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Propagation and Pinning of Travelling Wave for Nagumo Type Equation

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Abstract

In this paper, we study the propagation and its failure to propagate (pinning) of a travelling wave in a Nagumo type equation, an equation that describes impulse propagation in nerve axons that also models population growth with Allee effect. An analytical solution is derived for the traveling wave and the work is extended to a discrete formulation with a piecewise linear reaction function. We propose an operator splitting numerical scheme to solve the equation and demonstrate that the wave either propagates or gets pinned based on how the spatial mesh is chosen.

Keywords

Operator Splitting, Travelling Wave, Piecewise Reaction, Nagumo Equation, Pinning, Finite Differences

1. Introduction

The Nagumo equation is a well-studied mathematical model, especially, to understand impulse propagation in nerve axons and to describe the growth of a population with Allee effect [1] [2]. The equation is written as

$$u_t = u_{xx} + f(u), \tag{1}$$

where $f(u)$ is a cubic polynomial of the form $u(1-u)(u-a)$ with three roots, namely, $u=0, u=a$ and $u=1$, with $0<a<1/2$. Depending on the phenomenon that one is trying to model, $u(x,t)$ is either the action potential or the population density at the location x at time t. Similarly, depending on the phenomenon, a is either the doping parameter or the threshold value for the population to grow. If one considers (1) as an initial value problem in the infinite x domain, it is possible to look for a travelling wave solution. In fact, it can be shown that a travelling wave solution can be obtained analytically.

Let us introduce the transformation $z = x - ct$ where c is the speed of the solution profile with asymptotic conditions $u(-\infty) \to 1$ and $u(\infty) \to 0$. Now, suppose there is a solution $u(z)$, such that $u_z = u'$ is in the following form,

$$u' = Au + Bu^n. \tag{2}$$

Then, choosing $n = 2$, for $u_{zz} = u''$, we get,

$$u'' = A^2 u + 3ABu^2 + 2B^2 u^3. \tag{3}$$

Since $z = x - ct$, $u_t = -cu_z$ and $u_{xx} = u_{zz}$.

Rewriting Equation (1) in terms of $u(z)$ with its derivatives we have,

$$-cu_z = u_{zz} - u^3 + (1+a)u^2 - au.$$

This becomes,

$$u'' + cu' = u^3 - (1+a)u^2 + au. \tag{4}$$

Plugging in u' and u'' (with $n = 2$) into (4) gives,

$$2B^2 u^3 + (3AB + Bc)u^2 + \left(Ac + A^2\right)u = u^3 - (1+a)u^2 + au. \tag{5}$$

Solving for like term coefficients, we obtain two cases for A and B.

Case 1: When $B = \frac{1}{\sqrt{2}}$, we have $A = -\frac{1}{\sqrt{2}}$ and $c = \frac{1}{\sqrt{2}}$ (1-2a).

Case 2: When $B = -\frac{1}{\sqrt{2}}$, we have $A = \frac{1}{\sqrt{2}}$ and $c = \frac{1}{\sqrt{2}}$ (2a-1).

Now, the traveling wave solution in case 1 has the following analytical form [3],

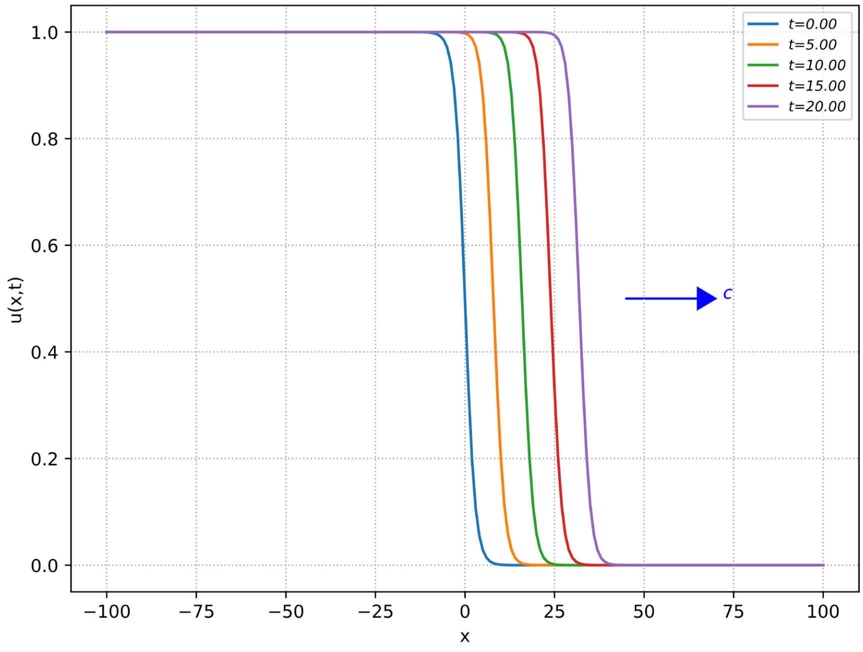
$$u(x,t) = \frac{1}{1 + \mathrm{e}^{\frac{x-ct}{\sqrt{2}}}}, \tag{6}$$

with wave speed, $c = \frac{1}{\sqrt{2}}(1-2a)$.

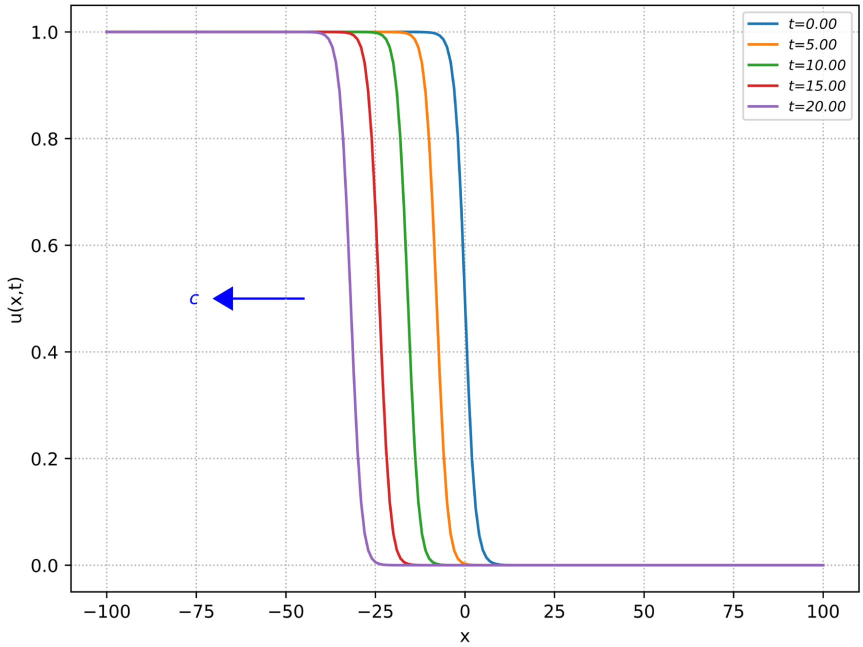
Note that the solution (6) matches the asymptotic Huxley solution [3] [4]. The propagation of the travelling wave (6) is presented in **Figure 1**. One could see that the wave travels to the right when $a < 0.5$ but reverses its direction of propagation when $a > 0.5$.

This brings up an interesting question—what happens to the wave when $a = 0.5$? From (6), it is obvious that the wave speed $c = 0$ when $a = 0.5$. That means that the Nagumo wave will be a standing wave that does not propagate. It should be noted that in recent years, a number of researchers have studied the discrete Nagumo equation with different cubic nonlinearities and have identified parameter regions for propagation failure, also known as pinning, of the Nagumo wave [5] [6] [7] [8] [9].

Our goal in this paper is to study the Nagumo equation both in the continuous and discrete forms with a piece-wise linear reaction term in order to better understand the propagation or failure to propagate of the Nagumo wave. One should note that the novel idea of using a piece-wise linear reaction term in place of a cubic reaction term was first introduced by McKean [9]. In a recent work,



(a)



(b)

Figure 1. Travelling wave profile when (a) $a = 0.25$ and (b) $a = 0.75$.

Fath [10] used the same idea to study the Nagumo equation with Fourier transform methods. One should note that in terms of nerve axons, any failure of impulse propagation can be a cause for concern. This could indicate to health professionals that possibly, there is a damage in a nerve axon that needs remedied. On the other hand, with regards to population growth with Allee effect, propagation failure could mean stalling of invasion waves or migration waves of either plant or animal species. This may indicate to ecologists which type of habitats may become inhospitable due to climate change for the survival of certain species.

2. Nagumo Equation with Piece-Wise Linear Reaction

Now, let us consider the Nagumo equation as an initial value problem as follows,

$$u_t = u_{xx} + f(u),\ -\infty < x < \infty, 0 < t < T$$
$$u(x,0) = h(x), \tag{7}$$

where the initial condition is given by $h(x)$ and the reaction term is the piece-wise linear function,

$$f(u) = \begin{cases} -u, & 0 \le u \le a \\ 1-u, & a \le u \le 1 \end{cases} \tag{8}$$

and note that $f'(u) < 0$ and $\int_0^1 f(u)\mathrm{d}u \ne 0$.

In order to find a travelling wave solution for (7), as shown in Section 1, we introduce the travelling wave coordinate $z = x - ct$. Then, in terms of z, one can write (7) as

$$u'' + cu' - u = 0,\ 0 \le u \le a. \tag{9}$$

and

$$u'' + cu' + (1-u) = 0,\ a \le u \le 1. \tag{10}$$

Note that ' means d/dz. Also, from (9) and (10), we see that u = 0 and u = 1 are the steady states in this model.

Solving the above ordinary differential equations in the two regions of u, in [0, a] and [a, 1] along with the asymptotic conditions $u(-\infty) = 1$ and $u(\infty) = 0$, we get

$$u(z) = \begin{cases} C\mathrm{e}^{r_2 z}, & 0 \le u \le a \\ D\mathrm{e}^{r_1 z} + 1, & a \le u \le 1 \end{cases} \tag{11}$$

where C and D, are constants to be determined.

If we are to have a smooth wave solution then, we need the values of both u and u' from the branches, $u \le a$ and $u \ge a$ to match at u = a. Then we obtain,

$$C = \frac{-c - \sqrt{c^2+4}}{2\sqrt{c^2+4}} + 1$$

$$D = \frac{-c - \sqrt{c^2+4}}{2\sqrt{c^2+4}}$$

$$a = \frac{r_1}{r_1 - r_2}$$

where $r_{1,2} = \frac{1}{2}\left(-c \pm \sqrt{c^2+4}\right)$.

Now, without loss of generality, letting $u = a$ at $z = 0$, gives $a = C$. Consequently, the wave speed c can be written as,

$$c = \frac{1-2a}{\sqrt{a(1-a)}} \tag{12}$$

and the travelling wave solution is,

$$u(z) = \begin{cases} a\mathrm{e}^{\frac{z}{c}}, & z \ge 0 \\ 1-(1-a)\mathrm{e}^{-\frac{z}{c}}, & z \le 0 \end{cases} \tag{13}$$

It is instructive to compare this wave solution obtained in (13) with the wave solution (6) found for the cubic reaction term. Even though the mathematical formulations look different, in both cases, the pinning of the wave will occur at $a = 0.5$. In the next section, we develop a discrete formulation, using finite differences, to study the initial value problem (7) further.

3. Operator Splitting Scheme

Here, we propose an operator splitting formulation [3], that is known to be a versatile scheme, where (3) is split into two sub-problems, based on the reaction and diffusion processes such that,

$$u_t = \begin{cases} -u, & 0 \le u \le a \\ (1-u), & a \le u \le 1 \end{cases} \tag{14}$$

and

$$u_t = u_{xx}. \tag{15}$$

We now proceed to solve the (14) and (15), in the time domain t, sequentially over subintervals $[t_n, t_{n+1}]$. Here, $t_n = nk$ is the time at time level n, where n is an integer and k is the time step in the discrete formulation. The numerical schemes that correspond to (14) and (15) can be written using an explicit finite difference method for the numerical approximation, of $u(x,t)$ as,

$$u_m^* = u_m^n + kf\left(u_m^n\right) \tag{16}$$

and

$$u_m^{n+1} = u_m^* + r\left(u_{m+1}^* - 2u_m^* + u_{m-1}^*\right) \tag{17}$$

where, $x_m = mh$ is the location at spatial level m, where m is an integer and h is the spatial mesh in the discrete formulation. In (16), u_m^* can be thought of as an intermediate value in the sub-interval $[t_n, t_{n+1}]$ and $r = k/h^2$. Here, our computational domain is [−100, 100].

Next, plug in the initial condition $u_m^0 = u(x,0) = h(x)$ into (16) to find u_m^*, *i.e.*, the estimate for the first-time step, such that,

$$u_m^* = u_m^0 + kf\left(u_m^0\right). \tag{18}$$

Our initial condition $h(x)$ is derived from (13).

Then, use (17) to obtain the numerical solution at the next time level, and the numerical solution at any time level can be obtained by repeating the process. Note that the numerical scheme (17) with $u_m^* = u_m^n$ is,

$$u_m^{n+1} = u_m^n + r\left(u_{m+1}^n - 2u_m^n + u_{m-1}^n\right) \tag{19}$$

We can look at the stability of this scheme using von Neumann stability analysis.

With the ansatz,

$$u_m^n = \mathrm{e}^{\alpha nk}\mathrm{e}^{i\beta mh}, \quad \left(i = \sqrt{-1}\right) \tag{20}$$

if $\left|\mathrm{e}^{\alpha k}\right| < 1$ for any *β*, then the method is said to be stable. Substituting (20) into (19) gives,

$$\mathrm{e}^{\alpha(n+1)k}\mathrm{e}^{i\beta mh} = \mathrm{e}^{\alpha nk}\mathrm{e}^{i\beta mh} - \frac{k}{h^2}\left[\mathrm{e}^{\alpha nk}\mathrm{e}^{i\beta(m-1)h} - 2\mathrm{e}^{\alpha nk}\mathrm{e}^{i\beta mh} + \mathrm{e}^{\alpha nk}\mathrm{e}^{i\beta(m+1)h}\right].$$

Simplifying the above equation yields,

$$\mathrm{e}^{\alpha k} = 1 + 2r + r\left[\mathrm{e}^{-i\beta h} + \mathrm{e}^{i\beta h}\right].$$

Recall that $r = k/h^2$. Further simplification leads

$$\mathrm{e}^{\alpha k} = 1 - 4r\sin^2\left(\frac{\beta h}{2}\right)$$

The goal is to verify that $\left|\mathrm{e}^{\alpha k}\right| < 1$, *i.e.*,

$$\left|1 - 4r\sin^2\left(\frac{\beta h}{2}\right)\right| < 1. \tag{21}$$

Solving the inequality (21), we get,

$$0 < r < \frac{1}{2}. \tag{22}$$

So, the von Neumann stability analysis shows that the numerical scheme (17) will be stable provided that the time step, *k*, and the spatial mesh, *h*, are chosen such that $0 < r < 1/2$. Also, making use of Taylor series expansions on the numerical schemes (16) and (17), it is easy to show that the method is first order accurate in time and second order accurate in space.

Propagation and Pinning

Now, consider the discrete formulation of our problem $u_t = u_{xx} + f(u)$ such that,

$$\left(u_m\right)_t = \frac{1}{h^2}\left(u_{m+1} - 2u_m + u_{m-1}\right) + f\left(u_m\right). \tag{23}$$

Then, following [10] and our analysis in Section 2, define the values of *a* as,

$$a = \frac{1}{2}\left(1 \pm \frac{h}{\sqrt{h^2 + 4}}\right). \tag{24}$$

We now proceed to study the pinning region by observing the relationship between the spatial mesh, *h*, and the value *a*. **Figure 2** shows the pinning region,

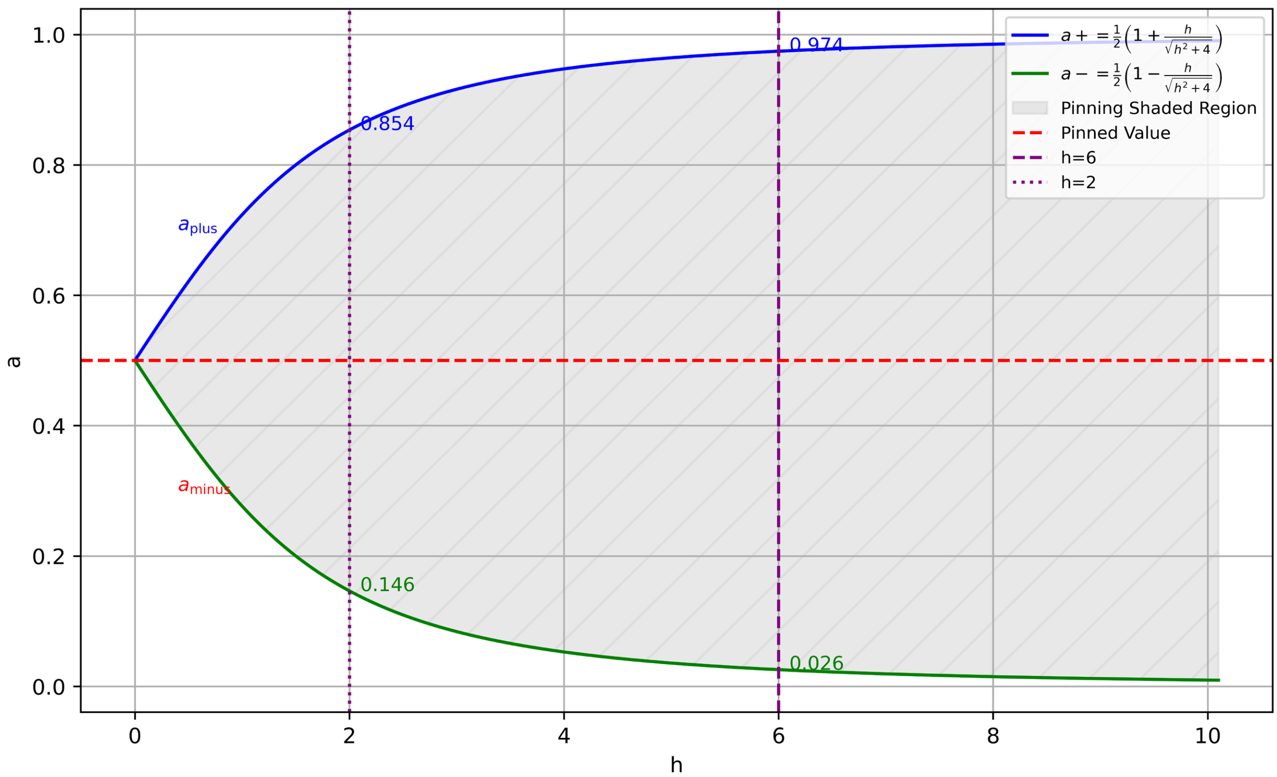


Figure 2. Pinning region for different ranges of h for $0 \le a \le 1$.

the shaded area, where the wave propagation fails. It should be noted that according to Keener [4] wave propagation fails in semi-discrete systems because in such systems many stationary solutions exist. As can be seen in **Figure 2**, regardless of the range of *h*, the pinning region falls in the shaded region for values of *a* below and above 0.5. At this juncture, it should be reminded that in the continuous model (as seen in Section 2), pinning happens only for one *a* value and that is ½.

Let us re-write (24) as,

$$a_- = \frac{1}{2}\left(1 - \frac{h}{\sqrt{h^2+4}}\right) \text{ and } a_+ = \frac{1}{2}\left(1 + \frac{h}{\sqrt{h^2+4}}\right).$$

As *h* grows very large, a_- approaches 0 and a_+ approaches 1, and when $h = 0$, it is clear that $a = 0.5$, which indicates the profile changes its direction. From the figure, when $h = 2$, the range of the pinning region is $0.146 < a < 0.854$, and when $h = 6$, the region is $0.026 < a < 0.974$.

We are able to verify the pinning behavior of the Nagumo wave solution using our operator splitting numerical scheme for appropriate choice of spatial meshes, *h*. In **Figure 3**, one can observe that when $h = 2$, the flat region coincides with the pinned region in **Figure 2**, regardless of chosen *a* values within the range. Additional computations were performed for various *h* values and a similar pattern was observed for propagation failure.

In **Figure 4**, we present numerical solutions for $a \approx 0.146$ and $h = 2$, and $a \approx 0.027$ and $h = 6$. In the first case, we are very close to the pinning region,

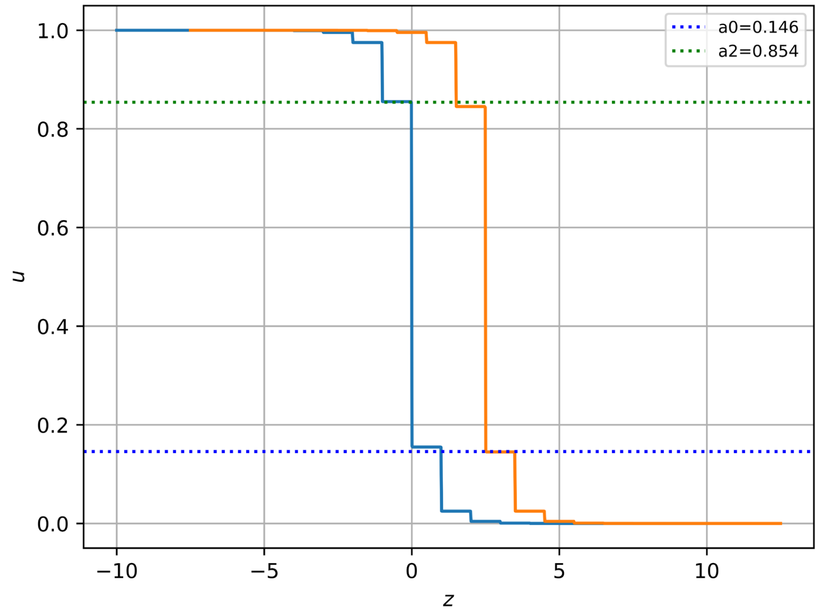
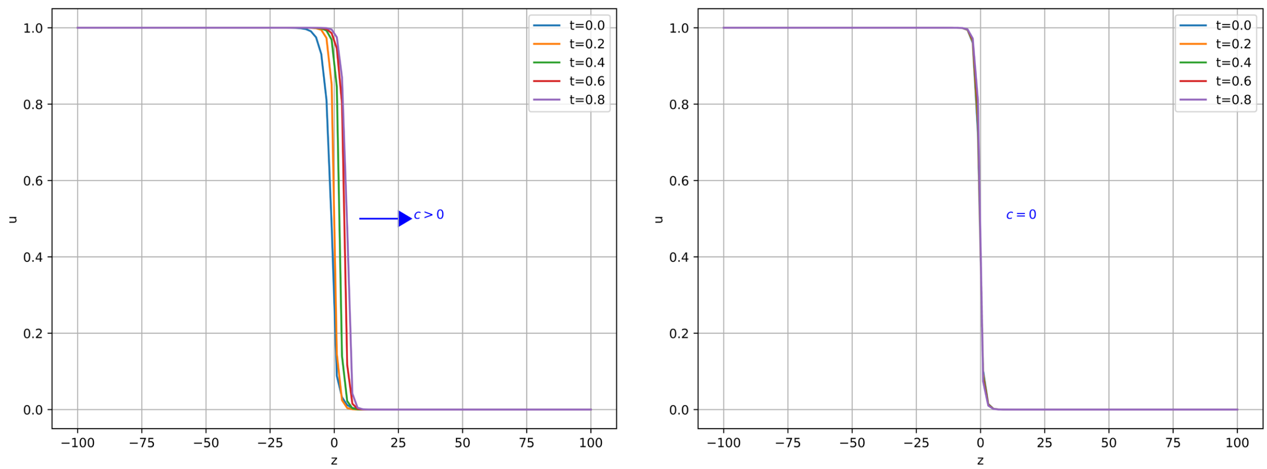


Figure 3. Travelling Nagumo wave profile propagation failure at $h = 2$.



(a) (b)

Figure 4. Numerical solution. (a) $a \approx 0.146$, $h = 2$. (b) $a \approx 0.027$ and $h = 6$.

and we would expect the propagation to exist weakly. Whereas, when choosing $a = 0.027$ and $h = 6$, we fall in the shaded region and thus, there should be propagation failure [7] [8]. **Figure 4** illustrates these expected behaviors.

4. Conclusions

In this paper, using analytical techniques and an operator splitting numerical scheme, we show that it is possible to have different solution behaviors when looking at a continuous problem and its discrete counterpart. In particular, we show that for the Nagumo model with a linear piecewise reaction term, the discrete formulation has a whole pinning region for a range of values for the doping parameter a where the wave solution fails to propagate.

This is in contrast to the continuous formulation where the wave solution fails to propagate only when the doping parameter *a* takes the *unique value* ½. In practical scenarios, depending on the experimental data available and the phenomena that one is trying to model, the resulting model can be either a continuous or a discrete model. Therefore, it is imperative that practitioners understand any contrasting behaviors that could exist (as in our model here) between the continuous and discrete cases, so that reasonable and correct management decisions could be made. Even though, for the ease of analysis and exposition, our study deals with the Nagumo model with a linear piecewise reaction term, the ideas presented in this paper and the references herein can be readily extended to other reaction diffusion models.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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