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Analytical Soliton-Like Solutions to Nonlinear Dirac Equation of Spinor Field in Spherical Symmetric Metric

Jonas Edou[1*], Valerie I. S. Hontinfinde[1,2], A. Adomou[1,3], Siaka Massou[1]

[1]Department of Theoretical Physics and Mathematics, Laboratory of Physics Mathematics and Theoretical Physics, University of Abomey-Calavi, Abomey-Calavi, Benin
[2]National Higher School of Mathematical Engineering and Modeling, National University of Sciences, Technology, Engineering and Mathematics of Abomey, Abomey, Benin
[3]National Higher Institute of Industrial Technology, INSTI-Lokossa, National University of Sciences, Technology, Engineering and Mathematics of Abomey, Abomey, Benin
Email: *edjonas131988@gmail.com

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Abstract

The present research work deals with an extension of a previous work entitled [Exact Soliton-like spherical symmetric solutions of the Heisenberg-Ivanenko type nonlinear spinor field equation in gravitational theory, Journal of Applied Mathematics and Physics, 2020, 8, 1236-1254] to Analytical Soliton-Like Solutions to Nonlinear Dirac Equation of Spinor Field in Spherical Symmetric Metric. The nonlinear terms in the Lagrangian density are functions of the invariant $I_S = S^2 = (\bar{\psi}\psi)^2$. Equations with power and polynomial nonlinearities are thoroughly scrutinized. It is shown that soliton is responsible for the deformation in the metric and hence in the geometry as well as gravitational field. The role of nonlinearity and the influence of the proper gravitational field of the elementary particles are also examined. The consideration of the nonlinear terms in the spinor Lagrangian, the own gravitational field of elementary particles and the geometrical properties of the metric are necessary and sufficient conditions in order to obtain soliton-like solutions with total charge and total spin in general relativity.

Keywords

Lagrangian, Gravitationnal Field, Flat Space-Time

1. Introduction

In the theory of General Relativity, the structure of elementary particles configuration is modeled by solitons corresponding to solutions of nonlinear differential equations. As mentioned in [1], the generalization of classical field theory remains one the possible ways to overcome the difficulties of the theory which considers elementary particles as mathematical points. So that the field equations possess regular solutions it is necessary to introduce nonlinear terms, describing the fields interactions. The role of nonlinear terms in the Lagrangian density of some classical field theories was examined in [2]. The exact static plane-symmetric soliton-like solutions of the nonlinear spinor field equations are investigated in a series articles [3] [4] [5]. In all these activities, the authors emphasized that the energy density T_0^0 is localized and the total energy of the system is bounded. Nevertheless, the total charge Q and the total spin S_1 of the system are unlimited. Therefore the metric considered presents some shortcomings. But thanks to the metric space-time admitting spherical symmetric, the problematic of the divergence of Q and S_1 is recently corrected in the remarkable articles [6] [7] [8] [9] [10]. The role of the geometrical symmetries in general relativity is introduced by Katzin, Lavine and Davis in a series of papers [11] [12] [13] [14], appearing between 1969 and 1977. They emphasized that the geometrical of the space-time are expressible through the vanishing of the Lie derivative of certain tensors with respect to a vector. This vector may be time-like, space-like or null. These research works have lead to the concept of Ricci solitons introduced by Hamilton. They are natural generalizations of Einstein metric, which have been a subject of intense study in differential geometry and geometric analysis [15].

The present work, considered as part II of all these investigated initiated in [9], aims and extending the results to analytical solutions to Dirac equation of nonlinear spinor field in spherical symmetric metric. Here also equations with power and polynomial nonlinearities are thoroughly scrutinized.

The purpose of the paper is to describe the configurations of the elementary particles by the nonlinear generalization of classical field theory and taking into account their own gravitational field.

The paper is organized as follows. Section 2 addresses the model with fundamental equations via the Lagrangian, the metric, basics equations and concepts. We consider a self-consistent system to obtain spherical-symmetric solutions, taking into account the own gravitational field of elementary particles. Section 3 deals with main results and their discussion; the solutions of the Einstein and nonlinear spinor field equations are derived. Besides, the regularity properties of the obtained solutions as well as the asymptotic behavior of the energy and charge densities are studied. Concluding remarks are outlined in Section 4.

2. Fields Equations and General Solutions

So that the field equations possess regular solutions it is necessary to introduce nonlinear terms, we consider the Lagrangian of the self-consistent system of spinor and gravitational fields as follows [1]:

$$L = \frac{R}{2\chi} + L_{Sp}. \tag{1}$$

with L_{Sp} the spinor field Lagrangian. It is defined as follows:

$$L_{Sp} = \frac{i}{2}\left(\bar{\psi}\gamma^{\mu}\nabla_{\mu}\psi - \nabla_{\mu}\bar{\psi}\gamma^{\mu}\psi\right) - m\bar{\psi}\psi + L_N. \tag{2}$$

Here L_N is the nonlinear terms of L_{Sp}, describing the self-interaction of the spinor field. $L_N = F\left(I_S\right)$ is an arbitrary function depending on the invariant $I_S = S^2 = \left(\bar{\psi}\psi\right)^2$. $R = R_{\mu\nu}g^{\mu\nu}$ is the scalar curvature. Then, $\chi = \frac{8\Pi G}{c^4}$ is Einstein's gravitational constant, G is Newton's gravitational constant and c is the speed of light in vacuum. ψ is the 4-components Dirac's spinor with $\bar{\psi}$ its conjugate. The following paragraph will address to the metric.

In this present analysis, the metric of space-time admitting spherical symmetric may be written as follows:

$$\mathrm{d}s^2 = \mathrm{e}^{2\gamma}\mathrm{d}t^2 - \mathrm{e}^{2\alpha}\mathrm{d}\xi^2 - \mathrm{e}^{2\beta}\left[\mathrm{d}\theta^2 + \sin^2\left(\theta\right)\mathrm{d}\varphi^2\right]. \tag{3}$$

Note that, the speed of light has been taken to be unity ($c = 1$). We define spatial variable as in [6] $\xi = \frac{1}{r}$, where r stands for the radial component of the spherical symmetric metric. We assume that the metric functions α, β and γ are stationnary and depend on ξ alone. In addition, they verify the harmonic coordinate condition as mentioned in [3]:

$$\alpha\left(\xi\right) = 2\beta\left(\xi\right) + \gamma\left(\xi\right). \tag{4}$$

Variation of (1) with respect to the spinor field ψ and its conjugate $\bar{\psi}$ gives nonlinear spinor field equations as follows:

$$i\gamma^{\mu}\nabla_{\mu}\psi - m\psi + 2\sqrt{I_S}\frac{\mathrm{d}F}{\mathrm{d}I_S}\psi = 0, \tag{5}$$

$$i\nabla_{\mu}\bar{\psi}\gamma^{\mu} + m\bar{\psi} - 2\sqrt{I_S}\frac{\mathrm{d}F}{\mathrm{d}I_S}\bar{\psi} = 0. \tag{6}$$

Then, varying of (1) with respect to the metric tensor $g_{\mu\nu}$ leads to the general form of Einstein's field equation as follows:

$$G_{\mu}^{\nu} = R_{\mu}^{\nu} - \frac{1}{2}\delta_{\mu}^{\nu}R = -\chi T_{\mu}^{\nu}, \tag{7}$$

where G_{μ}^{ν} is the Einstein's tensor; R_{μ}^{ν} is the Ricci's tensor; δ_{μ}^{ν} is the Kronecker's symbol and T_{μ}^{ν} is the metric energy-momentum tensor of the nonlinear spinor field. In the sequel, taking into account (1), we obtain the components of the tensor G_{μ}^{ν} in the metric (3) under the coordinate condition (4) as in [6]:

$$G_0^0 = \mathrm{e}^{-2\alpha}\left(2\beta'' - 2\gamma'\beta' - \beta'^2\right) - \mathrm{e}^{-2\beta} = -\chi T_0^0, \tag{8}$$

$$G_1^1 = \mathrm{e}^{-2\alpha}\left(2\beta'\gamma' + \beta'^2\right) - \mathrm{e}^{-2\beta} = -\chi T_1^1, \tag{9}$$

$$G_2^2 = \mathrm{e}^{-2\alpha}\left(\beta'' + \gamma'' - 2\beta'\gamma' - \beta'^2\right) = -\chi T_2^2, \tag{10}$$

$$G_2^2 = G_3^3, \quad T_2^2 = T_3^3. \tag{11}$$

Prime (ʹ) in previous equations denotes differentiation with respect to ξ.

The components of the metric energy-momentum tensor of the spinor field are:

$$T_{\mu}^{\nu} = \frac{i}{4} g^{\nu\rho} \left(\bar{\psi}\gamma_{\mu}\nabla_{\nu}\psi + \bar{\psi}\gamma_{\nu}\nabla_{\mu}\psi - \nabla_{\mu}\bar{\psi}\gamma_{\nu}\psi - \nabla_{\nu}\bar{\psi}\gamma_{\mu}\psi \right) - \delta_{\mu}^{\nu} L_{Sp}. \tag{12}$$

Using the spinor field Equations (5) and (6), L_{Sp} takes the following form:

$$L_{S_P} = \frac{1}{2}\bar{\psi}\left(i\gamma^{\mu}\nabla_{\mu}\psi - m\psi\right) - \frac{1}{2}\left(i\nabla_{\mu}\bar{\psi}\gamma^{\mu} + m\bar{\psi}\right)\psi + F\left(I_S\right), \tag{13}$$

$$= -2S^2 \frac{\partial F}{\partial I_S} + F\left(I_S\right), \tag{14}$$

$$= -2I_S \frac{\partial F}{\partial I_S} + F\left(I_S\right). \tag{15}$$

Taking into account (15), let us write the nontrivial components of the tensor T_{μ}^{ν}:

$$T_0^0 = T_2^2 = T_3^3 = -L_{Sp} = 2I_S \frac{\partial F\left(I_S\right)}{\partial I_S} - F\left(I_S\right), \tag{16}$$

$$T_1^1 = \frac{i}{2}\left(\bar{\psi}\gamma^1\nabla_1\psi - \nabla_1\bar{\psi}\gamma^1\psi\right) + 2I_S \frac{\partial F\left(I_S\right)}{\partial I_S} - F\left(I_S\right). \tag{17}$$

Let us emphasize that γ^{μ} represent Dirac's matrices in curved space-time. They are linked to Dirac's matrices in flat space-time $\bar{\gamma}_a$ by:

$$g_{\mu\nu}\left(\xi\right) = e_{\mu}^{a}\left(\xi\right) e_{\nu}^{b}\left(\xi\right) \eta_{ab}$$

$$\gamma_{\mu}\left(\xi\right) = e_{\mu}^{a}\left(\xi\right)\bar{\gamma}_a, \tag{18}$$

where $\eta_{ab} = diag\left(1,-1,-1,-1\right)$ is the metric of Minkowski and $e_{\mu}^{a}\left(\xi\right)$ are tetradic 4-vectors.

With the Relation (18), we have:

$$\gamma^0\left(\xi\right) = \mathrm{e}^{-\gamma}\bar{\gamma}^0,\ \gamma^1\left(\xi\right) = \mathrm{e}^{-\alpha}\bar{\gamma}^1,\ \gamma^2\left(\xi\right) = \mathrm{e}^{-\beta}\bar{\gamma}^2,\ \gamma^3\left(\xi\right) = \frac{\mathrm{e}^{-\beta}\bar{\gamma}^3}{\sin\theta},\ \gamma^5\left(\xi\right) = \bar{\gamma}^5. \tag{19}$$

The Dirac's matrices in flat space-time are defined in the following way [16] [17]:

$$\bar{\gamma}^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix};\ \bar{\gamma}^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\bar{\gamma}^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix};\ \bar{\gamma}^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\gamma^5 = \bar{\gamma}^5 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

In the Expressions (2), (5)-(6) and (12), ∇_μ represent the covariant derivative of the spinor meaning. It is connected to the spinor affine connection matrices $\Gamma_\mu(\xi)$ as in [18]:

$$\nabla_\mu \psi = \frac{\partial \psi}{\partial \xi^\mu} - \Gamma_\mu \psi \quad \text{or} \quad \nabla_\mu \bar{\psi} = \frac{\partial \bar{\psi}}{\partial \xi^\mu} + \Gamma_\mu \bar{\psi}. \tag{20}$$

The matrice Γ_μ has the following general form:

$$\Gamma_\mu(\xi) = \frac{1}{4} g_{\rho\mu}\left(\partial_\mu e^b_\sigma e^\rho_a - \Gamma^\rho_{\mu\sigma}\right)\gamma^\delta \gamma^\sigma. \tag{21}$$

In the Relation (21), $\Gamma^\rho_{\mu\sigma}$ are Christoffel's symbols. According to the Expression (21), we have the spinor affine connection matrices:

$$\begin{aligned} &\Gamma_0 = -\frac{1}{2} e^{-2\beta} \bar{\gamma}^0 \bar{\gamma}^1 \gamma', \quad \Gamma_1 = 0, \quad \Gamma_2 = \frac{1}{2} e^{-\beta-\gamma} \bar{\gamma}^2 \bar{\gamma}^1 \beta', \\ &\Gamma_3 = \frac{1}{2}\left(e^{-\beta-\gamma} \bar{\gamma}^3 \bar{\gamma}^1 \beta' \sin\theta + \bar{\gamma}^3 \bar{\gamma}^2 \cos\theta\right). \end{aligned} \tag{22}$$

In virtue of Einstein's convention sommation, we get:

$$\gamma^\mu \Gamma_\mu = -\frac{1}{2}\left(e^{-\alpha} \alpha' \bar{\gamma}^1 + \bar{\gamma}^2 e^{-\beta} \cot\theta\right). \tag{23}$$

When we substitute (20) and (23) into (5) and (6), we have

$$i e^{-\alpha} \bar{\gamma}^1 \left(\partial_\xi + \frac{1}{2}\alpha'\right)\psi + \frac{i}{2} \bar{\gamma}^2 e^{-\beta} \psi \cot\theta - \left(m - 2\sqrt{I_S}\frac{dF}{dI_S}\right)\psi = 0, \tag{24}$$

$$i e^{-\alpha} \bar{\gamma}^1 \left(\partial_\xi + \frac{1}{2}\alpha'\right)\bar{\psi} + \frac{i}{2} \bar{\gamma}^2 e^{-\beta} \bar{\psi} \cot\theta + \left(m - 2\sqrt{I_S}\frac{dF}{dI_S}\right)\bar{\psi} = 0. \tag{25}$$

By choosing the 4-component Dirac spinor under the form $\psi(\xi) = V_\delta(\xi)$ with $V_\delta(\xi) = \begin{pmatrix} V_1(\xi) \\ V_2(\xi) \\ V_3(\xi) \\ V_4(\xi) \end{pmatrix}$, from (24), we get the following set of equations:

$$V_4' + \frac{1}{2}\alpha' V_4 - \frac{i}{2} e^{\alpha-\beta} V_4 \cot\theta + i e^\alpha \left(m - 2\sqrt{I_S}\frac{dF}{dI_S}\right) V_1 = 0, \tag{26}$$

$$V_3' + \frac{1}{2}\alpha' V_3 + \frac{i}{2} e^{\alpha-\beta} V_3 \cot\theta + i e^\alpha \left(m - 2\sqrt{I_S}\frac{dF}{dI_S}\right) V_2 = 0, \tag{27}$$

$$V_2' + \frac{1}{2}\alpha' V_2 - \frac{i}{2} e^{\alpha-\beta} V_2 \cot\theta - i e^\alpha \left(m - 2\sqrt{I_S}\frac{dF}{dI_S}\right) V_3 = 0, \tag{28}$$

$$V_1' + \frac{1}{2}\alpha' V_1 + \frac{i}{2} e^{\alpha-\beta} V_1 \cot\theta - i e^\alpha \left(m - 2\sqrt{I_S}\frac{dF}{dI_S}\right) V_4 = 0. \tag{29}$$

The functions V_1, V_2, V_3 and V_4 are connected by the relation:

$$V_1^2 - V_2^2 - V_3^2 + V_4^2 = cste. \tag{30}$$

Summing the set of Equations (26)-(29), we obtain the first-order differential equations for the invariant function $I_S = S^2$ as follows:

$$\frac{\mathrm{d}I_S}{\mathrm{d}\xi} + 2\alpha'(\xi) I_S = 0. \tag{31}$$

The solution of the Equation (31) is:

$$I_S(\xi) = C_0 \exp\left[-2\alpha(\xi)\right], \quad C_0 = const. \tag{32}$$

The Expression (32) reflects the natural link between the nonlinear spinor field of elementary particles and their own gravitational field.

Using the spinor field equation in the Form (24) and the conjugate one, we obtain the following expression for the tensor T_1^1 from the Relation (17):

$$T_1^1 = m\sqrt{I_S} - F(I_S). \tag{33}$$

The following paragraph devotes to the resolution of Einstein's field equations. To this purpose, as the commponents T_0^0 and T_2^2 are equal, we have $G_0^0 - G_2^2 = 0$. This leads to the following equation:

$$\beta'' - \gamma'' = \mathrm{e}^{2\beta + 2\gamma}, \tag{34}$$

which can be transformed into a Liouville equation type (see [19], p. 30) having the solutions:

$$\beta(\xi) = \frac{A}{4}\left(1 + \frac{2}{D}\right) \ln\left[\frac{A}{DT^2(h, \xi + \xi_1)}\right] = \left(1 + \frac{2}{D}\right)\gamma(\xi), \tag{35}$$

$$\gamma(\xi) = \frac{A}{4} \ln\left[\frac{A}{DT^2(h, \xi + \xi_1)}\right], \tag{36}$$

where A and D are integration constants and T is a function. The function T has the following form:

$$T(h, \xi + \xi_1) = \begin{cases} \frac{1}{h}\sinh\left[h(\xi + \xi_1)\right], h > 0 \\ (\xi + \xi_1), h = 0 \\ \frac{1}{h}\sin\left[h(\xi + \xi_1)\right], h < 0 \end{cases} \tag{37}$$

where h and ξ_1 are another unknown integration constants.

By substituting the Expressions (35) and (36) into (4), we get the metric function $\alpha(\xi)$ as follows:

$$\alpha(\xi) = \frac{A}{2}\left(\frac{3}{2} + \frac{2}{D}\right) \ln\left[\frac{A}{DT^2(h, \xi + \xi_1)}\right]. \tag{38}$$

Finally we define the relations between the metric functions $\alpha(\xi)$, $\beta(\xi)$ and $\gamma(\xi)$:

$$\beta(\xi) = \frac{2 + D}{4 + 3D}\alpha(\xi); \quad \gamma(\xi) = \frac{D}{4 + 3D}\alpha(\xi). \tag{39}$$

Equation (9) look likes to the first integral of the Equations (8) and (10). It is also a first order differential equation. Then, introducing (33) and (39) into (9), we have

$$\left(\alpha'\right)^2=\frac{\left(4+3D\right)^2}{3D^2+8D+4}\mathrm{e}^{2\alpha}\left[\mathrm{e}^{\frac{-4-2D}{4+2D}\alpha}-\chi\left(m\sqrt{I_S}-F\left(I_S\right)\right)\right]. \tag{40}$$

Taking into account $\alpha'=-\frac{1}{2I_S}\frac{\mathrm{d}I_S}{\mathrm{d}\xi}$ and $I_S\left(\xi\right)=C_0\mathrm{e}^{-2\alpha(\xi)}$, from (40) we obtain

$$\frac{\mathrm{d}I_S}{\mathrm{d}\xi}=\pm\frac{2\sqrt{C_0}\left(4+3D\right)}{\sqrt{3D^2+8D+4}}\sqrt{I_S}\sqrt{\left[\left(\frac{I_S}{C_0}\right)^{\frac{2+D}{4+3D}}-\chi\left(m\sqrt{I_S}-F\left(I_S\right)\right)\right]}. \tag{41}$$

The general solutions of the Equation (41) are given by:

$$\int\frac{\mathrm{d}I_S}{\sqrt{I_S}\sqrt{\left[\left(\frac{I_S}{C_0}\right)^{\frac{2+D}{4+3D}}-\chi\left(m\sqrt{I_S}-F\left(I_S\right)\right)\right]}}=\pm\frac{2\sqrt{C_0}\left(4+3D\right)}{\sqrt{3D^2+8D+4}}\left(\xi+\xi_0\right). \tag{42}$$

Setting a concrete form of the function $F\left(I_S\right)$, from (42) we can determine explicitly $I_S\left(\xi\right)$. Then, if $I_S\left(\xi\right)$ is known, we can find the metric function $\alpha\left(\xi\right)$ from (32). Finally, we can completely determine the solutions of Einstein equations from the Expression (39).

Considering the invariant $I_S\left(\xi\right)=C_0\mathrm{e}^{-2\alpha(\xi)}$, we can establish the regularity properties of the solutions obtained. Studying the distribution of the energy per unit invariant volume $T_0^0\sqrt{-3_g}$, we can establish the localization properties of the solutions.

Let us determine the concrete analytical form of the functions $V_\delta\left(\xi\right)$. To doing so, we must solve the set of Equations (26)-(29) in more compacte form if we pass to the functions $W_\delta\left(\xi\right)=\mathrm{e}^{\frac{\alpha}{2}}V_\delta\left(\xi\right)$, with $\delta=1,2,3,4$. In this perspective, we obtain:

$$W_4'-\frac{i}{2}\mathrm{e}^{\alpha-\beta}W_4\cot\theta+i\mathrm{e}^{\alpha}\left(m-2\sqrt{I_S}\frac{\mathrm{d}F}{\mathrm{d}I_S}\right)W_1=0, \tag{43}$$

$$W_3'+\frac{i}{2}\mathrm{e}^{\alpha-\beta}W_3\cot\theta+i\mathrm{e}^{\alpha}\left(m-2\sqrt{I_S}\frac{\mathrm{d}F}{\mathrm{d}I_S}\right)W_2=0, \tag{44}$$

$$W_2'-\frac{i}{2}\mathrm{e}^{\alpha-\beta}W_2\cot\theta-i\mathrm{e}^{\alpha}\left(m-2\sqrt{I_S}\frac{\mathrm{d}F}{\mathrm{d}I_S}\right)W_3=0, \tag{45}$$

$$W_1'+\frac{i}{2}\mathrm{e}^{\alpha-\beta}W_1\cot\theta-i\mathrm{e}^{\alpha}\left(m-2\sqrt{I_S}\frac{\mathrm{d}F}{\mathrm{d}I_S}\right)W_4=0, \tag{46}$$

where the derivative of the function $W_\rho\left(\xi\right)$ has the form:

$$W_\rho'=\left(V_\rho'+\frac{1}{2}\alpha' V_\rho\right)\mathrm{e}^{\frac{1}{2}\alpha}. \tag{47}$$

With the set of Equations (43)-(46) where $W=W_\delta\left(\xi\right)$ let us pass to the system of equations depending on functions of the argument I_S, *i.e.*

$W_\delta(I_S)=W_\delta(\xi)$, $I_S(\xi)=C_0\mathrm{e}^{-2\alpha(\xi)}$. We obtain for $W_\delta(I_S)$ the set of equations as follows:

$$\frac{\mathrm{d}W_4}{\mathrm{d}I_S}-iE(I_S)W_4+iK(I_S)W_1=0, \tag{48}$$

$$\frac{\mathrm{d}W_3}{\mathrm{d}I_S}+iE(I_S)W_3+iK(I_S)W_2=0, \tag{49}$$

$$\frac{\mathrm{d}W_2}{\mathrm{d}I_S}-iE(I_S)W_2-iK(I_S)W_3=0, \tag{50}$$

$$\frac{\mathrm{d}W_1}{\mathrm{d}I_S}+iE(I_S)W_1-iK(I_S)W_4=0, \tag{51}$$

where $E(I_S)$ and $K(I_S)$ are defined by the following expressions:

$$E(I_S)=\frac{1}{2}\frac{\left(\sqrt{\frac{C_0}{I_S}}\right)^{\frac{2+2D}{4+3D}}\cot\theta}{\frac{2\sqrt{C_0}(4+3D)}{\sqrt{3D^2+8D+4}}\sqrt{I_S}\sqrt{\left[\left(\frac{I_S}{C_0}\right)^{\frac{2+D}{4+3D}}-\chi\left(m\sqrt{I_S}-F(I_S)\right)\right]}}; \tag{52}$$

$$K(I_S)=\frac{\left(\sqrt{\frac{C_0}{I_S}}\right)\left(m-2\sqrt{I_S}\frac{\mathrm{d}F}{\mathrm{d}I_S}\right)}{\frac{2\sqrt{C_0}(4+3D)}{\sqrt{3D^2+8D+4}}\sqrt{I_S}\sqrt{\left[\left(\frac{I_S}{C_0}\right)^{\frac{2+D}{4+3D}}-\chi\left(m\sqrt{I_S}-F(I_S)\right)\right]}}. \tag{53}$$

In sequel, we shall transform the Equation (48)-(51) to the second order differential equations. In this perspective, differentiating Equation (48) and substituting the expression of the function $W_1(I_S)$ and the expression of its derivative into the result, we obtain:

$$W_4''-\frac{K'(I_S)}{K(I_S)}W_4'+\left[E^2(I_S)-K^2(I_S)+i\frac{K'(I_S)E(I_S)-K(I_S)E'(I_S)}{K(I_S)}\right]W_4=0. \tag{54}$$

Similarly differentiating the Equation (51) and introducing into the result the expression of $W_4(I_S)$ and the expression of its derivative, we obtain the second-order differential equation for the function $W_1(I_S)$:

$$W_1''-\frac{K'(I_S)}{K(I_S)}W_1'+\left[E^2(I_S)-K^2(I_S)+i\frac{K(I_S)E'(I_S)-K'(I_S)E'(I_S)}{K(I_S)}\right]W_1=0. \tag{55}$$

Doing the same operating on the Equations (49)-(50), we find the second-order differential equations obeyed by the functions $W_2(I_S)$ and $W_3(I_S)$ as follows:

$$W_3''-\frac{K'(I_S)}{K(I_S)}W_3'+\left[E^2(I_S)-K^2(I_S)+i\frac{K(I_S)E'(I_S)-K'(I_S)E'(I_S)}{K(I_S)}\right]W_3=0. \tag{56}$$

$$W_2'' - \frac{K'(I_S)}{K(I_S)} W_2' + \left[E^2(I_S) - K^2(I_S) + i \frac{K'(I_S)E(I_S) - K(I_S)E'(I_S)}{K(I_S)} \right] W_2 = 0. \tag{57}$$

By summing (54)-(55) and setting $U = W_1 + W_4$, we obtain the following second-order differential equations of the function $U(I_S)$:

$$U''(I_S) - \frac{K'(I_S)}{K(I_S)} U'(I_S) + 2\left[E^2(I_P) - K^2(I_S)\right] U(I_S) = 0. \tag{58}$$

The Equation (58) may be transformed to:

$$\frac{1}{K(I_S)\sqrt{2\varepsilon}} \frac{\mathrm{d}}{\mathrm{d}I_S} \left[\frac{U'(I_S)}{K(I_S)\sqrt{2\varepsilon}} \right] - U(I_S) = 0, \tag{59}$$

under the condition $E^2(I_S) = (1-\varepsilon)K^2(I_S)$ with $0 < \varepsilon \le 1$.

The first integral of the Equation (59)

$$U'(I_S) = \pm\sqrt{U^2(I_S) + C_1} K(I_S)\sqrt{2\varepsilon}, \quad C_1 = const. \tag{60}$$

If $C_1 = a_1^2 > 0$, then the Equation (60) has the solution

$$U(I_S) = a_1 \sinh N_1(I_S). \tag{61}$$

If $C_1 = -b_1^2 < 0$, the solution of the Equation (60) is given by:

$$U(I_S) = b_1 \cosh N_1(I_S), \tag{62}$$

with

$$N_1(I_S) = \sqrt{2\varepsilon}\int K(I_S)\mathrm{d}I_S + R_1, \quad R_1 = const. \tag{63}$$

The difference of Equations (48) and (51), taking into account of (61) and (62), gives:

$$X(I_S) = W_1 - W_4 = -ia_1\left(\frac{\sqrt{1-\varepsilon}-1}{\sqrt{2\varepsilon}}\right)\cosh N_1(I_S), \tag{64}$$

or

$$X(I_S) = W_1 - W_4 = -ib_1\left(\frac{\sqrt{1-\varepsilon}-1}{\sqrt{2\varepsilon}}\right)\sinh N_1(I_S), \tag{65}$$

where a_1 and b_1 are integration constants.

Solving analogously the Equations (56) and (57), we obtain the following expressions for $Y(I_S) = W_2 + W_3$ as follows:

$$Y(I_S) = a_2 \sinh N_2(I_S), \quad \text{for } C_2 = a_2^2 > 0 \tag{66}$$

or

$$Y(I_S) = b_2 \cosh N_2(I_S), \quad \text{for } C_2 = -b_2^2 < 0. \tag{67}$$

In these conditions, it then follows from the Expressions (66) and (67) that:

$$V(I_S) = W_2 - W_3 = ia_2\left(\frac{\sqrt{1-\varepsilon}-1}{\sqrt{2\varepsilon}}\right)\cosh N_2(I_S), \tag{68}$$

or

$$V(I_S) = W_2 - W_3 = ib_2\left(\frac{\sqrt{1-\varepsilon}-1}{\sqrt{2\varepsilon}}\right)\sinh N_2(I_S), \tag{69}$$

$$N_2\left(I_P\right)=\sqrt{2\varepsilon}\int H\left(I_P\right)\mathrm{d}I_P+R_2, \tag{70}$$

where a_2, b_2 and R_2 are integration constants.

Considering the cases where $C_1=a_1^2>0$ and $C_2=-b_2^2<0$, let us determine the expressions of the functions $W_\delta\left(I_S\right)$. We get for the functions $W_\delta\left(I_S\right)$ the following expressions:

$$W_1\left(I_S\right)=a_0\left[\sinh N_1\left(I_S\right)-i\left(\frac{\sqrt{1-\varepsilon}-1}{\sqrt{2\varepsilon}}\right)\cosh N_1\left(I_S\right)\right], \tag{71}$$

$$W_2\left(I_S\right)=b_0\left[\cosh N_2\left(I_S\right)+i\left(\frac{\sqrt{1-\varepsilon}-1}{\sqrt{2\varepsilon}}\right)\sinh N_2\left(I_S\right)\right], \tag{72}$$

$$W_3\left(I_S\right)=b_0\left[\cosh N_2\left(I_S\right)-i\left(\frac{\sqrt{1-\varepsilon}-1}{\sqrt{2\varepsilon}}\right)\sinh N_2\left(I_S\right)\right], \tag{73}$$

$$W_4\left(I_S\right)=a_0\left[\sinh N_1\left(I_S\right)+i\left(\frac{\sqrt{1-\varepsilon}-1}{\sqrt{2\varepsilon}}\right)\cosh N_1\left(I_S\right)\right], \tag{74}$$

with $a_0=\frac{1}{2}a_1$ and $b_0=\frac{1}{2}b_2$.

Let us note that we can also obtain the expressins of the funstions $W_\delta\left(I_S\right)$ considering $C_1=-b_1^2<0$ and $C_2=a_2^2>0$. Furthermore, in the relations (63) and (70), without loss of generality we can use the minus sign before the integral. Let us pass to the functions $V_\delta\left(\xi\right)$ by multiplying the functions $W_\delta\left(\xi\right)$ obtained in the Expressions (71)-(74) by $\mathrm{e}^{-\frac{1}{2}\alpha(\xi)}$ as follows:

$$\begin{aligned}V_1\left(\xi\right)=a_0&\left[\sinh N_1\left(\xi\right)-i\left(\frac{\sqrt{1-\varepsilon}-1}{\sqrt{2\varepsilon}}\right)\cosh N_1\left(\xi\right)\right]\\&\times\exp\left\{-\frac{A}{4}\left(\frac{3}{2}+\frac{2}{D}\right)\ln\left[\frac{A}{DT^2\left(h,\xi+\xi_1\right)}\right]\right\},\end{aligned} \tag{75}$$

$$\begin{aligned}V_2\left(\xi\right)=b_0&\left[\cosh N_2\left(\xi\right)+i\left(\frac{\sqrt{1-\varepsilon}-1}{\sqrt{2\varepsilon}}\right)\sinh N_2\left(\xi\right)\right]\\&\times\exp\left\{-\frac{A}{4}\left(\frac{3}{2}+\frac{2}{D}\right)\ln\left[\frac{A}{DT^2\left(h,\xi+\xi_1\right)}\right]\right\},\end{aligned} \tag{76}$$

$$\begin{aligned}V_3\left(\xi\right)=b_0&\left[\cosh N_2\left(\xi\right)-i\left(\frac{\sqrt{1-\varepsilon}-1}{\sqrt{2\varepsilon}}\right)\sinh N_2\left(\xi\right)\right]\\&\times\exp\left\{-\frac{A}{4}\left(\frac{3}{2}+\frac{2}{D}\right)\ln\left[\frac{A}{DT^2\left(h,\xi+\xi_1\right)}\right]\right\},\end{aligned} \tag{77}$$

$$\begin{aligned}V_4\left(\xi\right)=a_0&\left[\sinh N_1\left(\xi\right)+i\left(\frac{\sqrt{1-\varepsilon}-1}{\sqrt{2\varepsilon}}\right)\cosh N_1\left(\xi\right)\right]\\&\times\exp\left\{-\frac{A}{4}\left(\frac{3}{2}+\frac{2}{D}\right)\ln\left[\frac{A}{DT^2\left(h,\xi+\xi_1\right)}\right]\right\}.\end{aligned} \tag{78}$$

The following section deals with the analysis of the general results obtained previously by considering the concrete nonlinear terms of the arbitrary function $F(I_S)$ in the Lagrangian density.

3. Analysis of Principal Results and Discussions

Let us consider a concrete type nonlinear spinor field equations when:

$$F(I_S) = \lambda I_S^n = \lambda S^{2n} \tag{79}$$

where λ is nonlinearity parameter and n power nonlinearity. Consider two cases $n=1$ and $n>1$.

- Firstly, we should aborde the Ivanenko-Heisenberg type nonlinear spinor field equation corresponding to $n=1$, $F(I_S)=\lambda I_S$ and $\lambda \neq 0$. But this study is intensively dealt with in [9]. Let us now pass to the generalization of the analysis.
- Secondly, we deal with the case where $n>1$ and $\lambda > 0$. According to (79), the Equation (42) takes the form:

$$\int \frac{\mathrm{d}I_S}{\sqrt{I_S}\sqrt{\left[\left(\frac{I_S}{C_0}\right)^{\frac{2+D}{4+3D}} - \chi\left(m\sqrt{I_S} - \lambda I_S^n\right)\right]}} = \pm\frac{2\sqrt{C_0}\left(4+3D\right)}{\sqrt{3D^2+8D+4}}\left(\xi+\xi_0\right). \tag{80}$$

In this section, we can consider massless spinor field *i.e.* $m=0$ without loss of the generality. Indeed, as one sees from (80), for $m\neq 0$ at no value of ξ the invariant becomes trivial. Since as $I_S(\xi)\to 0$, the denominator of the integrant beginning from some value of $I_S(\xi)$ becomes imaginary. Thus, in order to avoid the imaginary value at denominator of the integrant and evident value of the invariant function $I_S(\xi)$, it is necessary to choose massless spinor field setting $m=0$ in the Equation (80). Similarly, it should be emphasized that, in the unified nonlinear spinor theory of Heisenberg, the massive terme is absent. So, according to Heisenberg theory, the particle mass should be obtained as a result of quantization of spinor prematter [20].

Substituting $m=0$, without loss of generality, in the Equation (80) and taking $n>1$ and $\lambda>0$, we get:

$$I_S(\xi) = \left\{\frac{1}{\chi\lambda C_0 \sinh^2\left[\frac{4+3D}{\sqrt{3D^2+8D+4}}(n-1)(\xi+\xi_0)\right]}\right\}^{\frac{1}{2(n-1)}} ;n>1. \tag{81}$$

Introducing (79) and (80) into (16), the energy density of the nonlinear spinor field T_0^0 is

$$T_0^0(\xi) = \lambda(2n-1)\left\{\frac{1}{\chi\lambda C_0^2 \sinh^2\left[\frac{4+3D}{\sqrt{3D^2+8D+4}}(n-1)(\xi+\xi_0)\right]}\right\}^{\frac{n}{2(n-1)}} ;n>1. \tag{82}$$

We note from (81) that $I_S(\xi)$ is not bounded. Moreover from (82) the energy density T_0^0 is not localized. The soliton-like solutions are absent. Therefore, the nonlinear form $F(I_s)=\lambda I_S^n$, $n>1$ and $\lambda>0$ is not plausible to get the soliton-like configurations with localized energy density. In the following paragraph, let us discuss the case where $\lambda=-\Lambda^2<0$.

To this end, when $\lambda=-\Lambda^2$, from (80), we obtain:

$$I_S(\xi)=\left\{\frac{1}{\chi\Lambda^2 C_0\cosh^2\left[\frac{4+3D}{\sqrt{3D^2+8D+4}}(n-1)(\xi+\xi_0)\right]}\right\}^{\frac{1}{2(n-1)}};n>1. \quad (83)$$

From (16), the energy density T_0^0 is defined as follows:

$$T_0^0(\xi)=-M\left\{\frac{1}{\chi\Lambda^2 C_0\cosh^2\left[\frac{4+3D}{\sqrt{3D^2+8D+4}}(n-1)(\xi+\xi_0)\right]}\right\}^{\frac{n}{2(n-1)}};n>1. \quad (84)$$

with $M=\Lambda^2(2n-1)=const$.

We conclude from (84) that the energy density of a nonlinear spinor field is negative and is localized in space [2].

As for the distribution of the spinor field energy density per unit invariant volume $\varepsilon(\xi)=T_0^0(\xi)\sqrt{3_{-g}}$, it's given by

$$\varepsilon(\xi)=-M\left\{\frac{1}{\chi\Lambda^2 C_0\cosh^2\left[\frac{4+3D}{\sqrt{3D^2+8D+4}}(n-1)(\xi+\xi_0)\right]}\right\}^{\frac{n}{2(n-1)}}\mathrm{e}^{\zeta(\xi)}\sin\theta, \quad (85)$$

where

$$\zeta(\xi)=\left(\frac{8+5D}{4D}\right)A\sqrt{C_0}\left\{\chi\Lambda^2 C_0\cosh^2\left[\frac{4+3D}{\sqrt{3D^2+8D+4}}(n-1)(\xi+\xi_0)\right]\right\}^{\frac{1}{2(n-1)}}. \quad (86)$$

The total energy is defined by

$$E=-M\sin(\theta)\int_0^{\xi_C}\left\{\frac{1}{\sqrt{\chi\Lambda^2 C_0^2}\cosh\left[\frac{4+3D}{\sqrt{3D^2+8D+4}}(n-1)(\xi+\xi_0)\right]}\right\}^{\frac{n}{2(n-1)}}\mathrm{e}^{\zeta(\xi)}\mathrm{d}\xi<\infty \quad (87)$$

The obtained solutions describe a nonlinear spinor field configuration with regular localized energy density $T_0^0(\xi)$ and energy density per unit volume $\varepsilon(\xi)$, negative total energy E and the metric functions are regular and stationary. The solutions possess the properties of soliton-like solutions as mentioned in [19]. They may be used in order to describe the properties of the field configuration of the elementary particles.

Considering $F(I_S) = -\Lambda^2 I_S^2$, we can get an explicit form of the function $V_\delta(\xi)$. We have:

$$V_4(\xi) = a_0 \left\{ \sinh\left[f(\xi)\right] + i\frac{1}{\sqrt{1-\varepsilon}}\left[\sinh\left[f(\xi)\right] - \sqrt{2\varepsilon}\cosh\left[f(\xi)\right]\right]\right\}\exp\left[-\phi(\xi)\right], \quad (88)$$

$$V_3(\xi) = a_0 \left\{ \sinh\left[f(\xi)\right] + i\frac{1}{\sqrt{1-\varepsilon}}\left[-\sinh\left[f(\xi)\right] + \sqrt{2\varepsilon}\cosh\left[f(\xi)\right]\right]\right\}\exp\left[-\phi(\xi)\right], \quad (89)$$

$$V_2(\xi) = \beta_0 \left\{ \cosh\left[f(\xi)\right] - i\frac{1}{\sqrt{1-\varepsilon}}\left[\sqrt{2\varepsilon}\sinh\left[f(\xi)\right] + \cosh\left[f(\xi)\right]\right]\right\}\exp\left[-\phi(\xi)\right], \quad (90)$$

$$V_1(\xi) = \beta_0 \left\{ \cosh\left[f(\xi)\right] + i\frac{1}{\sqrt{1-\varepsilon}}\left[\sqrt{2\varepsilon}\sinh\left[f(\xi)\right] + \cosh\left[f(\xi)\right]\right]\right\}\mathrm{expp}\left[-\phi(\xi)\right], \quad (91)$$

where

$$f(\xi) = N_{1,2}(\xi) = -\frac{2n\sqrt{3D^2+8D+4}}{(4+3D)\chi C_0^2}\tanh\left[\frac{4+3D}{\sqrt{3D^2+8D+4}}(n-1)(\xi+\xi_0)\right] + R_{1,2}, \quad (92)$$

and

$$\exp\left[-\phi(\xi)\right] = \left[\frac{\left\{\dfrac{1}{\sqrt{\chi\Lambda^2 C_0^2}\cosh\left[\dfrac{4+3D}{\sqrt{3D^2+8D+4}}(n-1)(\xi+\xi_0)\right]}\right\}^{\frac{1}{2(n-1)}}}{\sqrt{C_0}}\right]. \quad (93)$$

We can then finally write the components of the function $\psi(\xi)$ of the spinor field under the form:

$$\psi(\xi) = \begin{pmatrix} \beta_0 \left\{ \cosh\left[f(\xi)\right] + i\frac{1}{\sqrt{1-\varepsilon}}\left[\sqrt{2\varepsilon}\sinh\left[f(\xi)\right] + \cosh\left[f(\xi)\right]\right]\right\}\exp\left[-\phi(\xi)\right] \\ \beta_0 \left\{ \cosh\left[f(\xi)\right] - i\frac{1}{\sqrt{1-\varepsilon}}\left[\sqrt{2\varepsilon}\sinh\left[f(\xi)\right] + \cosh\left[f(\xi)\right]\right]\right\}\exp\left[-\phi(\xi)\right] \\ a_0 \left\{ \sinh\left[f(\xi)\right] + i\frac{1}{\sqrt{1-\varepsilon}}\left[-\sinh\left[f(\xi)\right] + \sqrt{2\varepsilon}\cosh\left[f(\xi)\right]\right]\right\}\exp\left[-\phi(\xi)\right] \\ a_0 \left\{ \sin\left[f(\xi)\right] + i\frac{1}{\sqrt{1-\varepsilon}}\left[\sinh\left[f(\xi)\right] - \sqrt{2\varepsilon}\cosh\left[f(\xi)\right]\right]\right\}\exp\left[-\phi(\xi)\right] \end{pmatrix}. \quad (94)$$

From (94), we conclude that the function $\psi(\xi) = V_\delta(\xi)$ is regular.

The obtained solutions describe the configuration of nonlinear spinor field with localized energy density. The energy density per unit invariant volume $\varepsilon(\xi)$ is also a localized function. The total energy has a finite and negative quantity. The metrics functions are stationnary and regular. Then, they are soliton-like solutions and must be used to describe the configuration of elementary particles.

The following paragraph deals with the total charge and the total spin. Let us start with the components of spinor current vector.

Using the solutions (88)-(91) we can determine the components of the spinor current vector $j^\mu = \bar{\psi}\gamma^\mu\psi$:

$$\begin{aligned} j^0 &= 2\mathrm{e}^{-\alpha-\gamma}\left\{a_0^2\left[\sinh^2 N_1 + \left(\frac{-1+\sqrt{1-\varepsilon}}{\sqrt{2\varepsilon}}\right)^2\cosh^2 N_1\right]\right. \\ &\left.+\beta_0^2\left[\cosh^2 N_2 + \left(\frac{-1+\sqrt{1-\varepsilon}}{\sqrt{2\varepsilon}}\right)^2\sinh^2 N_2\right]\right\} \end{aligned} \tag{95}$$

$$\begin{aligned} j^1 &= 2\mathrm{e}^{-2\alpha}\left\{a_0^2\left[\sinh^2 N_1 - \left(\frac{-1+\sqrt{1-\varepsilon}}{\sqrt{2\varepsilon}}\right)^2\cosh^2 N_1\right]\right. \\ &\left.+\beta_0^2\left[\cosh^2 N_2 - \left(\frac{-1+\sqrt{1-\varepsilon}}{\sqrt{2\varepsilon}}\right)^2\sinh^2 N_2\right]\right\} \end{aligned} \tag{96}$$

$$\begin{aligned} j^2 &= 4\mathrm{e}^{-\alpha-\beta}\left[a_0^2\frac{-1+\sqrt{1-\varepsilon}}{\sqrt{2\varepsilon}}\cosh N_1(\xi)\sinh N_1(\xi)\right. \\ &\left.-\beta_0^2\frac{-1+\sqrt{1-\varepsilon}}{\sqrt{2\varepsilon}}\cosh N_2(\xi)\sinh N_2(\xi)\right] \end{aligned} \tag{97}$$

$$j^3 = 0. \tag{98}$$

As in this study the configuration is assumed static, the components j^1, j^2 and j^3 are evident. The requirement of existence of one component of the vector j^0 leads to $a_0 = b_0 = a$, $N_1(\xi) = N_2(\xi) = N(\xi)$ and $\varepsilon = 1$. From the component j^0, we define the charge density or the chronometric invariant of the spinor field as follows:

$$\rho(\xi) = \left(j_0 j^0\right)^{\frac{1}{2}} = 3a^2\vartheta(\xi)\cosh 2N(\xi) \tag{99}$$

where $N(\xi)$ is defined by the Expression (92) and

$$\vartheta(\xi) = \mathrm{e}^{-\alpha(\xi)} = \left[\frac{\left\{\dfrac{1}{\chi\Lambda^2 C_0\cosh^2\left[\dfrac{4+3D}{\sqrt{3D^2+8D+4}}(n-1)(\xi+\xi_0)\right]}\right\}^{\frac{1}{4(n-1)}}}{\sqrt{C_0}}\right]. \tag{100}$$

The charge density is localized when $\xi \in [0, \xi_c]$. The total charge nonlinear spinor field equation is:

$$Q = \int_0^{\xi_c} \rho \sqrt{-3_g} \, \mathrm{d}\xi = 3a^2 \sin(\theta) \int_0^{\xi_c} \cosh 2N(\xi) \mathrm{e}^{\alpha-\gamma} \mathrm{d}\xi < \infty, \tag{101}$$

ξ_C being the center of the field configuration and

$$\mathrm{e}^{\alpha-\gamma} = \left[\frac{C_0}{\left\{ \frac{1}{\chi \Lambda^2 C_0 \cosh^2 \left[\frac{4+3D}{\sqrt{3D^2+8D+4}} (n-1)(\xi+\xi_0) \right]} \right\}^{\frac{1}{2(n-1)}}} \right]. \tag{102}$$

From (101) the total charge is finite as the charge density is continous and localized.

Let us deal with the spin tensor of the spinor field. Its general form is:

$$S^{\mu\nu,\lambda} = \frac{1}{4} \bar{\psi} \left\{ \gamma^\lambda \sigma^{\mu\nu} + \sigma^{\mu\nu} \gamma^\lambda \right\} \psi. \tag{103}$$

From the Expression (102), the spatial density of the spin tensor $S^{ik,0}, i, k = 1;2;3$ is:

$$S^{ik,0} = \frac{1}{4} \bar{\psi} \left\{ \gamma^0 \sigma^{ik} + \sigma^{ik} \gamma^0 \right\} \psi = \frac{1}{2} \bar{\psi} \gamma^0 \sigma^{ik} \psi. \tag{104}$$

Thus, we have

$$S^{12,0} = S^{13,0} = 0. \tag{105}$$

$$S^{23,0} = \frac{3}{2} a^2 \cosh 2N(\xi) \mathrm{e}^{-\alpha}. \tag{106}$$

The Relation (106) leads to the definition of the chronometric invariant of the spatial density as follows:

$$S_{chI}^{23,0} = \left(S_{23,0} S^{23,0} \right)^{\frac{1}{2}} = \frac{3}{2} a^2 \cosh 2N(\xi) \mathrm{e}^{-\alpha}. \tag{107}$$

Thus, the projection of the spin vector on the radial axis has the form:

$$S_1 = \int_0^{\xi_c} S_{chI}^{23,0} \sqrt{-3_g} \, \mathrm{d}\xi = \frac{3}{2} a^2 \sin(\theta) \int_0^{\xi_c} \cosh 2N(\xi) \mathrm{e}^{\alpha-\gamma} \mathrm{d}\xi. \tag{108}$$

Note that the spin tensor of the spinor field has a finite value.

We can conclude that Dirac's nonlinear equation has configuration with finite value of the total charge and the total spin.

It is necessary to clarify the role of the nonlinear terms in the nonlinear field equations in the formation of regular localized soliton-like solutions. In this case, we must resolve Dirac's equation and compare its solutions with solutions to nonlinear spinor equations. For detail refer to [9].

4. Concluding Remarks

Taking into account the proper gravitational field of elementary particles, the solutions that we have obtained in this research work are soliton-like solutions. They are regular with a localized energy density and limited total energy. The metric functions are stationary, the total charge and the total spin have finite quantities else. The soliton-like solutions exist in flat espace-time and absent in linear case. The nonlinear terms, the proper gravitaional field and the geometrical form of the metric play an important role in the obtaining of the soliton-like solutions. Note that in static plane symmetric metric the charge Q and spin have not finite value but in the spherical static symmetric metric, those are finite. In the forthcoming paper, we shall deal with Spherical symmetric solitons of interacting spinor and scalar fields in general relativity theory.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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