Journal of Applied Mathematics and Physics, 2024, 12, 877-896
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Comparative Studies between Picard's and Taylor's Methods of Numerical Solutions of First Ordinary Order Differential Equations Arising from Real-Life Problems

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How to cite this paper: Elnour, K.A.E.A.A. (2024) Comparative Studies between Picard's and Taylor's Methods of Numerical Solutions of First Ordinary Order Differential Equations Arising from Real-Life Problems. *Journal of Applied Mathematics and Physics*, **12**, 877-896.
https://doi.org/10.4236/jamp.2024.123055

Received: January 25, 2024
Accepted: March 26, 2024
Published: March 29, 2024

Abstract

To solve the first-order differential equation derived from the problem of a free-falling object and the problem arising from Newton's law of cooling, the study compares the numerical solutions obtained from Picard's and Taylor's series methods. We have carried out a descriptive analysis using the MATLAB software. Picard's and Taylor's techniques for deriving numerical solutions are both strong mathematical instruments that behave similarly. All first-order differential equations in standard form that have a constant function on the right-hand side share this similarity. As a result, we can conclude that Taylor's approach is simpler to use, more effective, and more accurate. We will contrast Rung Kutta and Taylor's methods in more detail in the following section.

Keywords

First-Order Differential Equations, Picard Method, Taylor Series Method, Numerical Solutions, Numerical Examples, MATLAB Software

1. Introduction

Differential equations play a crucial role in advancing various natural and social sciences by finding solutions to many problems humans face. This is achieved through modeling problems, which often involve rates. Due to the difficulty of finding solutions for some types of these equations, mathematicians have turned to alternative methods known as approximate numerical solutions. These solutions take various forms, such as Taylor's method, Picard's method, Euler's method, Runge-Kutta's method, and others.

These methods have garnered significant interest from researchers by applying them to solve differential equations, especially those of the first order. Researchers have also focused on the idea of comparing these methods in terms of the quality of the approximate solutions. Previous studies [1] [2] [3] [4] addressed the comparison topic and a good agreement with the exact solution in favor of the Runge-Kutta method for accurate results and minimum amount of error, and in [5] using Euler's method for higher orders. In studies [6] and [7] the authors used Picard and Taylor methods to solve motion and atmospheric pressure problems and observed that Taylor's method converged faster than Picard's method and for [8] the authors showed that Euler's method is less accurate than Runge-Kotta method. In contrast, the current study focused on using Picard and Taylor's methods to study the phenomena of free-fall motion of bodies and Newton's law of cooling. The uniqueness of the current study lies in its connection between these two phenomena, which many studies have overlooked. Additionally, the study used the MATLAB program for the comparative analysis.

Through careful and investigative study in this field, the researcher addressed the scarcity of research that dealt with physical and natural issues by finding appropriate solutions to them, especially using the Picard and Taylor methods, and from here the problem of the study emerged, which was represented in the following questions.

1.1. Questions of the Study

1) Is there a difference between the numerical solutions of Picard's and Taylor's methods compared to the exact solution of the first-order differential equation?

2) Is there a difference between the numerical solutions of Picard's and Taylor's method compared to the exact solution of the first-order differential equation arising from the problem of free-falling objects under the influence of Earth's gravity?

3) Is there a difference between the numerical solutions of Picard's and Taylor's methods compared to the exact solution of the first-order differential equation arising from the problem of Newton's law of cooling?

1.2. Research Objectives

This research target to:

1) Investigate the difference between the numerical solutions of Picard's and Taylor's methods compared to the exact solution of the first-order differential equation.

2) Compare the numerical solutions of Picard's and Taylor's series method for finding the solution of the first-order differential equation arising from the problem of free-falling objects under the influence of Earth's gravity.

3) Investigate the difference between the numerical solutions of Picard's and Taylor's methods compared to the exact solution of the first-order differential equation arising from the problem of Newton's law of cooling.

1.3. Formulation of the Problem

For obtaining the approximate solutions to the initial value problem of an ordinary differential equation, we consider two numerical methods consisting of the form:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y),\ y(x_0) = y_0 \tag{1}$$

Such that $y(x)$ is the solution of Equation (1).

1.4. Lipschitz Condition

A function $f(x,y)$ defined on a domain D of xy plane is satisfying Lipschitz condition w.r.t y in D if there exists a constant K such that:

$$\left|f(x,y_2) - f(x,y_1)\right| \le k(y_2 - y_1),\ \text{ for all } (x,y_1),(x,y_2) \in D.$$

1.5. Theorem

Let D be z region in $\mathbb{R}^2$, such that $D = \{(x,y):|x-x_0| \le a, (x,y):|y-y_0| \le b\}$ and $f: D \to \mathbb{R}$ be a real function such that:

1) $f(x,y)$ is continuous on D.
2) $f(x,y)$ is bounded on D.
3) $f(x,y)$ satisfy Lipschitz condition.

Then the initial value problem of first-order differential equation has a unique solution $y = y(x)$ for which $y(x_0) = y_0$ in the interval $|x - x_0| \le h$, where $h = \min\left\{a, \frac{b}{M}\right\}$.

1.5.1. Picard Iteration Method

It is an important iterative method for generating a sequence of increasingly accurate algebraic approximations of the specific exact solution of the first-order differential equation with initial value (1). So, the integration of Equation (1) yields the following:

$$y_1(x) = y(x_0) + \int_{s=x_0}^{x} f(x,y_0)\mathrm{d}s$$

$$y_2(x) = y(x_0) + \int_{s=x_0}^{x} f(x,y_1)\mathrm{d}s$$

$$y_3(x) = y(x_0) + \int_{s=x_0}^{x} f(x,y_2)\mathrm{d}s$$

$$\vdots$$

To continue this process, we get a sequence of functions of x *i.e.* $y_1, y_2, y_3, \cdots, y_n$, so, to keep them going, we use the following iterative formula:

$$y_{n+1}(x) = y(x_0) + \int_{s=x_0}^{x} f(x,y_n)\mathrm{d}s \tag{2}$$

Therefore, $\lim_{n\to\infty} y_{n+1} = y(x)$, which is the exact solution.

1.5.2. Maclaurin Series

The general form:

$$g(x) = g(0) + g'(0)x + \frac{g''(0)x^2}{2!} + \frac{g'''(0)x^3}{3!} + \cdots + \frac{g^{(n)}(0)x^n}{n!} + \cdots$$

It is not working to: $g(x) = \ln(x)$, since $g(0) = \ln(0)$ which is undefined.

1.5.3. Taylor's Series Method

Taylor's series is a numerical method used to approximate the value of a function $f(x)$ at a specific point $x = a$, by using a series of terms that are derived from the function's derivatives evaluated at that point.

Let $f(x+a) = g(x)$, *i.e.*, replace x with $x+a$, $a \neq 0$, then

$$f(x+a) = f(a) + f'(a)x + \frac{f''(a)x^2}{2!} + \frac{f'''(a)x^3}{3!} + \cdots + \frac{f^{(n)}(a)x^n}{n!} + \cdots$$

Again, by replacing x with $x-a$, we get:

$$f(x) = f(a) + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} + \cdots + \frac{f^{(n)}(a)(x-a)^n}{n!} + \cdots$$

Which it can be written in a sigma notation as follows:

$$f(x) = \sum_{n=0}^{\infty} \frac{(x-a)^n}{n!} f^{(n)}(a). \tag{3}$$

Note that any function can be represented by Taylor's Series about the position (a) if it is continuous and differentiable near a.

Let $f(x) = \sin x$, then

$$f'(x) = \cos x$$
$$f''(x) = -\sin x$$
$$f'''(x) = -\cos x$$
$$f''''(x) = \sin x$$

and so on

The Taylor series for $\sin x$ at $x = 0$, is given by:

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

1.6. Comparison Criteria

The criteria used to compare the effectiveness and accuracy of numerical solutions of any first-order differential equation with the exact solution are as follows:

1) The type of equation to be solved contributes, highlighting the contrast between the numerical solutions used and the exact solution.

2) The numerical results of the absolute error calculated are one of the most important criteria used to compare the effectiveness and accuracy

3) Computing more terms of the sequence of solution, increasing accuracy.

4) Using small steps provides better approximations that are close to the exact solution.

5) The graphical representation of the numerical solutions compared with the graph representations of the exact solution of the first-order differential equation contributes to determining the most effective and accurate numerical solution by identifying the curve close to the exact solution curve.

6) The limit of the sequence of numerical solutions tends to the exact solution when the independent variable tends to infinity.

1.7. Some Limitations and Assumptions

1) Lip itches condition is satisfied as the basis for the existence of a unique solution of the first-order differential equation.

2) If the R. H. S. of the standard first differential equation is constant, then the numerical solutions of both Picard and Taylor are identical.

3) It is difficult to find the exact solution of a first-order differential equation at which the degree of the right-hand side is greater than four.

4) The distinction of one numerical method over another one in terms of effectiveness and accuracy depends on dividing the specific interval used into small subintervals, more of them increasing efficiency and accuracy.

2. Applications and Comparisons

2.1. Problem

Consider the first-order differential equation: $\frac{\mathrm{d}y}{\mathrm{d}x} = x^3 - 2y$, with $y(0) = 1$.

First, we solve it by classical method and then by numerical methods.

2.2. Classical Method

It classified as linear differential equation comparing with $\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x)$, here $p(x) = 2$, $q(x) = x^3$, then the integrating factor = $I = \mathrm{e}^{\int 2\mathrm{d}x} = \mathrm{e}^{2x}$, so its solution is:

$$y \cdot I = \int QI\mathrm{d}x + c$$

$$y\mathrm{e}^{2x} = \int x^3 \mathrm{e}^{2x} \mathrm{d}x$$

Then after using the Tanique of integration by parts three times, we get:

$$y\mathrm{e}^{2x} = \frac{x^3}{2}\mathrm{e}^{2x} - \frac{3x^2}{2}\mathrm{e}^{2x} + \frac{3x}{4}\mathrm{e}^{2x} - \frac{3}{8}\mathrm{e}^{2x} + c.$$

Using the initial condition $y(0) = 1$, then $c = 1$, so the particular (exact) solution is:

$$y(x) = \frac{x^3}{2} - \frac{3x^2}{2} + \frac{3x}{4} - \frac{3}{8} + \frac{11}{8}\mathrm{e}^{-2x}. \tag{4}$$

Now we apply the above two numerical methods to our given differential equation one by one.

2.3. Solution by Picard's Method

The iteration scheme is:

$$y_{k+1}(x) = 1 + \int_{s=0}^{x} \left(x^3 - 2y_k\right) \mathrm{d}s,$$

where $k = 0,1,2,3,\cdots$, By putting $y_0 = 1$, then we get the first approximation solution as:

$$y_1(x) = 1 - 2x + \frac{1}{4}x^4.$$

and the second approximation solution is:

$$y_2(x) = 1 - 2x + 2x^2 + \frac{1}{4}x^4 - \frac{1}{10}x^5.$$

So,

$$y_3(x) = 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{1}{4}x^4 - \frac{1}{10}x^5 + \frac{1}{60}x^6.$$

$$y_4(x) = 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{11}{12}x^4 - \frac{1}{10}x^5 + \frac{1}{30}x^6 - \frac{1}{210}x^7.$$

Then after the four iterations we have the following approximation:

$$y(x) = -\frac{1}{210}x^7 + \frac{1}{30}x^6 - \frac{1}{10}x^5 + \frac{11}{12}x^4 - \frac{4}{3}x^3 + 2x^2 - 2x + 1 \quad (5)$$

2.4. Solution by Taylor's Series Method

For the equation $\frac{\mathrm{d}y}{\mathrm{d}x} = x^3 - 2y$, we have: (Figure 1 and Figure 2 and Table 1)

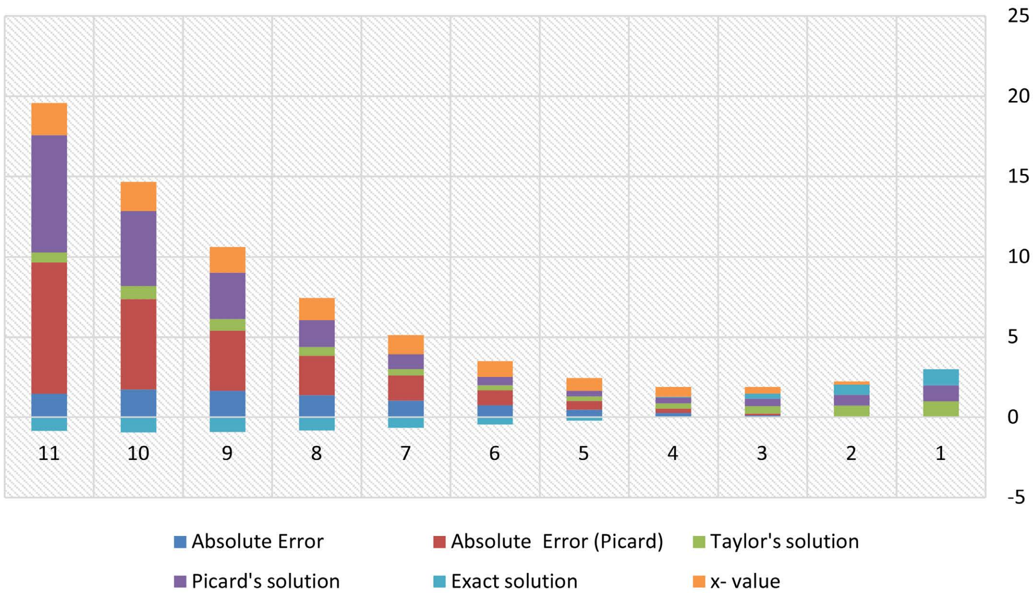

Figure 1. Comparison of exact solution and numerical solution of picard and taylor.

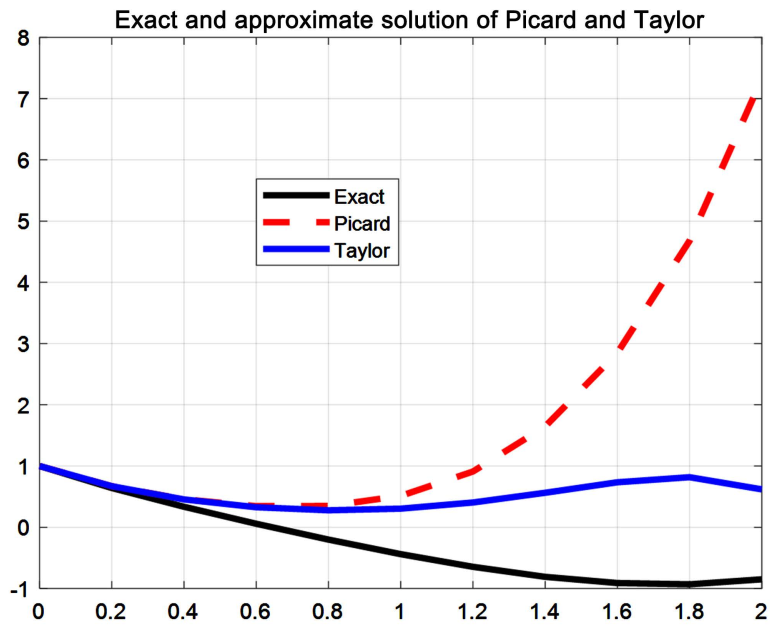

Figure 2. Comparison of the Exact Solution and the Numerical Solution of Picard and Taylor The following table gives the values of y from $x = 0$ to 2 taking $h = 0.2$.

Table 1. Numerical comparison between exact and approximate solutions of proposed two methods.

x-value	Exact solution	Picard's solution	Taylor's solution	Absolute Error (Picard)	Absolute Error (Taylor)
0	1.0000	1.0000	1.0000	0.0000	0.0000
0.2000	0.6407	0.6708	0.6707	0.0301	0.0300
0.4000	0.3348	0.4572	0.4548	0.1224	0.1200
0.6000	0.0571	0.3444	0.3270	0.2873	0.2699
0.8000	−0.2014	0.3478	0.2774	0.5492	0.4788
1.0000	−0.4389	0.5119	0.3040	0.9508	0.7429
1.2000	−0.6463	0.9104	0.4042	1.5567	1.0505
1.4000	−0.8094	1.6458	0.5629	2.4551	1.3723
1.6000	−0.9110	2.8690	0.7345	3.7599	1.6455
1.8000	−0.9314	4.6794	0.8175	5.6109	1.7490
2.0000	−0.8498	7.3238	0.6190	8.1736	1.4689

$$y'(x) = x^3 - 2y \Rightarrow y'(0) = -2$$

$$y''(x) = 3x^2 - 2y' \Rightarrow y''(0) = 4$$

$$y'''(x) = 6x - 2y'' \Rightarrow y'''(0) = -8$$

$$y^{(4)}(x) = 6 - 2y''' \Rightarrow y^{(4)}(0) = 22$$

$$y^{(5)}(x) = -2y^{(4)} \Rightarrow y^{(5)}(0) = -44$$

$$y^{(6)}(x) = -2y^{(5)} \Rightarrow y^{(6)}(0) = 88$$

$$y^{(7)}(x) = -2y^{(6)} \Rightarrow y^{(7)}(0) = -176$$

Therefore, Taylor's series expansion is:

$$y(x) = -\frac{11}{315}x^7 + \frac{11}{90}x^6 - \frac{11}{30}x^5 + \frac{11}{12}x^4 - \frac{4}{3}x^3 + 2x^2 - 2x + 1 \quad (6)$$

3. Falling Objects with Air Resistance

Many of the laws of nature are statements or relations involving rates, at which things happen, when such the statement is expressed in mathematical terms, then it becomes an equation describing a physical process. Suppose that an object of mass (m) is under falling motion which was influenced by the gravity of the earth (F_g) and the air resistance (F_r) in an opposite direction of the object, so we have two forces acting on it (**Figure 3**).

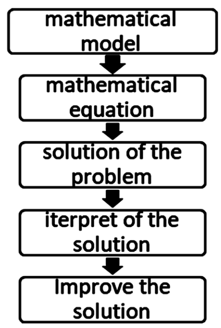

Figure 3. Modelling.

1) Gravity force: It is equal to the multiplication of the mass object times the acceleration due to gravity (g):

$$F_g = mg \quad (7)$$

2) Air resistance (drag) (F_r): This force is proportional to the square of the velocity (v) of the object, so

$$F_r \propto v$$

$$F_r = \beta v \quad (8)$$

Modelling: Now by using Newton's second law of motion, we perform the following substitution: (**Figure 4**)

$$\sum F = ma \quad (9)$$

where (a) represents the acceleration of the object.

$$F_g - F_r = ma$$

$$mg - \beta v = ma$$

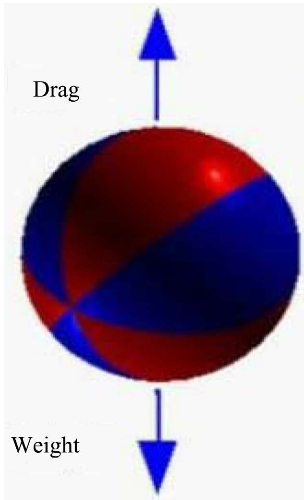

Figure 4. Free falling object.

Then the differential equation of the falling object is given by:

$$\frac{\mathrm{d}v}{\mathrm{d}t}+\frac{\beta}{m}v=g \tag{10}$$

With initial condition $v(0)=0$, therefore: $f(t,v)=g-\frac{\beta}{m}v$. Which is a linear differential equation, since

$$\left|f(t,v_2)-f(t,v_1)\right|=\left|g-\frac{k}{m}v_2-g+\frac{k}{m}v_1\right|=\frac{k}{m}\left|v_2-v_1\right|.$$

Therefore, $f(t,v)$ has a unique solution $v(t)$ for which $v(0)=0$. Therefore $f(t,v)$ has a unique solution $y(t)$ for which $v(0)=0$ (see [1.5]).

3.1. Picard Iteration Method

By integrating (10), we get:

$$\int_0^t \mathrm{d}v=v_0+\int_0^t f(t,v)\mathrm{d}s \tag{11}$$

So, the iteration scheme is:

$$v_{k+1}(t)=v_0+\int_0^t f(t,v_k)\mathrm{d}s \tag{12}$$

Therefore, the first approximation to the solution in $f(t,v)$ by putting $v=v_0$ and integrate (12), so

$$v_1=v(0)+\int_0^t\left(g-\frac{k}{m}v_0\right)\mathrm{d}s$$

For the second approximation solution, we put $v=v_1$ in $f(t,v_k)$, and again integral (12), we get

$$v_2=v(0)+\int_0^t\left(g-\frac{k}{m}v_1\right)\mathrm{d}s$$

Continue this process, a sequence of functions of t *i.e.* $v_1,v_2,v_3,\cdots$ is obtained giving a better approximation of a desired solution in the preceding one.

3.2. Taylor's Series Method

Consider the first-order differential Equation (10), and by differentiating it we get:

$$v'' = v' \tag{13}$$

Differentiate (13) successively we get v''', $v^{(4)}$, $v^{(5)}$, etc. When we put $t = 0$ and $v = 0$, the values of $v_0', v_0'', v_0''', \cdots$ can be obtained. Therefore, Taylor's series follows:

$$v(t) = v_0 + (t - t_0)v'(0) + \frac{(t-t_0)^2}{2!}v''(0) + \frac{(t-t_0)^3}{3!}v'''(0) + \cdots \tag{14}$$

and this gives the values of v for every value of t for which (14) converge.

3.3. Numerical Observations

The exact general solution for Equation (10) is $v(t) = \frac{gm}{k}\left(1 - \mathrm{e}^{-\frac{k}{m}t}\right)$ now, we discuss two cases:

Case 1: let $k = m$, then Equation (10) becomes:

$$\frac{\mathrm{d}v}{\mathrm{d}t} + v = g \tag{15}$$

Equation (15) was classified as a linear equation with integration factor $I = \mathrm{e}^t$, then the solution is:

$$v \cdot I = \int g\mathrm{e}^t \mathrm{d}t$$

$$v\mathrm{e}^t = g\mathrm{e}^t + cg$$

Then, $v = g + cg\mathrm{e}^{-t}$ Using the initial condition $v(0) = 0$, we get $c = 1$, then the exact solution (particular) of (15) is given by: $V(t) = g - g\mathrm{e}^{-t}$.

3.4. For Picard Iteration Method

$$v_1 = \int_{s=0}^{t} g\mathrm{d}s = gt,$$

$$v_2 = gt - \frac{gt^2}{2}$$

$$v_3 = gt - \frac{gt^2}{2} + \frac{gt^3}{6}$$

$$v_4 = gt - \frac{gt^2}{2} + \frac{gt^3}{6} - \frac{gt^4}{24}$$

$$v_5 = g\left(t - \frac{t^2}{2} + \frac{t^3}{6} - \frac{t^4}{24} + \frac{t^5}{120}\right)$$

After the fifth iteration, we have the approximation:

$$v(t) = gt - g\frac{t^2}{2} + g\frac{t^3}{6} - g\frac{t^4}{24} + g\frac{t^5}{120}. \tag{16}$$

3.5. For Taylor's Series Method

$$v'(t) = g - v \Rightarrow v'(0) = g$$

$$v''(t) = -v' \Rightarrow v''(0) = -g$$

$$v'''(t) = -v'' \Rightarrow v'''(0) = g$$

$$v^{(4)}(t) = -v''' \Rightarrow v^{(4)}(0) = -g$$

$$v^{(5)}(t) = -v'''' \Rightarrow v^{(5)}(0) = g$$

Then the fifth order Taylor's formula is: see (Figure 5 and Figure 6)

$$v(t) = gt - g\frac{t^2}{2} + g\frac{t^3}{6} - g\frac{t^4}{24} + g\frac{t^5}{120} \tag{17}$$

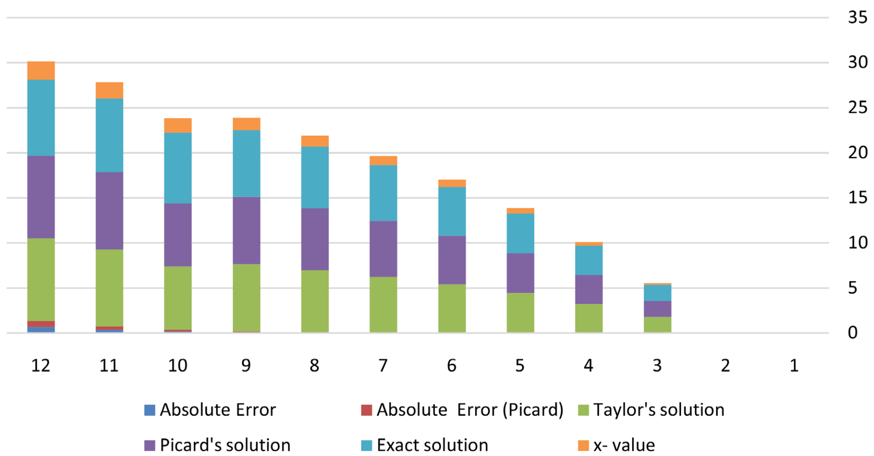

Figure 5. Comparative analysis of the estimated and ideal solutions for the proposed methods addressing the free-falling body problem ($k = m$).

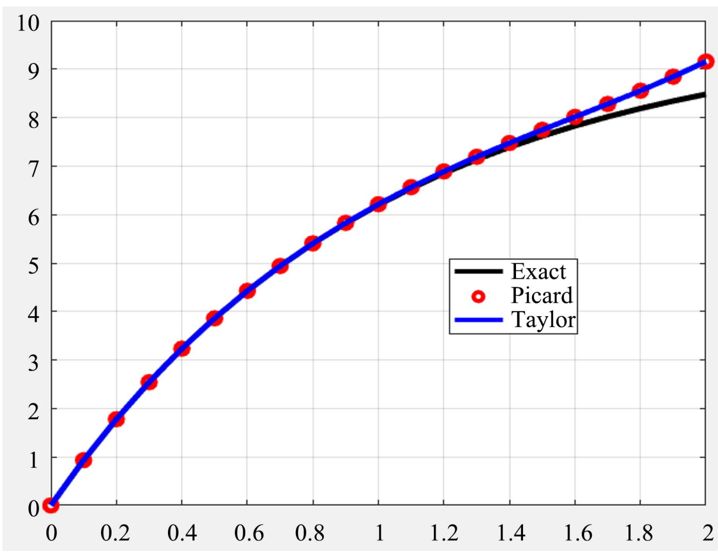

Figure 6. Comparative analysis of the estimated and ideal solutions for the proposed methods addressing the free-falling body problem ($k = m$).

The following **Table 2** gives the values of v from $t = 0$ to 2 taking $h = 0.2$.

Table 2. Numerical comparison of the approximate and perfect solutions to suggested techniques for the free-falling body problem ($k = m$).

x-value	Exact solution	Picard's solution	Taylor's solution	Absolute Error (Picard)	Absolute Error (Taylor)
0	0.0000	0.0000	0.0000	0.0000	0.0000
0.2000	1.7783	1.7783	1.7783	0.0000	0.0000
0.4000	3.2342	3.2342	3.2342	0.0001	0.0001
0.6000	4.4262	4.4267	4.4267	0.0006	0.0006
0.8000	5.4021	5.4053	5.4053	0.0032	0.0032
1.0000	6.2011	6.2130	6.2130	0.0119	0.0119
1.2000	6.8553	6.8899	6.8899	0.0346	0.0346
1.4000	7.3909	7.4761	7.4761	0.0852	0.0852
1.6000	7.8294	7.0146	7.0146	0.1852	0.1852
1.8000	8.1884	8.5549	8.5549	0.3665	0.3665
2.0000	8.4824	9.1560	9.1560	0.6736	0.6736

Case 2: Let $k \neq m$, $v(0) = 0$

$$f(t,v) = g - \frac{k}{m}v$$

$$v_1 = \int_{s=0}^{t}\left(g - \frac{k}{m}v_0\right)\mathrm{d}s = gt,$$

$$v_2 = \int_{s=0}^{t}\left(g - \frac{k}{m}v_1\right)\mathrm{d}s = gt - \frac{kgt^2}{2m}$$

$$v_3 = \int_{s=0}^{t}\left(g - \frac{k}{m}v_2\right)\mathrm{d}s = gt - \frac{kgt^2}{2m} + \frac{k^2gt^3}{6m^2}$$

$$v_4 = \int_{s=0}^{t}\left(g - \frac{k}{m}v_3\right)\mathrm{d}s = gt - \frac{kgt^2}{2m} + \frac{k^2gt^3}{6m^2} - \frac{k^3gt^4}{24m^3}$$

$$v_5 = \int_{s=0}^{t}\left(g - \frac{k}{m}v_4\right)\mathrm{d}s = gt - \frac{gt^2}{2} + \frac{gt^3}{6} - \frac{kgt^4}{24m} + \frac{k^2gt^5}{120m^2}$$

$$V(t) = gt - \frac{kgt^2}{2m} + \frac{k^2gt^3}{6m^2} - \frac{k^3gt^4}{24m^3} + \frac{k^4gt^5}{120m^4} \tag{18}$$

Now let: $m = 2, k = 4$, then Equation (10) becomes:

$$\frac{\mathrm{d}v}{\mathrm{d}t} + 2v = g \tag{19}$$

With initial condition $v(0) = 0$, so, the exact solution of such a problem is:

$$V(t) = \frac{1}{2}g - \frac{1}{2}g\mathrm{e}^{-2t}$$

3.6. For Picard Iteration Method

$$v_1(t)=\int_{s=0}^{t}(g-2v_0)\,ds=gt,$$

$$v_2(t)=\int_{s=0}^{t}(g-2v_1)\,ds=gt-gt^2$$

$$v_3(t)=\int_{s=0}^{t}(g-2v_2)\,ds=gt-gt^2+\frac{2gt^3}{3}$$

$$v_4(t)=\int_{s=0}^{t}(g-2v_3)\,ds=gt-gt^2+\frac{2gt^3}{3}-\frac{gt^4}{3}$$

$$v_5(t)=\int_{s=0}^{t}(g-2v_4)\,ds=gt-gt^2+\frac{2gt^3}{3}-\frac{4gt^4}{9}+\frac{2gt^5}{15}.$$

$$V(t)=gt-gt^2+\frac{2gt^3}{3}-\frac{gt^4}{3}+\frac{2gt^5}{15} \tag{20}$$

3.7. For Taylor's Series Method

$$v'(t)=g-2v\Rightarrow v'(0)=g$$

$$v''(t)=-2v'\Rightarrow v''(0)=-2g$$

$$v'''(t)=-2v''\Rightarrow v'''(0)=4g$$

$$v^{(4)}(t)=-2v'''\Rightarrow v^{(4)}(0)=-8g$$

$$v^{(5)}(t)=-2v^{(4)}\Rightarrow v^{(5)}(0)=16g$$

Therefore, Taylor's Series expansion is: see (Figure 7 and Figure 8)

$$v(t)=v(0)+tv(0)+\frac{t^2v''(0)}{2!}+\frac{t^3v'''(0)}{3!}+\frac{t^4v^{(4)}(0)}{4!}+\cdots$$

$$v(t)=gt-\frac{2gt^2}{2!}+\frac{4gt^3}{3!}-\frac{8gt^4}{4!}+\frac{16gt^5}{5!}$$

$$V(t)=gt-gt^2+\frac{2gt^3}{3}-\frac{gt^4}{3}+\frac{2gt^5}{15} \tag{21}$$

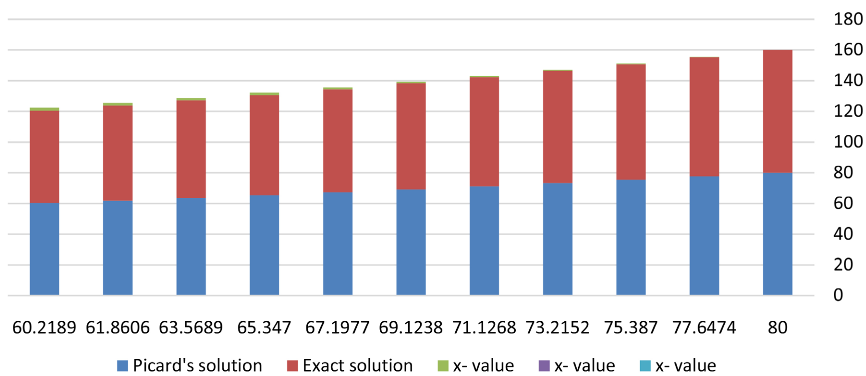

Figure 7. Comparison of Exact solution and numerical solution for ($k \neq m$) related to the equation of free-falling body.

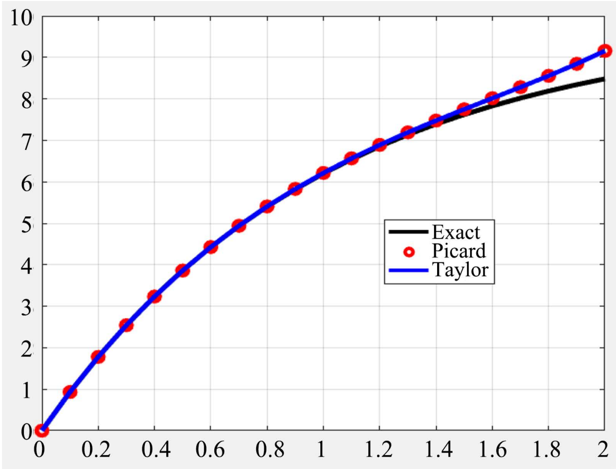

Figure 8. Comparative analysis of the estimated and ideal solutions for the proposed methods addressing the free-falling body problem ($k = m$).

The following **Table 3** gives the values of v from $t = 0$ to 2 taking $h = 0.2$.

Table 3. Presents a numerical comparison of the exact and approximate solutions of the proposed approaches to the equation of a free-falling body with $k \neq m$.

x-value	Exact solution	Picard's solution	Taylor's solution	Absolute Error (Picard)	Absolute Error (Taylor)
0	0.0000	0.0000	0.0000	0.0000	0.0000
0.2000	1.6171	1.6171	1.6171	0.0000	0.0000
0.4000	2.7010	2.7026	2.7026	0.0016	0.0016
0.6000	3.4276	3.4450	3.4450	0.0173	0.0173
0.8000	3.9147	4.0073	4.0073	0.0926	0.0926
1.0000	4.2412	4.5780	4.5780	0.3368	0.3368
1.2000	4.6400	5.4208	5.4208	0.9607	0.9607
1.4000	4.6067	6.9249	6.9249	2.3181	2.3181
1.6000	4.7051	9.6553	9.6553	4.9503	4.9503
1.8000	4.7710	14.4033	14.4033	9.6323	9.6323
2.0000	4.8152	22.2360	22.2360	17.4208	17.4208

3.8. Corollary

For the standard form of a first-order ordinary differential equation if the left-hand side is a constant then a numerical solution of both Picard and Taylor methods are identical.

3.9. Example

Find the numerical solutions if: $\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - 2y$, with $y(0) = 1$ using Picard and

Taylor methods.

Solution

For Picard method:	For Taylor's method
	$y' = -1$
$y_1 = 1 - x$	$y'' = -2y' \Rightarrow y''(0) = 2$
$y_2 = 1 - x + x^2$	$y''' = -2y'' \Rightarrow y'''(0) = -4$
$y_3(x) = 1 - x + x^2 - \frac{2}{3}x^3$	$y_3(x) = 1 - x + x^2 - \frac{2}{3}x^3$

4. Newton's Law of Cooling

The Newton law of cooling states that: the rate of change of the temperature of an object is proportional to the difference between its temperature and the temperature of a surrounding, so, the rate of loss of heat from any object is directly proportional to the difference in the temperature of the object and its surrounding.

We assume that there is a hot object whose temperature is T and the surrounding temperature is T_s, which is of lower temperature, then

$$-\frac{\mathrm{d}Q}{\mathrm{d}t} \propto T - T_s$$

$$-\frac{\mathrm{d}Q}{\mathrm{d}t} = k(T - T_s) \tag{22}$$

Since the formula of the heat loss is:

$$\mathrm{d}Q = ms\mathrm{d}T \tag{23}$$

Then from (22) and (23), we get:

$$-\frac{ms\mathrm{d}T}{\mathrm{d}t} = k(T - T_s)$$

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{-k}{ms}(T - T_s) \tag{24}$$

By putting: $c = \frac{-k}{ms}$, we get: $\frac{\mathrm{d}T}{\mathrm{d}t} = c(T_k - T_s)$, $c < 0$, where $\frac{\mathrm{d}T}{\mathrm{d}t}$ is the rate of heat lost, and c is the coefficient of heat transfer. Then we get the following linear differential equation:

$$\frac{\mathrm{d}T}{\mathrm{d}t} - cT = -cT_s \tag{25}$$

With $T(0) = T_0$, here we have the following function: $f(t,T) = cT - cT_s$.

4.1. Picard Iteration Method

By integrating (25), we get

$$\int_{T_0}^{T} \mathrm{d}T = \int_{t_0}^{t} f(t,T)\mathrm{d}t \tag{26}$$

So,

$$T_{k+1} = T_0 + c\int_{t_0}^{t} (T_k - T_s)\mathrm{d}s \tag{27}$$

for first approximation T_1 to the solution, we put $T = T_0$, and integral (26), we get:

$$T_1 = T_0 + c\int_0^t (T_0 - T_s)\,\mathrm{d}s\,,$$

For a second approximation T_2, we put $T = T_1$ in $f(t,T)$, and integral (26), we obtain:

$$T_2 = T_0 + c\int_0^t (T_1 - T_s)\,\mathrm{d}s$$

Continuing this process, a sequence of functions of *t*, *i.e.* $T_1, T_2, T_3, \cdots$ is obtained, and each giving a better approximation of the desire solution than the preceding one.

4.2. Taylor's Series Method

We have: $f(t,T) = cT - cT_s$, then buy differentiating we get:

$$f'(t,T) = cT' - c\frac{\partial f}{\partial T_s}\cdot\frac{\mathrm{d}T}{\mathrm{d}t}$$

differentiating this successively, we can get: T''', $T^{(4)}$, $T^{(5)}$, $\cdots$, etc.

Put $t = 0$, then the values of: $T'(0)$, $T''(0)$, $T'''(0)$ can be obtained, therefore the Taylor's series:

$$\begin{aligned} T(t) &= T_0 + (t - t_0)T'T'' \\ &= cf_t' - cf_{Ts}'\cdot f \\ &= T(0) + \frac{t-t_0}{1!}T'(0) + \frac{(t-t_0)^2}{2!}T''(0) + \frac{(t-t_0)^3}{3!}T'''(0) + \cdots \end{aligned} \tag{28}$$

Gives the value of T for every value of t for which converges.

4.3. Numerical Observations

Case 1: Substitute $c = -0.2$ in the Equation (25) we get: $\frac{\mathrm{d}T}{\mathrm{d}t} + 0.2T = 0.2T_s$, with the initial value $T(o) = T_0$, then the exact general solution is: $T(t) = T_s + \lambda \mathrm{e}^{-0.2t}$, where λ is a constant, and $T(o) = T_s + \lambda$, then $\lambda = T(o) - T_s$, so

$$T(t) = T_s + (T_0 - T_s)\mathrm{e}^{-0.2t}\,,$$

Let $T_0 = 80$ and $T_s = 20$, then we get the exact solution:

$$T(t) = 20 + 60\mathrm{e}^{-0.2t}\,.$$

1) Picard Iteration Method

From Equation (26) we have the following approximate solutions:

$$T_1 = 80 - 12t$$

$$T_2 = 80 - 12t + \frac{6}{5}t^2$$

$$T_3 = 80 - 12t + \frac{6}{5}t^2 - \frac{2}{25}t^3$$

$$T_4 = 80 - 12t + \frac{6}{5}t^2 - \frac{2}{25}t^3 + \frac{1}{250}t^4$$

$$T_5 = 80 - 12t + \frac{6}{5}t^2 - \frac{2}{25}t^3 + \frac{1}{250}t^4 - \frac{1}{6250}t^5$$

2) For Taylor's Series Method (Figure 9 and Figure 10)

Since $c = -0.2$, then

$$T'(t) = -0.2T + 0.2T_s \Rightarrow T'(0) = -12$$

$$T''(t) = -0.2T'(t) \Rightarrow T''(0) = \frac{12}{5}$$

$$T'''(t) = -0.2T''(t) \Rightarrow T'''(0) = -\frac{12}{25}$$

$$T''''(t) = -0.2T'''(t) \Rightarrow T''''(0) = -\frac{12}{125}$$

$$T'''''(t) = -T''''(t) \Rightarrow T'''''(0) = -\frac{12}{625}$$

$$T_5 = 80 - 12t + \frac{6}{5}t^2 - \frac{2}{25}t^3 + \frac{1}{250}t^4 - \frac{1}{6250}t^5.$$

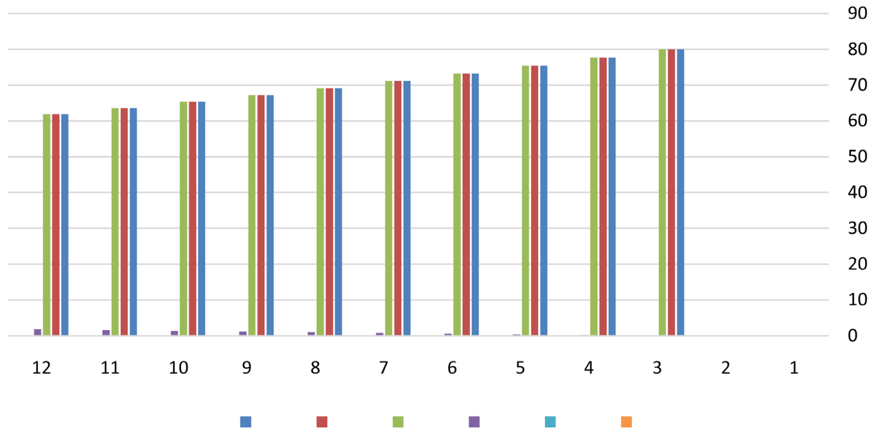

Figure 9. Compare the exact and approximate solutions of two proposed methods related to Newton's law of cooling.

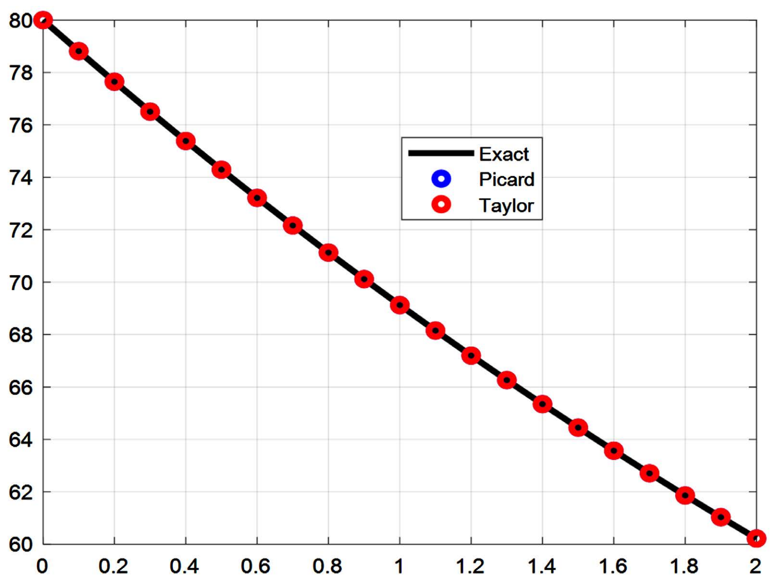

Figure 10. Compare the exact and the approximate solutions of two proposed methods related to Newton's law of cooling.

The following Table 4 gives the values of v from $t = 0$ to 2 taking $h = 0.2$.

Table 4. Numerical comparison between exact and approximate solutions of proposed two methods related to the equation of Newton's law of cooling.

x-value	Exact solution	Picard's solution	Taylor's solution	Absolute Error (Picard)	Absolute Error (Taylor)
0	80.0000	80.0000	80.0000	0.0000	0.0000
0.2000	77.6474	77.6474	77.6474	−0.0000	−0.0000
0.4000	75.3870	75.3870	75.3870	−0.0000	−0.0000
0.6000	73.2152	73.2152	73.2152	−0.0000	−0.0000
0.8000	71.1268	71.1268	71.1268	−0.0000	−0.0000
1.0000	69.1238	69.1238	69.1238	−0.0000	−0.0000
1.2000	67.1977	67.1977	67.1977	−0.0000	−0.0000
1.4000	65.3470	65.3470	65.3470	−0.0000	−0.0000
1.6000	63.5689	63.5689	63.5689	−0.0001	−0.0001
1.8000	61.8606	61.8606	61.8606	−0.0002	−0.0002
2.0000	60.2192	60.2189	60.2189	−0.0003	−0.0003

5. Discussion and Results

The results of problem one is displayed in Table 1 and graphically in Figure 1, and Figure 2 and the approximate solutions and the error estimation are calculated using the MATLAB program, we observed that the numerical solutions are more effective and reveal that amount of error is maximum for Picard's than Taylor's method, so there is a difference between these two methods compared to the exact solution in Favor of Taylor's method. The results of the equation that a rising from the problem of free—a falling body are displayed in Table 2 and Figure 5 and Figure 6, also the numerical solutions and the absolute error were calculated using the MATLAB program and we noted that the numerical solutions of Picard and Taylors's are identical and a powerful mathematical tool and this identity is attributed to all first order differential equations of constant function on the right-hand side in case of standard form.

The comparison between the numerical solutions of Picard and Taylor's Methods to the third problem a rising from Newton's law of cooling was displayed in Table 4 with their graph in Figure 8 and Figure 9 respectively, and we noticed that these solutions were closely matched to the exact solution especially in a case of constant function at the right-hand side of the first order ordinary differential equation.

Taylor's method is simple, more effective, and more accurate for the following reasons:

1) This method relies on calculating the values of the upper-order derivatives of the function, which are easy to find.

2) By looking at the three curves, we found that the curve of the numerical solution according to Taylor's method is closer to the curve of the exact solution.

3) By calculating the error committed, which represents the absolute value of the difference between the numerical solution and the exact solution, we found that the value of the error committed using the Taylor method is less than the value of the error committed using the Picard method.

For example, consider the following three solutions:

1) For the exact solution: $y(0.4) = 0.3348$.

2) For the Taylor solution: $y(0.4) = 0.4548$.

3) For the Picard solution: $y(0.4) = 0.4572$.

6. Conclusion

In this paper, Picard and Taylor's method is used for solving the First Order differential equations with initial value problems, especially in some applications arising in science. The numerical solutions obtained by the two proposed methods are in good agreement with the exact solution and the accuracy depends on the equation, and these come from the computational view of point, we can conclude that Taylor's method is more reliable, efficient and easy to use. So, in our subsequent, we shall examine the comparison of Taylor's method with Rung Kutta, because of similarity, efficiency and well suited for I.V.P OF O.D.E.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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