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Fuzzy Henstock-Kurzweil Triple Integral on a Type 1 Quasi-Fuzzy Parallelepipedal Domain

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Abstract

In this article, we propose by using the Hausdorff distance Simpson's rule for the triple integral of a fuzzy-valued function and the error bound of this method, one of the variables of which is fuzzy. In addition, thin δ-fine partitions are introduced. The integration domain is a quasi-fuzzy parallelepiped. A numerical example is presented in order to show the application and the significance of the method.

Keywords

Fuzzy-Valued Function, Hausdorff Distance, Quasi-Fuzzy Parallelepiped, Henstock Integral

1. Introduction

The term fuzzy integral was introduced by Sugeno [1]. In order to evaluate a fuzzy set, some methods digital ones have recently been proposed. Wu [2] [3], Allahviranloo [4] and Fariborzi [5] [6] have developed some numerical methods to evaluate fuzzy integrals using quadratic methods and the definition of level sets. Wu and Gong [7] proposed the Henstock integral of a fuzzy-valued function and developed this work by applying the concept of fuzzy function differentiability. Bede and Gal [8] applied the quadrature rule to evaluate the integral of a fuzzy-valued function.

In our previous paper we developed the Henstock-Kurzweil triple integral of a fuzzy function on a classic parallelepiped [9].

In the present article, we develop this idea for a fuzzy-valued function (function with three variables of which one is fuzzy) by applying Simpson's rule which is powerful tool for numerical integration, especially when dealing with curves and polynomial functions and introducing Henstock-Kurzweil's triple integral. Certain domains of space can present random deformations. Thus, determining the centers of gravity of these domains will require taking into account the different variations. These variations can be done according to the three components of space. In this paper, we model these transformations by fuzzy variables and a particular case is treated. It involves taking into account a fuzzy variable among the three variables and treating the other two classically.

In section two, we present some preliminary notions on fuzzy sets as well as some fundamental theorems that we will use later.

In section three, we introduce Simpson's rule to compute the fuzzy triple integral of Henstock on a quasi-fuzzy parallelepiped. The calculations will be based on Simpson's rule for the fuzzy triple integral over an almost fuzzy three-dimensional domain. This rule generalizes that used for the calculation of the double fuzzy integral on a quasi-fuzzy two-dimensional domain, Didier and Zerbo [10]. The method involves the approximation of a fuzzy integral on a quasi-fuzzy parallelepiped. To calculate this integral, we use a fuzzy approximation formula based on fuzzy quadratic interpolation polynomials in 3 dimensions, one of which is fuzzy and the other two crisp.

Finally, in order to explain an application of the proposed method, in Section 4, a fuzzy triple integral of a fuzzy function which depends on three variables one of which is fuzzy is evaluated in order to show the effectiveness of the mentioned method.

2. Preliminaries

In this section, we talk about some basic definitions of fuzzy sets theory which are being used in the following.

Definition 2.1. [11] Let X be a non-empty reference set. A nonempty subset $\left\{\left(x,\tilde{A}(x)\right): x\in X\right\}$ of $X\times[0,1]$ such that $\tilde{A}: X\to[0,1]$ is a fuzzy subset (fuzzy set) of X. The function $\tilde{A}$ is itself called fuzzy set. $\tilde{A}(x)$ denotes the degree of membership of the element x in the fuzzy set $\tilde{A}$.

We denote by X_F the collection of all fuzzy-subsets of X.

Definition 2.2. [11] [12] Let $\mathbb{R}$ be a real set. Given a fuzzy-subset $\tilde{u}:\mathbb{R}\to[0,1]$ satisfying the properties below:

1) $\tilde{u}$ is normal, i.e. $\exists x_0\in\mathbb{R}$ such that $\tilde{u}(x_0)=1$,

2) $\tilde{u}$ is a convex fuzzy set, i.e.,

$$\tilde{u}\left(\lambda x+(1-\lambda)y\right)\geq \min\left\{\tilde{u}(x),\tilde{u}(y)\right\} \quad \forall x,y\in\mathbb{R}, \lambda\in[0,1],$$

3) $\tilde{u}$ is upper semi-continuous on $\mathbb{R}$, i.e. $\forall\varepsilon>0$, $\exists\delta>0$ such that $|x-x_0|<\delta\Rightarrow\tilde{u}(x)-\tilde{u}(x_0)<\varepsilon$.

4) The set $\overline{\left\{x\in\mathbb{R}:\tilde{u}(x)>0\right\}}$ is compact, where $\overline{B}$ denotes the closure of B.

This function $\tilde{u}$ is called a fuzzy number.

We denote by $\mathbb{R}_F$ the set of all fuzzy real numbers. We define $[\tilde{u}]^\alpha=\left\{x\in\mathbb{R}:\tilde{u}(x)\geq\alpha\right\}$ and $[\tilde{u}]^0=\left\{x\in\mathbb{R}:\tilde{u}(x)>0\right\}$, for $0<\alpha\leq 1$, as the α-cut and respectively the support of a fuzzy number such as $\tilde{u}$. Moreover, we define $\tilde{u}_\alpha^- = \inf[\tilde{u}]^\alpha$ and $\tilde{u}_\alpha^+ = \sup[\tilde{u}]^\alpha$.

A triangular fuzzy number $\tilde{u} = (a,b,c)$ where, $a < b < c$ and $a,b,c \in \mathbb{R}$ is defined by $\tilde{u}_\alpha^- = a + (b-a)\alpha$ and $\tilde{u}_\alpha^+ = c - (c-b)\alpha$.

Definition 2.3. [11] [12] A set of level α of a fuzzy set satisfies the following properties $\forall \tilde{u}, \tilde{v} \in \mathbb{R}_F^n$ and $\alpha, \beta \in I = (0,1]$,

1) $\tilde{u}_0 = \mathbb{R}^n$,
2) $\alpha \le \beta \Rightarrow \tilde{u}_\beta \subset \tilde{u}_\alpha$,
3) $\tilde{u}_\alpha = \bigcap_{\beta < \alpha} \tilde{u}_\beta$,
4) $(\tilde{u} \vee \tilde{v})_{\alpha = \tilde{u}_\alpha \cup \tilde{v}_\alpha}$ and $(\tilde{u} \wedge \tilde{v})_{\alpha = \tilde{u}_\alpha \cap \tilde{v}_\alpha}$.

The theorem below shows that any fuzzy set can be represented by a family of its α cuts $\{\tilde{u}_\alpha : \alpha \in I\}$, and it can be represented by its countable sets of level α denoted by $\{\tilde{u}_\alpha : \alpha \in I \cap \mathbb{Q}\}$.

Theorem 2.4. [12] Let $\tilde{u} \in \mathbb{R}_F^n$ and let $\{\tilde{u}_\alpha : \alpha \in I\}$ be a family of its sets of level α. For all $x \in \mathbb{R}^n$, we have:

$$\tilde{u}(x) = \sup_{\alpha \in I}\left[\alpha \cdot I_{\tilde{u}_\alpha}(x)\right] = \sup_{\alpha \in I \cap \mathbb{Q}}\left[\alpha \cdot I_{\tilde{u}_\alpha}(x)\right]$$

which can also be written

$$\tilde{u}(x) = \sup\{\alpha \in I : x \in \tilde{u}_\alpha\} = \sup\{\alpha \in I \cap \mathbb{Q} : x \in \tilde{u}_\alpha\}$$

Let $\{M_\alpha : \alpha \in I \cap \mathbb{Q}\}$ be a family of nonempty closed sets of $\mathbb{R}^n$ such that $M_\alpha \supset M_\beta$ $\forall \alpha, \beta \in I \cap \mathbb{Q}$ with $\alpha < \beta$.

Then the function $\tilde{u}$ defined by

$$\tilde{u}(x) = \sup\{\alpha \in I \cap \mathbb{Q} : x \in M_\alpha\},$$

is upper semi-continuous. Additionally, it checks

$$\tilde{u}_\alpha = \bigcap_{\beta < \alpha, \beta \in \mathbb{Q} \cap (0,1]} M_\beta, \alpha \in (0,1].$$

Remark 2.5. A fuzzy set $\tilde{u}$ is said to be convex if for all $\alpha \in I$, the level set α denoted by $\tilde{u}_\alpha$ is a sub convex set of $\mathbb{R}^n$.

We denote by $\mathbb{R}_{F_c}^n$ the family of all convex fuzzy sets.

Definition 2.6. [11] [12] Let $\tilde{u}, \tilde{v} \in \mathbb{R}_F^n$ we define for all $x \in \mathbb{R}^n$

$$(\tilde{u} \oplus \tilde{v})(x) = \sup\{\alpha \in I : x \in \tilde{u}_\alpha \oplus_{int} \tilde{v}_\alpha\}$$

and

$$(\lambda \odot \tilde{u})(x) = \begin{cases} \tilde{u}\left(\dfrac{x}{\lambda}\right) & \text{si } \lambda \ne 0 \\ I_0(x) & \text{si } \lambda = 0 \end{cases}$$

where I_0 is the characteristic function of the singleton {0}.

For $\tilde{u}, \tilde{v} \in \mathbb{R}_F^n$ and $\lambda \in \mathbb{R}$, we can define the sum $\tilde{u} \oplus \tilde{v}$ and the product $\lambda \odot \tilde{u}$ by

$$[\tilde{u} \oplus \tilde{v}]^\alpha = [\tilde{u}]^\alpha \oplus_{int} [\tilde{v}]^\alpha \text{ and } [\lambda \odot \tilde{u}]^\alpha = \lambda \odot_{int} [\tilde{u}]^\alpha \quad \forall \alpha \in [0,1].$$

Given $\tilde{u}, \tilde{v}$ two convex fuzzy sets, another definition of addition is given by

$$\left(\tilde{u}\oplus\tilde{v}\right)(x)=\sup_{x=x_1+x_2}\min\left\{\tilde{u}(x_1),\tilde{v}(x_2)\right\},\ x\in\mathbb{R}^n.$$

Definition 2.7 [11] The three metrics below generalize the Hausdorff metric: for all $\tilde{u},\tilde{v}\in\mathbb{R}^n_{F_c}$

$$\mathbf{D}_\infty\left(\tilde{u},\tilde{v}\right)=\sup_{\alpha\in(0,1]}D\left(\tilde{u}_\alpha,\tilde{v}_\alpha\right);$$

$$\mathbf{D}_1\left(\tilde{u},\tilde{v}\right)=\int_0^1 D\left(\tilde{u}_\alpha,\tilde{v}_\alpha\right)\mathrm{d}\alpha;$$

and

$$\mathbf{D}_p\left(\tilde{u},\tilde{v}\right)=\left\{\int_0^1\left[D\left(\tilde{u}_\alpha,\tilde{v}_\alpha\right)\right]^p\mathrm{d}\alpha\right\}^{\frac{1}{p}};\text{ for }\ p>1.$$

We note by

$$\|\tilde{u}\|_F=\mathbf{D}_\infty\left(\tilde{u},I_0\right)=\sup_{\alpha>0}\|\tilde{u}_\alpha\|_K$$

3. Triple Simpson's Rule for the Fuzzy Henstock-Kurzweil Triple Integrals

The concept of the Henstock integral for a fuzzy number-valued function were introduced by Wu and Gong [7]. We introduce this definition for a three-dimensional fuzzy number-valued function in which one of the three variables is fuzzy.

Let $\tilde{f}:[a,b]\times[c,d]\otimes[\tilde{p},\tilde{q}]\to\mathbb{R}_F$ and $\Delta_m: a=x_0<x_1<x_2<\cdots<x_m=b$, $\Delta_n: c=y_0<y_1<y_2<\cdots<y_n=d$ and $\Delta_s:\tilde{p}=\tilde{z}_0\prec\tilde{z}_1\prec\tilde{z}_2\prec\cdots\prec\tilde{z}_s=\tilde{q}$ be the partitions of the intervals $[a,b],[c,d]$ and $[\tilde{p},\tilde{q}]$ respectively.

Consider the points $\xi_i\in[x_{i-1},x_i]$, $i=1,2,\cdots,m$; $\eta_j\in\left[y_{j-1},y_j\right]$, $j=1,2,\cdots,n$; $\tilde{\zeta}_k\in\left[\tilde{z}_{k-1},\tilde{z}_k\right]$, $k=1,2,\cdots,s$.

Let π be the partition defined by

$$\pi=\left\{\left(\left(\xi_i,\eta_j,\tilde{\zeta}_k\right),[x_{i-1},x_i]\times\left[y_{j-1},y_j\right]\otimes\left[\tilde{z}_{k-1},\tilde{z}_k\right]\right)\right\}$$

In what follows we will define the fuzzy Henstock triple integral.

Let the fuzzy gauge function defined on the type 1 quasi-fuzzy parallelepiped al domain $[a,b]\times[c,d]\otimes[\tilde{p},\tilde{q}]$ by

$$\tilde{\delta}:[a,b]\times[c,d]\otimes[\tilde{p},\tilde{q}]\to\mathbb{R}_F.$$

The partition π is said to be $\tilde{\delta}$-fine if

$$\left(\xi_i,\eta_j,\tilde{\zeta}_k\right)\in[x_{i-1},x_i]\times\left[y_{j-1},y_j\right]\otimes\left[\tilde{z}_{k-1},\tilde{z}_k\right]\subseteq\Delta\left(\left(\xi_i,\eta_j,\tilde{\zeta}_k\right),\tilde{\delta}\left(\xi_i,\eta_j,\tilde{\zeta}_k\right)\right),$$

where $\Delta\left(\left(\xi_i,\eta_j,\tilde{\zeta}_k\right),\tilde{\delta}\left(\xi_i,\eta_j,\tilde{\zeta}_k\right)\right)$ is the open quasi-fuzzy ball with center $\left(\xi_i,\eta_j,\tilde{\zeta}_k\right)$ and positive radius $\tilde{\delta}\left(\xi_i,\eta_j,\tilde{\zeta}_k\right)$.

The function $\tilde{f}:[a,b]\times[c,d]\otimes[\tilde{p},\tilde{q}]\to\mathbb{R}_F$ is said to be Henstock integrable if $\exists\tilde{I}\in\mathbb{R}_F$ such that $\forall\varepsilon>0$, there exists $\tilde{\delta}$ (called fuzzy gauge function) such as for any subdivision $\tilde{\zeta}_k$-fine (cfr the partition π defined above), we have:

$$D\left(\bigoplus_{i=1}^{m}\bigoplus_{j=1}^{n}\bigoplus_{k=1}^{s}(x_i-x_{i-1})(y_j-y_{j-1})\odot(\tilde{z}_k\ominus\tilde{z}_{k-1})\otimes\tilde{f}(\xi_i,\eta_j,\tilde{\zeta}_k),\tilde{I}\right)<\varepsilon,$$

where $\tilde{I}=(FHTI)\int_a^b\int_c^d\int_{\tilde{p}}^{\tilde{q}}\tilde{f}(x,y,\tilde{z})\mathrm{d}\tilde{z}\mathrm{d}y\mathrm{d}x$ is the Fuzzy Henstock Triple Integral.

Definition 3.1. Let $\tilde{f}:[a,b]\times[c,d]\otimes[\tilde{p},\tilde{q}]\to\mathbb{R}_F$ a bounded fuzzy function. So the function $\omega_{[a,b]\times[c,d]\otimes[\tilde{p},\tilde{q}]}(\tilde{f},.,.,.):\mathbb{R}^+\cup\{\infty\}\to\mathbb{R}^+$ such that

$$\begin{aligned}&\omega_{[a,b]\times[c,d]\otimes[\tilde{p},\tilde{q}]}(\tilde{f},\delta_1,\delta_2,\delta_3)\\&=\sup\Big\{D\big(\tilde{f}(x_1,y_1,\tilde{z}_1),\tilde{f}(x_2,y_2,\tilde{z}_2)\big);(x_1,y_1,\tilde{z}_1),(x_2,y_2,\tilde{z}_2)\in[a,b]\\&\qquad\times[c,d]\otimes[\tilde{p},\tilde{q}]:|x_1-x_2|\le\delta_1,|y_1-y_2|\le\delta_2,\|\tilde{z}_1\ominus\tilde{z}_2\|_{\mathbb{R}_F}\le\delta_3\Big\}.\end{aligned}$$

where

$$\begin{aligned}\|\tilde{z}_1\ominus\tilde{z}_2\|_{\mathbb{R}_F}&=\left\|\left[\tilde{z}^L_{1_\alpha},\tilde{z}^U_{1_\alpha}\right]\ominus_{int}\left[\tilde{z}^L_{2_\alpha},\tilde{z}^U_{2_\alpha}\right]\right\|\\&=\inf\left\{|t-u|,t\in\left[\tilde{z}^L_{1_\alpha},\tilde{z}^U_{1_\alpha}\right],u\in\left[\tilde{z}^L_{2_\alpha},\tilde{z}^U_{2_\alpha}\right];\forall\alpha\in[0,1]\right\}.\end{aligned}$$

For this reason,

$$\begin{aligned}&\omega_{[a,b]\times[c,d]\otimes[\tilde{p},\tilde{q}]}(\tilde{f},\delta_1,\delta_2,\delta_3)\\&=\sup\Big\{D\Big(\tilde{f}\big(x_1,y_1,\tilde{z}^L_{1_\alpha}\big),\tilde{f}\big(x_1,y_1,\tilde{z}^U_{1_\alpha}\big),\tilde{f}\big(x_2,y_2,\tilde{z}^L_{2_\alpha}\big),\tilde{f}\big(x_2,y_2,\tilde{z}^U_{2_\alpha}\big)\Big);\\&\quad|x_1-x_2|\le\delta_1,|y_1-y_2|\le\delta_2,\inf\{|t-u|,t\in\left[\tilde{z}^L_{1_\alpha},\tilde{z}^U_{1_\alpha}\right],u\in\left[\tilde{z}^L_{2_\alpha},\tilde{z}^U_{2_\alpha}\right];\forall\alpha\in[0,1]\Big\}.\\&\qquad\forall(x_1,y_1,\tilde{z}_1),(x_2,y_2,\tilde{z}_2)\in[a,b]\times[c,d]\otimes[\tilde{p},\tilde{q}].\end{aligned}$$

ω is the modulus of oscillation of the function $\tilde{f}$ on $[a,b]\times[c,d]\otimes[\tilde{p},\tilde{q}]$. If furthermore, $\tilde{f}$ is continuous on $[a,b]\times[c,d]\otimes[\tilde{p},\tilde{q}]$, then ω is called the uniform continuity modulus of $\tilde{f}$.

We can prove the following theorem from the definition 3.1.

Theorem 3.2. The following statements, concerning the modulus of oscillation are true.

1)
$$D\big(\tilde{f}(x_1,y_1,\tilde{z}_1),\tilde{f}(x_2,y_2,\tilde{z}_2)\big)\le\omega_{([a,b]\times[c,d]\otimes[\tilde{p},\tilde{q}])}\big(\tilde{f},|x_1-x_2|,|y_1-y_2|,\|\tilde{z}_1\ominus\tilde{z}_2\|_{\mathbb{R}_F}\big)$$
$$\forall(x_1,y_1,\tilde{z}_1),(x_2,y_2,\tilde{z}_2)\in[a,b]\times[c,d]\otimes[\tilde{p},\tilde{q}],$$

2) $\omega_{([a,b]\times[c,d]\otimes[\tilde{p},\tilde{q}])}(\tilde{f},\delta_1,\delta_2,\delta_3)$ is a non-decreasing mapping in δ_1,δ_2 and δ_3,

3) $\omega_{([a,b]\times[c,d]\otimes[\tilde{p},\tilde{q}])}(\tilde{f},0,0,0)=0$,

4) $\omega_{([a,b]\times[c,d]\otimes[\tilde{p},\tilde{q}])}(\tilde{f},m\delta_1,n\delta_2,s\delta_3)\le mns\,\omega_{([a,b]\times[c,d]\otimes[\tilde{p},\tilde{q}])}(\tilde{f},\delta_1,\delta_2,\delta_3)$ $\forall\delta_1,\delta_2,\delta_3\ge0$ and $m,n,s\in\mathbb{N}$,

5) $$\begin{aligned}&\omega_{([a,b]\times[c,d]\otimes[\tilde{p},\tilde{q}])}(\tilde{f},\lambda_1\delta_1,\lambda_2\delta_2,\lambda_3\delta_3)\\&\le(\lambda_1+1)(\lambda_2+1)(\lambda_3+1)\omega_{([a,b]\times[c,d]\otimes[\tilde{p},\tilde{q}])}(\tilde{f},\delta_1,\delta_2,\delta_3)\end{aligned}$$ for any $\delta_1,\delta_2,\delta_3,\lambda_1,\lambda_2,\lambda_3\ge0$.

6) If $[e,f]\subseteq[a,b];[g,h]\subseteq[c,d]$ and $[\tilde{i},\tilde{j}]\preccurlyeq[\tilde{p},\tilde{q}]$, then

$$\omega_{([e,f]\times[g,h]\otimes[\tilde{i},\tilde{j}])}\left(\tilde{f},\delta_1,\delta_2,\delta_3\right)\le\omega_{([a,b]\times[c,d]\otimes[\tilde{p},\tilde{q}])}\left(\tilde{f},\delta_1,\delta_2,\delta_3\right).$$

Proof (6) According to the hypothesis,

$$\begin{aligned}&\sup\Big\{D\left(\tilde{f}(x_1,y_1,\tilde{z}_1),\tilde{f}(x_2,y_2,\tilde{z}_2)\right);(x_1,y_1,\tilde{z}_1),(x_2,y_2,\tilde{z}_2)\in[e,f]\\&\quad\times[g,h]\otimes[\tilde{i},\tilde{j}],|x_1-x_2|\le\delta_1,|y_1-y_2|\le\delta_2,\|\tilde{z}_1\ominus\tilde{z}_2\|_{\mathbb{R}_F}\le\delta_3\Big\}\\&\le\sup\Big\{D\left(\tilde{f}(x_1,y_1,\tilde{z}_1),\tilde{f}(x_2,y_2,\tilde{z}_2)\right);(x_1,y_1,\tilde{z}_1),(x_2,y_2,\tilde{z}_2)\in[a,b]\\&\quad\times[c,d]\otimes[\tilde{p},\tilde{q}],|x_1-x_2|\le\delta_1,|y_1-y_2|\le\delta_2,\|\tilde{z}_1\ominus\tilde{z}_2\|_{\mathbb{R}_F}\le\delta_3\Big\}\end{aligned}$$

which is prove the relation.

We can prove similarly the other statements.

□

Lemma 3.3.

1) If $\tilde{f}$ and $\tilde{h}$ are Henstock triple integrable mappings and if $D\left(\tilde{f}(x,y,\tilde{z}),\tilde{h}(x,y,\tilde{z})\right)$ is Lebesgue integrable, then

$$\begin{aligned}&D\left((FHTI)\int_a^b\int_c^d\int_{\tilde{p}}^{\tilde{q}}\tilde{f}(x,y,\tilde{z})\,\mathrm{d}\tilde{z}\mathrm{d}y\mathrm{d}x,(FHTI)\int_a^b\int_c^d\int_{\tilde{p}}^{\tilde{q}}\tilde{h}(x,y,\tilde{z})\,\mathrm{d}\tilde{z}\mathrm{d}y\mathrm{d}x\right)\\&\le(L)\int_a^b\int_c^d\int_{\tilde{p}}^{\tilde{q}}D\left(\tilde{f}(x,y,\tilde{z}),\tilde{h}(x,y,\tilde{z})\right)\mathrm{d}\tilde{z}\mathrm{d}y\mathrm{d}x.\end{aligned}$$

2) Let $\tilde{f}:[a,b]\times[c,d]\otimes[\tilde{p},\tilde{q}]\to\mathbb{R}_F$ be a Henstock triple integrabe bounded mapping.

Then, $\forall(u,v,\tilde{w})\in[a,b]\times[c,d]\otimes[\tilde{p},\tilde{q}]$, the function $\varphi_{(u,v,\tilde{w})}:[a,b]\times[c,d]\otimes[\tilde{p},\tilde{q}]\to\mathbb{R}^+$ defined by $\varphi_{(u,v,\tilde{w})}(x,y,\tilde{z})=D\left(\tilde{f}(u,v,\tilde{w}),\tilde{f}(x,y,\tilde{z})\right)$ is Lebesgue integrable on $[a,b]\times[c,d]\otimes[\tilde{p},\tilde{q}]$.

Proof (2) If $\tilde{f}$ is Henstock integrable and bounded on $[a,b]\times[c,d]\otimes[\tilde{p},\tilde{q}]$, then it follows that $\tilde{f}_-^\alpha(x,y,\tilde{z})$ and $\tilde{f}_+^\alpha(x,y,\tilde{z})$ are Henstock triple integrable with $\alpha\in[0,1]$. Therefore, $\tilde{f}_-^\alpha(x,y,\tilde{z})$ and $\tilde{f}_+^\alpha(x,y,\tilde{z})$ are Lebesgue measurable and uniformly bounded $\forall\alpha\in[0,1]$, [7]. Moreover,

$$\begin{aligned}&\varphi(x,y,\tilde{z})=D\left(\tilde{f}(x_1,y_1,\tilde{z}_1),\tilde{f}(x_2,y_2,\tilde{z}_2)\right)\\&=\sup_{\alpha\in[0,1]}\max\left\{\left|\tilde{f}_-^\alpha(x_1,y_1,\tilde{z}_1)-\tilde{f}_-^\alpha(x_2,y_2,\tilde{z}_2)\right|,\left|\tilde{f}_+^\alpha(x_1,y_1,\tilde{z}_1)-\tilde{f}_+^\alpha(x_2,y_2,\tilde{z}_2)\right|\right\}\\&=\sup_{\alpha_n\in[0,1]}\max\left\{\left|\tilde{f}_-^{\alpha_n}(x_1,y_1,\tilde{z}_1)-\tilde{f}_-^{\alpha_n}(x_2,y_2,\tilde{z}_2)\right|,\left|\tilde{f}_+^{\alpha_n}(x_1,y_1,\tilde{z}_1)-\tilde{f}_+^{r_n}(x_2,y_2,\tilde{z}_2)\right|\right\},\end{aligned}$$

where the $\alpha_n(n\in\mathbb{N})$ are the rational numbers in $[0,1]$. According to Lebesgue's dominated convergence theorem, it follows that $\varphi(x,y,\tilde{z})$ is Lebesgue integrable over $[a,b]\times[c,d]\otimes[\tilde{p},\tilde{q}]$ and what completes the proof.

□

Keeping now three integrals we reach the following definitions.

Definition 3.4. A function $\tilde{f}:[a,b]\times[c,d]\otimes[\tilde{p},\tilde{q}]\to\mathbb{R}_F$ is said to be (L_1,L_2,L_3) Lipschitz if for any $(x_1,y_1,\tilde{z}_1),(x_2,y_2,\tilde{z}_2)\in[a,b]\times[c,d]\otimes[\tilde{p},\tilde{q}]$,

$$D\left(\tilde{f}(x_1,y_1,\tilde{z}_1),\tilde{f}(x_2,y_2,\tilde{z}_2)\right)\le L_1|x_1-x_2|+L_2|y_1-y_2|+L_3\|\tilde{z}_1\ominus\tilde{z}_2\|_{\mathbb{R}_F}.$$

In order to introduce triple Simpson's rule for evaluating *FHTI*, firstly we prove the following theorem.

Theorem 3.5. Let $\tilde{f}:[a,b]\times[c,d]\otimes[\tilde{p},\tilde{q}]\to\mathbb{R}_F$ be a fuzzy Henstock integrable, bounded mapping. Then, for any subdivision $a=x_0<x_1<x_2<\cdots<x_m=b$, $c=y_0<y_1<y_2<\cdots<y_n=d$, $\tilde{p}=\tilde{z}_0\prec\tilde{z}_1\prec\tilde{z}_2\prec\cdots\prec\tilde{z}_s=\tilde{q}$ and any points $\xi_i\in[x_{i-1},x_i]$, $\eta_j\in[y_{j-1},y_j]$, $\tilde{\zeta}_k\in[\tilde{z}_{k-1},\tilde{z}_k]$ we have

$$D\Bigg((FHTI)\int_a^b\int_c^d\int_{\tilde{p}}^{\tilde{q}}\tilde{f}(x,y,\tilde{z})\mathrm{d}\tilde{z}\mathrm{d}y\mathrm{d}x,$$

$$\bigoplus_{i=1}^m\bigoplus_{j=1}^n\bigoplus_{k=1}^s(x_i-x_{i-1})(y_j-y_{j-1})\|\tilde{z}_k\ominus\tilde{z}_{k-1}\|_{\mathbb{R}_F}\odot\tilde{f}(\xi_i,\eta_j,\tilde{\zeta}_k)\Bigg)$$

$$\le\sum_{i=1}^m\sum_{j=1}^n\sum_{k=1}^s(x_i-x_{i-1})(y_j-y_{j-1})\|\tilde{z}_k\ominus\tilde{z}_{k-1}\|_{\mathbb{R}_F}$$

$$\omega_{([x_{i-1},x_i]\times[y_{j-1},y_j]\otimes[\tilde{z}_{k-1},\tilde{z}_k])}\left(\tilde{f},(x_i-x_{i-1}),(y_j-y_{j-1}),\|\tilde{z}_k\ominus\tilde{z}_{k-1}\|_{\mathbb{R}_F}\right).$$

Proof Since that the Henstock integral is additive related to interval [13], hence,

$$D\Bigg((FHTI)\int_a^b\int_c^d\int_{\tilde{p}}^{\tilde{q}}\tilde{f}(x,y,\tilde{z})\mathrm{d}\tilde{z}\mathrm{d}y\mathrm{d}x,$$

$$\bigoplus_{i=1}^m\bigoplus_{j=1}^n\bigoplus_{k=1}^s(x_i-x_{i-1})(y_j-y_{j-1})\|\tilde{z}_k\ominus\tilde{z}_{k-1}\|_{\mathbb{R}_F}\odot\tilde{f}(\xi_i,\eta_j,\tilde{\zeta}_k)\Bigg)$$

$$=D\Bigg(\bigoplus_{i=1}^m\bigoplus_{j=1}^n\bigoplus_{k=1}^s(FHTI)\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j}\int_{\tilde{z}_{k-1}}^{\tilde{z}_k}\tilde{f}(x,y,\tilde{z})\mathrm{d}\tilde{z}\mathrm{d}y\mathrm{d}x,$$

$$\bigoplus_{i=1}^m\bigoplus_{j=1}^n\bigoplus_{k=1}^s(x_i-x_{i-1})(y_j-y_{j-1})\|\tilde{z}_k\ominus\tilde{z}_{k-1}\|_{\mathbb{R}_F}\odot\tilde{f}(\xi_i,\eta_j,\tilde{\zeta}_k)\Bigg).$$

Since it's clear that $(FHTI)\int_a^b\int_c^d\int_{\tilde{p}}^{\tilde{q}}\tilde{k}\mathrm{d}\tilde{z}\mathrm{d}y\mathrm{d}x=(b-a)(d-c)\odot(\tilde{q}\ominus\tilde{p})\otimes\tilde{k}$ for any fuzzy constant $\tilde{k}\in\mathbb{R}_F$, we obtain

$$D\Bigg((FHTI)\int_a^b\int_c^d\int_{\tilde{p}}^{\tilde{q}}\tilde{f}(x,y,\tilde{z})\mathrm{d}\tilde{z}\mathrm{d}y\mathrm{d}x,$$

$$\bigoplus_{i=1}^m\bigoplus_{j=1}^n\bigoplus_{k=1}^s(x_i-x_{i-1})(y_j-y_{j-1})\|\tilde{z}_k\ominus\tilde{z}_{k-1}\|_{\mathbb{R}_F}\odot\tilde{f}(\xi_i,\eta_j,\tilde{\zeta}_k)\Bigg)$$

$$=D\Bigg(\bigoplus_{i=1}^m\bigoplus_{j=1}^n\bigoplus_{k=1}^s(FHTI)\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j}\int_{\tilde{z}_{k-1}}^{\tilde{z}_k}\tilde{f}(x,y,\tilde{z})\mathrm{d}\tilde{z}\mathrm{d}y\mathrm{d}x,$$

$$\bigoplus_{i=1}^m\bigoplus_{j=1}^n\bigoplus_{k=1}^s(FHTI)\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j}\int_{\tilde{z}_{k-1}}^{\tilde{z}_k}\tilde{f}(\xi_i,\eta_j,\tilde{\zeta}_k)\mathrm{d}\tilde{z}\mathrm{d}y\mathrm{d}x\Bigg).$$

By the fourth property of the theorem 3.2, we have

$$D\Bigg(\bigoplus_{i=1}^m\bigoplus_{j=1}^n\bigoplus_{k=1}^s(FHTI)\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j}\int_{\tilde{z}_{k-1}}^{\tilde{z}_k}\tilde{f}(x,y,\tilde{z})\mathrm{d}\tilde{z}\mathrm{d}y\mathrm{d}x,$$

$$\bigoplus_{i=1}^m\bigoplus_{j=1}^n\bigoplus_{k=1}^s(FHTI)\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j}\int_{\tilde{z}_{k-1}}^{\tilde{z}_k}\tilde{f}(\xi_i,\eta_j,\tilde{\zeta}_k)\mathrm{d}\tilde{z}\mathrm{d}y\mathrm{d}x\Bigg)$$

$$\le\bigoplus_{i=1}^m\bigoplus_{j=1}^n\bigoplus_{k=1}^s D\Bigg((FHTI)\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j}\int_{\tilde{z}_{k-1}}^{\tilde{z}_k}\tilde{f}(\xi_i,\eta_j,\tilde{\zeta}_k)\mathrm{d}\tilde{z}\mathrm{d}y\mathrm{d}x,$$

$$(FHTI)\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j}\int_{\tilde{z}_{k-1}}^{\tilde{z}_k}\tilde{f}(\xi_i,\eta_j,\tilde{\zeta}_k)\mathrm{d}\tilde{z}\mathrm{d}y\mathrm{d}x\Bigg)$$

Since the functions $D\left(\tilde{f}(x,y,\tilde{z}),\tilde{f}(\xi_i,\eta_j,\tilde{\zeta}_k)\right)$ are Lebesgue integrable for $i=1,\cdots,m;\ j=1,\cdots,n$ and $k=1,\cdots,s$ from lemma 3.3 we have

$$D\Bigg((FHTI)\int_a^b\int_c^d\int_{\tilde{p}}^{\tilde{q}}\tilde{f}(x,y,\tilde{z})\mathrm{d}\tilde{z}\mathrm{d}y\mathrm{d}x,$$

$$\bigoplus_{i=1}^{m}\bigoplus_{j=1}^{n}\bigoplus_{k=1}^{s}(x_i-x_{i-1})(y_j-y_{j-1})\|\tilde{z}_k\ominus\tilde{z}_{k-1}\|_{\mathbb{R}_F}\odot\tilde{f}(\xi_i,\eta_j,\tilde{\zeta}_k)\Bigg)$$

$$\leq\bigoplus_{i=1}^{m}\bigoplus_{j=1}^{n}\bigoplus_{k=1}^{s}(L)\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j}\int_{\tilde{z}_{k-1}}^{\tilde{z}_k}D\left(\tilde{f}(x,y,\tilde{z}),\tilde{f}(\xi_i,\eta_j,\tilde{\zeta}_k)\right)\mathrm{d}\tilde{z}\mathrm{d}y\mathrm{d}x.$$

From the first property of the theorem 3.2 applied to each of the above integrals we have

$$D\Bigg((FHTI)\int_a^b\int_c^d\int_{\tilde{p}}^{\tilde{q}}\tilde{f}(x,y,\tilde{z})\mathrm{d}\tilde{z}\mathrm{d}y\mathrm{d}x,$$

$$\bigoplus_{i=1}^{m}\bigoplus_{j=1}^{n}\bigoplus_{k=1}^{s}(x_i-x_{i-1})(y_j-y_{j-1})\|\tilde{z}_k\ominus\tilde{z}_{k-1}\|_{\mathbb{R}_F}\odot\tilde{f}(\xi_i,\eta_j,\tilde{\zeta}_k)\Bigg)$$

$$\leq\bigoplus_{i=1}^{m}\bigoplus_{j=1}^{n}\bigoplus_{k=1}^{s}(L)\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j}\int_{\tilde{z}_{k-1}}^{\tilde{z}_k}\omega_{([x_{i-1},x_i]\times[y_{j-1},y_j]\otimes[\tilde{z}_{k-1},\tilde{z}_k])}$$

$$\left(\tilde{f},(x_i-x_{i-1}),(y_j-y_{j-1}),\|\tilde{z}_k\ominus\tilde{z}_{k-1}\|_{\mathbb{R}_F}\right)\mathrm{d}\tilde{z}\mathrm{d}y\mathrm{d}x$$

$$=\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s}(x_i-x_{i-1})(y_j-y_{j-1})\|\tilde{z}_k\ominus\tilde{z}_{k-1}\|_{\mathbb{R}_F}$$

$$\omega_{([x_{i-1},x_i]\times[y_{j-1},y_j]\otimes[\tilde{z}_{k-1},\tilde{z}_k])}\left(\tilde{f},(x_i-x_{i-1}),(y_j-y_{j-1}),\|\tilde{z}_k\ominus\tilde{z}_{k-1}\|_{\mathbb{R}_F}\right),$$

which completes the proof.

□

Corollary 3.6. Let $\tilde{f}:[a,b]\times[c,d]\otimes[\tilde{p},\tilde{q}]\to\mathbb{R}_F$ be a Henstock triple integrable, bounded mapping. Then, for $m=n=s=2$ we have

$$D\Bigg((FHTI)\int_a^b\int_c^d\int_{\tilde{p}}^{\tilde{q}}\tilde{f}(x,y,\tilde{z})\mathrm{d}\tilde{z}\mathrm{d}y\mathrm{d}x,$$

$$\bigoplus_{i=1}^{2}\bigoplus_{j=1}^{2}\bigoplus_{k=1}^{2}(x_i-x_{i-1})(y_j-y_{j-1})\|\tilde{z}_k\ominus\tilde{z}_{k-1}\|_{\mathbb{R}_F}\odot\tilde{f}(\xi_i,\eta_j,\tilde{\zeta}_k)\Bigg)$$

$$\leq(\alpha-a)(\beta-c)\|\tilde{\gamma}\ominus\tilde{p}\|_{\mathbb{R}_F}\,\omega_{[a,\alpha]\times[c,\beta]\otimes[\tilde{p},\tilde{\gamma}]}\left(\tilde{f},(\alpha-a),(\beta-c),\|\tilde{\gamma}\ominus\tilde{p}\|_{\mathbb{R}_F}\right)$$

$$+(\alpha-a)(\beta-c)\|\tilde{q}\ominus\tilde{\gamma}\|_{\mathbb{R}_F}\,\omega_{[a,\alpha]\times[c,\beta]\otimes[\tilde{\gamma},\tilde{q}]}\left(\tilde{f},(\alpha-a),(\beta-c),\|\tilde{q}\ominus\tilde{\gamma}\|_{\mathbb{R}_F}\right)$$

$$+(\alpha-a)(d-\beta)\|\tilde{\gamma}\ominus\tilde{p}\|_{\mathbb{R}_F}\,\omega_{[a,\alpha]\times[\beta,d]\otimes[\tilde{p},\tilde{\gamma}]}\left(\tilde{f},(\alpha-a),(d-\beta),\|\tilde{\gamma}\ominus\tilde{p}\|_{\mathbb{R}_F}\right)$$

$$+(\alpha-a)(d-\beta)\|\tilde{q}\ominus\tilde{\gamma}\|_{\mathbb{R}_F}\,\omega_{[a,\alpha]\times[\beta,d]\otimes[\tilde{\gamma},\tilde{q}]}\left(\tilde{f},(\alpha-a),(d-\beta),\|\tilde{q}\ominus\tilde{\gamma}\|_{\mathbb{R}_F}\right)$$

$$+(b-\alpha)(\beta-c)\|\tilde{\gamma}\ominus\tilde{p}\|_{\mathbb{R}_F}\,\omega_{[\alpha,b]\times[c,\beta]\otimes[\tilde{p},\tilde{\gamma}]}\left(\tilde{f},(b-\alpha),(\beta-c)\|\tilde{\gamma}\ominus\tilde{p}\|_{\mathbb{R}_F}\right)$$

$$+(b-\alpha)(\beta-c)\|\tilde{q}\ominus\tilde{\gamma}\|_{\mathbb{R}_F}\,\omega_{[\alpha,b]\times[c,\beta]\otimes[\tilde{\gamma},\tilde{q}]}\left(\tilde{f},(b-\alpha),(\beta-c)\|\tilde{q}\ominus\tilde{\gamma}\|_{\mathbb{R}_F}\right)$$

$$+(b-\alpha)(d-\beta)\|\tilde{\gamma}\ominus\tilde{p}\|_{\mathbb{R}_F}\,\omega_{[\alpha,b]\times[\beta,d]\otimes[\tilde{p},\tilde{\gamma}]}\left(\tilde{f},(b-\alpha),(d-\beta),\|\tilde{\gamma}\ominus\tilde{p}\|_{\mathbb{R}_F}\right)$$

$$+(b-\alpha)(d-\beta)\|\tilde{q}\ominus\tilde{\gamma}\|_{\mathbb{R}_F}\,\omega_{[\alpha,b]\times[\beta,d]\otimes[\tilde{\gamma},\tilde{q}]}\left(\tilde{f},(b-\alpha),(d-\beta),\|\tilde{q}\ominus\tilde{\gamma}\|_{\mathbb{R}_F}\right),$$

for any $\alpha \in [a,b], \beta \in [c,d]$ and $\tilde{\gamma} \in [\tilde{p},\tilde{q}]$, $(u,v,\tilde{w}) \in [a,\alpha]\times[c,\beta]\otimes[\tilde{p},\tilde{\gamma}]$ and $(u',v',\tilde{w}') \in [\alpha,b]\times[\beta,d]\otimes[\tilde{\gamma},\tilde{q}]$ where $\xi_1 = u$, $\xi_2 = u'$; $\eta_1 = v$, $\eta_2 = v'$; $\tilde{\zeta}_1 = \tilde{w}$, $\tilde{\zeta}_2 = \tilde{w}'$.

Proof It's clear that for $m = 2, n = 2$ and $s = 2$ in the theorem 3.5 the inequality stated above is obtained. □

The corollary below gives the fuzzy variant of Simpson's threefold rule with a new error bound.

Corollary 3.7. Let $\tilde{f} : [a,b]\times[c,d]\otimes[\tilde{p},\tilde{q}] \mapsto \mathbb{R}_F$ be a Henstock triple integrable, bounded mapping. Then,

$$D\Bigg((ITFHK)\int_a^b\int_c^d\int_{\tilde{p}}^{\tilde{q}} \tilde{f}(x,y,\tilde{z})\,\mathrm{d}\tilde{z}\mathrm{d}y\mathrm{d}x, \frac{(b-a)(d-c)\|\tilde{q}\ominus\tilde{p}\|_{\mathbb{R}_F}}{216}$$
$$\odot\Bigg\{\Bigg[\Big(\tilde{f}(a,c,\tilde{p})\oplus\tilde{f}(a,c,\tilde{q})\Big)\oplus 4\odot\Big(\tilde{f}\Big(\frac{a+b}{2},c,\tilde{p}\Big)\oplus\tilde{f}\Big(\frac{a+b}{2},c,\tilde{q}\Big)\Big)\oplus\Big(\tilde{f}(b,c,\tilde{p})\oplus\tilde{f}(b,c,\tilde{q})\Big)\Bigg]$$
$$\oplus 4\odot\Bigg\{\Bigg[\tilde{f}\Big(a,c,\frac{\tilde{p}\oplus\tilde{q}}{2}\Big)\oplus 4\odot\tilde{f}\Big(\frac{a+b}{2},c,\frac{\tilde{p}\oplus\tilde{q}}{2}\Big)\oplus\tilde{f}\Big(b,c,\frac{\tilde{p}\oplus\tilde{q}}{2}\Big)\Bigg]$$
$$\oplus\Bigg[\tilde{f}\Big(a,\frac{c+d}{2},\tilde{p}\Big)\oplus 4\odot\tilde{f}\Big(\frac{a+b}{2},\frac{c+d}{2},\tilde{p}\Big)\oplus\tilde{f}\Big(b,\frac{c+d}{2},\tilde{p}\Big)\Bigg]$$
$$\oplus\Bigg[\tilde{f}\Big(a,\frac{c+d}{2},\frac{\tilde{p}\oplus\tilde{q}}{2}\Big)\oplus 4\odot\tilde{f}\Big(\frac{a+b}{2},\frac{c+d}{2},\frac{\tilde{p}\oplus\tilde{q}}{2}\Big)\oplus\tilde{f}\Big(b,\frac{c+d}{2},\frac{\tilde{p}\oplus\tilde{q}}{2}\Big)\Bigg]$$
$$\oplus\Bigg[\tilde{f}\Big(a,\frac{c+d}{2},\tilde{q}\Big)\oplus 4\odot\tilde{f}\Big(\frac{a+b}{2},\frac{c+d}{2},\tilde{q}\Big)\oplus\tilde{f}\Big(b,\frac{c+d}{2},\tilde{q}\Big)\Bigg]$$
$$\oplus\Bigg[\tilde{f}\Big(a,d,\frac{\tilde{p}\oplus\tilde{q}}{2}\Big)\oplus 4\odot\tilde{f}\Big(\frac{a+b}{2},d,\frac{\tilde{p}\oplus\tilde{q}}{2}\Big)\oplus\tilde{f}\Big(b,d,\frac{\tilde{p}\oplus\tilde{q}}{2}\Big)\Bigg]\Bigg\}$$
$$\oplus\Bigg[\Big(\tilde{f}(a,d,\tilde{p})\oplus\tilde{f}(a,d,\tilde{q})\Big)\oplus 4\odot\Big(\tilde{f}\Big(\frac{a+b}{2},d,\tilde{p}\Big)\oplus\tilde{f}\Big(\frac{a+b}{2},d,\tilde{q}\Big)\Big)\oplus\Big(\tilde{f}(b,d,\tilde{p})\oplus\tilde{f}(b,d,\tilde{q})\Big)\Bigg]\Bigg\}\Bigg)$$
$$\le 27(b-a)(d-c)\|\tilde{q}\ominus\tilde{p}\|_{\mathbb{R}_F}\,\omega_{[a,b]\times[c,d]\otimes[\tilde{p},\tilde{q}]}\Bigg(\tilde{f},\frac{b-a}{6},\frac{d-c}{6},\frac{\|\tilde{q}\ominus\tilde{p}\|_{\mathbb{R}_F}}{6}\Bigg)$$

Proof

This inequality follows from the previous corollary by setting $\alpha = \frac{5a+b}{6}$, $\beta = \frac{5c+d}{6}$, $\tilde{\gamma} = \frac{5\odot\tilde{p}\oplus\tilde{q}}{6}$, $u = a$, $v = \frac{a+b}{2}$, $w = b$, $u' = c$, $v' = \frac{c+d}{2}$, $w' = d$, $\tilde{u} = \tilde{p}$, $\tilde{v} = \frac{\tilde{p}\oplus\tilde{q}}{2}$ and $\tilde{w} = \tilde{q}$.

□

Theorem 3.8. Let $\tilde{f} : [a,b]\times[c,d]\otimes[\tilde{p},\tilde{q}] \to \mathbb{R}_F$ be a Lipschitz mapping with the constants L_1, L_2 and L_3. Then, for any subdivision

$$\Delta_m : a = x_0 < x_1 < x_2 < \cdots < x_m = b,\quad \Delta_n : c = y_0 < y_1 < y_2 < \cdots < y_n = d$$

and

$$\Delta_s : \tilde{p} = \tilde{z}_0 \prec \tilde{z}_1 \prec \tilde{z}_2 \prec \cdots \prec \tilde{z}_s = \tilde{q}.\ \ \forall \xi_i \in [x_{i-1}, x_i],\ i = 1,2,\cdots,m;$$

$$\eta_j \in [y_{j-1}, y_j],\ j = 1,2,\cdots,n \ \text{ and } \ \tilde{\zeta}_k \in [\tilde{z}_{k-1}, \tilde{z}_k],\ k = 1,2,\cdots,s;$$

we have

$$D\left(\left(FHTI\right)\int_a^b\int_c^d\int_{\tilde{p}}^{\tilde{q}}\tilde{f}\left(x,y,\tilde{z}\right)\mathrm{d}\tilde{z}\mathrm{d}y\mathrm{d}x,\right.$$

$$\left.\bigoplus_{i=1}^{m}\bigoplus_{j=1}^{n}\bigoplus_{k=1}^{s}\left(x_i-x_{i-1}\right)\left(y_j-y_{j-1}\right)\left\|\tilde{z}_k\ominus\tilde{z}_{k-1}\right\|_{\mathbb{R}_F}\odot\tilde{f}\left(\xi_i,\eta_j,\tilde{\zeta}_k\right)\right)$$

$$\leq\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s}\Big(L_1\left(y_j-y_{j-1}\right)\left\|\tilde{z}_k\ominus\tilde{z}_{k-1}\right\|_{\mathbb{R}_F}\left(x_i-x_{i-1}\right)^2$$

$$+L_2\left(x_i-x_{i-1}\right)\left\|\tilde{z}_k\ominus\tilde{z}_{k-1}\right\|_{\mathbb{R}_F}\left(y_j-y_{j-1}\right)^2$$

$$+L_3\left(x_i-x_{i-1}\right)\left(y_j-y_{j-1}\right)\left\|\tilde{z}_k\ominus\tilde{z}_{k-1}\right\|_{\mathbb{R}_F}^2\Big).$$

Proof Similar to the proof of theorem 3.5 we have

$$D\left(\left(FHTI\right)\int_a^b\int_c^d\int_{\tilde{p}}^{\tilde{q}}\tilde{f}\left(x,y,\tilde{z}\right)\mathrm{d}\tilde{z}\mathrm{d}y\mathrm{d}x,\right.$$

$$\left.\bigoplus_{i=1}^{m}\bigoplus_{j=1}^{n}\bigoplus_{k=1}^{s}\left(x_i-x_{i-1}\right)\left(y_j-y_{j-1}\right)\left\|\tilde{z}_k\ominus\tilde{z}_{k-1}\right\|_{\mathbb{R}_F}\odot\tilde{f}\left(\xi_i,\eta_j,\tilde{\zeta}_k\right)\right)$$

$$\leq\bigoplus_{i=1}^{m}\bigoplus_{j=1}^{n}\bigoplus_{k=1}^{s}\left(L\right)\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j}\int_{\tilde{z}_{k-1}}^{\tilde{z}_k}D\left(\tilde{f}\left(x,y,\tilde{z}\right),\tilde{f}\left(\xi_i,\eta_j,\tilde{\zeta}_k\right)\right)\mathrm{d}\tilde{z}\mathrm{d}y\mathrm{d}x.$$

We obtain by the definition of a Lipschitz mapping

$$\bigoplus_{i=1}^{m}\bigoplus_{j=1}^{n}\bigoplus_{k=1}^{s}\left(L\right)\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j}\int_{\tilde{z}_{k-1}}^{\tilde{z}_k}D\left(\tilde{f}\left(x,y,\tilde{z}\right),\tilde{f}\left(\xi_i,\eta_j,\tilde{\zeta}_k\right)\right)\mathrm{d}\tilde{z}\mathrm{d}y\mathrm{d}x.$$

$$\leq\bigoplus_{i=1}^{m}\bigoplus_{j=1}^{n}\bigoplus_{k=1}^{s}\left(L_1\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j}\int_{\tilde{z}_{k-1}}^{\tilde{z}_k}\left|x-\xi_i\right|\mathrm{d}\tilde{z}\mathrm{d}y\mathrm{d}x\right.$$

$$\left.\oplus L_2\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j}\int_{\tilde{z}_{k-1}}^{\tilde{z}_k}\left|y-\eta_j\right|\mathrm{d}\tilde{z}\mathrm{d}y\mathrm{d}x\oplus L_3\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j}\int_{\tilde{z}_{k-1}}^{\tilde{z}_k}\left\|\tilde{z}\ominus\tilde{\zeta}_k\right\|_{\mathbb{R}_F}\mathrm{d}\tilde{z}\mathrm{d}y\mathrm{d}x\right)$$

It follows by direct computation that

$$\bigoplus_{i=1}^{m}\bigoplus_{j=1}^{n}\bigoplus_{k=1}^{s}\left(L_1\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j}\int_{\tilde{z}_{k-1}}^{\tilde{z}_k}\left|x-\xi_i\right|\mathrm{d}\tilde{z}\mathrm{d}y\mathrm{d}x\right.$$

$$\left.\oplus L_2\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j}\int_{\tilde{z}_{k-1}}^{\tilde{z}_k}\left|y-\eta_j\right|\mathrm{d}\tilde{z}\mathrm{d}y\mathrm{d}x\oplus L_3\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j}\int_{\tilde{z}_{k-1}}^{\tilde{z}_k}\left\|\tilde{z}\ominus\tilde{\zeta}_k\right\|_{\mathbb{R}_F}\mathrm{d}\tilde{z}\mathrm{d}y\mathrm{d}x\right)$$

$$=\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s}(L_1\left(y_j-y_{j-1}\right)\left\|\tilde{z}_k\ominus\tilde{z}_{k-1}\right\|_{\mathbb{R}_F}\left[\left(x_i-\xi_i\right)^2-\left(x_{i-1}-\xi_i\right)^2\right]$$

$$+L_2\left(x_i-x_{i-1}\right)\left\|\tilde{z}_k\ominus\tilde{z}_{k-1}\right\|_{\mathbb{R}_F}\left[\left(y_j-\eta_j\right)^2-\left(y_{j-1}-\eta_j\right)^2\right]$$

$$+L_3\left(x_i-x_{i-1}\right)\left(y_j-y_{j-1}\right)\left[\left\|\tilde{z}_k\ominus\tilde{\zeta}_k\right\|_{\mathbb{R}_F}^2-\left\|\tilde{z}_{k-1}\ominus\tilde{\zeta}_k\right\|_{\mathbb{R}_F}^2\right]\Big)$$

$$\leq\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s}(L_1\left(y_j-y_{j-1}\right)\left\|\tilde{z}_k\ominus\tilde{z}_{k-1}\right\|_{\mathbb{R}_F}\left(x_i-x_{i-1}\right)^2$$

$$+L_2\left(x_i-x_{i-1}\right)\left\|\tilde{z}_k\ominus\tilde{z}_{k-1}\right\|_{\mathbb{R}_F}\left(y_j-y_{j-1}\right)^2$$

$$+L_3\left(x_i-x_{i-1}\right)\left(y_j-y_{j-1}\right)\left\|\tilde{z}_k\ominus\tilde{z}_{k-1}\right\|_{\mathbb{R}_F}^2\Big).$$

□

4. Numerical Example

Let $\tilde{f}:[0,1]\times[1,2]\otimes\left[\tilde{1},\tilde{2}\right]\to\mathbb{R}_F$, $\tilde{f}(x,y,\tilde{z})=x^2+3y\oplus\tilde{z}$ where $\tilde{1}=(0,1,2)$; $\tilde{2}=(1,2,3)$; $\tilde{z}=(z-1,z,z+1)$, and where (a_1,a_2,a_3) is a triangular fuzzy number such that

$$\mu(x)=\begin{cases}\dfrac{x-a_1}{a_2-a_1} & a_1\le x\le a_2\\ \dfrac{a_3-x}{a_3-a_2} & a_2\le x\le a_3\\ 0 & \text{otherwise}\end{cases}$$

We must compute the integral

$$(FHTI)\int_0^1\int_1^2\int_{\tilde{1}}^{\tilde{2}}\tilde{f}(x,y,\tilde{z})\mathrm{d}\tilde{z}\mathrm{d}y\mathrm{d}x$$

numerically.

Firstly let's define the different α-cuts below:

$$\left(\tilde{f}(x,y,\tilde{z})\right)^\alpha=\left[x^2+3y+z-1+\alpha,x^2+3y+z+1-\alpha\right],$$

$$\tilde{z}^\alpha=\left[z-1+\alpha,z+1-\alpha\right],$$

$$\tilde{1}^\alpha=\left[\alpha,2-\alpha\right],$$

$$\tilde{2}^\alpha=\left[1+\alpha,3-\alpha\right],\text{ and}$$

$$\tilde{I}^\alpha=\left((FHTI)\int_0^1\int_1^2\int_{\tilde{1}}^{\tilde{2}}\tilde{f}(x,y,\tilde{z})\mathrm{d}\tilde{z}\mathrm{d}y\mathrm{d}x\right)^\alpha=\left[\min S,\max S\right].$$

where

$$\begin{aligned}S=\Big\{&\int_0^1\int_1^2\int_\alpha^{1+\alpha}\tilde{f}_-^\alpha(x,y,\tilde{z})\mathrm{d}\tilde{z}_-^\alpha\mathrm{d}y\mathrm{d}x;\int_0^1\int_1^2\int_\alpha^{1+\alpha}\tilde{f}_-^\alpha(x,y,\tilde{z})\mathrm{d}\tilde{z}_+^\alpha\mathrm{d}y\mathrm{d}x;\\
&\int_0^1\int_1^2\int_\alpha^{3-\alpha}\tilde{f}_-^\alpha(x,y,\tilde{z})\mathrm{d}\tilde{z}_-^\alpha\mathrm{d}y\mathrm{d}x;\int_0^1\int_1^2\int_\alpha^{3-\alpha}\tilde{f}_-^\alpha(x,y,\tilde{z})\mathrm{d}\tilde{z}_+^\alpha\mathrm{d}y\mathrm{d}x;\\
&\int_0^1\int_1^2\int_{2-\alpha}^{1+\alpha}\tilde{f}_-^\alpha(x,y,\tilde{z})\mathrm{d}\tilde{z}_-^\alpha\mathrm{d}y\mathrm{d}x;\int_0^1\int_1^2\int_{2-\alpha}^{1+\alpha}\tilde{f}_-^\alpha(x,y,\tilde{z})\mathrm{d}\tilde{z}_+^\alpha\mathrm{d}y\mathrm{d}x;\\
&\int_0^1\int_1^2\int_{2-\alpha}^{3-\alpha}\tilde{f}_-^\alpha(x,y,\tilde{z})\mathrm{d}\tilde{z}_-^\alpha\mathrm{d}y\mathrm{d}x;\int_0^1\int_1^2\int_{2-\alpha}^{3-\alpha}\tilde{f}_-^\alpha(x,y,\tilde{z})\mathrm{d}\tilde{z}_+^\alpha\mathrm{d}y\mathrm{d}x;\\
&\int_0^1\int_1^2\int_\alpha^{1+\alpha}\tilde{f}_+^\alpha(x,y,\tilde{z})\mathrm{d}\tilde{z}_-^\alpha\mathrm{d}y\mathrm{d}x;\int_0^1\int_1^2\int_\alpha^{1+\alpha}\tilde{f}_+^\alpha(x,y,\tilde{z})\mathrm{d}\tilde{z}_+^\alpha\mathrm{d}y\mathrm{d}x;\\
&\int_0^1\int_1^2\int_\alpha^{3-\alpha}\tilde{f}_+^\alpha(x,y,\tilde{z})\mathrm{d}\tilde{z}_-^\alpha\mathrm{d}y\mathrm{d}x;\int_0^1\int_1^2\int_\alpha^{3-\alpha}\tilde{f}_+^\alpha(x,y,\tilde{z})\mathrm{d}\tilde{z}_+^\alpha\mathrm{d}y\mathrm{d}x;\\
&\int_0^1\int_1^2\int_{2-\alpha}^{1+\alpha}\tilde{f}_+^\alpha(x,y,\tilde{z})\mathrm{d}\tilde{z}_-^\alpha\mathrm{d}y\mathrm{d}x;\int_0^1\int_1^2\int_{2-\alpha}^{1+\alpha}\tilde{f}_+^\alpha(x,y,\tilde{z})\mathrm{d}\tilde{z}_+^\alpha\mathrm{d}y\mathrm{d}x;\\
&\int_0^1\int_1^2\int_{2-\alpha}^{3-\alpha}\tilde{f}_+^\alpha(x,y,\tilde{z})\mathrm{d}\tilde{z}_-^\alpha\mathrm{d}y\mathrm{d}x;\int_0^1\int_1^2\int_{2-\alpha}^{3-\alpha}\tilde{f}_+^\alpha(x,y,\tilde{z})\mathrm{d}\tilde{z}_+^\alpha\mathrm{d}y\mathrm{d}x\Big\}.\end{aligned}$$

We must calculate these sixteen integrals while noting that they are two by two equal.

We remark that

$$\begin{aligned}&D\left(\tilde{f}(x_1,y_1,\tilde{z}_1),\tilde{f}(x_2,y_2,\tilde{z}_2)\right)\\
&=\sup_{\alpha\in[0,1]}\max\left\{\left|\tilde{f}_-^\alpha(x_1,y_1,\tilde{z}_1)-\tilde{f}_-^\alpha(x_2,y_2,\tilde{z}_2)\right|,\left|\tilde{f}_+^\alpha(x_1,y_1,\tilde{z}_1)-\tilde{f}_+^\alpha(x_2,y_2,\tilde{z}_2)\right|\right.\\
&=\sup_{\alpha\in[0,1]}\max\left\{\left|\left(x_1^2-x_2^2\right)+3(y_1-y_2)+(z_1-z_2)+(\alpha-1)\right|,\right.\\
&\left.\left|\left(x_1^2-x_2^2\right)+3(y_1-y_2)+(z_1-z_2)+(1-\alpha)\right|\right\}\end{aligned}$$

$$\leq \sup_{\alpha\in[0,1]} \max\left\{|x_1-x_2|\left(|x_1+x_2|,|x_1+x_2|\right)\right\}+\sup_{\alpha\in[0,1]} \max\left\{|y_1-y_2|(3,3)\right\}$$
$$+\sup_{\alpha\in[0,1]} \max\left\{|z_1-z_2|(1,1)\right\}$$
$$\leq 2|x_1-x_2|+3|y_1-y_2|+|z_1-z_2|.$$

i.e. $\tilde{f}$ is a Lipschitz mapping with $L_1=2, L_2=3$ and $L_3=1$.

Indeed,

$$\int_0^1\int_1^2\int_\alpha^{1+\alpha} \tilde{f}_-^\alpha(x,y,\tilde{z})\mathrm{d}\tilde{z}_-^\alpha \mathrm{d}y\mathrm{d}x=\int_0^1\int_1^2\int_\alpha^{1+\alpha} \tilde{f}_-^\alpha(x,y,\tilde{z})\mathrm{d}\tilde{z}_+^\alpha \mathrm{d}y\mathrm{d}x, \tag{4.1}$$

because $\mathrm{d}\tilde{z}_-^\alpha=\mathrm{d}\tilde{z}_+^\alpha=\mathrm{d}z$

From where, 4.1 becomes

$$\begin{aligned}\int_0^1\int_1^2\int_\alpha^{1+\alpha} \tilde{f}_-^\alpha(x,y,\tilde{z})\mathrm{d}z\mathrm{d}y\mathrm{d}x &= \int_\alpha^{1+\alpha}\int_1^2\int_0^1 \tilde{f}_-^\alpha(x,y,\tilde{z})\mathrm{d}x\mathrm{d}y\mathrm{d}z\\
&=\int_\alpha^{1+\alpha}\int_1^2\int_0^1\left(x^2+3y+z-1+\alpha\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z\\
&=\int_\alpha^{1+\alpha}\int_1^2\left[\frac{x^3}{3}+(3y+z-1+\alpha)\right]_{x=0}^1 \mathrm{d}y\mathrm{d}z\\
&=\int_\alpha^{1+\alpha}\int_1^2\left(3y-\frac{2}{3}+\alpha+z\right)\mathrm{d}y\mathrm{d}z\\
&=\int_\alpha^{1+\alpha}\left[\frac{3}{2}y^2+\left(-\frac{2}{3}+\alpha+z\right)y\right]_{y=1}^2 \mathrm{d}z\\
&=\int_\alpha^{1+\alpha}\left[\left(\frac{23}{6}+\alpha\right)+z\right]\mathrm{d}z\\
&=\left[\left(\frac{23}{6}+\alpha\right)z+\frac{1}{2}z^2\right]_{z=\alpha}^{1+\alpha}\\
&=\frac{13}{3}+2\alpha.\end{aligned}$$

Proceeding in the same way for the other integrals we obtain

$$\begin{aligned}\int_0^1\int_1^2\int_\alpha^{3-\alpha} \tilde{f}_-^\alpha(x,y,\tilde{z})\mathrm{d}\tilde{z}_-^\alpha\mathrm{d}y\mathrm{d}x &= \int_0^1\int_1^2\int_\alpha^{3-\alpha} \tilde{f}_-^\alpha(x,y,\tilde{z})\mathrm{d}\tilde{z}_+^\alpha\mathrm{d}y\mathrm{d}x\\
&=\int_0^1\int_1^2\int_\alpha^{3-\alpha} \tilde{f}_-^\alpha(x,y,\tilde{z})\mathrm{d}z\mathrm{d}y\mathrm{d}x\\
&=-2\alpha^2-\frac{23}{3}\alpha+16\end{aligned}$$

$$\begin{aligned}\int_0^1\int_1^2\int_{2-\alpha}^{1+\alpha} \tilde{f}_-^\alpha(x,y,\tilde{z})\mathrm{d}\tilde{z}_-^\alpha\mathrm{d}y\mathrm{d}x &= \int_0^1\int_1^2\int_{2-\alpha}^{1+\alpha} \tilde{f}_-^\alpha(x,y,\tilde{z})\mathrm{d}\tilde{z}_+^\alpha\mathrm{d}y\mathrm{d}x\\
&=\int_0^1\int_1^2\int_{2-\alpha}^{1+\alpha} \tilde{f}_-^\alpha(x,y,\tilde{z})\mathrm{d}z\mathrm{d}y\mathrm{d}x\\
&=\frac{6\alpha^2+29\alpha-16}{3}\end{aligned}$$

$$\begin{aligned}\int_0^1\int_1^2\int_{2-\alpha}^{3-\alpha} \tilde{f}_-^\alpha(x,y,\tilde{z})\mathrm{d}\tilde{z}_-^\alpha\mathrm{d}y\mathrm{d}x &= \int_0^1\int_1^2\int_{2-\alpha}^{3-\alpha} \tilde{f}_-^\alpha(x,y,\tilde{z})\mathrm{d}\tilde{z}_+^\alpha\mathrm{d}y\mathrm{d}x\\
&=\int_0^1\int_1^2\int_{2-\alpha}^{3-\alpha} \tilde{f}_-^\alpha(x,y,\tilde{z})\mathrm{d}z\mathrm{d}y\mathrm{d}x\\
&=\frac{19}{3}\end{aligned}$$

$$\int_0^1\int_1^2\int_\alpha^{1+\alpha}\tilde{f}_+^\alpha(x,y,\tilde{z})\mathrm{d}\tilde{z}_-^\alpha\mathrm{d}y\mathrm{d}x=\int_0^1\int_1^2\int_\alpha^{1+\alpha}\tilde{f}_+^\alpha(x,y,\tilde{z})\mathrm{d}\tilde{z}_+^\alpha\mathrm{d}y\mathrm{d}x$$
$$=\int_0^1\int_1^2\int_\alpha^{1+\alpha}\tilde{f}_+^\alpha(x,y,\tilde{z})\mathrm{d}z\mathrm{d}y\mathrm{d}x$$
$$=\frac{19}{3}$$

$$\int_0^1\int_1^2\int_\alpha^{3-\alpha}\tilde{f}_+^\alpha(x,y,\tilde{z})\mathrm{d}\tilde{z}_-^\alpha\mathrm{d}y\mathrm{d}x=\int_0^1\int_1^2\int_\alpha^{3-\alpha}\tilde{f}_+^\alpha(x,y,\tilde{z})\mathrm{d}\tilde{z}_+^\alpha\mathrm{d}y\mathrm{d}x$$
$$=\int_0^1\int_1^2\int_\alpha^{3-\alpha}\tilde{f}_+^\alpha(x,y,\tilde{z})\mathrm{d}z\mathrm{d}y\mathrm{d}x$$
$$=2\alpha^2-\frac{53}{3}\alpha+22$$

$$\int_0^1\int_1^2\int_{2-\alpha}^{1+\alpha}\tilde{f}_+^\alpha(x,y,\tilde{z})\mathrm{d}\tilde{z}_-^\alpha\mathrm{d}y\mathrm{d}x=\int_0^1\int_1^2\int_{2-\alpha}^{1+\alpha}\tilde{f}_+^\alpha(x,y,\tilde{z})\mathrm{d}\tilde{z}_+^\alpha\mathrm{d}y\mathrm{d}x$$
$$=\int_0^1\int_1^2\int_{2-\alpha}^{1+\alpha}\tilde{f}_+^\alpha(x,y,\tilde{z})\mathrm{d}z\mathrm{d}y\mathrm{d}x$$
$$=\frac{-6\alpha^2+47\alpha-22}{3}$$

$$\int_0^1\int_1^2\int_{2-\alpha}^{3-\alpha}\tilde{f}_+^\alpha(x,y,\tilde{z})\mathrm{d}\tilde{z}_-^\alpha\mathrm{d}y\mathrm{d}x=\int_0^1\int_1^2\int_{2-\alpha}^{3-\alpha}\tilde{f}_+^\alpha(x,y,\tilde{z})\mathrm{d}\tilde{z}_+^\alpha\mathrm{d}y\mathrm{d}x$$
$$=\int_0^1\int_1^2\int_{2-\alpha}^{3-\alpha}\tilde{f}_+^\alpha(x,y,\tilde{z})\mathrm{d}z\mathrm{d}y\mathrm{d}x$$
$$=\frac{25}{3}-2\alpha.$$

Finally S becomes

$$S=\left\{\frac{13}{3}+2\alpha,-2\alpha^2-\frac{23}{3}\alpha+16,\frac{6\alpha^2+29\alpha-16}{3},\frac{19}{3},2\alpha^2-\frac{53}{3}\alpha+22,\frac{-6\alpha^2+47\alpha-22}{3},\frac{25}{3}-2\alpha\right\}.$$

We have for $\alpha=1$ that $\tilde{I}^1=[6,3333;6,3333]$, for $\alpha=0,9$

$$S=\{6,1333;7,4800;4,9867;6,3333;7,7200;5,1467;6,5333\}$$

this is shows that $\tilde{I}^{0,9}=[4,9867;7,7200]$.

In Table 1 below we summarize the results for the different values of α.

Table 1. The results of example.

α	$\underline{I}_\alpha^{m,n,s}$	$\overline{I}_\alpha^{m,n,s}$
1	6.3333	6.3333
0.9	4.9867	7.7200
0.8	3.6800	9.1467
0.7	2.4133	10.6133
0.6	1.1867	12.1200
0.5	0.0000	13.6667
0.4	−1.3867	15.2533
0.3	−2.8133	16.8800
0.2	−4.2800	18.5467
0.1	−5.7867	20.2533
0.0	−7.3333	22.0000

5. Conclusion

In this article, it was a question of introducing and evaluating the fuzzy triple integral of Henstock-Kurzweil whose one of the variables is fuzzy, using Simpson's rule and Hausdorff distance. Therefore, we compute the integral on a quasi-fuzzy parallelepiped domain. In this direction, we have established and demonstrated a theorem which shows the upper limit of the distance between the exact values and approximate. For the rest, it would be possible to do the same analyzes for the integral of a fuzzy function with three variables of which two (or three) are fuzzy on a fuzzy parallelepipedal domain.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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