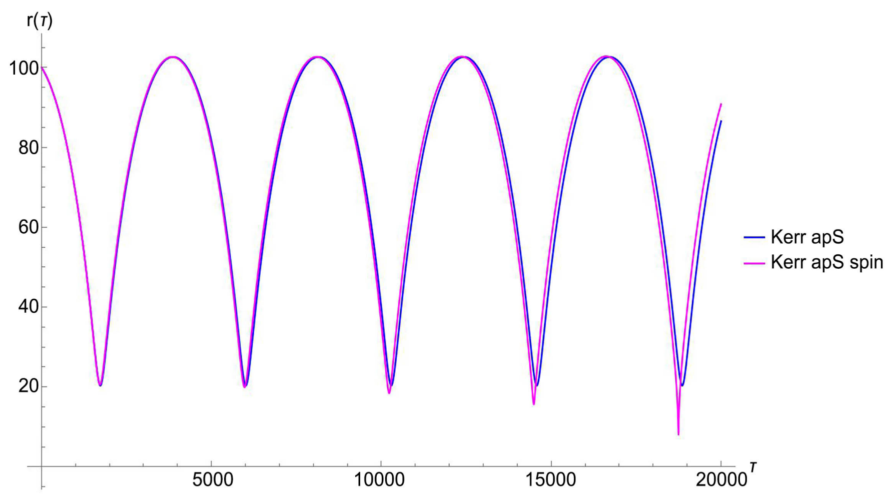

Figure 6. Orbit radius $r(\tau)$ without and with spin: Kerr apS Schwarzschild-approx.

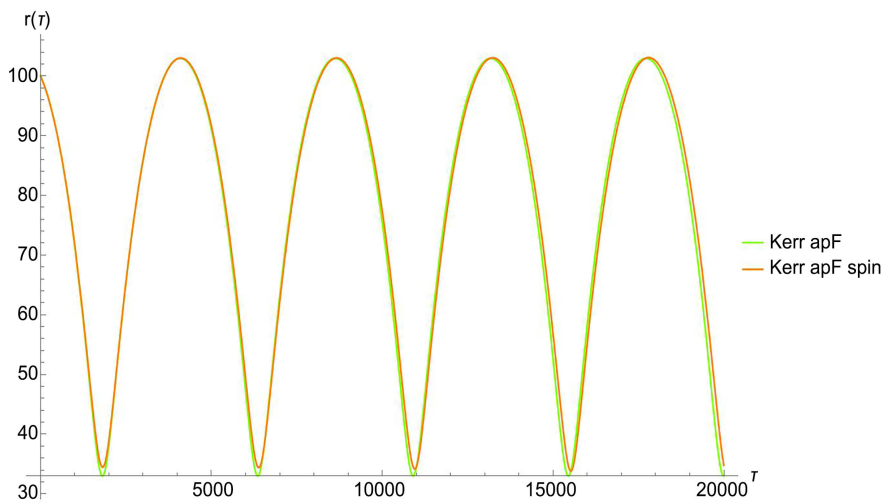

Figure 7. Orbit radius $r(\tau)$ without and with spin: Kerr apF full Kerr-approx.

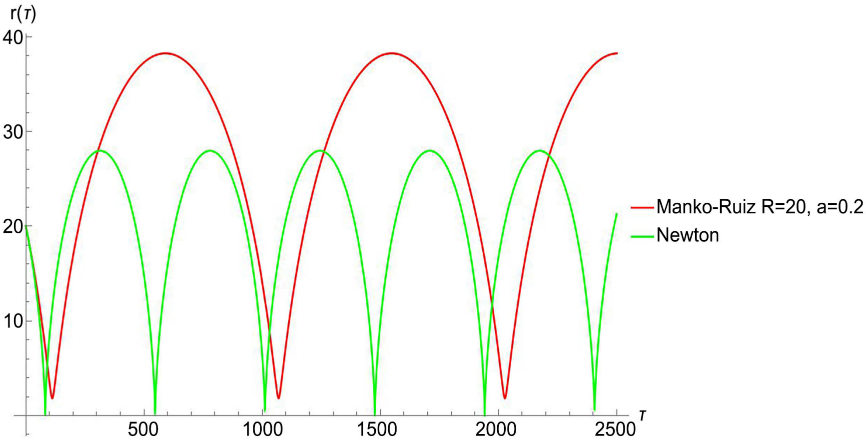

Figure 8. The Manko-Ruiz $r_c = 20$ orbit and corresponding Newton orbit.

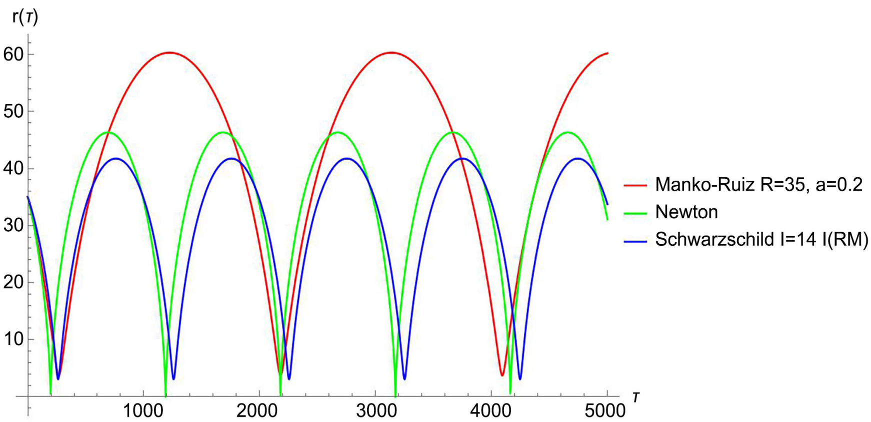

Figure 9. The Manko-Ruiz $r_c = 35$ orbit and corresponding Newton and Schwarzschild orbit.

Table 2. Orbit parameters of observer orbits in its own field, $r_c = 20$.

Parameters $\mu = 2$ $a = 0.2$	$r_{\min}$	$r_{\max}$	period T	energy ε_c	ang.m l_c
Manko Ruiz	1.8	38.2	957.3	0.018	0.13
Newton	0.018	27.9	547.3	0.018	0.13
Schwarzschild no solution					

Table 3. Orbit parameters of observer orbits in its own field, $r_c = 35$.

Parameters $\mu = 2$ $a = 0.2$	$r_{\min}$	$r_{\max}$	period T	energy ε_c	ang.m l_c
Manko Ruiz	4.52	61.4	1755.4	0.010	0.15
Newton	0.022	46.3	991.6	0.010	0.15
Schwarzschild min. l_c	3.07	41.7	996	0.010	13.8×0.15

The Kepler law $T \simeq r_c^{3/2}$ does not apply anymore, we have

$$\frac{T(r_c = 35)}{T(r_c = 20)} = 1.83 = \left(\frac{35}{20}\right)^{1.1}.$$

Also, the formula for the eccentricity is different from the Keplerian $e = \sqrt{1 - 8\varepsilon_c l_c^2}$: e is larger for Manko-Ruiz, and the energy-radius relation $\varepsilon_c \approx \frac{1.45}{4r_c}$ deviates from the Newtonian case.

However, we have to keep in mind that in the limit $r \gg 1$ (*i.e.* here $r_{\min} \gg 1$) Manko-Ruiz metric becomes the first Schwarzschild and then flat Minkowski metric and the orbit become the first Schwarzschild and then Newtonian orbits.

9. Conclusion

The central theme here is the gravitational two-body problem.

New results are:

- Precise formulation and calculation

The two-body problem is precisely formulated and calculated as (single) rotator with a single mass based on the Newtonian, Schwarzschild, and Kerr effective potential $V_{eff}(r)$ in the energy (radial) equation.

- Adaptation of the Kerr and Manko-Ruiz spacetime for observer orbit

Kerr and Manko-Ruiz spacetime are adapted to the Newtonian rotator correction for the observer orbit, *i.e.* parameters are fitted resp. calculated in order to describe the Newtonian correction in order $O\left(\frac{1}{r^4}\right)$.

In the case of the Manko-Ruiz spacetime, the Newtonian-limit-principle in GR guarantees that with adaptation to the Newtonian correction, the Manko-Ruiz spacetime $M_{br} = \left(g_{\mu\nu}\left(m_1, m_2, R(r_c), a(r_c)\right)\right)$ for observer orbit is the exact GR-solution for an orbit in the rotator potential, since it is a solution of the Einstein equation with exactly 4 independent parameters matching the 4 parameters (m_1, m_2, r_c, l_c) of the two-body problem, and becomes the Newtonian solution in the limit $r \to \infty$.

- Calculation of rotator orbit in its own field

The verification of Manko-Ruiz Einstein equations, the symbolic calculation of the Manko-Ruiz general effective potential $\tilde{V}_{eff}(r)$, its development in $\frac{1}{r}$-powers are new results.

Furthermore, we derive a formula for the reduced angular momentum $l = l(R, a, \mu)$ of a Manko-Ruiz rotator in its own field from the rotational parameter a, the orbit radius R, and the mass ratio μ, and for the relative total energy $\varepsilon = \varepsilon(R, \mu)$. These two parameters determine the rotator orbit, and we calculate this orbit in two examples and present the results in table and graphical form.

- Numerical calculation of Manko-Ruiz orbits and of the effective potential

The presented numerical calculation of observer orbits in the Newton-adapted Manko-Ruiz potential is also a new result.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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Appendix A1

Schwarzschild spacetime in matrix form (dimensionless, r in r_s units)

$$\begin{pmatrix} 1-\frac{1}{r} & & & \\ & -\frac{1}{1-\frac{1}{r}} & & \\ & & -r^2 & \\ & & & -r^2\sin(\theta)^2 \end{pmatrix}$$

Kerr spacetime in matrix form (Boyer-Lindquist coordinates), Kerr parameter

$$a=\frac{L}{mc}$$

$$\begin{pmatrix} 1-\frac{r}{r^2+\alpha^2\cos^2\theta} & & & \frac{r\alpha\sin^2\theta}{r^2+\alpha^2\cos^2\theta} \\ & -\frac{r^2+\alpha^2\cos^2\theta}{r^2-r+\alpha^2} & & \\ & & -\left(r^2+\alpha^2\cos^2\theta\right) & \\ \frac{r\alpha\sin^2\theta}{r^2+\alpha^2\cos^2\theta} & & & -\sin^2\theta\left(r^2+\alpha^2+\frac{r\alpha^2\sin^2\theta}{r^2+\alpha^2\cos^2\theta}\right) \end{pmatrix}$$

Christoffel symbols $\Gamma^{\kappa}_{\mu\nu}$ (Schwarzschild) have the values

$$\Gamma^0_{\mu\nu}=\left(\left(0,\frac{1}{2\left(1-\frac{1}{r}\right)r^2},0,0\right),\left(\frac{1}{2\left(1-\frac{1}{r}\right)r^2},0,0,0\right),(0,0,0,0),(0,0,0,0)\right)$$

$$\Gamma^1_{\mu\nu}=\left(\left(\frac{r-1}{2r^3},0,0,0\right),\left(0,\frac{1}{2r^2\left(1-\frac{1}{r}\right)},0,0\right),(0,0,(r-1),0),\left(0,0,0,(r-1)\sin^2\theta\right)\right)$$

$$\Gamma^2_{\mu\nu}=\left((0,0,0,0),\left(0,0,\frac{1}{r},0\right),\left(0,\frac{1}{r},0,0\right),(0,0,0,(0,0,0,-\cos\theta\sin\theta))\right)$$

$$\Gamma^3_{\mu\nu}=\left((0,0,0,0),\left(0,0,0,\frac{1}{r}\right),(0,0,0,\cot\theta),\left(0,\frac{1}{r},\cot\theta,0\right)\right)$$