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Critical Equatorial Trajectories of Massless Particles in a Kerr Spacetime

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Abstract

In the presented work a closed analytical expression is obtained that describes the critical photon orbits in the equatorial plane of a spinning Kerr black hole (BH). A significant difference in the behavior of photons with prograde and retrograde directions of rotation is shown. The photons with prograde rotation exhibit an exponential increase in the deflection angle together with the number of rotations around the BH as its spin parameter increases. The number of rotations exceeds 10^3 when spin parameter of the BH reaches 0.999. At the same time this value decreases insignificantly for reverse rotating photons and is less than that for the non-spinning Schwarzschild BH. The transition to a zero spin limit made it possible to determine the number of photon rotations along the critical trajectory for such BHs.

Keywords

Black Holes, Gravitation, Light Deflection, Photon Trajectories

1. Introduction

Despite the fact that the study of recently discovered gravitational waves offers a new method of obtaining knowledge about the black holes (BH), the analysis of the trajectories of massive and massless particles is still the most important source of information about these largely mysterious physical objects.

Recently obtained by the collaboration of the event horizon telescope (EHT) data on light bands in the vicinity of the supermassive object in the heart of the galaxy M87 [1] once again demonstrated the importance of solutions for null geodesics in the Schwarzschild and Kerr spacetimes. Even though the analysis of temporal and null (lightlike) geodesics has been going on for almost 100 years, starting with Ref. [2], many researches are now focused on this topic. The most significant pioneering works in the field have been carried out by C. Darwin [3] (in the Schwarzschild spacetime) and S. Chandrasekhar [4] (in the Kerr spacetime). The results of their works are described in detail in Chandrasekhar's own book as well as in books of other authors [5] [6]. A review over the last decade can be found in [7]).

All calculations accomplished so far have been performed through transformation of radial coordinate r into an auxiliary intermediate function $u = 1/r$. This technique originates in classical celestial mechanics long before the relativistic era and is explained by quite clear reason. It is a desire to remove the singularity at the origin of coordinates to infinity and, at the same time, to include the physical infinity, which is a source and (often) an outlet for test bodies, and very often the position of the observer, in the field of events.

The most general results for equatorial orbits in the Kerr spacetime were obtained by Chandrasekhar, who also used the inverse radius method. However, although he presented the result for critical trajectories in integral form, he could not reach a closed analytical expression.

As a consequence, this section has been presented by him with only very limited numerical results. Whereas we regard critical trajectories as most important, while they separate the photons which undergo deflection and retreat back to infinity from those falling into the events horizon.

This gap was largely filled by the authors [8], who found, in particular, a significant difference in behavior between photons with two directions of rotation around the BH, coinciding with its rotation and opposite to it.

In the present work, calculations of critical photon trajectories in the Kerr spacetimes are limited to equatorial planes only. Furthermore, they have been performed without the previously mentioned preliminary transformation and were carried out in the natural radius-azimuth angle coordinate system instead. This has allowed us to find the analytical solution for critical trajectories for the Kerr metric that has not been known previously and to obtain both a new and a simpler presentation of the solution for the Schwarzschild spacetime.

The paper consists of two parts. The first part is devoted to the calculation of the critical photon orbits in the spinning Kerr spacetime, but exclusively in the equatorial plane. More general cases which are not limited in this way and include not only flat, but also spatial orbits are considered in other works [9] [10] [11] [12]. The second part deals with the same subject in the Schwarzschild metric using the result obtained in the previous part.

In both cases, only angular geodesics are considered, since radial ones are covered in sufficient detail in the book [4].

It is also assumed that the particles source is located at a sufficiently large distance from the event horizon, in any case, many times greater than the Schwarzschild radius.

2. Kerr Spacetime

The Kerr line element (using Boyer-Lindquist coordinates) can be written ($c = G = 1$) [6],

$$\mathrm{d}s^2 = \left(1 - \frac{2Mr}{\Sigma}\right)\mathrm{d}t^2 + \frac{4aMr\sin^2\theta}{\Sigma}\mathrm{d}t\mathrm{d}\varphi - \frac{\Sigma}{\Delta}\mathrm{d}r^2 - \Sigma\mathrm{d}\theta^2 - \left(r^2 + a^2 + \frac{2Mra^2\sin^2\theta}{\Sigma}\right)\sin^2\theta\mathrm{d}\varphi^2 \tag{2.1}$$

Here $\Sigma = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - 2Mr + a^2$, $a = \frac{J}{M}$ is the angular momentum per unit mass.

Taking into account that for massless particles $\mathrm{d}s^2 = 0$, the angular momentum L and energy E are integrals of motion, for the equatorial plane $\theta = \pi/2$ in Ref. [13] the following equation is derived:

$$\left(\frac{\mathrm{d}r}{d\varphi}\right)^2 = \frac{\Delta^2\left[r^3 - r\left(b^2 - a^2\right) + 2M\left(b - a\right)^2\right]}{r\left[br + 2M\left(a - b\right)\right]^2}, \tag{2.2}$$

where $b = L/E$ is an impact parameter.

It should be emphasized, that the Kerr metric has only axial symmetry instead of the spherical one for the Schwarzschild metric, therefore, not only equatorial, but also other types of orbits are possible. The orbits of massless particles in other planes, as well as spherical orbits, can be found in other works mentioned above.

The cubic polynomial in Equation (2.2) can be transformed, taking into account that the critical trajectory is characterized by two identical roots, and the third is twice as large and has the opposite sign. Thus, the sum of all roots is zero.

Thereby

$$r^3 - r\left(b_c^2 - a^2\right) + 2M\left(b_c - a\right)^2 = \left(r - r_c\right)^2\left(r + 2r_c\right), \tag{2.3}$$

where b_c is the critical impact parameter [4]

$$b_c^2 = 3r_c^2 + a^2$$

It is well known that critical radii for Kerr metric are [14]

$$r_c = 2M\left[1 + \cos\left(\frac{2}{3}\arccos\left(-\frac{a}{M}\right)\right)\right], \tag{2.4}$$

where $a > 0$ for prograde spinning photons and $a < 0$ for retrograde ones. Thus

$$1M \le r_c\left(a > 0\right) \le 3M \le r_c\left(a < 0\right) \le 4M, \tag{2.5}$$

so Schwarzschild case $a = 0$ corresponds to the intermediate position $r_c = 3M$.

Integration of the Equation (2.2) allows us to find the equatorial null trajectories, as follows

$$\varphi = \int_r^\infty \frac{\left[br + 2M\left(a - b\right)\right]r}{\left(r^2 - 2Mr + a^2\right)\left(r - r_c\right)\sqrt{r\left(r + 2r_c\right)}}\mathrm{d}r. \tag{2.6}$$

The limits of integration were chosen so as to automatically satisfy the boundary condition:

when $r \to \infty$, $\varphi = 0$.

As a result of sufficiently long transformations and integrations, the final expression of the trajectory has the following form

$$\varphi(r)=\frac{b_c r_c+2M(a-b_c)}{\sqrt{3}\Delta_c}\left[Z(r)-\frac{r_c Y_2(r)}{\Delta_c}+\frac{s\Delta_c\left(Y_2(r)-Y_1(r)\right)}{(r_c-r_+)\sqrt{M^2-a^2}}\right]+\frac{b_c}{\sqrt{3}}\left[\frac{Y_2(r)}{\Delta_c}-\frac{s\left(Y_2(r)-Y_1(r)\right)}{\sqrt{M^2-a^2}}\right] \tag{2.7}$$

Here, to shorten the expression, we introduced the notations:

$$Z(r)=-\ln\left[\frac{\sqrt{3\left[1+\frac{4r_c}{r-r_c}+\frac{3r_c^2}{(r-r_c)^2}\right]}+\frac{3r_c^2}{(r-r_c)^2}+2}{2+\sqrt{3}}\right] \tag{2.8}$$

$$Y_{1,2}(r)=\frac{1}{p_{1,2}}\ln\left[\frac{\sqrt{1-2q_{1,2}(r_c-r_\pm)+p_{1,2}^2(r_c-r_\pm)^2}+2p_{1,2}(r_c-r_\pm)-\frac{q_{1,2}}{p_{1,2}}}{\sqrt{1-2q_{1,2}u_{1,2}(r)+p_{1,2}^2u_{1,2}^2(r)}+2p_{1,2}u_{1,2}(r)-\frac{q_{1,2}}{p_{1,2}}}\right] \tag{2.9}$$

$$\Delta_c=r_c^2-2Mr_c+a^2, \tag{2.10}$$

$$s=\frac{r_+}{2r_c(r_c-r_+)},\quad p_{1,2}=\frac{\sqrt{r_\pm(r_\pm+2r_c)}}{\sqrt{3}r_c(r_c-r_\pm)},\quad q_{1,2}=\frac{r_c+2r_\pm}{3r_c(r_c-r_\pm)}, \tag{2.11}$$

$$u_{1,2}(r)=\frac{(r-r_c)(r_c-r_\pm)}{r-r_\pm} \tag{2.12}$$

along with the well known expressions [4]

$$r_\pm=M\pm\sqrt{M^2-a^2}\,. \tag{2.13}$$

Formulae (2.7)-(2.13) allow plotting critical photon trajectories around a spinning black hole (**Figure 1**). The direction of rotation of the black hole is conventionally taken to be clockwise. The radius of each final section of the orbit is $10M$. The figures show parts of the orbit near the BH for photons with spin parameters $a/M=-0.9$ (a), 0.9 (b), 0.8 (c), and 0.95 (d).

It is clearly seen that the number of the rotations for prograde rotating photons increases with the growth of the spin parameter. On the contrary, the number of revolutions for photons with the opposite direction of rotation does not increase. On **Figure 2** these results are shown more precisely. The Figure shows a section of the trajectory for photons of both directions of rotation at distances $100M$ and $10M$ from the BH (2.a—with reverse rotation and 2.b and 2.c—with direct one). The influence of the BH gravitational field is practically negligible at a large distance for both directions of photons. The dotted lines show the asymptotic values at large distances calculated by formula (3.5). The influence of gravitational field at a closer distance leads to a strong exponential increase in the azimuthal angle for photons of direct rotation. At the same time,

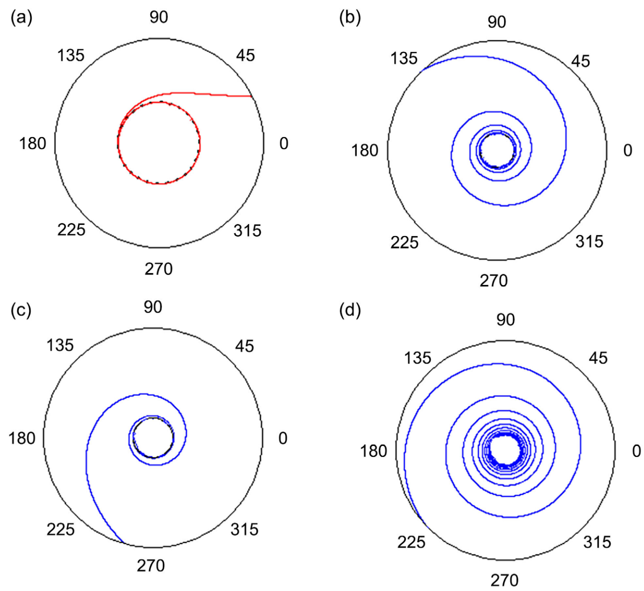
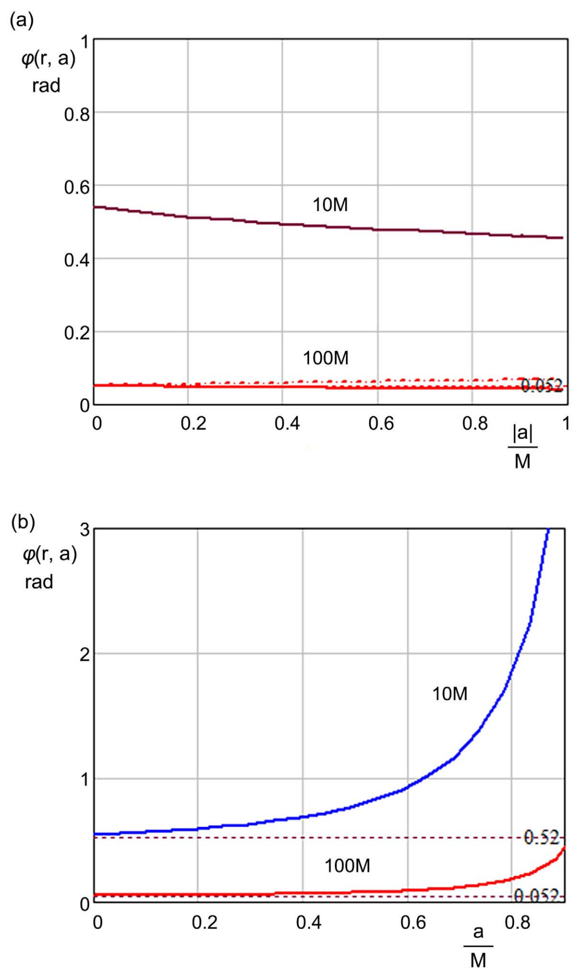


Figure 1. Critical photon orbits in vicinity of the BH. (a) $a/M = -0.9$; (b) $a/M = 0.9$; (c) $a/M = 0.8$; (d) $a/M = 0.95$. $a < 0$ describes retrograde spin, $a > 0$ indicates prograde spin. BH spinning direction is conventionally taken clockwise. Dotted line indicates the critical photon ring. Radius of each shown area corresponds to $10M$.



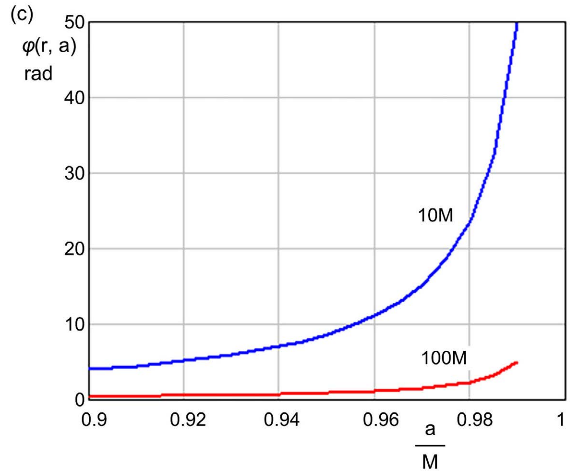


Figure 2. Plot of photon rotation angle versus spin parameter. Distances from the BH are 100*M* and 10*M*. Dotted lines indicate asymptotic values according Equation (3.5). (a) Retrograde spin; (b) and (c) Prograde spin.

however, the effect of the action for photons of reverse rotation is completely different. The azimuthal angle decreases slightly. Figure 3 shows the number of photon revolutions around the BH calculated according to our formulas the depending on the spin parameter. It also exhibits an exponential increase for prograde spinning photons and a slight decrease for retrograde spinning ones. The number of rotations exceeds 10^3 when spin parameter reaches 0.999. It should be emphasized that the calculated number of revolutions means their number from an infinitely distant source up to the photon circle, and not on it. The latter means that calculation is possible only up to the distance $r_c\left(1+10^{-15}\right)$, since at smaller distances, as well as on the circle itself, the integral diverges. This is a mathematical manifestation of the instability of a circular orbit. At the same time, this fact can be used as proof of the instability.

Our results made it possible to determine the number of rotations also for non-spinning Schwarzschild black hole. This is also possible up to the same limiting distance to the circle stated above and $a/M = 10^{-4}$. The number of rotations N_{sch}, calculated from the presented expressions, is 5.57.

It should also be noted that, although all spatial dimensions of the orbit are proportional to the BH mass *M*, which sets the scale, the azimuthal angles, as it follows from expression (2.7), do not depend on *M*.

3. Schwarzschild Spacetime

The calculation of the critical trajectory for the Schwarzschild metric is certainly possible, starting from the integral (2.6), if we set $a = 0$ in it. However, this is completely redundant, since we have already calculated it in a general form. Therefore, in our result (2.7) it suffices to put $a = 0$. It turns out that terms 2 and 4, as well as 3 and 5, cancel each other out. As a result

$$\varphi(r) = -Z(r), \tag{3.1}$$

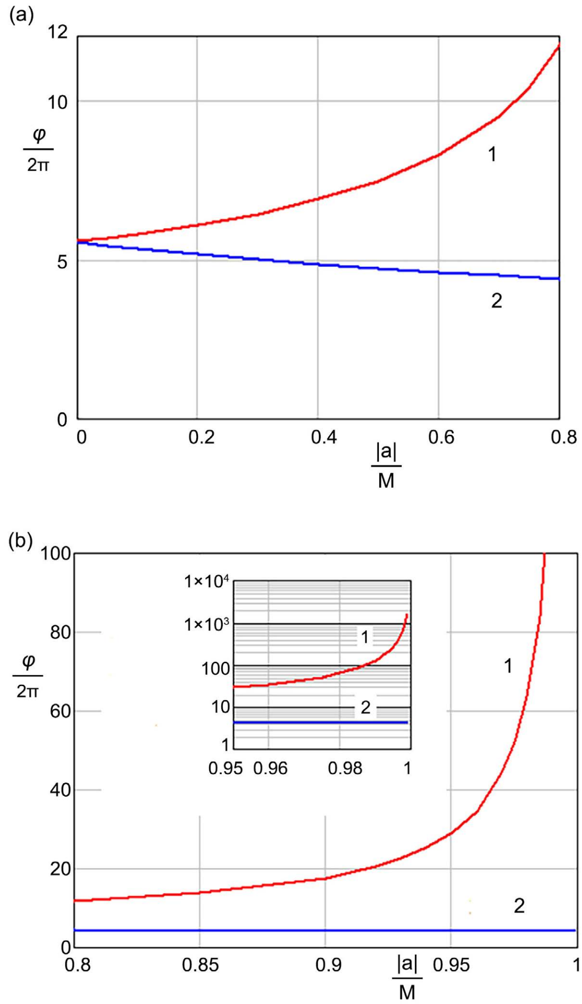


Figure 3. Plot of the number of revolutions around the BH on a trajectory from infinity to the critical photon circle $r_c\left(1+10^{-15}\right)$ depending on the spin parameter. 1—prograde and 2—retrograde spinning photons.

or after substitution $r_c\left(0\right)=3M$ we get

$$\exp\left(-\varphi\right)=\frac{\sqrt{3\left[\frac{27M^2}{\left(r-3M\right)^2}+\frac{12M}{r-3M}+1\right]}+2+\frac{9M}{r-3M}}{2+\sqrt{3}}. \tag{3.2}$$

After obvious transformations we obtain the final result

$$r=3M+\frac{9M}{2}\left(ch\left(\varphi\right)-1-\frac{\sqrt{3}}{2}sh\left(\varphi\right)\right)^{-1}. \tag{3.3}$$

Traditionally quoted [4] [15] result for critical photon orbit in the Schwarzschild spacetime belongs to sir C.G. Darwin [3], who used the auxiliary coordinate variable method $u = 1/r$ and found that

$$u=-\frac{1}{6M}+\frac{1}{2M}\tanh^2\left(\frac{\varphi-\varphi_0}{2}\right), \tag{3.4}$$

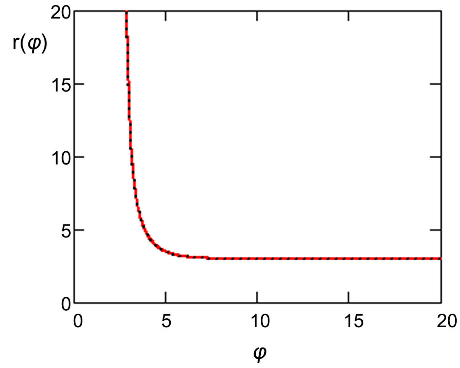


Figure 4. Comparison of the classical expression (3.4) (dotted line) and the derived dependence (3.3) (solid line).

where

$$\tanh^2 \frac{\varphi_0}{2} = \frac{1}{3}.$$

This result looks clearly different from the expression (3.3). Despite the apparent difference between the two solutions, it can be shown that both of them are identical. The simplest way to verify this statement without tedious proof is to plot both expressions. They match perfectly (**Figure 4**). Nevertheless, from the expression (3.3) the well-known properties of the orbit are immediately obvious, while from the classical result (3.4) it is not so clear.

Equation (3.3) implies that when $r \gg r_c$ and $\varphi \ll 1$ we get quite understandable approximation for large distances

$$\varphi = \frac{3\sqrt{3}M}{r} = \frac{b_c}{r}, \tag{3.5}$$

which is used in **Figure 2(a)** and **Figure 2(b)**.

Equation (3.3) also implies the law of approaching the circular orbit. Assuming $\varphi \gg 1$ we obtain

$$r - r_c \sim \exp(-\varphi) \tag{3.6}$$

It is easy to notice that $\exp(-2\pi N_{sch}) = 0.6 \times 10^{-15}$ coincides with the smallest distance to the critical radius which it was possible to achieve when we calculated the number of revolutions.

4. Conclusion

Closed analytical expressions are derived that describe the full critical trajectory of photons motion in the equatorial plane of a Kerr black hole. Our detailed formulas make it possible an accurate description of the trajectories for both directions of rotation for an arbitrary value of the BH spin parameter. Our results show that the orbits of photons with direct rotation can include many more revolutions (up to hundreds of times) around the BH than for photons with the opposite rotation direction. This conclusion may be of particular importance for the lensing effect. These photons, travelling very close to the unstable circular orbit, generate an infinite sequence of additional images that contribute to the total flux received by the observer [16].

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Johnson, M.D., Lupsasca, A., Strominger, A., Wong, G.N., Hadar, S., Kapec, D., *et al*. (2020) Universal Interferometric Signatures of a Black Hole's Photon Ring. *Science Advances*, **6**, Article No. eaaz1310. https://doi.org/10.1126/sciadv.aaz1310

[2] Hagihara, Y. (1931) *Theory of the Relativistic Trajeetories in a Gravitational Field of Schwarzschild*. *Japanese Journal of Astronomy and Geophysics*, **8**, 67-176. https://articles.adsabs.harvard.edu//full/1930JaJAG...8...67H/0000067.000.html

[3] Darwin, C. (1959) The Gravity Field of a Particle. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, **249**, 180-194. https://doi.org/10.1098/rspa.1959.0015

[4] Chandrasekhar, S. (1992) The Mathematical Theory of Black Holes. Clarendon Press, Oxford.

[5] Misner, C.W., Thorn, K.S. and Weeler, J.A. (1973) Gravitation. Freeman, San Francisco.

[6] Shapiro, S.L. and Teukolsky, S.A. (1983) Black Holes, White Dwarfs and Neutron Stars. The Physics of Compact Objects. Whiley-Interscience Publication, New York. https://doi.org/10.1002/9783527617661

[7] Hackmann, E. and Lämmerzahl, C. (2014) Analytical Solution Methods for Geodesic Motion. *AIP Conference Proceedings*, **1577**, 78-88. arXiv:1506.00807v1 [gr-qc]. https://doi.org/10.1063/1.4861945

[8] Iyer, S.V. and Hansen, E.C. (2009) Light's Bending Angle in the Equatorial Plane of a Kerr Black Hole. *Physical Review D*, **80**, Article ID: 124023. https://doi.org/10.1103/PhysRevD.80.124023

[9] Teo, E. (2003) Spherical Photon Orbits around a Kerr Black Hole. *General Relativity and Gravitation*, **35**, 1909-1926. https://doi.org/10.1023/A:1026286607562

[10] Teo, E. (2021) Spherical Orbits Around a Kerr Black Hole. *General Relativity and Gravitation*, **53**, Article No. 10. arXiv: 2007-04022v2 [gr-qc]. https://doi.org/10.1007/s10714-020-02782-z

[11] Dexter, J. and Agol, E. (2009) A Fast New Public Code for Computing Photon Orbits in a Kerr Spacetime. *The Astrophysical Journal*, **696**, 1616-1629. https://doi.org/10.1088/0004-637X/696/2/1616

[12] Gralla, S.E. and Lipsasca, A. (2020) Null Geodesics of the Kerr Exterior. *Physical Review D*, **101**, Article ID: 044032. https://doi.org/10.1103/PhysRevD.101.044032

[13] Cramer, C.R. (1997) Using the Uncharged Kerr Black Hole as a Gravitational Mirror. *General Relativity and Gravitation*, **29**, 445-454. https://doi.org/10.1023/A:1018878515046

[14] Bardeen, J.M., Press, W.H. and Teukolsky, S.A. (1972) Rotating Black Holes: Locally Non-rotating Frames, Energy Extraction, and Scalar Synchrotron Radiation. *Astrophysical Journal*, **178**, 347-370. https://doi.org/10.1086/151796

[15] Munoz , G. (2014) Orbits of Massless Particles in the Schwarzschild Metric: Exact solutions. *American Journal of Physics*, **82**, 564-573. https://doi.org/10.1119/1.4866274

[16] Bozza, V. and Scarpetta, G. (2007) Strong Deflection Limit of Black Hole Gravitational Lensing with Arbitrary Source Distances. *Physical Review D*, **76**, Article ID: 083008. https://doi.org/10.1103/PhysRevD.76.083008