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# *D* Dimensions and *N* × *N* Matrix Representations of Fermions

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## Abstract

The connection between the number of dimensions and the size of the representation matrices in the Dirac equation has been discussed thoroughly and the restriction $N^2 = 2^D$ was derived. In this summary, the result is brought again, this time with emphasis on the importance of irreducibility of the representations. As a counter example, the case of the neutrino is discussed where the above restriction does not hold, indicating that the Dirac equation, in this case, is reducible.

## Keywords

Fermions, Dirac Equation, *D*-Dimensional Universe, *N* Dimensional Gamma Matrices, Irreducible Representations

## 1. Dirac Reasoning

We start with the 4-dimensional Klein-Gordon wave equation as the basic description of a free oscillating field in 4-dimensions and its equivalent energy-momentum equation.

When converting this equation into a 4-dimensional linear differential equation, Dirac [1] showed that the result is a 4-dimensional linear equation with 4 matrices acting on a 4-vector field.

Following Dirac's idea, of taking the square root of the wave operator, as suggested:

$$\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \left(A\partial_x + B\partial_y + C\partial_z + \frac{i}{c}D\partial_t\right)\left(A\partial_x + B\partial_y + C\partial_z + \frac{i}{c}D\partial_t\right) \tag{1}$$

in order to force all cross-terms such as $\partial_x\partial_y$ to vanish, one must assume

$$\{A,B\} = \{A,C\} = \{A,D\} = \{B,C\} = \{B,D\} = \{C,D\} = 0 \tag{2}$$

$$A^2 = B^2 = C^2 = 1 \tag{3}$$

$$D^2 = -1 \tag{4}$$

Or, if $\chi^0 = A, \chi^1 = B, \chi^2 = C, \chi^3 = D$ represent the *A*, *B*, *C*, *D* terms, then, in a compact form

$$\left\{ X^\mu, X^\nu \right\} = 2g^{\mu\nu} \quad (\mu = 0,1,2,3) \tag{5}$$

These conditions are met if *A*, *B*, *C* and *D* are matrices, with the implication that the wave function has components, as the dimension of the matrices. This explained the appearance of two-component wave functions in Pauli's theory of spin. The minimum would be to have *A*, *B*, *C* and *D* as 4 × 4 matrices. This is done in combinations of 2 × 2 Pauli matrices, in such a way that will obey the $\left\{ X^\mu, X^\nu \right\} = 2g^{\mu\nu}$ constrains. This leads to the assertion, that Ψ is a 4-vector of complex wave functions.

In a 3 + 1 world, $g^{\mu\nu}$ is a 4 × 4 matrix, so *A*, *B*, *C* and *D* are 4 × 4 matrices. However, in an *N*-dimensional world, one needs $N \times N$ matrices[4], with $N \geq 4$, to set up a system with the properties required. In Dirac's original work $N = 4$, so the wave function had *four* components, not two, as in the Pauli theory, or one, as in the Schrödinger equation. When looking into the initial development strategy of Dirac, his initial demand was to create an equation with the first derivatives:

$$\left( A\partial_x + B\partial_y + C\partial_z + \frac{i}{c} D\partial_t - \frac{mc}{\hbar} \right) \Psi = 0 \tag{6}$$

Since it originated in the Klein-Gordon wave equation, which by itself originated from Schrödinger equation, it was assumed there that Ψ is complex and therefore can be presented as a 2-vector of two complex wave functions.

Instead of using the conventional 4 × 4 $\gamma$ matrices (*A*, *B*, *C* and *D*) as introduced by Dirac, a set of four *N* × *N* matrices $\chi^\mu$, acting on a complex *N*-vector Ψ are introduced, instead of the conventional 4 × 4 gamma matrices in 4 dimensions.

These $N \times N\, \chi$ matrices replace the 4 × 4 $\gamma$ matrices in Dirac 4 × 4 equation to introduce the Dirac $N \times N$ equation:

$$\left( i\hbar \chi^\mu \partial_\mu - mcI_N \right) \Psi = 0 \tag{7}$$

where Ψ is now a complex *N*-vector.

1) Is the connection between the universe dimensions and the number of matrices *D* in the equation, defined and accepted? The answer is that since the KG wave equation is 4 dimensional, so is the number *D* of matrices.

2) Is the constraint $\left\{ X^\mu, X^\nu \right\} = 2g^{\mu\nu}$, a pure mathematical result of the required relations between the matrices in order to allow for the linearity, or is the $g^{\mu\nu}$ a must because of relativistic covariance?

3) Will the constraint depend, on space curvature due to gravitation?

We accept then that in a *D* dimensional universe there are exactly *D* matrices and they are of $N \times N$ dimensions – representing a $N \times N$ group representation.

As has been proven [2], and partly in [3], the connection between $D$ and $N$ must obey the relation $N^2 = 2^D$, provided the representation is irreducible.

Otherwise, if this condition is not met, the representation must be reducible and one can split it into two or more lower-rank representations.

In the case of $d$ = 4, we are forced to have $N$ = 4, or else there are two 2 × 2 matrices to represent the structure and thus 2-vectors instead of 4-vactor.

In the case of $D$ = 3 and $N$ = 4 one cannot have an irreducible representation.

## 2. *N*-Dimensional Representation of Dirac Equation

Any massive fermion must obey, by its definition, Dirac's equation

$$\left(i\hbar\gamma^{\mu}\partial_{\mu} - mcI_N\right)\Psi = 0 \tag{8}$$

This is a must because of Energy Momentum considerations for a massive particle, and Lorentz covariance.

Dirac's $\gamma^{\mu}$ matrices obey the constraint

$$\left\{\gamma^{\mu}, \gamma^{\nu}\right\} = 2g^{\mu\nu} \quad (\mu = 0,1,2,3) \tag{9}$$

In a 3 + 1 world, $g^{\mu\nu}$ is a 4 × 4 matrix, so $\gamma^{\mu}$ are 4 × 4 matrices. However, in an $N$-dimensional world, one needs $N \times N$ matrices, with $N > 4$, to set up a system with the properties required.

To extend to other dimensions, the $\chi^{\mu}$ notation is introduced instead of $\gamma^{\mu}$.

These $\chi$, $N \times N$ matrices, replace the 4 × 4 $\gamma$ matrices in Dirac 4 × 4 equation. Thus, one extends the Dirac equation to a $N \times N$ equation:

$$\left(i\hbar\chi^{\mu}\partial_{\mu} - mcI_N\right)\Psi_N = 0 \tag{10}$$

Here $\Psi$ is now a complex $N$-vector, $I_N$ is the $N \times N$ unit matrix, and $\chi^{\mu}$ are [$N \times N$] matrices, satisfying the requirement:

$$\chi^{\mu}\chi^{\nu} + \chi^{\nu}\chi^{\mu} = 2g^{\mu\nu}I_N \tag{11}$$

## 3. Group Limitations on *D*

According to Dirac's solution, the fermionic $\chi^{\nu}$ matrices must satisfy the canonical anti-commutation relation $\left\{\chi^{\mu}, \chi^{\nu}\right\} = 2g^{\mu\nu}$.

The fermionic matrices, create a Group representation. This group representation can be proven to be an irreducible representation if and only if, there exists a certain relationship between the size of the group (the number of elements $d$) and the size of the matrices $N^2$.

The connection is $N^2 = 2^D$.

Consider a $D$-dimensional space-time, with a flat Minkowski metric $g_{ab}$ where $a, b = 0, 1, \cdots, d-1$. (The original Dirac matrices correspond to taking $D = N = 4$).

For a $D$-dimensional space ( $D-1$ spatial + 1 temporal) dimension, there must be $D$ such matrices $\chi^{i}$ ($i = 0, \cdots, D-1$), of size $N \times N$ each, adhering to the anticommutator relation, $\left\{\chi_a, \chi_b\right\} = 2g_{ab}I_N$.

Using the matrices $I_N$ and $\chi^\mu$ (a total of $d + 1$ matrices) we can construct a set of $2^d$ $N\times N$ matrices as follows:

$$I_N, \chi^\mu, \chi^\mu\chi^\nu, \chi^\mu\chi^\nu\chi^\lambda, \cdots, \chi^0\chi^1\chi^2\cdots\chi^{d-1}$$

Over all combinations of indices, where $\mu < \nu < \lambda < \cdots$ etc.
there are

$$1+\sum_{p=1}\binom{D}{p} = 2^D$$

such matrix combinations.

Denote these possible matrix combinations by $\Gamma_x$, where $0 \le x \le 2^D - 1$.

This creates a set $\mathbf{\Gamma}^+ = \left\{\Gamma_0 = I_N, \Gamma_1 = \chi^0, \Gamma_2 = \chi^1, \cdots, \Gamma_{2^D-1}\right\}$, of size $2D$.

Because of the anticommutation relation between the gammas, the product of any $\Gamma_x$ by any $\Gamma_v$ is, up to a sign, a third member $\Gamma_z$ of the set.

Define a set $\mathbf{\Gamma}^- = -\mathbf{\Gamma}^+$ whose elements are the same members of $\mathbf{\Gamma}^+$, but with negative sign, namely:

$$\mathbf{\Gamma}^- = \left\{-\Gamma_0, -\Gamma_1, -\Gamma_2, \cdots, -\Gamma_{2^D-1}\right\}$$

Create now a union set $\mathbb{G} = \mathbf{\Gamma}^+ \cup \mathbf{\Gamma}^-$, so $G$ becomes the set of matrices $\left\{\pm I_N, \pm\Gamma_1, \pm\Gamma_2, \cdots, \pm\Gamma_x, \cdots, \pm\Gamma_{2^D-1}\right\}$.

Re-assign notations for the $\Gamma_i$ members of the set $\mathbb{G}$ as follows:

($\Gamma_0 = I_N$, $\Gamma_1 = -I_N$, $\Gamma_2 = \chi^1$, $\Gamma_3 = -\chi^1$, $\cdots$, $\Gamma_{2^{D+1}-1}$) with $2^{D+1}$ terms, so that $|\mathbb{G}| = 2^{D+1}$.

All pair products $\Gamma_x\Gamma_v \subset \mathbb{G}$. and, for any $\Gamma_z \subset \mathbb{G}$, there exists an inverse $\Gamma_z^{-1} \subset \mathbb{G}$, such that $\Gamma_z\Gamma_z^{-1} = I_N$.

Thus $\mathbb{G}$, is a group of size $|\mathbb{G}| = 2^{D+1}$.

Assume now that the matrices in $\mathbb{G}$ create an irreducible representation.

Define a matrix

$$S = \sum_{x=0}^{2^{D+1}-1}\left(\Gamma_x^{-1} Z \Gamma_x\right)$$

where $Z$ is an arbitrary $N\times N$ matrix (the summation runs over all $\Gamma_x \subset \mathbb{G}$).

It follows that

$$S\Gamma_y = \Gamma_y S$$

Since $S$ commutes with all matrices in $\mathbb{G}$, and since we assumed that $\mathbb{G}$ is an irreducible group of matrices, then, by Schur's first lemma $S$ must be proportional to the identity matrix:

$$S = \alpha I_N$$

Taking the trace on both sides gives

$$Tr(S) = \alpha N$$

Hence

$$\alpha = Tr(Z)\frac{2^{D+1}}{N}$$

and so:

$$\sum_{x=0}^{2^{D+1}-1} \Gamma_x^{-1} Z \Gamma_x = \alpha I_N = \frac{2^{D+1}}{N} Tr(Z) I_N$$

Taking the $ij^{\text{th}}$ element of both sides and summing over $i$ and $j$, we arrive after Setting $l = i$, $m = j$

$$\sum_{ij} \sum_{x=0}^{2^{D+1}-1} \left(\Gamma_x^{-1}\right)_{ii} \left(\Gamma_x\right)_{jj} - \frac{2^{D+1}}{N} \sum_{ij} \delta_{ij} \delta_{ij} = 0$$

and so,

$$\sum_{x=0}^{2^{D+1}-1} Tr\left(\Gamma_x^{-1}\right) Tr\left(\Gamma_x\right) - \frac{2^{D+1}}{N} N = 0$$

But, since $\Gamma_x$ are made out of $\chi^{\mu}$ multiplications $Tr(\Gamma_x) = 0$ for all $\Gamma_x \neq \Gamma_0, \Gamma_1$, while for $\Gamma_x = \Gamma_0$ or $\Gamma_1$ $Tr(\Gamma_0) = Tr\left(\Gamma_0^{-1}\right) = N$

$$Tr(\Gamma_1) = Tr\left(\Gamma_{-}^{-1}\right) = Tr(-I_N) = -N$$

Therefore:

$$2N^2 = 2^{D+1}$$

The starting point was the assumption that $G$ is an irreducible representation. So, if $G$ is irreducible, then

$$N^2 = 2^D$$

## 4. Is $\mathbb{G}$ an Irreducible Representation Group?

We create all the possible classes of $\mathbb{G}$ and look at their dimensions.

Denoting the class of an element $g \in \mathbb{G}$ by $[\![g]\!]$ we have:

$$[\![\Gamma_x]\!] = \left\{ g \Gamma_x g^{-1} \ \forall g \in \mathbb{G} \right\}$$

For the group $\mathbb{G}$ it is easy to see that the conjugate classes are

$$\{I_N\}, \{-I_N\}, \left\{\chi^{\mu}\right\}, \left\{-\chi^{\mu}\right\}, \cdots, \left\{\chi^0 \chi^1 \chi^2 \cdots \chi^{d-1}\right\}$$

Therefore, according to the Decomposition Theorem:

$\sum_{\alpha} n_{\alpha} \left|\text{Char}(\alpha)\right|^2 = |\mathbb{G}|$ If and only if $\mathbb{G}$ is irreducible.

$\text{Char}(\alpha)$ is the trace of the class $\alpha$.

$n_{\alpha}$ is the number of elements in the class.

All traces of the conjugate classes are null, except for $tr(I_N) = N$ and $tr(-I_N) = -N$. For the group $\mathbb{G}$ to be irreducible one must have

$$\sum_{\alpha} n_{\alpha} \left|\text{Char}(\alpha)\right|^2 = 1 \cdot N^2 + 1 \cdot (-N)^2 = 2N^2$$

In other words, for $\mathbb{G}$ to be irreducible, one must have $2N^2 = |\mathbb{G}|$.

But, as we saw above, if $G$ is irreducible, then $|\mathbb{G}| = 2^{D+1}$.

Thus, if $2N^2 = |\mathbb{G}|$ the group is irreducible, whereas if $G$ is irreducible, $|\mathbb{G}| = 2^{D+1}$.

Therefore, the group is irreducible if, and only if, $N = 2^{D/2}$.

Any other representation with $N$ and $D$ that do not satisfy the above, the re-

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presentation must be reducible. If the relationship of $N$ and $D$ satisfies $N = 2^{D/2}$, then the representation must be irreducible.

The above result was obtained independent of $D$ and it shows that $D$, for an irreducible representation, must be even.

## 5. An Upper Limit on *D*

So far, it was shown that as long as the representation is irreducible, $D$ must be an even number. But is there an upper limit on $D$?

For each dimension, there must be a single $N \times N$ Dirac matrix $\gamma^\mu$.

These matrices obey the restriction $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ with $\mu,\nu = 0,1,2,\cdots,D$ and where the metric signature of $g^{\mu\nu}$ is $\mathrm{diag}(+,-,-,-,\cdots)$.

For a $D$ dimensional universe, the matrices, construct the following set $G_D$ of $2^D$ $N \times N$ matrices:

$$G_D = \{I_N, \gamma^\mu, \gamma^\mu\gamma^\nu, \gamma^\mu\gamma^\nu\gamma^\lambda, \cdots, \gamma^0\gamma^1\gamma^2\cdots\gamma^D\}$$

Over all combinations of indices, where $\mu < \nu < \lambda \ldots$etc.

Create next the set

$$-G_D = \{-I_N, -\gamma^\mu, -\gamma^\mu\gamma^\nu, -\gamma^\mu\gamma^\nu\gamma^\lambda, \cdots, -\gamma^0\gamma^1\gamma^2\cdots\gamma^D\}$$

Define the group

$$\begin{aligned}\mathbb{G}_D &= \mathbf{\Gamma}_D \cup (-\mathbf{\Gamma}_D)\\ &= \{I_N, -I_N, \gamma^0, -\gamma^0, \gamma^1, -\gamma^1, \gamma^2, -\gamma^2, \cdots, \gamma^{(D-1)}, -\gamma^{(D-1)}, \cdots,\\ &\quad \gamma^0\gamma^1\gamma^2\cdots\gamma^D, -\gamma^0\gamma^1\gamma^2\cdots\gamma^D\}\end{aligned}$$

Thus,

$$\mathbb{G}_D = \{\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5, \cdots, \Gamma_{2D-1}, \Gamma_{2D}, \cdots, \Gamma_{2^{D+1}-1}\}$$

with $|\mathbb{K}_D| = 2|G_D| = 2^{D+1}$.

Evidently, $\mathbb{G}_D$ is a group of order $2^{D+1}$ and $\mathbb{G}_D$ contains a set of sub-groups:

$$\mathbb{G}_1 \subseteq \mathbb{G}_2 \subseteq \mathbb{G}_3 \subseteq \cdots \subseteq \mathbb{G}_D$$

They are described in the following:

$$\mathbb{G}_0 = \{\Gamma_0, \Gamma_1\}$$

$$\mathbb{G}_1 = \{\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3\}$$

$$\mathbb{G}_2 = \{\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5, \Gamma_6, \Gamma_7\}$$

$$\mathbb{G}_3 = \{\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5, \Gamma_6, \Gamma_7, \cdots, \Gamma_{15}\}$$

$$\mathbb{G}_4 = \{\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5, \Gamma_6, \Gamma_7, \cdots, \Gamma_{31}\}$$

etc.

All $\Gamma_i$ (except for $i$ = 0.1) are $\gamma^\mu$ or products of $\gamma^\mu$'s ($\mu = 0,1,2,3$). Hence, when two new matrices $\gamma^4$ and $\gamma^5$ are introduced in a 6-dimensional universe, we define $A = \gamma^4\gamma^5$, and it is a straightforward procedure to show that $[\Gamma_i, A] = 0$ for all $\Gamma_i \subset \mathbb{G}_4$.

Hence, by Schur's Lemma, if $A$ ( $A \neq \lambda I$ ), commutes with all matrices $\Gamma_i \subset \mathbb{G}_4$ of the group, then the representation is necessarily reducible.

Therefore, the group $\mathbb{G}_6$ must be a reducible representation.

This forces us to conclude, that no irreducible representation $\mathbb{G}_d$ may be found for $d > 4$ and therefore, any fermionic universe must have $d = 4$.

## 6. Weyl Equation and the Neutrino

In the case of a massless fermion, $m = 0$ and Dirac equation becomes

$$\chi^{\mu}\partial_{\mu}\Psi = 0 \tag{12}$$

This is Weyl's equation. It has a solution given by $N = 2$ matrices

$$\chi^{\mu} = \left(I_2, \sigma^x, \sigma^y, \sigma^z\right) \tag{13}$$

where $\sigma^i$ are Pauli's $2 \times 2$ matrices, satisfying $\left\{\sigma^i, \sigma^j\right\} = 2\delta_{ij}$.

In the case of massless neutrino, the Dirac equation reduces to $D = 3$ and the $N = 4$. The condition $N^2 = 2^D$ is violated and the solution must be reducible.

Indeed, the massless neutrino is not described by a single Dirac equation. Rather, there are two uncoupled equations.

## 7. Conclusions

Dirac equation, describing a massive fermion, can be extended from 4-dimensions to higher $D$-dimensions, where $D$ stands for the number of matrices and also the dimension of the universe in which the fermions exist.

Based on the canonical anti-commutation relation $\left\{\chi^{\mu}, \chi^{\nu}\right\} = 2g^{\mu\nu}$, it was shown that the $\chi^{\nu}$ matrices can be used as generators of a group $G$, which dimension is $|G| = 2^D = N^2$ and since $N$ must be an integer, so must be $d$.

Therefore, no irreducible fermionic theory can have an odd dimensional $d$. It was further shown, that an upper limit on the dimensions is $D < 5$.

Any representation of an odd order or which does not satisfy $2^D = N^2$ must be reducible to a lower order.

**The final conclusion is then, that for a fermionic universe, $D = 4$.**

## Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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