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Gravito-Electrodynamics in Galaxy Rotation Curves. ENG: An Extended Newtonian Gravitation

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Abstract

An extended Newtonian gravitation (ENG) model was developed to explain the rotation curves in galaxies and galaxy clusters. ENG requires the knowledge of a parameter that is a function of the mass of the gravitational source. An approximate eq. for that parameter was obtained (for disk galaxies) that yields asymptotic speeds close to binned measured data. ENG yielded larger circular speeds for galaxy clusters when compared with the MOND results. A classical gravito-electromagnetic model (which neither is based on Einstein GR, nor on gravito-magnetism only) was developed which yielded asymptotic circular speeds very small compared to experimental results. However when ENG was used to develop an extended gravito-electromagnetic model, it yielded results compatible with MOND results for simulated galaxies and larger than MOND results for a simulated galaxy cluster. This model showed measurable increase in the circular speed when compared to ENG alone in the galaxy cluster. The need for modifying the Einstein field equation to address the dark matter problem in the framework of the ENG model was illustrated.

Keywords

Dark Matter, Galaxies: Kinematics and Dynamics, Cosmology

1. Introduction

It has passed about 90 years since Zwicky published a paper [1] where an inconsistency between the galaxy's circular speeds and the mass of the Coma galaxy cluster was noticed. He referred to the existence of dark matter as a possibility for the explanation of such a discrepancy. Since no dark matter has been found (despite intensive efforts in this endeavor) and current alternatives that modify Newton's gravitation have not solved completely the missing mass problem (e.g. in galaxy clusters) it was thought of addressing this problem with Gravito-Electrodynamics despite that references can be found in the scientific literature that use Gravito-Magnetism (classical and in the frame of Einstein's general relativity) to face this problem.

What is distinctive in this paper is that classical (not based on Einstein GR) Gravito-Electrodynamics (GED) was used where the impact of the magnetic field is completely neglected (due to its very small impact in comparison to the electric field at the range of velocities of the problems in question) in all numerical models and examples and the superposition of the fields of moving point masses (around the center of the cosmic structure) was used to determine the field at the test particle location. It did not work (very small value of asymptotic speeds were obtained) for typical values of galaxy masses where MOND paradigm works remarkable well without assuming the existence of dark matter [2] [3], but that paradigm still has a missing mass problem in galaxy clusters [4].

The combination of this GED (as the gravitational model) and MOND (as the inertial model) was then tried. This hybrid approach did not get rid of the missing mass problem in galaxy clusters.

An extended GED (EGED) model was then developed in such a way that it worked for typical galaxies and yielded larger values of circular speeds (in comparison with MOND results) for galaxy clusters.

To do that, an extended Newtonian/Coulomb gravitational model (ENG) was developed that yielded circular speeds compatible with values for typical masses of galaxies and yields larger values for galaxy clusters when compared to MOND results. This ENG model adds a new term to the Newtonian gravitation which is inversely proportional to the separation distance between the gravitational source and the test particle and contains a parameter which is function of the total mass of the gravitational source. That parameter was determined (for disk galaxies) and yielded results, for disks galaxies, close to the calculations using binned measured data.

Note that the classical GED has (intrinsically) the impact of the speed and acceleration of the source along with the speed of the test particle (if the GM field is considered) unlike other theories in which it needs to be assumed.

This paper has the following structure: Section 2 describes a GED algorithm for rotation curve calculations. Section 3 describes the combined GED-MOND model. In section 4 the ENG and EGED (GED-ENG) models are described. In section 5 example calculations for the developed models are presented and inter compared. Section 6 provides a summary with some concluding remarks. The first 2 appendices provide equations of the gravito-electromagnetic (GEM) fields produced by a point mass source moving in an arbitrary direction that were referenced and extended in this paper. Appendix C shows the need for modifying the Einstein field equation to address the dark matter problem in the framework of the ENG model.

2. Classical Gravito-Electromagnetism (GEM) in Galaxy Rotation Curves

The gravito-electric (GE) and gravito-magnetic (GM) field, assuming that the speed of the gravitational force propagation and the speed of the gravitational waves are the same and equal to the speed of light in vacuum can be written as (see Appendix A)

$$\vec{E}=\frac{GMc}{d_v^3}\left(d_a\left(\vec{r}-\frac{r}{c}\vec{v}\right)-d_v\frac{r}{c}\vec{a}\right),\quad \vec{B}=\frac{GM}{c}\frac{1}{d_v^3}\left(d_v\vec{a}\times\vec{r}+d_a\vec{v}\times\vec{r}\right)$$

$$d_v=r\left(c-\hat{r}\cdot\vec{v}\right),\quad d_a=c^2-v^2+\vec{r}\cdot\vec{a}$$

where

r: Distance from the position of the point mass to the observation point (separation),

$\vec{r}$, $\vec{v}$, $\vec{a}$: Separation vector, velocity and acceleration of the source of the fields.

$$\vec{a}=\vec{a}_N\left(1-\beta^2\right)^3$$

$\vec{a}_N$: Newtonian acceleration.

The radial balance between the inertial acceleration and the GE field for a thin disk galaxy (ignoring the retarded time and the GM force) is written as:

$$a\approx\frac{G}{g}\sum_{i=1}^{N_r}\sum_{j=1}^{N_a}\Delta f\left(r_i,\theta_j\right),\quad g=\left(1-\left(v/c\right)^2\right)^3$$

where,

a: Inertial acceleration (circular) of the test particle.

G: Newton's gravitational constant.

g: Correction to a that yields the perihelion precession of the planets of the solar system.

N_r: Number of rings (radial partitions).

N_a: Number of angles (azimuthal partition).

r_i: Radius from the center of the galaxy to the i ring.

θ_j: Angle between the radial axis and the j disk segment (azimuthal angle).

v: Circular speed of the test particle.

c: Speed of light in vacuum.

$\Delta f\left(r_i,\theta_j\right)=\Delta M_{i,j}E_{i,j}$: Projection of the GE Field to the radial axis.

$\Delta M_{i,j}$: Mass contained in the disk segment represented by i, j.

$$E_{i,j}=\frac{\rho_{i,j}}{d_{v,i,j}^2}a_i g_i k_j+c\frac{d_{a,i,j}}{d_{v,i,j}^3}\rho_{i,j}\alpha_{i.j}$$

$\rho_{i,j}$: Distance from the source to the test particle.

$a_i g_i$: Corrected acceleration of the source.

v_i: Circular speed of the source.

$$d_{v,i,j} = \rho_{i,j}c - \rho_{i,j}v_i\hat{\rho}_{i,j}\cdot\hat{\phi}_i :$$

$$d_{a,i,j} = c^2 - v_i^2 + \rho_{i,j}a_i\hat{\rho}_{i,j}\cdot\hat{a}_i g_i :$$

$$\alpha_{i.j} = \left(r - r_i\cos\theta_j\right)/\rho_{i,j}, \quad k_j = \cos\theta_j, \quad \rho_{i,j} = \sqrt{r_i^2 + r^2 - 2r_i r\cos\theta_j}$$

Note that the total field at the observation point is determined using the superposition of the individual segments (considered as a gravitational mass point source) of the ring in question (rotating at a constant angular speed).

The circular speed is then calculated from

$$v^2 g = Gr\sum_{i=1}^{N_r}\sum_{j=1}^{N_a}\Delta M_{i,j}E_{i,j} \tag{1}$$

Because the ring in question is assumed to spin with constant angular speed the following equations are used:

$$\hat{\rho}_{i,j}\cdot\hat{\phi}_i = \frac{r - r_i\cos\theta_j}{\rho_{i,j}}\sin\theta_j + \frac{r_i\sin\theta_j}{\rho_{i,j}}\cos\theta_j$$

$$\hat{\rho}_{i,j}\cdot\hat{a}_i = \frac{r - r_i\cos\theta_j}{\rho_{i,j}}\cos\theta_j - \frac{r_i\sin\theta_j}{\rho_{i,j}}\sin\theta_j$$

To find a potential asymptotic behavior for the rotational speed, distances far from the source are to be considered which leads to:

$$E_{i,j} = \frac{a_i g_i}{d_{v/r}^2}k_j\frac{1}{r} + c\frac{d_{a,i,j}}{d_{v/r}^3}\frac{1}{r^2}, \quad d_{v/r} = d_{v.i.j}/\rho_{i,j}, \quad \alpha_{i.j} = 1.$$

For very large r the 2nd term is neglected. Substituting into Equation (1):

$$v^2 g = G\sum_{i=1}^{N_r}\sum_{j=1}^{N_a}\Delta M_{i,j}\frac{a_i g_i}{d_2^{v/r}}k_j \tag{2}$$

Note that for very high speeds $d_{v/r}$ becomes small and therefore Equation (2) would yield asymptotic speeds which cannot be obtained using pure Newtonian gravitation. This at least will lessen the need of hypothesizing the existence of dark matter at those speed levels.

Assuming non-relativistic speeds and only one ring, Equation (2) becomes

$$v^2 \approx GM\frac{a_S}{c^2}\sum_{j=1}^{N_a}k_j = 0 \tag{3}$$

Equation (2) yields results very small compared with measurements of asymptotic circular speeds in many disk galaxies. This is expected (see Equation (3)).

3. Gravito-Electrodynamics with MOND as an Inertial Acceleration

Because Equation (2) yields very small asymptotic speed (at non relativistic regime) it is of interest to use MOND as inertia (because of its remarkable success in reproducing rotation curves of galaxies) to establish the balance between the GE field and the inertial acceleration.

3.1. Balance between the Newtonian Gravitational Field and the Inertial Acceleration Using MOND

For a thin galaxy, the radial balance between the Newtonian gravitational field and the MOND's inertial acceleration, using the simple interpolating function, is written as

$$a\mu(a,a_0) = G\sum_{i=1}^{N_r}\sum_{j=1}^{N_a}\Delta M_{i,j}\frac{\alpha_{i.j}}{\rho_{i,j}^2}, \quad \mu(a,a_0) = \frac{a}{a+a_0}$$

where,

$a_0 \approx \frac{1}{6}cH_0 \approx 1.16$: Characteristic acceleration of MOND.

H_0: Hubble constant.

In terms of the circular acceleration that balance can be written as

$$\frac{v^4}{r^2} = Gf(r)\left(\frac{v^2}{r}+a_0\right) \tag{4}$$

$$f(r) = \sum_{i=1}^{N_r}\sum_{j=1}^{N_a}\Delta M_{i,j}\frac{\alpha_{i.j}}{\rho_{i,j}^2}$$

$$\text{For} \quad r \gg r_i \quad \text{and} \quad \frac{v^2}{r} \ll a_0 \quad v = (GMa_0)^{1/4} \tag{4a}$$

3.2. Gravito-Electrodynamics with MOND as the 2nd Law of Newton

The radial balance between the GE force and the 2nd law of Newton using MOND is written as:

$$v^4 g = G\sum_{i=1}^{N_r}\sum_{j=1}^{N_a}\Delta M_{i,j}\left(E_{i,j}rv^2 + E_{i,j}r^2 a_0\right) \tag{5}$$

$$E_{i,j} = \frac{\rho_{i,j}}{d_{v,i,j}^2}\frac{(a_i g_i)^2}{a_i g_i + a_0}k_j + c\frac{d_{a,i,j}}{d_{v,i,j}^3}\rho_{i,j}\alpha_{i.j}, \quad d_{a,i,j} = c^2 - v_i^2 + \rho_{i,j}\frac{(a_i g_i)^2}{a_i g_i + a_0}\hat{\rho}_{i,j}\cdot\hat{a}_i$$

For $a_i \ll a_0$, and $a \ll a_0 \Rightarrow v^4 g = G\sum_{i=1}^{N_r}\sum_{j=1}^{N_a}\Delta M_{i,j}E_{i,j}r^2 a_0$, $r \gg r_i \Rightarrow$

$$E_{i,j} = \frac{(a_i g_i)^2}{d_{v/r,i,j}^2 a_0}k_j\frac{1}{r} + c\frac{d_{a,i,j}}{d_{v,/r.i,j}^3}\frac{1}{r^2} \Rightarrow$$

$$v^4 g = G\sum_{i=1}^{N_r}\sum_{j=1}^{N_a}\Delta M_{i,j}\left(\frac{(a_i g_i)^2}{d_{v/r}^2}k_j r + c\frac{d_{a,i,j}}{d_{v/r}^3}a_0\right) \tag{5a}$$

Assuming non-relativistic speeds and only one ring, Equation (5a) becomes:

$$v^4 \approx GM\left(\frac{a_S^2}{c^2}r\sum_{j=1}^{N_a}k_j + a_0\right) = GMa_0 \tag{6}$$

Equation (5) yields speeds that are still inconsistent with the results in galaxy clusters (as it will be seen later).

4. Extended Classical Gravito-Electrodynamics

Because Equation (5) still has a missing mass problem in galaxy clusters and the potential problems with the 1st term of Equation (5a) it was thought of modifying the GE field in such a way that an asymptotic value of the circular speed could be obtained that could be compatible with experiments involving low speed. For that purpose, the Newton/Coulomb (Gravitation/Electrostatics) static field equation was modified. Note that this modification does not impact the classical GEM wave theory.

4.1. Extended Newtonian Gravitation (ENG)

As an alternative to MOND as inertia and to the existence of dark matter, the Newtonian gravitation force field is extended as

$$\vec{E} = G\frac{M}{r^2}\hat{r} + G_1\frac{M}{r}\hat{r} \tag{7}$$

where

$G_1 = f(M) \ll G$,

r: Separation distance between the source of the field and the observation point.

The radial balance between the extended gravitational force and the 2nd law of Newton for a thin disk galaxy is then written as

$$v^2 = Gr\sum_{i=1}^{N_r}\sum_{j=1}^{N_a}\Delta M_{i,j}\frac{\alpha_{i,j}}{\rho_{i,j}^2} + G_1 r\sum_{i=1}^{N_r}\sum_{j=1}^{N_a}\Delta M_{i,j}\frac{\alpha_{i,j}}{\rho_{i,j}} \tag{8}$$

Here r is the distance from the center of the galaxy to the location of the test particle.

$$r \gg r_i \Rightarrow v^2 = GM\frac{1}{r} + G_1 M\ . \quad r \to \infty \Rightarrow v = \sqrt{G_1 M} \tag{9}$$

In [5] the relation $v = \sqrt{2\pi GM/(4R)}$ was obtained for a Mestel disk. Equating both equations $\Rightarrow G_1 = \frac{\pi G}{2R}$.

This eq. is the result of a very strong correlation between a galaxy with a constant circular speed in the interior (using Newtonian dynamics) and a galaxy with an asymptotic speed outside its edge, both having the same total mass.

In ref. [6] it was found that $\frac{M}{R^2} \approx 0.5$ for 26 disk galaxies, therefore

$$G_1 = \frac{\pi G}{2\sqrt{2M}} \tag{9a}$$

Equation (9) (using Equation (9a)) yields results in good agreement with binned measurement data of disk galaxies as will be seen later.

Substituting Equation (9a) into Equation (9)

$$\Rightarrow v^4 = \left(\frac{\pi G}{2}\right)^2 \frac{1}{2}M\ . \tag{9b}$$

This eq. is also consistent with the slope of 4 in the power law (Tully-Fisher relation) of binned measure data.

4.2. Extended GEM (EGEM)

The gravitational static potential usually defined as $V(r) \equiv -\int_{\infty}^{r} \vec{E} \cdot \mathrm{d}\vec{l}$ leads, when equation 7 is used, to $V(r) = -\left(-GM\frac{1}{r} + G_1 MLn(r)\right)\Big|_{\infty}^{r}$ for a point mass. This yields a term equal to ∞.

To avoid that, the potential is defined as $V(r) \equiv -\int_{r_0}^{r} \vec{E} \cdot \mathrm{d}\vec{l}$, $r_0 \gg r$. The potential equation is then $V(r) = G\frac{M}{r} - G\frac{M}{r_0} + G_1 MLn(r_0) - G_1 MLn(r)$. Note that $V(r_0) = 0$ and that $\vec{E} = -\nabla V \Rightarrow$ Equation (7).

That eq. can be written as:

$$V(r) = \frac{G}{r}\int \rho(r')\mathrm{d}v' - \frac{G}{r_0}\int \rho(r')\mathrm{d}v' + G_1 Ln(r_0)\int \rho(r')\mathrm{d}v' - G_1 Ln(r)\int \rho(r')\mathrm{d}v'$$

G_1 should be expressed in terms of integrals also, but a dependency valid for the whole mass range of interest does not yet exist, therefore this detail is not considered in this work.

The potential for GED is obtained by generalizing the gravito-static equation. Therefore

$$V(r,t) = \frac{G}{r}\int \rho(r',t_r)\mathrm{d}v' - \frac{G}{r_0}\int \rho(r',t_r)\mathrm{d}v' + G_1 Ln(r_0)\int \rho(r',t_r)\mathrm{d}v' - G_1 Ln(r)\int \rho(r',t_r)\mathrm{d}v'$$

t_r: Retarded time (the time elapsed between the time when the source moves and the time when the information of that movement reaches an observation point).

It can be shown that $\int \rho(r',t_r)\mathrm{d}v' = \dfrac{M}{1 - \hat{r}\cdot\vec{v}/c}$ [7] therefore

$$V(r,t) = G\frac{Mc}{d_v} - \frac{G}{r_0}\frac{Mc}{d_{v/r}} + G_1 Ln(r_0)\frac{Mc}{d_{v/r}} - G_1\frac{Mc}{d_{v/r}}Ln(r) = V_1 + V_2 + V_3 + V_4$$

$$d_v = r(c - \hat{r}\cdot\vec{v}),\quad d_{v/r} = d_v/r$$

To calculate the GE field, the gradient of the scalar potential is needed:

$$\vec{\nabla}V_1 = GMc\vec{\nabla}(1/d_v) = GMc\frac{d_v\vec{v} - d_a\vec{r}}{d_v^3} \quad [7]$$

$$d_a = c^2 - v^2 + \vec{r}\cdot\vec{a}$$

Choosing r_0 such that $\dfrac{G}{r_0} = G_1 Ln(r_0) \Rightarrow V_2 + V_3 = 0$ (see additional remarks at the end of this section).

The gradient of the last term is calculated as

$$\vec{\nabla}V_4 = -G_1Mc\frac{\vec{\nabla}\left(Ln(r)\right)d_{v/r} - Ln(r)\vec{\nabla}d_{v/r}}{d_{v/r}^2}.$$

where $\vec{\nabla}\left(Ln(r)\right) = \frac{c\vec{r}}{rd_v}$, $\vec{\nabla}\left(d_{v/r}\right) = \left(\frac{d_a}{rd_v} - \frac{c}{r^2}\right)\vec{r} - \frac{\vec{v}}{r}$. The relation $\vec{\nabla}\left(d_{v/r}\right) = \vec{\nabla}\left(\frac{1}{r}d_v\right) = \vec{\nabla}\left(\frac{1}{r}\right)d_v + \frac{1}{r}\vec{\nabla}\left(d_v\right)$ was used (with $\vec{\nabla}\left(\frac{1}{r}\right) = -\frac{c}{r^2}\frac{\vec{r}}{d_v}$). The calculation of $\vec{\nabla}\left(d_v\right)$ is described in [7].

The total gradient is therefore:

$$\vec{\nabla}V = GMc\frac{d_v\vec{v} - d_a\vec{r}}{d_v^3} - G_1Mc^2\frac{\vec{r}}{d_v^2} + G_1Mc\frac{Ln(r)\vec{\nabla}\left(d_{v/r}\right)}{d_{v/r}^2}$$

The GE and the GM fields, using the results shown in Appendix A, are then written as

$$\vec{E} = \frac{GMc}{d_v^3}\left(d_a\left(\vec{r} - \frac{r}{c}\vec{v}\right) - d_v\frac{r}{c}\vec{a}g\right) + G_1Mc^2\frac{\vec{r}}{d_v^2} - G_1Mc\frac{Ln(r)}{d_{v/r}^2}K$$

$$K = \left(\frac{d_a}{rd_v} - \frac{c}{r^2}\right)\vec{r} - \frac{\vec{v}}{r}$$

$$\vec{B} = \frac{GM}{c}\frac{1}{d_v^3}\left(d_vg\vec{a}\times\vec{r} + d_a\vec{v}\times\vec{r}\right)$$

where the terms containing G_1 in the GE field represent the contribution from the extended Newtonian gravitation presented in this work.

Appendix B shows the GE field equation extended to consider the parameter concerning the speed of the gravitational force propagation.

The radial balance between the GE field and the test particle's inertial acceleration in a thin disc galaxy is then written as:

$$v^2g = r\sum_{i=1}^{N_r}\sum_{j=1}^{N_a}\Delta M_{i,j}\left(GE_{i,j} + G_1E_{i.j}^1 - G_1E_{i.j}^2\right) \tag{10}$$

where,

$$E_{i,j} = c\left(\frac{1}{c}\frac{\rho_{i,j}}{d_{v,i,j}^2}a_ig_ik_j + \frac{d_{a,i,j}}{d_{v,i,j}^3}\rho_{i,j}\alpha_{i.j}\right)$$

$$E_{i.j}^1 = c^2\frac{\rho_{i,j}}{d_{v,i,j}^2}\alpha_{i.j}$$

$$E_{i.j}^2 = c\frac{Ln\left(\rho_{i,j}\right)}{d_{v/r,i,j}^2}\left(\frac{d_{a.i.j}}{d_{v.i.j}} - \frac{c}{\rho_{i,j}}\right)\alpha_{i.j}$$

For $r \gg r_i$ and making $d_{v/r} = d_{v/i,j}/r$:

$$E_{i,j} = \frac{a_ig_i}{d_{v/r}^2}k_j\frac{1}{r} + c\frac{d_{a,i,j}}{d_{v/r}^3}\frac{1}{r^2},\quad E_{i.j}^1 = \frac{c^2}{d_{v/r}^2}\frac{1}{r}$$

$$E_{i.j}^2 = Kc\frac{Ln(r)}{r}, \quad K = \frac{d_{a.i.j}}{d_{v/r}^3} - \frac{c}{d_{v/r}^2}$$

For very large r (the term $\propto \frac{1}{r^2}$ is neglected):

$$v^2 g = \sum_{i=1}^{N_r}\sum_{j=1}^{N_a}\Delta M_{i,j}\left(G\frac{a_i g_i}{d_{v/r}^2}k_j + G_1\frac{c^2}{d_{v/r}^2} - G_1 KcLn(r)\right) \tag{11}$$

Note that $K = 0$ for low speeds of the source and for very massive system $G_1 \approx 0$.

For only one ring (an average one):

$$v^2 g = GMa_s g_s \sum_{j=1}^{N_a}\frac{k_j}{d_{v/r}^2} + G_1 Mc^2\sum_{j=1}^{N_a}\frac{1}{d_{v/r}^2} - G_1 McLn(r)\sum_{j=1}^{N_a}K.$$

Note that for small circular speed of the source and the test particle the equation $v = \sqrt{G_1 M}$ (Equation (9)) is recovered.

It is noted that, for example, for $G_1 = 10^{-32} \Rightarrow r_0 \approx 1.5\times10^{20}$ m which could appear as a restriction on the range of validity of the calculation. But note that $V_2 \ll V_1$ because $r_0 \gg r$ any way, and that the substitution $-Ln(r_0) \to Ln(r)$ can be made in the electric field (for the same reason). Taking $r_0 = 10^{26}$ m (which is about the radius of the visible universe) in the numerical example (galaxy cluster) shown in the next section made no impact on the results. So $\vec{\nabla}V_2 + \vec{\nabla}V_3$ could reasonably be ignored.

For completeness the GE field is written next without neglecting any term of the full potential:

$$\vec{E} = \frac{GMc}{d_v^3}\left(d_a\left(\vec{r} - \frac{r}{c}\vec{v}\right) - d_v\frac{r}{c}\vec{a}g\right) + G_1 Mc^2\frac{\vec{r}}{d_v^2} + \delta$$

$$\delta = -Mc\left(\frac{G}{r_0} + G_1\left(Ln(r) - Ln(r_0)\right)\right)\frac{\left(\frac{d_a}{rd_v} - \frac{c}{r^2}\right)\vec{r} - \frac{\vec{v}}{r}}{d_{v/r}^2}$$

Then $E_{i.j}^2$ in Equation (10) can be written as

$$E_{i.j}^2 = c\left(\frac{G}{G_1 r_0} + G_1\left(Ln(\rho_{i,j}) - Ln(r_0)\right)\right)\frac{K_{i.j}\alpha_{i.j}}{d_{v/r,i,j}^2}, \quad K_{i.j} = \frac{d_{a.i.j}}{d_{v.i.j}} - \frac{c}{\rho_{i,j}}.$$

Note that for low speed and/or large distances $K_{i.j} \approx 0$.

5. Computational Results and Analysis

Table 1 shows binned measurements of circular speeds of galaxies and clusters of galaxies along with the total mass (from luminosity and gas data) of the cosmic structures which were reported in ref [8]. The circular speeds represent asymptotic values [9]. That Table also shows v_a predicted by ENG (Equation 9(a)) and by MOND (Equation 4(a)). **Table 1** also shows the values of G_1 using the binned experimental data ($G_1 = v_c^2/M_b$) and Equation 9(a).

Note the relatively closeness (for spiral and gas disk galaxies) between the measured circular speeds and the results of ENG and MOND (see **Figure 1** also).

Table 1. Asymptotic circular speed for ENG (Equations ((9), (9a))) and MOND models. v_c and M_b (baryonic mass) were calculated from the binned measurement data (power law of $v_c = f(M_b)$) reported in [8].

System	M_b (M_{sun})	V_c (km/s)	V_a (Equation (9))	V_m MOND	G_1 Binned	G_1 (Equation (9a))
Cluster	1.00E+14	1.66E+03	1.02E+03	1.11E+03	1.38E−32	5.25E−33
Cluster	3.72E+13	1.26E+03	7.98E+02	8.70E+02	2.14E−32	8.62E−33
Cluster	1.38E+13	9.12E+02	6.23E+02	6.79E+02	3.03E−32	1.41E−32
Cluster	5.75E+12	6.92E+02	5.01E+02	5.46E+02	4.18E−32	2.19E−32
Cluster	2.88E+12	5.13E+02	4.21E+02	4.59E+02	4.59E−32	3.09E−32
Spiral	2.09E+11	2.51E+02	2.19E+02	2.38E+02	1.52E−31	1.15E−31
Spiral	9.77E+10	2.09E+02	1.81E+02	1.97E+02	2.25E−31	1.68E−31
Spiral	4.27E+10	1.70E+02	1.47E+02	1.60E+02	3.40E−31	2.54E−31
Spiral	1.82E+10	1.41E+02	1.19E+02	1.29E+02	5.51E−31	3.89E−31
Spiral	1.00E+10	1.20E+02	1.02E+02	1.11E+02	7.27E−31	5.25E−31
Gas Disk	7.08E+09	1.17E+02	9.38E+01	1.02E+02	9.80E−31	6.24E−31
Spiral	6.17E+09	1.07E+02	9.06E+01	9.87E+01	9.36E−31	6.69E−31
Spiral	2.04E+09	8.32E+01	6.87E+01	7.49E+01	1.70E−30	1.16E−30
Gas Disk	1.62E+09	7.59E+01	6.49E+01	7.07E+01	1.78E−30	1.30E−30
Gas Disk	4.17E+08	6.03E+01	4.62E+01	5.03E+01	4.38E−30	2.57E−30
Gas Disk	1.74E+08	4.47E+01	3.71E+01	4.04E+01	5.77E−30	3.98E−30
Gas Disk	1.91E+07	2.34E+01	2.14E+01	2.33E+01	1.45E−29	1.20E−29
Dwarf	4.68E+06	1.95E+01	1.50E+01	1.64E+01	4.09E−29	2.43E−29
Dwarf	3.98E+05	1.45E+01	8.12E+00	8.85E+00	2.64E−28	8.33E−29
Dwarf	6.46E+03	8.71E+00	2.90E+00	3.16E+00	5.91E−27	6.54E−28

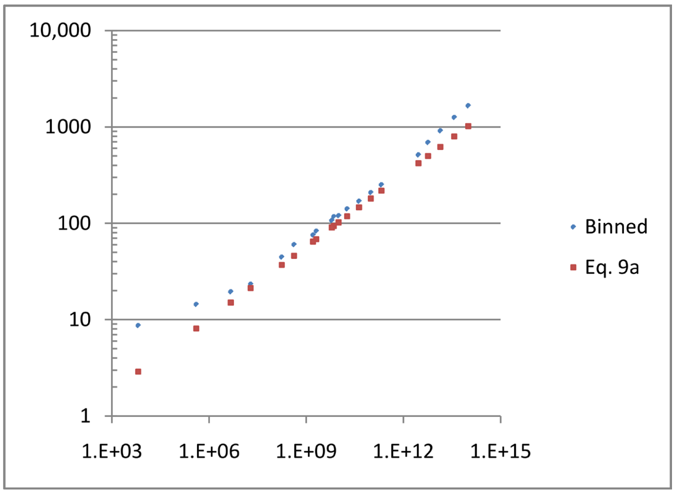

Figure 1. Asymptotic circular speed (km/s) vs. Mass (solar mass) for the ENG (Equation (9a)) model and the binned measurement data (the reference). Log10 scale is used on both axes.

The MOND results are closer than the ENG (using Equation (9a) for G_1), but using the G_1 values obtained from the binned data in ENG obviously reproduce the binned circular speeds which are significantly larger than MOND values for galaxy clusters. The ENG model slope (power law) is 4.0.

The significant deviation from the results of the Dwarf galaxies (quasi-spherical shape) and clusters of galaxies should not be a surprise since Equation (9a) was obtained for a set 26 disk galaxies. Additionally, it is noted that the binned data set for the dwarf galaxies is accompanied by large uncertainties and it includes the local group that some of them appear not to be isolated [10].

Figure 2(a) shows a mass distribution resembling the one reported in [11] for NGC-4736 but truncated at 10 kpc and normalized (total mass used was $M_T = 3.43\times10^{10} M_{Sun}$). Figure 2(b) shows the circular speed profile for the mass distribution of Figure 2(a).

Figure 3(a) shows a mass distribution according to $M(r) = M_T(2/\pi)\text{arctang}(r)$. Note that the plot shows only values up to 10 kpc

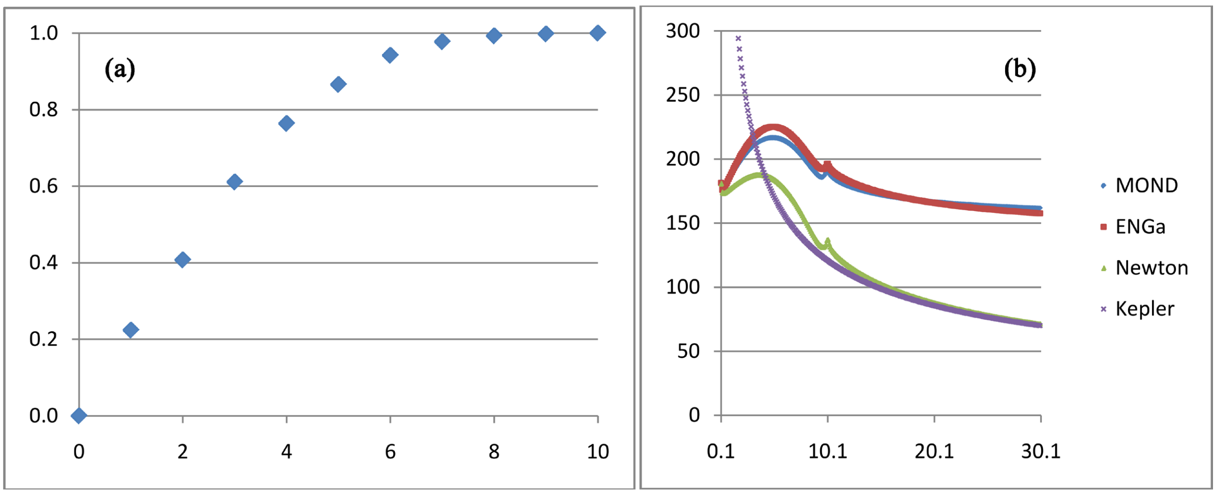

Figure 2. (a) Normalized mass vs. r (kpc); (b) V (km/s) vs. r (kpc). For (a) mass function.

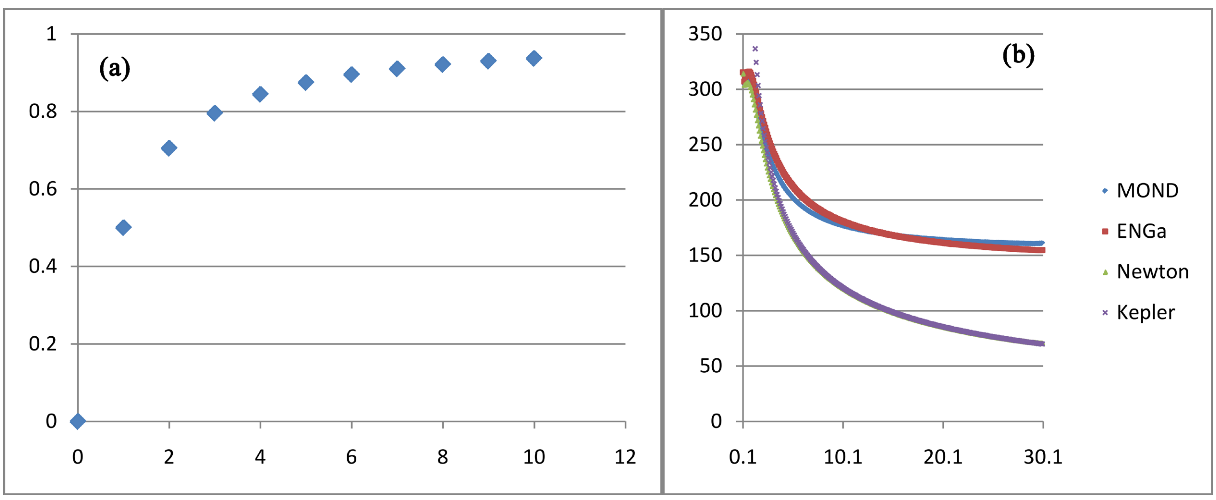

Figure 3. (a) Normalized mass vs. r (kpc). $M = (2/\pi)\tan^{-1}(r)$; (b) V (km/s) vs. r (kpc). For (a) mass function.

but it spans up to about 30 kpc. Note the radical difference of the circular speed profile between **Figure 2(b)** and **Figure 3(b)** which illustrates the importance of using a very accurate mass distribution. Notice in **Figure 3(b)** that the Newtonian results are, in that plot, indistinguishable from the Keplerian ones. This is expected since at $r = 3$ kpc about 80% of the mass is inside that radius.

Ref [11] calculated the mass distribution of the galaxy GNC-4736 without the need of dark matter or modified gravity. An explanation for that feature of that galaxy is still not known. Could the method of determining the circular speeds from star luminosity and gas data be not unique in such a way that mass discrepancy could be missed in some cases?

Figure 4 shows the circular speed profile for a galaxy having a total mass of $M_T = 1.0\times10^{11} M_{sun}$ and a radius of about $R_g = 16\text{ kpc}$ using a mass distribution corresponding to constant mass density with a thickness of $10^{-4} R_g$. Note that even using Newtonian dynamics the Keplerian behavior (like in the solar system) is not obtained for this disc galaxy up to about 30 kpc. Note also that about 50 kpc the speed yielded by ENGa (and MOND) is about twice the Newtonian result and the ratio ENGb/Newton is even greater.

A straightforward plot of the circular speed for a hypothetical compact galaxy cluster with $M_T = 1.0\times10^{14} M_{sun}$ and $R_g = 16\text{ kpc}$ does not provide a clear comparison of the different gravitational models (due to the large speeds involved). But **Table 2** shows the circular speed for different models: MOND, ENGa (using Equation (9a)), ENGb (using binned measured data ($G_1 = v_c^2/M_b$) shown in **Table 1**) and Newtonian. That Table indicates that MOND yields a significant larger speed than the Newtonian results (mainly for large r outside the cluster edge). But what is more relevant is the difference between the MOND results (smaller values) and the ENGb results (larger values): The differences are about 57, 142, 261 and 392 km/s for 4, 16.1, 22.5 and 47.9 kpc respectively. That MOND

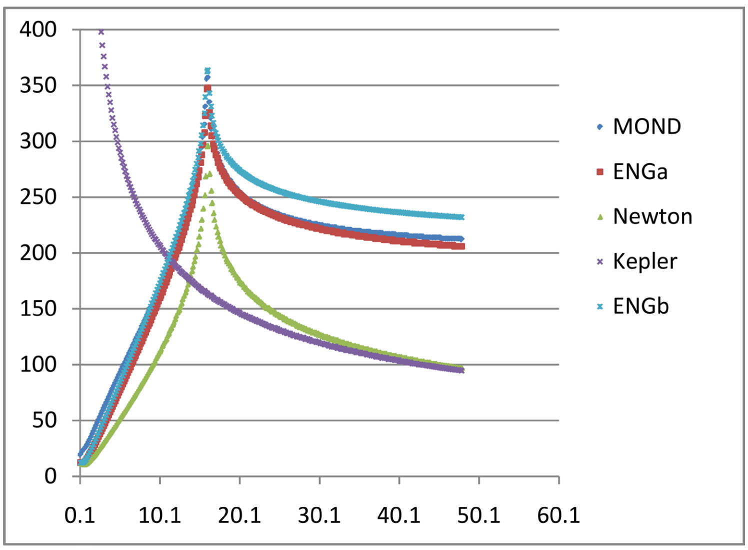

Figure 4. v (km/s) vs. r (kpc) $M_T = 10^{11} M_{sun}$, $R_g = 16\text{ kpc}$. At 30 kpc: top (ENGb). Bottom (Newton/Kepler).

under predicts the circular speeds of galaxy clusters can also be inferred directly from Table 1 as was previously noted by [12].

It is noticed that in this application and others the MOND model yields larger values than what ENG yields at some distance near the galaxy center, farther on ENG exceeds significantly the values of the MOND model.

Table 3 shows the results of MOND and ENG (using the same mass of Table 2) in the framework of the GE (Equation (5) and Equation (10)). It can be seen a small impact on the circular speed. The major impact (between 18 and 19 km/s increase) happens at about the edge of the galaxy as expected because the speed is greater at that location. The differences between the models are about the same as in the case of Table 2. Note that the results shown in Table 3 contain the small impact of g, g_i (test particle and the source, see Equation (10)). For Non-GE calculation (previous examples) g was ignored.

Note that in Equation (5) and Equation (10) v (the test particle circular speed) depends on the speed and acceleration of the source (r. h. s.) which are unknowns. An outer iteration (source iteration) algorithm was implemented that stops when $\Delta v = |v - v_s|$ inside the galaxy is less than 1 km/s. An inner iteration was performed to calculate v (g is a 6th degree polynomial on v). The calculation stops when $\Delta v < 10^{-6}$ m/s.

Neither ENG nor MOND showed significant impact on the circular speed due to the GE terms proportional to $Ln(r)$ and r respectively up to about c/H_0

Table 2. V vs. r for different gravitational models.

r (kpc)	V (km/s)			
	MOND	ENGa (Equation (9a))	ENGb (Binned)	NEWTON
0.2	385.6	384.9	385.0	384.9
4.0	1284.1	1302.9	1341.5	1278.6
16.1	9370.5	9423.0	9512.7	9367.4
22.5	4966.1	5062.3	5227.5	4958.0
47.9	3087.4	3226.0	3479.6	3059.8

Table 3. V vs. r for different GE models.

r (kpc)	V (km/s)		
	MOND	ENGa (Equation (9a))	ENGb (Binned)
0.2	385.6	384.9	385.0
4.0	1284.6	1303.7	1342.7
16.1	9388.4	9441.2	9531.5
22.5	4969.7	5066.1	5231.6
47.9	3088.8	3227.6	3481.4

for ENG and up to about $10^{-2} c/H_0$ for MOND (the numerical algorithm did not converge for larger distances).

Using $r_0 = 10^{26}$ m which is about the radius of the visible universe, in the eq. for δ, made no impact on the results (up to one decimal place after the point in a floating point format (in km/s)) of this application.

There could be more than one theory that could reproduce the binned circular speed of **Table 1** at large distances from the edge of the cosmic structure. ENG does it. If we allow a_0 to vary with the mass of the cosmic structure then MOND also will reproduce the binned circular speed of **Table 1**. **Table 4** shows the circular speeds of MOND and ENG with values of a_0 (5.72×10^{-10} m/s^2, Equation (6)) and G_1 (1.38×10^{-32} (m/s)^2/kg, Equation (9)) calculated from the binned measured values of the mass and circular speed shown in **Table 1**. In that way not only ENG should reproduce the binned speeds but MOND also should. From that Table can be seen that at 0.16 kpc MOND yields about 3 km/s larger than the ENG value, but ENG yields significantly larger values farther on (*i.e.* about 37, 130, 239, 295 km/s larger than MOND values for 4, 16.1, 22.5 and 47.9 kpc respectively). This shows that even allowing a_0 to change with the mass, MOND yields significant smaller speed than what ENG yields in this galaxy cluster application.

Table 5 shows the circular speeds at the edge of a hypothetical galaxy having a mass of $M = 2.0 \times 10^{14} M_{sun}$ for 3 values of the galaxy's edge. That mass is close to the mass of the super-giant elliptical galaxy IC 1101 reported in [13] where the size of the galaxy is also reported as being more than 50 times the size of the Milky Way. Notice that increasing the radius of the edge, the average mass density decreases cause the total mass is the same. The circular speed at the last

Table 4. Circular speed vs. distance for MOND (mass dependent a_0) and ENG.

r **(kpc)**	***V*** **(km/s)**	
	MOND Binned a0	**ENG Binned G1**
0.16	388.4	385
4.00	1304.8	1341.5
16.14	9382.5	9512.7
22.53	4997.4	5227.5
47.94	3185.1	3479.6

Table 5. Circular speed at the galaxy edge vs. the galaxy radius.

	R_g **(kpc)** ***V*** **(km/s)**		
	MOND	**ENGa**	**Newton**
5.0	23,698.2	23,728.9	23,697.8
16.1	13,200.0	13,300.0	13,200.0
807.0	2320.0	2230.0	1870.0

point (about 50 times the radius of the Milky Way) is very far from the value of 23,295 km/s reported in [14] if it is assumed that it corresponds to a location near the edge of the galaxy. However at the 1st point (~1/3 of the Milky Way radius) the circular speed is close to that value. This again illustrates the importance of accurate information of the mass distribution of a galaxy.

It is curious in that Table that at the last point the circular speed of MOND (Equation (4a)) is greater than the one of ENGa (Equation (9a)). But note that ENGb (binned G_1) was not used in this example (the mass is outside the range of the binned data of **Table 1**).

It is expected that the GE-MOND and the GE-ENG model will yield, for this case, larger differences than the ones obtained in **Table 3** since the mass in **Table 5** is twice the one in **Table 3** and therefore larger circular speeds are involved.

Any new theory that extends an old one which has been experimentally verified before should reproduce the old one as a limited case (like the principle of correspondence of Quantum Mechanics). There are many examples of that: GTR vs. Newtonian theory, STR vs. classical mechanical kinetics, Classical GEM vs. Newtonian gravitation, etc. In this sense if MOND does not allow a variable characteristic acceleration (e.g. mass dependent) it will not comply with this extended correspondence principle when used in the solar system unless for example, it is experimentally found that a very isolated star (like the sun) has satellites at very large distances that have significant non-zero asymptotic circular speeds. Note that in this case an important condition is present: very low acceleration (deep MOND regime) with small mass (solar system). Note also that if it is assumed that $a_0 \approx \frac{1}{6} c H_0$ is not just a numerical coincidence then making the speed of light infinite will not recover classical dynamics. However in that case STR becomes classical kinematics and classical GEM (EM) becomes Newtonian gravitation (Coulomb electrostatic).

In order for ENG (and therefore EGEM) to comply with this extended correspondence principle G_1 should have, for example, a bell shape (as function of mass) where the left branch (lower values of mass) asymptotically goes to zero and the right branch (larger values of mass) should cover the binned values shown in **Table 1** with a tail approaching 0. Note that eq. 11 (1st term) could yield a significant asymptotic speed for relativistic speeds even if G_1 is zero, this is an intrinsic feature of GEM for a bound test particle in circular motion with constant angular speed. However if $a_0 = 0$ is made in Equation (5a), no asymptotic behavior will be present.

It is curious that using $M_u = c^3 / (2 G H_0)$ as the universe mass [15] (≈8.57 × 10^{52} kg) in the eq. c^2 / M_u, it yields a value of ≈10^{-36} (m/s)^2/kg. Could this be an approximate value of G_1 at the edge of a visible rotating universe? In this case $G_1 = \frac{2 G H_0}{c}$, if c is made infinite $\Rightarrow G_1 = 0$, however in MOND $a_0 \approx \frac{1}{6} c H_0 = \infty$.

Note that ref. [16] calculated H_0 (close to published experimental results) for a non-expanding universe based on a tired light theory and that ref [17] reports that the predictions of galaxy size and surface brightness based on an expanding universe contradict observations.

6. Summary and Concluding Remarks

An extended Newtonian gravitation (ENG) theory was developed to explain the missing mass problem in galaxy and galaxy clusters as an alternative to MOND (as a modification of the Newtonian inertial acceleration) and to the potential existence of dark matter.

Because of the remarkable success of MOND reproducing the rotation curve of galaxies it was used as a reference for comparison with ENG in simulated galaxies and cluster of galaxies.

MOND has 2 free parameters (the characteristic acceleration and the interpolation function), ENG requires only the knowledge of a mass dependent parameter which was calculated straightforwardly from published binned measured data (asymptotic circular speeds and baryonic mass) in a broad range (about 11 orders of magnitude) of masses.

Both theories were extended in the work frame of the classical GEM to consider relativistic speeds of the source. The GEM extension of ENG required the derivation of terms which are not present in classical EM theory while the extension of MOND was straightforward (correct the acceleration terms of the source in the GEM equation with the interpolation function).

It was shown that the ENG results are relatively close to MOND's values in galaxies while are significantly larger in galaxy cluster which is where MOND is known to still have needs for more mass to reproduce the circular speeds in galaxy clusters.

It could be worthy to extend (if possible) the baryon content of cosmic structures all the way to the solar system and to very massive structures comparable to the visible universe.

While waiting for the review results of this paper an application of the ENG on GR was developed (Appendix C). It turned out that the Einstein field equation needs to be modified to obtain the new Poisson equation corresponding to ENG.

It was called to my attention by the reviewer that extended theories of relativity, in principle, could take care of the dark matter problem in galaxies and galaxy clusters (see [18] for example). Appendix C is a confirmation of that statement.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendix A

The GEM fields of a point mass moving in an arbitrary motion can be written as [19]:

$$\vec{E} = G\frac{c_d^2}{c}\frac{M}{b_1^3}\left[b_1\left(\vec{v}\left(k-\frac{c}{c_d}\right)-kr\frac{\vec{a}}{c_d}\right)+b_2\left(\frac{c}{c_d}\vec{r}-k\frac{r}{c_d}\vec{v}\right)\right]$$

$$\vec{B} = G\frac{k_w c_d}{c^2}\frac{M}{b_1^3}\left(b_1\vec{a}\times\vec{r}+b_2\vec{v}\times\vec{r}\right),\quad \vec{F} = -m\left(\vec{E}+\vec{v}_m\times\vec{B}\right)$$

$$b_1 = rc_d - \vec{r}\cdot\vec{v},\quad b_2 = c_d^2 - v^2 + \vec{r}\cdot\vec{a},\quad k = k_w/k_d,\quad c_d = c/k_d$$

where

M: Mass of the gravitational source.

k_d: Coefficient to consider the impact of the speed (delay) of the GEM force on the orbit decay.

k_w: Coefficient to consider the impact of the GM permeability on the orbit decay $\Rightarrow c_w = c/\sqrt{k_w}$.

c_w: Gravitational wave speed.

r: Distance from the position of the point mass source to the observation point (separation)

$\vec{r}$, $\vec{v}$, $\vec{a}$: Separation vector, velocity and acceleration of the source of the fields.

$\vec{a} = \vec{a}_N\left(1-(v/c)^2\right)^3$ $\vec{a}_N$: Newtonian acceleration. This correction yields the perihelion precession of planets

If $\vec{a} = \vec{a}_N\left(1-(v/c)^2\right)$ is used instead, the correct deflection of light near massive bodies is obtained [20].

$m, \vec{v}_m$: Mass and velocity of the test particle at the field point.

All the terms are to be evaluated at the retarded time (except for $G, c_d, c, M, k, m, \vec{v}_m$).

Appendix B

The extended GE field ($\vec{E}_e$) can be written in term of $\vec{E}$ (Appendix A) as

$$\vec{E}_e = \vec{E} + G_1Mc_d^2\frac{\vec{r}}{b_1^2} - G_1Mc_d\frac{Ln(r)}{d_{v/r}^2}K,\quad K = \left(\frac{b_2}{rb_1}-\frac{c_d}{r^2}\right)\vec{r}-\frac{\vec{v}}{r}$$

Appendix C

It is noted that the ENG potential could be used directly in the week limit of Einstein field equation (EFE). Just substitute the new potential in $g_{0.0} = -\left(1+\frac{2\phi}{c^2}\right)$ to obtain the Newtonian relation between gravitational field and the potential [21].

When considering the full EFE the energy-momentum tensor needs to be modified in such a way that in the week limit of Einstein GR (small mass density and speed) the following extended Poisson equation is obtained:

$$\nabla^2\phi = 4\pi G\rho + \frac{16\pi}{3}G_1\rho r$$

ϕ is the gravitational potential. Positive sign in the right hand side was used to follow a common use in GR. That equation can be obtained from $\nabla\cdot\vec{E} = \nabla\cdot(-\nabla\phi) = -\nabla^2\phi$ using Equation (7).

The EFE for matter can be written as

$$R_{\mu.\nu} - 1/2\, g_{\mu.\nu}R = \frac{8\pi G}{c^2}T_{\mu.\nu} \quad \text{or} \quad R_{\mu.\nu} = \frac{8\pi G}{c^2}\left(T_{\mu.\nu} - 1/2\, g_{\mu.\nu}T\right) \quad [21]$$

where

$R_{\mu.\nu}$: Ricci tensor

$g_{\mu.\nu}$: Metric tensor

R: Ricci curvature

$T_{\mu.\nu}$: Energy-Momentum tensor

$T = g^{\mu.\nu}T_{\mu.\nu}$

Following the weak limit approach [21], in order to get that extended Poisson equation the following is needed: $T_{\mu.\nu} - 1/2\, g_{\mu.\nu}T \approx \frac{\rho}{2}\delta_{\mu.\nu} + \frac{2\rho}{3}r\delta_{\mu.\nu}$

For $\mu = \nu = 0 \Rightarrow R_{0.0} = \frac{1}{c^2}\nabla^2\phi = \frac{8\pi G}{c^2}\left(\frac{\rho}{2} + \frac{2\rho}{3}r\right) \Rightarrow$ $\nabla^2\phi = 4\pi G\rho + \frac{16\pi}{3}G_1\rho r.$

Because $T_{\mu.\nu} - 1/2\, g_{\mu.\nu}T = \frac{\rho}{2}\delta_{\mu.\nu}$ in EFE, to get the desired Poisson equation the Energy-Momentum tensor needs to be modified or an extra tensor needs to be added to EFE e.g.

$R_{\mu.\nu} = \frac{8\pi G}{c^2}\left(P_{\mu.\nu} + Q_{\mu.\nu}\right)$ Extended Einstein field equation

$P_{\mu.\nu} = T_{\mu.\nu} - 1/2\, g_{\mu.\nu}T_P$, $T_P = g^{\mu.\nu}T_{\mu.\nu}$ EFE tensors

$Q_{\mu.\nu} = T_{Q\mu.\nu} - 1/2\, g_{\mu.\nu}T_Q$, $T_Q = g^{\mu.\nu}T_{Q\mu.\nu}$ New tensors

For $\mu = \nu = 0$ (weak limit) $\Rightarrow R_{0.0} = \frac{1}{c^2}\nabla^2\phi$, $P_{0.0} = \frac{\rho}{2}$, $Q_{0.0} = \frac{2\rho}{3}r$