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Modeling a Periodic Signal Using Fourier Series

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Abstract

This paper covers the concept of Fourier series and its application for a periodic signal. A periodic signal is a signal that repeats its pattern over time at regular intervals. The idea inspiring is to approximate a regular periodic signal, under Dirichlet conditions, via a linear superposition of trigonometric functions, thus Fourier polynomials are constructed. The Dirichlet conditions, are a set of mathematical conditions, providing a foundational framework for the validity of the Fourier series representation. By understanding and applying these conditions, we can accurately represent and process periodic signals, leading to advancements in various areas of signal processing. The resulting Fourier approximation allows complex periodic signals to be expressed as a sum of simpler sinusoidal functions, making it easier to analyze and manipulate such signals.

Keywords

Fourier Series, Signal, Periodic, Period, Frequency, Sine, Cosine, Convergence

1. Introduction

In the eighteenth century, a significant problem emerged that aided both mathematical analysis and other areas, including physics. The problem was to express a periodic function as an infinite series of sine and cosine functions. In mathematics, infinite series are crucial and extensively used in calculators and computers to evaluate values of various functions. The Fourier series, an infinite series initiated by attempts to solve heat conduction problems in a bar, is of great importance. The theory of Fourier's series, fascinating in itself and vital in countless applications in both pure and applied mathematics, is addressed in almost all treatises on higher analysis.

Fourier series is a fundamental tool in the field of mathematics and signal processing. It allows us to decompose complex periodic functions into an infinite sum of simpler periodic functions (sine and cosine waves) providing a powerful framework for analyzing and understanding various phenomena.

In the world of signals, Fourier series plays a crucial role in analyzing and manipulating signals. By decomposing a complex signal into its constituent sinusoidal components using Fourier series analysis, engineers and scientists can gain valuable insights into the frequency content and characteristics of a signal. Signals are often represented as time-varying functions, and understanding their frequency content is vital for tasks such as filtering, compression, and modulation. Fourier series provides a systematic approach to extract the frequency components of a signal, allowing us to analyze its spectral characteristics and make informed decisions about signal processing techniques.

For instance, in Medical imaging, such as MRI (Magnetic Resonance Imaging) and CT (Computed Tomography) scans, can benefit from Fourier series-based image compression techniques. By compressing the images, medical professionals can efficiently store and transmit patient data, enabling rapid diagnoses and effective treatment planning while minimizing data storage requirements. This is particularly critical in telemedicine, where medical images need to be transmitted remotely for consultation and analysis.

In telecommunications, Fourier analysis is used to decompose an electrical signal, such as a voice or data signal, into its different frequency components. For example, when you make a phone call, Fourier analysis allows the signal to be transmitted through the network by breaking it down into different frequencies that can be sent and received. This decomposition helps improve the quality and reliability of the transmitted signal.

In audio Engineering, when you adjust the equalizer settings on a music player or sound system, Fourier analysis is employed to manipulate and balance different frequency components of the audio signal. This allows you to control the bass, treble, and other aspects of the sound to achieve the desired audio output.

In optics, when light passes through a narrow slit or encounters an obstacle, it diffracts and creates a pattern of bright and dark areas. Fourier analysis can be applied to analyze this diffraction pattern and determine the size and shape of the slit or obstacle. This understanding is crucial in various fields, such as designing optical systems for microscopes or telescopes and fabricating gratings for spectroscopy applications.

Fourier analysis finds significant applications in various areas of our lives specially for periodic signals where we are interested. Our aim in this work is to introduce the Fourier series approximation of a periodic signal. First, the definition of a periodic signal is illustrated in Section 2, followed by a representation of the Fourier series in Section 3. Additionally, we discuss applications of Fourier series in the context of heartbeat signals in Section 4.1, concluding in the last section, Section 5.

2. Signal

Signals are like messages that we send from one place to another. They are functions that convey information about a phenomenon. Signals can represent various types of data, such as sound, images, video, or any other measurable quantity that changes over time or space. Signals can be classified into various types based on their characteristics and applications.

- **Analog signals** are used in audio systems to reproduce sounds or in measurement systems, such as temperature sensors and pressure gauges, to accurately determine physical quantities.
- **Digital signals** are extensively used in modern communication systems, such as the internet and mobile networks.
- **Electromagnetic signals** are used in a vast range of applications, including wireless communication, radar systems, and satellite communication
- **Optical signals** are used in fiber optic communication systems, where they enable high-speed data transfer over long distances with minimal loss and interference.
- …etc

The representation of a signal as a plot of amplitude versus time constitutes the waveform. It is said to be a periodic signal if it has a definite pattern and repeats itself at regular intervals of time. Whereas, the signal which does not repeat at regular intervals is known as an aperiodic signal or non-periodic signal. For example, the heartbeat is periodic, whereas the human vocal mechanism that produces speech is aperiodic. This work sheds light on periodic signals, as clarified in the following Subsection 2.1.

2.1. Periodic Signals

A periodic signal is a repetitive motion that occurs in fixed time intervals. So, the signal returns to its initial point after a fixed amount of time. Motions of ponies in a go-round, forces on the needle in a sewing machine are periodic signals.

A signal $f(t)$ is periodic if there is a number T such that for all t, we have:

$$f(t+T)=f(t) \tag{1}$$

and every integer multiple of the fundamental period is also a period:

$$f(t+nT)=f(t) \quad n=0,\pm1,\pm2,\cdots \tag{2}$$

The smallest positive number T that satisfies Equations (1) and (2) is the period, defining the duration of one complete cycle. The periodicity condition means that the shape of one cycle determines the graph everywhere; the shape is repeated over and over. Figure 1 illustrates the periodic signals graphically.

Focusing on periodic signals, we need to define some characteristics as frequency, angular frequency and amplitude.

2.1.1. Signal Frequency *f*

Signal frequency is the main tool for describing the oscillatory behavior of signals. The frequency is a measure of how rapidly a signal oscillates or repeats within a given unit of time. The unit of measurement for frequency is hertz (Hz), where 1 Hz represents one cycle per second. For example, if a signal has a frequency of 100 Hz, it means that 100 cycles of the signal occur in one second.

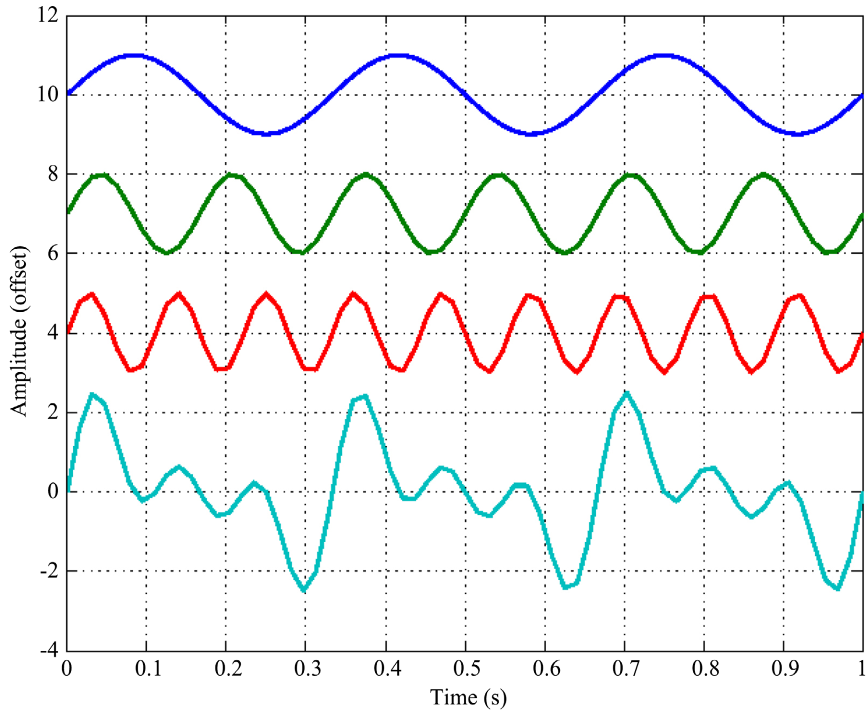

Figure 1. Periodic signals.

$$f = \frac{1}{T} \begin{cases} T \text{ in s} \\ f \text{ in Hz} \end{cases} \quad \Leftrightarrow \quad T = \frac{1}{f} \tag{3}$$

The smaller the period, the higher the frequency and vice versa. If a signal does not change at all, it never completes a cycle, so its frequency is 0 Hz. If a signal changes instantaneously, its period is zero, and the frequency, being the inverse of the period, is infinite or unbounded. Our work does not consider these cases.

Figure 2 illustrated the high and low frequency graphically.

2.1.2. Signal Angular Frequency *w*

Angular frequency, denoted by w (omega), is a measure of how quickly an object rotates or oscillates in circular motion[1]. It is often used in the context of oscillatory and wave phenomena. Angular frequency is related to the regular frequency f by the equation:

$$w = \frac{2\pi}{T} \begin{cases} 2\pi \text{ in rad} \\ T \text{ in s} \end{cases} \quad \Leftrightarrow \quad T = \frac{2\pi}{w} \text{ and } w = 2\pi f \tag{4}$$

Angular frequency is measured in radians per second (rad/s). One complete revolution or cycle corresponds to 2π radians. When the angular frequency increases, the signal moves more quickly, and the distance it covers from its starting point gets largers.

Figure 3 illustrated angular frequency graphically.

[1]https://www.pinterest.jp/pin/891501688711568234/

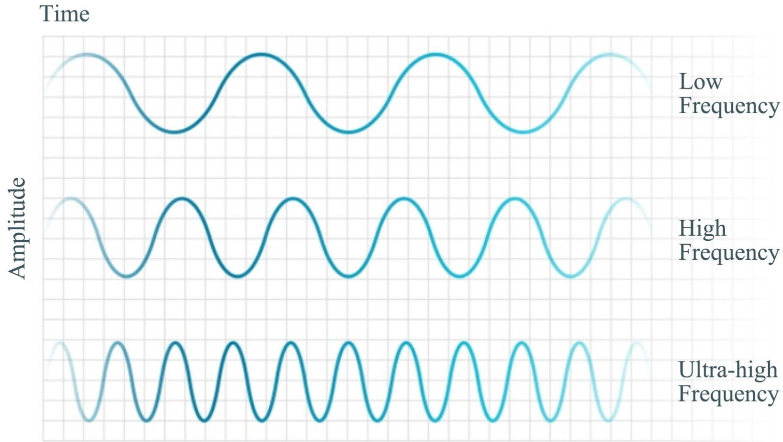

Figure 2. Low and high frequency.

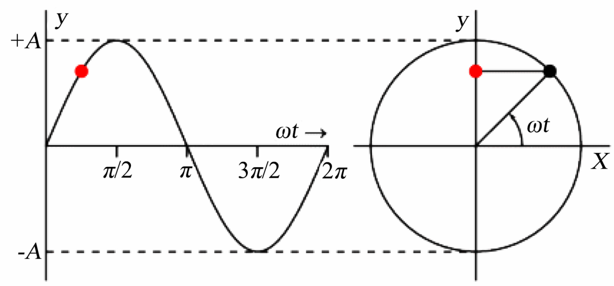

Figure 3. Angular frequency.

2.1.3. Signal Amplitude

The amplitude is the height, force or power of the signal. It is regarded as the maximum displacement of a variable from its mean value. Amplitudes can be either positive or negative. It refers to how strong or loud a signal is. A higher amplitude means a stronger signal, while a lower amplitude means a weaker signal

Figure 4 illustrated amplitude graphically.

Understanding periodic signals is important in the context of Fourier series. Fourier series, along with the generalizations examined in the below Section 3 will take place.

3. Fourier Series

Fourier series were introduced by Joseph Fourier following his research on the heat equation around 1830. They have since inspired much work by the greatest mathematicians (Dirichlet, Cantor, Lebesgue), and are still used today. He published his initial results in his 1807 Memoire Sur la Propagation de la Chaleur dans les Corps Solids (treatise on the propagation of heat in solid bodies) and Theories analytique de la chaleur in 1822 [1] [2]. Although his study later became widely used in solving an array of mathematical, engineering, and physical problems, especially those involving linear differential equations with constant coefficients.

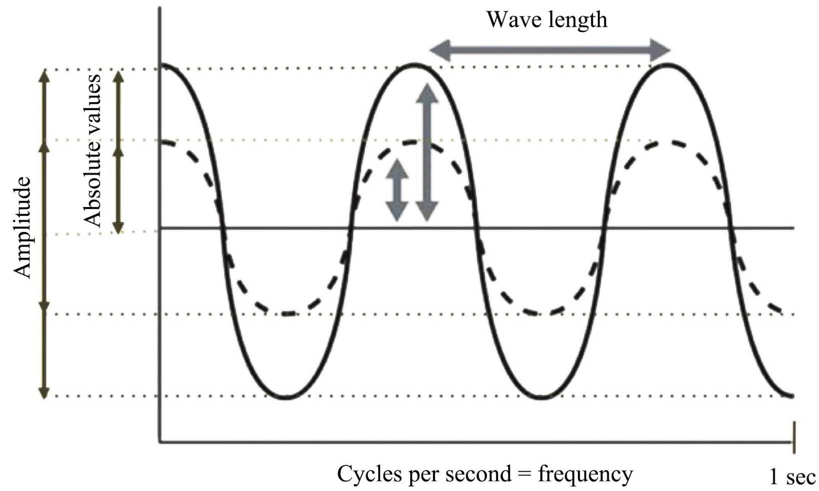

Figure 4. Amplitude change.

Basically, classical Fourier Analysis consists of two main areas: the Fourier Series, and the Fourier Transform [3] [4] [5]. The difference between the Fourier transform and the Fourier series is that the Fourier transform is applicable for non-periodic signals, while the Fourier series is applicable to periodic signals where we are interested to represent in this article.

The foundation of Fourier Series lies in the idea that any periodic signal can be approximated by a sum of simpler functions (sine and cosine) that have different frequencies. These frequencies are multiples of the fundamental frequency (lowest frequency = $\frac{1}{T}$) of the signal. This approximation is achieved by finding the coefficients that determine the amplitude of each components in the series. These coefficients can be calculated using integral calculus and are known as Fourier coefficients. The more terms we include in the series, the closer the approximation becomes to the original signal. By using a sufficient number of terms, we can accurately represent even highly irregular and complex periodic functions.

In order to apply the Fourier Series, we suppose that we are working with periodic signals that are well-behaved, meaning that various properties arise. To prove the below properties, it is crucial to recall mathematical representation of Fourier series and some mathematical conditions to proceed our work in Section 3.1.

3.1. Fourier Series Representation

There are two common forms of the Fourier Series, "Trigonometric" and "Exponential." Both are equivalent to each other [6]. Depending on the type of signal, most convenient representation is chosen.

In this paper, we limit our work to the trigonometric form, and in all the subsequent work, we denote by $f(t)$ the signal studied and $S_n(f(t))$ the Fourier series representation.

The Fourier series of a periodic signal $f(t)$ having a period T defined in $[0,T]$ is:

$$S_n(f(t)) = \frac{a_0}{2} + \sum_{i=1}^{+\infty}\left(a_n \cos(nwt) + b_n \sin(nwt)\right) \tag{5}$$

where n integer and $w = \frac{2\pi}{T}$ is the angular frequency of the signal.

The Fourier coefficients a_0, a_n and b_n are independent of time and are given by the following integrals:

$$\begin{cases} a_n = \frac{2}{T}\int_0^T f(t)\cos\left(\frac{2\pi nt}{T}\right)\mathrm{d}t & n = 0,1,2,3,\cdots \\ b_n = \frac{2}{T}\int_0^T f(t)\sin\left(\frac{2\pi nt}{T}\right)\mathrm{d}t & n = 1,2,3,\cdots \end{cases} \tag{6}$$

Or

$$\begin{cases} a_n = \frac{1}{L}\int_{-L}^L f(t)\cos\left(\frac{n\pi t}{L}\right)\mathrm{d}t & n = 0,1,2,3,\cdots \\ b_n = \frac{1}{L}\int_{-L}^L f(t)\sin\left(\frac{n\pi t}{L}\right)\mathrm{d}t & n = 1,2,3,\cdots \end{cases} \tag{7}$$

for $T = 2L$ and $f(t)$ defined on $[-L,L]$ instead of $[0,T]$.

- **Mean Value Term a_0**

a_0 is the average value of the signal $f(t)$. When $a_0 = 0$ then the signal $f(t)$ is alternative.

- **Coefficient Term a_n, b_n**

a_n, b_n are simply the amplitudes of the sinusoidal components(cosine and sine)at different frequencies. The higher the value of n, the higher the frequency of the corresponding sinusoidal term.

Note that The frequencies of the sines and cosines are $\frac{1}{T}, \frac{2}{T}, \frac{3}{T}, \cdots$, *i.e.*, they are multiples of the fundamental frequency $\frac{1}{T}$. Therefore the frequency $\frac{n}{T}$ is called the n^{th} harmonic. The name harmonic stems from the fact for the human ear frequencies with integer ratios sound "nice".

A graphical representation of Fourier series is given in the following **Figure 5**.

As seen in **Figure 5**, the more we increase n, meaning the more terms of cosine and sine we add, the Fourier series becomes closer to the black graph. Graphically, the convergence of the Fourier series refers to how well the colored graph fits the black initial signal. We say that the series approximates the original signal. This convergence depends on the properties of the signal that should match the Fourier condition. For smooth signals, the series converges quickly. However, for signals with sharp corners or discontinuities, the convergence may be slower or even fail at certain points. This is why we are interested in defining the Fourier condition or Dirichlet conditions in Subsection 3.2. Based on these conditions, we can ensure that the Fourier approximation fits the signal for a given integer n.

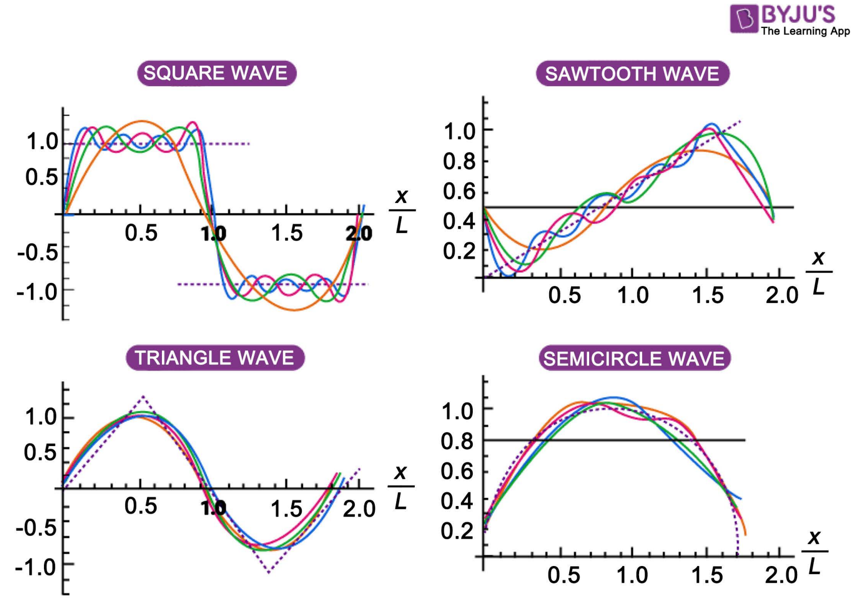

Figure 5. Fourier series for an increasing n.

3.2. Dirichlet's Condition for Existence of Fourier Series

Dirichlet conditions are a set of mathematical conditions that ensure the convergence of the Fourier series representation of a periodic function. They are named after the German mathematician Peter Gustav Lejeune Dirichlet, who first formulated these conditions in the 19th century. The Dirichlet conditions are crucial for establishing the convergence of Fourier series and ensuring that the series accurately represents the original periodic function.

These conditions are as follows:

1) Periodic and Integrable over period

$f(t)$ is a periodic and absolutely integrable over one period *i.e.*

$$\int_0^T |f(t)| < \infty$$

this guarantees that each coefficient a_k, b_k will be finite.

2) Finite Number of Extrema

For a finite integral, $f(t)$ is bounded variation, that is there are no more than a finite number of maximum or minimum during any single period of the signal. This condition ensures that the function does not oscillate too rapidly within each period and it is smooth, allowing for convergence of the Fourier series.

3) Finite Number of Discontinuities

For any finite interval, there are only a finite number of discontinuities. A discontinuity in a function occurs when there is a sudden jump or break in the function's values. This condition ensures that the function is well-behaved and can be accurately represented by a Fourier series.

4) Piecewise continuous function

The function $f(t)$ must be piecewise continuous on a finite interval. This means that while the function may have points where it jumps or changes abruptly (discontinuities), these points are isolated and do not affect the overall continuity of the function on each subinterval.

To better understand these conditions, let us define some periodic examples that violate them.

Example 1: $f(t)=\frac{1}{t}, 0<t\le 1$ violates condition 1 illustrated in Figure 6.

Example 2: $f(t)=\sin\left(\frac{2\pi}{t}\right), 0<t\le 1$ violates condition 2 but meets with condition 1 illustrated in Figure 7.

$\int_0^1\left|\sin\left(\frac{2\pi}{t}\right)\right|<\infty$ however an infinite number of maximum and minimum in the interval.

Example 3: $f(t)$ defined in (8) violates condition 3, illustrated in Figure 8. The signal is composed of an infinite number of sections, each of which is half of the height and half of the width of the previous section. Thus, the area under one period is less than 8. However, there is an infinite number of finite discontinuities, thereby violating condition 3.

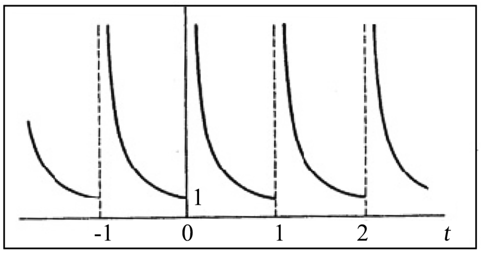

Figure 6. Graph of $\frac{1}{t}$.

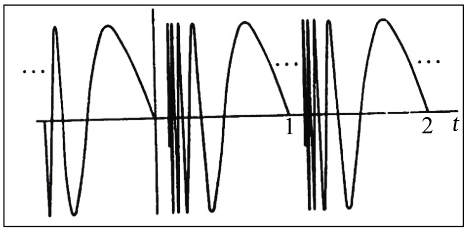

Figure 7. Graph of $\sin\left(\frac{2\pi}{t}\right)$.

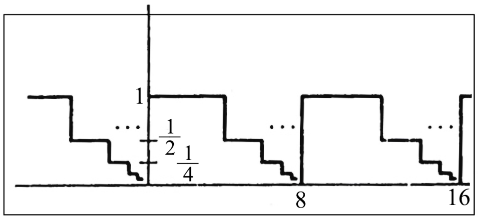

Figure 8. Graph of $f(t)$.

$$f(t)=\begin{cases}1 & \text{for } 0\le t<4\\ \frac{1}{2} & \text{for } 4\le t<6\\ \frac{1}{4} & \text{for } 6\le t<7\\ \frac{1}{8} & \text{for } 7\le t<7.5 \text{ etc}\end{cases} \tag{8}$$

These conditions form the theoretical basis for the practical utility of Fourier series in modeling and analyzing periodic phenomena across different disciplines. By satisfying the Dirichlet conditions, mathematicians and scientists can confidently apply Fourier series techniques in various fields, including signal processing, heat transfer, vibration analysis, quantum mechanics, and more. Here are some key points highlighting its significance under Dirichlet conditions

1) Signal Reconstruction: In signal processing, signals can often be represented as periodic functions. The Fourier series provides an accurate method for decomposing these signals into their constituent sinusoidal components, enabling efficient signal reconstruction and analysis.

2) Compression: Fourier series representation facilitates signal compression by capturing essential information about a signal using a relatively small number of coefficients. This is crucial in applications such as audio and image compression, where reducing data size without significant loss of information is desirable.

3) Mathematical Modeling: Many physical phenomena and systems exhibit periodic behavior, making Fourier series an essential tool for mathematical modeling and analysis. By accurately representing these periodic functions, engineers and scientists can gain insights into the underlying dynamics of systems and make predictions about their behavior.

4) Numerical Methods: The accuracy of Fourier series representation under Dirichlet conditions is also essential in numerical methods for solving partial differential equations. Fourier series can be used to represent the solution of certain boundary value problems, providing a powerful technique for solving differential equations numerically.

5) Control Systems: Fourier series are utilized in control systems engineering for system analysis and design. Accurate representation of periodic signals and system responses allows engineers to design control systems that effectively regulate and manipulate signals in various applications.

Overall, the accuracy of Fourier series representation under Dirichlet conditions is crucial for a wide range of practical scenarios where understanding and manipulating periodic functions are essential. By adhering to these conditions and leveraging the convergence properties of Fourier series, engineers, scientists, and mathematicians can achieve precise and efficient solutions in their respective fields.

The rate of convergence of the Fourier series approximation depends on the smoothness of the function. If the function is highly smooth (*i.e.*, has many continuous derivatives), then the Fourier series converges rapidly, and a small number of terms can provide a good approximation. On the other hand, if the function has sharp discontinuities or lacks smoothness, then the convergence may be slower, requiring more terms in the series for an accurate representation. The convergence of Fourier series refers to the behavior of the partial sums of the series as the number of terms increases. The convergence properties depend on the nature of the function being approximated.

If the function satisfies these conditions, then the Fourier series representation converges to the function in a least squares sense, meaning the sum of the squared errors between the function and its Fourier series approximation approaches zero as the number of terms in the series approaches infinity.

However, it's important to note that there may be cases where the Fourier series struggles to accurately represent functions with specific characteristics, such as highly oscillatory behavior or discontinuities that violate the Dirichlet conditions. The question of convergence for Fourier series outside Dirichlet conditions will not play a significant role in the remainder of this article. More details about the convergence of Fourier series are discussed in the following Subsection 3.2.1.

3.2.1. Convergence Theorem

Theorem 1. $f(t)$ is a periodic function that varies continuously (satisfies the Dirichlet conditions), then the Fourier series $S_n(f(t))$ converges (pointwise) to the signal everywhere expect the isolated points of discontinuities at which the series converge to the average of the left-hand and right-hand limits as follows:

$$S_n(f(t)) = \begin{cases} \dfrac{f(t_i^+) + f(t_i^-)}{2} & \text{for discontinuity points} \\ f(t) & \text{for continuity points} \end{cases} \tag{9}$$

This theorem establishes the convergence behavior of the Fourier series for functions that satisfy Dirichlet's conditions. Since the signal differ only at discontinuities points, then the integral of both signals are identical over any interval. For this reason the two signals behave identically under convolution and consequently will see this identification in the application part.

In other words, under the conditions specified by Dirichlet's Convergence Theorem, for a piecewise continuous periodic function, Fourier series converges to the function itself almost everywhere, except possibly at the points of discontinuity where it converges to the average of the left and right limits.

Additionally, there are other convergence theorems and results related to Fourier series, such as the Fejér's theorem, the Riemann-Lebesgue lemma, and the Parseval's theorem, which provide further insights into the convergence properties and behavior of Fourier series under various conditions. These theorems are often used to analyze the convergence and properties of Fourier series in specific contexts.

The calculation of coefficients a_0, a_n and b_n for Fourier representation can be a lot easier with knowledge of even and odd functions. A zero coefficient may be predicted with the use of these functions without performing the integration. The next Subsection 3.2.1, lists the odd and even functions.

3.2.2. Odd and Even Functions

Real functions can either be odd or even. In mathematics, even functions are symmetric with respect to the y-axis $f(-t) = f(t)$. In contrast, odd functions are symmetric with respect to the origin $f(-t) = -f(t)$.

By the definition, it is easy to see that the sum/difference/product of two even functions is even, the sum/difference of two odd functions is odd, the product of two odd functions is even, the product of an even function and an odd function is odd, etc. Note that the sine function is odd and the cosine function is even.

Also, we have the following integral identities:

- For odd functions

$$\int_{-L}^{L} f(t)\,\mathrm{d}t = 2\int_{0}^{L} f(t)\,\mathrm{d}t$$

- For even functions

$$\int_{-L}^{L} f(t)\,\mathrm{d}t = 0$$

By applying the properties of odd and even functions during the calculation of coefficients, as defined in Equation (7), the Fourier representation takes the following form:

- For even functions

$$\begin{aligned} &S_n(f(t)) = \sum_{n=1}^{+\infty} b_n \sin\left(\frac{n\pi t}{L}\right) \\ &a_n = 0 \\ &b_n = \frac{2}{L}\int_{0}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)\mathrm{d}t \end{aligned} \tag{10}$$

- For odd functions

$$\begin{aligned} &S_n(f(t)) = \sum_{n=0}^{+\infty} a_n \cos\left(\frac{n\pi t}{L}\right) \\ &a_n = \frac{2}{L}\int_{0}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)\mathrm{d}t \\ &b_n = 0 \end{aligned} \tag{11}$$

Understanding whether a function is odd, even, or a combination of both can simplify the computation of its Fourier series. Another interesting aspect of the Fourier representation is the orthogonality of the sine and cosine functions, as illustrated in the following Subsection 3.2.3.

3.2.3. Orthogonality Conditions for the Sine and Cosine Functions

Two functions f and g are said to be orthogonal over the interval [*a*, *b*] if:

$$\int_{a}^{b} f(t)g(t) = 0 \tag{12}$$

The functions in the Equation (5), $\cos(nx)$ and $\sin(nx)$, are orthogonal over the interval $[c, c+T]$ in other words, meaning that when multiplied together and integrated over this interval, the result is zero. If m and n are two nonnegative integers, then

$$\int_0^T \cos(nwt)\cos(mwt)\,\mathrm{d}t = \begin{cases} 0 & \text{if } m \neq n \\ T & \text{if } m = n \end{cases}$$
$$\int_0^T \sin(nwt)\sin(mwt)\,\mathrm{d}t = \begin{cases} 0 & \text{if } m \neq n \\ T & \text{if } m = n \end{cases} \tag{13}$$
$$\int_0^T \sin(nwt)\cos(mwt)\,\mathrm{d}t = 0 \text{ for any positive integers } m \text{ and } n.$$

To determine the coefficient a_m, b_m for any given integer m, we can multiply $S_n(f(t))$ in Equation (5) throughout by $\cos(mwt)$ and then integrate[2] over $[0, T]$, we can write

$$\int_0^T f(t)\cos(kwt)\,\mathrm{d}t = \int_0^T S_n(f(t))\cos(mwt)\,\mathrm{d}t$$
$$= \int_0^T \frac{a_0}{2}\cos(mwt)\,\mathrm{d}t \tag{14}$$
$$+\sum_{i=1}^{+\infty}\int_0^T a_n \cos(nwt)\cos(mwt)\,\mathrm{d}t$$
$$+\sum_{i=1}^{+\infty}\int_0^T b_n \sin(nwt)\cos(mwt)\,\mathrm{d}t \tag{15}$$

Then all the terms on the right-hand side vanish except when $n = m$, and we can solve for the coefficient a_m. Repeating the process with $\sin(kwt)$ we can similarly obtain the coefficient b_m. In the event the required formulas turn out to be as given in Equation (6).

This property is the basis for Fourier series. By decomposing a signal into its orthogonal components, we can analyze its frequency content and manipulate it more easily. More explanation for calling these orthogonality conditions is given in [7].

3.2.4. Differentiation and Integration of Fourier Series

Differentiation and integration of Fourier series can be justified by using some theorems, as discussed in [7] and [8]. It must be emphasized, however, that these lemmas provide sufficient conditions but not necessary ones. The following lemma for integration is especially useful.

Differentiation and integration of Fourier series can be justified by using some theorems as in [7] and [8]. It must be emphasized, however, that those lemmas provide sufficient conditions but not necessary. The following lemma for integration is especially useful.

- **Integration of Fourier Series**

Lemma 2. If $f(t)$ is piecewise continuous on $[a, b]$ and has a finite number of maxima and minima within that interval, then the Fourier series can be

[2] Assuming that it is permissible to integrate the infinite series term by term.

integrated term by term. The resulting series converges to the integral of $f(t)$ at every point where $f(t)$ is continuous, and it converges to the average of the left-hand and right-hand limits at points of discontinuity, then

$$\begin{aligned}\int_a^b S_n\left(f(t)\right)\mathrm{d}t &= S_n\left(\int_a^b f(t)\mathrm{d}t\right)\\ &= C+\sum_{i=1}^{+\infty}\left[\frac{T}{2\pi n}a_n\sin\left(2\pi\frac{nt}{T}\right)-\frac{T}{2\pi n}b_n\cos\left(2\pi\frac{nt}{T}\right)\right]\end{aligned} \tag{16}$$

Here, C is a constant of integration. The properties of $f(t)$ and the convergence behavior of the Fourier series play a significant role in determining the validity of term-wise integration. Additional conditions may be needed for convergence in specific cases.

- **Differentiation of Fourier Series**

Lemma 3. Suppose $f(t)$ is a periodic, continuous signal over $[0,T]$, its derivative $f'(t)$ is piecewise continuous over $[0,T]$ then the Fourier series of the signal can be differentiated term by term and the result is the Fourier series of the derivative $S_n\left(f'(t)\right)$.

$$S_n\left(f'(t)\right)=S_n'\left(f(t)\right)=\begin{cases}\sum_{i=1}^{+\infty}-\frac{2\pi n}{T}a_n\sin\left(2\pi\frac{nt}{T}\right)+\frac{2\pi n}{T}b_n\cos\left(2\pi\frac{nt}{T}\right)\\ \frac{f'\left(t_0^+\right)+f'\left(t_0^-\right)}{2} \qquad \text{for discontinuity points}\end{cases} \tag{17}$$

As with integration, It's important to note that the conditions for term-wise differentiation are crucial, and these conditions are usually related to the smoothness and continuity properties of the signal $f(t)$ and its derivative $f'(t)$. The convergence behavior can vary, and in some cases, additional conditions may be needed for convergence.

3.2.5. Parseval Theorem

Parseval's theorem states that the integral of the square of its signal $f(t)$ is equal to the square of the function's Fourier components. In other words, let us define the following theorem

Theorem 4. Suppose that a_n and b_n are the Fourier coefficients corresponding to $f(t)$ satisfing the Dirichlet conditions, then Parseval identity is as follows:

$$\frac{2}{T}\int_0^T f(t)^2\,\mathrm{d}t=\frac{a_0^2}{2}+\sum_{n=1}^{+\infty}\left(a_n^2+b_n^2\right) \tag{18}$$

Parseval's Theorem establishes a vital relationship between the energy of a signal in the time domain and its energy in the frequency domain, ensuring that energy is conserved during the transformation. Figure 9 illustrates the time and frequency domains with different harmonics.

In medicine applications, this formula can be interpreted as follows: Parseval's identity relates the energy or power of the heartbeat signal in the time domain to the energy or power of its frequency components in the frequency domain.

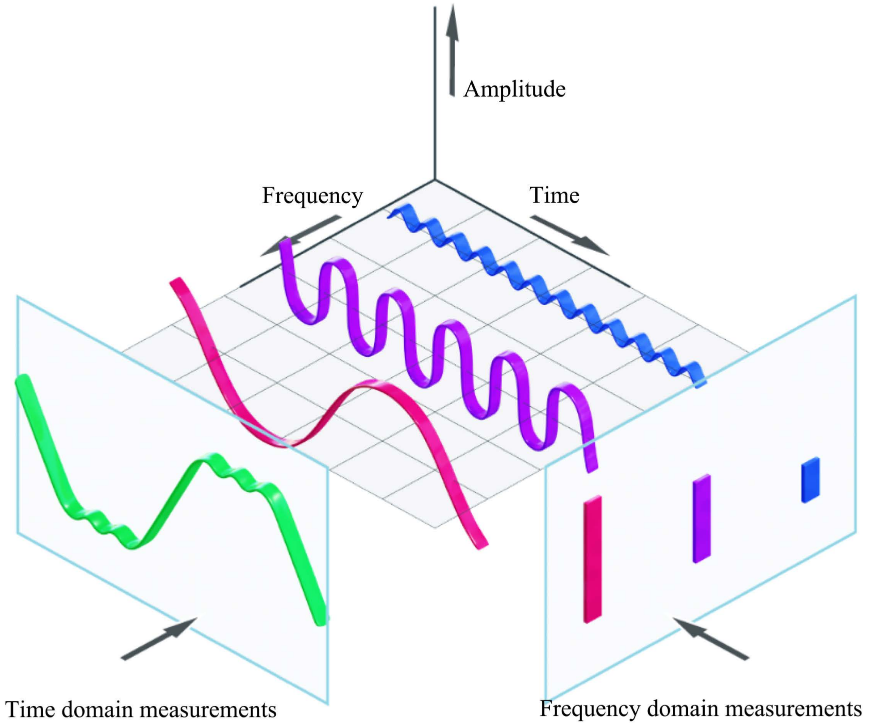

Figure 9. Frequency domain and time domain.

The application of Fourier Series on electrocardiogram (ECG) for heartbeat signals takes place in the following Section 4.1. Fourier Series application on electrocardiogram (ECG) for hearbeat signals take a place in the following Section 4.1.

4. Fourier Series Applications

Fourier series has played a crucial role in various specific applications and fields, including:

- Audio Engineering and Music Production:

Fourier series is used in synthesizing musical tones by combining sine waves of different frequencies and amplitudes to recreate complex sounds.

- Electrical Engineering and Signal Processing:

In telecommunications, Fourier series is used in modulation techniques like amplitude modulation (AM) and frequency modulation (FM) to encode information onto carrier signals.

- Digital Image Processing:

In image compression algorithms like JPEG, Fourier series is utilized to transform image data from the spatial domain to the frequency domain, where high-frequency components (representing fine details) can be compressed or discarded to reduce file size while preserving image quality.

- Physics and Engineering:

In mechanical and structural engineering, Fourier series is applied in analyzing vibrations and oscillations of systems h.

- Medical Imaging

In MRI, Fourier transform techniques like Fourier imaging and Fourier reconstruction are employed to convert raw data from MRI scanners into images.

These specific applications and fields highlight the diverse range of areas where Fourier series has played a crucial role, demonstrating its broad significance across various domains of science, engineering, and technology.

In the following part we are focusing on medical imaging techniques such as electrocardiogram (ECG) [9].

Fourier Series with ECG and Human Heart Beat

The electrocardiogram (ECG) is a first-line test used by your Cardiologist to obtain valuable information about your heart health by measuring the electrical activity of the heart. The ECG allows your Cardiologist to detect heart attacks, heart rhythm problems, and other heart-related conditions.

An ECG involves the use of small electrodes attached to your chest, arms and legs. These electrodes detect electrical signals produced by the heart, which are then translated and recorded by a machine, showing your heart rate, heart rhythm and electrical waveforms. A normal resting heart rate for adults is between 60 and 100 beats per minute. A normal ECG rhythm is described as sinus rhythm (regular) and without significant pauses or extra beats.

A normal ECG waveform includes the:

- P wave: the electrical activity that causes the atria to contract.
- Q RS complex: the electrical activity that causes the ventricles to contract.
- Q RS complex; the electrical activity that causes the ventricles to contract.
- T wave: this represents the ventricles returning to their resting state

As seen in **Figure 10**, the periodicity of heartbeats is crucial for the overall well-being of the cardiovascular system. Using its periodicity, the Fourier series can be applied. This allows researchers and medical professionals to analyze the frequency content of the heartbeat and identify various components.

Figure 10. Normal heartbeat signal.

For simplicity, we will only model the R wave for this article. To get a more accurate model for the heartbeat, we would just need to do a similar process for the P, Q, S and T waves and add them to my model.

A real example is given as follows:

A heart rate was about 60 beats per minute or 1 beat per second. So the period T = 1 second = 1000 milliseconds.

We observed that R wave was about 2.5 mV (millivolts) high and lasted for a total of 40 ms. The shape of the R wave is approached by a polynomial function, so the model is as follows (the time units are milliseconds): (Figure 11)

$$f(t) = -0.0000156(t-20)^4 + 2.5 \quad \text{over} \quad [0;40]$$

$$T = 1000\text{ ms} \quad \text{and} \quad L = 500$$

Then the coefficients of Fourier Series defined in Equation (7) are given by:

- **Mean Value Term a_0**

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)\mathrm{d}t = \frac{1}{500}\int_{-500}^{500} f(t)\mathrm{d}t$$
$$= \frac{1}{500}\int_0^{40}\left(-0.0000156(t-20)^4 + 2.5\right)\mathrm{d}t$$
$$= 0.16$$

- **First Coefficient Term a_n**

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}\mathrm{d}t$$
$$= \frac{1}{500}\int_{-500}^{500} f(t)\cos\frac{n\pi t}{500}\mathrm{d}t$$
$$= \frac{1}{500}\int_0^{40}\left(-0.0000156(t-20)^4 + 2.5\right)\cos\frac{n\pi t}{500}\mathrm{d}t$$

- **Second Coefficient Term b_n**

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}\mathrm{d}t$$
$$= \frac{1}{500}\int_{-500}^{500} f(t)\sin\frac{n\pi t}{500}\mathrm{d}t$$
$$= \frac{1}{500}\int_0^{40}\left(-0.0000156(t-20)^4 + 2.5\right)\sin\frac{n\pi t}{500}\mathrm{d}t$$

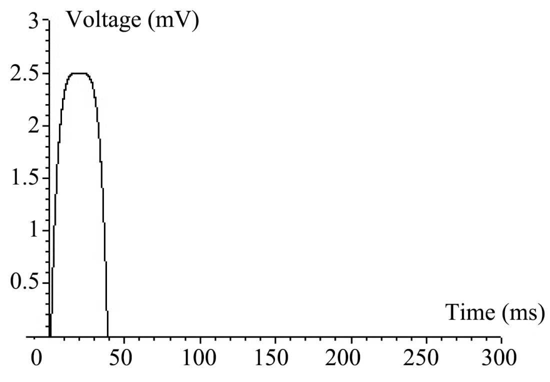

Figure 11. R heartbeat signal.

The results of a_n and b_n are too large, so to simplify, we will not explicitly state them here. Then our Fourier Series is:

$$f(t)=\frac{0.16}{2}+\sum_{n=1}^{\infty}\left(\frac{1}{500}\int_0^{40}\left(-0.0000156(t-20)^4+2.5\right)\cos\frac{n\pi t}{500}\mathrm{d}t\right)\cos\frac{n\pi t}{500}$$
$$+\sum_{n=1}^{\infty}\left(\frac{1}{500}\int_0^{40}\left(-0.0000156(t-20)^4+2.5\right)\sin\frac{n\pi t}{500}\mathrm{d}t\right)\sin\frac{n\pi t}{500}$$

Graphically, **Figure 12** and **Figure 13** shows the Fourier series for $n = 5$ and $n = 100$ terms.

Figure 13, where ($n = 100$ terms), provides a reasonable approximation for a regular R wave with a period of 1 second. The figure accurately fits the R wave. By adding the T wave to this next model, the function $f(t)$ will be defined as a piecewise continuous function. Graphically, the initial R and T signals together are presented in **Figure 14**, and the Fourier series for 100 terms is depicted in **Figure 15**. We could continue this process by adding the P, Q, and S waves to create an even better model.

It's important to note that the human heart is a complex organ, and the Fourier series is a simplification for analysis. In practice, more advanced signal processing techniques, such as Fourier transform or wavelet analysis, may be used for a more detailed and accurate representation of the heart's activity.

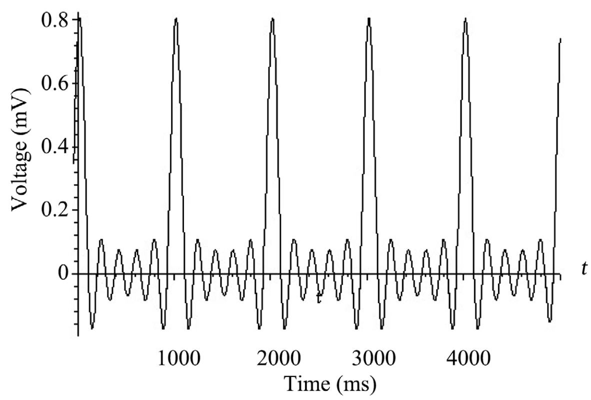

Figure 12. Fourier series n = 5.

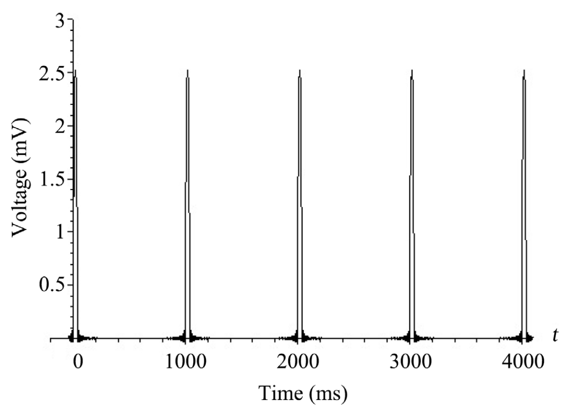

Figure 13. Fourier series n = 100.

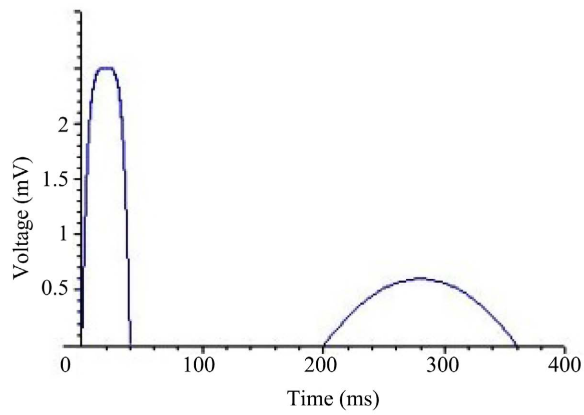

Figure 14. Graph of new $f(t)$ including R and T waves.

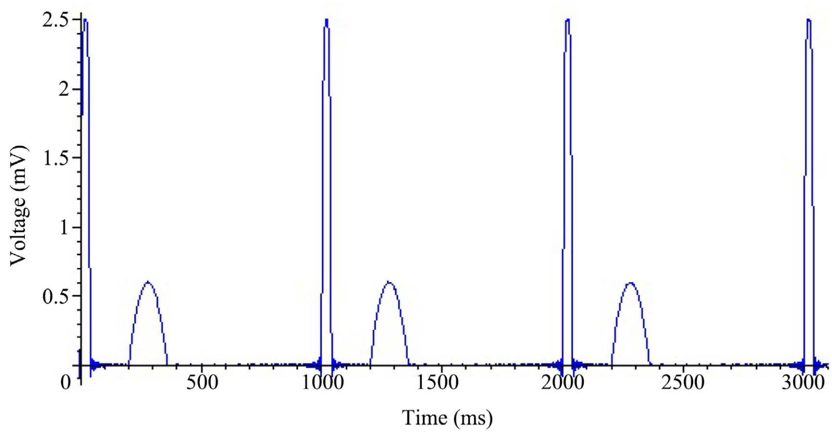

Figure 15. Fourier series n = 100 for R wave and T wave.

5. Conclusion

In conclusion, Fourier series is a powerful mathematical tool with a wide range of applications. It is a particularly promising approach to quantify periodic variability in underlying phenomena, achieving this by converting periodic signals into frequencies driven by sums of sines and cosines. From signal processing to image manipulation, and from physics to engineering, Fourier series helps us analyze and understand the periodic nature of various phenomena. This article is concerned with representing and analyzing periodic phenomena via Fourier series. In fact, nonperiodic phenomena (and thus just about any general function) provide a pathway from Fourier series to the Fourier transform… the spectrum is born. With it comes the most important principle of the future subject.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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