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Global Existence of the Solution for a Reduced Model of the Vectorial Quantum Zakharov System

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Abstract

In this paper, we study the global existence of the smooth solution for a reduced quantum Zakharov system in two spatial dimensions. Using energy estimates and the logarithmic type Sobolev inequality, we show the global existence of the solution to this system without any small condition on the initial data.

Keywords

Quantum Zakharov System, Global Existence, Logarithmic Sobolev Inequality

1. Introduction

In this paper, we consider the vectorial quantum Zakharov system

$$\begin{cases} \mathrm{i}E_t - \alpha\nabla\times(\nabla\times E) + \nabla(\nabla\cdot E) - \Gamma\nabla(\Delta(\nabla\cdot E)) = nE, \\ \lambda^{-2}n_{tt} - \Delta n + \Gamma\Delta^2 n = \Delta|E|^2, \\ E(x,0) = E_0(x), n(x,0) = n_0(x), n_t(x,0) = n_1(x), \end{cases} \tag{1}$$

where $E:\mathbb{R}^2\times\mathbb{R}\to\mathbb{C}^2$ is the slowly varying envelope of the rapidly oscillating electric field, and $n:\mathbb{R}^2\times\mathbb{R}\to\mathbb{R}$ is the deviation of the ion density from its mean value. $\lambda\in[1,\infty)$ denotes the ionic speed of sound, the parameter α defined as the square ratio of the light speed and the electron Fermi velocity is usually large, and the coefficient Γ that measures the influence of quantum effects is usually very small. This model describes the nonlinear interaction between high-frequency quantum Langmuir waves and low-frequency quantum ion-acoustic waves, we refer to [1] for more physical background.

Most of the known results are concerned on the scalar quantum Zakharov system which reads

$$\begin{cases} \mathrm{i}E_t + \Delta E - \Gamma\Delta^2 E = nE, \\ \lambda^{-2} n_{tt} - \Delta n + \Gamma\Delta^2 n = \Delta|E|^2. \end{cases} \tag{2}$$

For example, the references [2] [3] proved the local or global well-posedness results for (2), and for scattering results, we refer to [4] [5]. When $\Gamma = 0$, we can obtain the classical Zakharov system ([6])

$$\begin{cases} \mathrm{i}E_t + \Delta E = nE, \\ \lambda^{-2} n_{tt} - \Delta n = \Delta|E|^2, \end{cases} \tag{3}$$

which has been extensively studied for the local and global well-posedness [7]-[15].

By isolating the light dispersive term and the quantum dispersive term for E, the linear part of the first equation of (1) can be transformed into the form

$$\begin{cases} \mathrm{i}F_{1t} + \alpha\Delta F_1 = 0, \\ \mathrm{i}F_{2t} + \Delta F_2 - \Gamma\Delta^2 F_2 = 0. \end{cases} \tag{4}$$

The detailed computations are given in the appendix. Then we are interested in the following reduced model of the vectorial quantum Zakharov system

$$\begin{cases} \mathrm{i}F_{1t} + \Delta F_1 = nF_1, \\ \mathrm{i}F_{2t} + \Delta F_2 - \Delta^2 F_2 = nF_2, \\ n_{tt} - \Delta n + \Delta^2 n = \Delta|F|^2, \\ F_1(0) = F_{1,0}, F_2(0) = F_{2,0}, \\ n(0) = n_0, n_t(0) = n_1. \end{cases} \tag{5}$$

Here, we have set $\alpha = \Gamma = \lambda = 1$ for simplicity. Yang-Zhang-Jiang [16] proved local existence of the solution and the limit behavior for this system. In this work, our aim is to show the global existence of (5) in a two-dimensional case.

Before stating the main result, we first introduce some notations that will be used in the paper. For $m \in \mathbb{Z}^+$, we denote $H^m(\mathbb{R}^2)$ the usual inhomogeneous Sobolev space. If $u \in W^{m,p}(\mathbb{R}^2)$, we define its norm to be

$$\|u\|_{W^{m,p}} = \left(\sum_{|\alpha|\le m} \int_{\mathbb{R}^2} |D^\alpha u|^p \,\mathrm{d}x\right)^{\frac{1}{p}} (1 \le p < \infty),$$

or

$$\|u\|_{W^{m,\infty}} = \sum_{|\alpha|\le m} \operatorname{esssup}|D^\alpha u|,$$

where ess sup $f(x)$ denotes the essential supremum of a set of functions.

The homogeneous Sobolev space $\dot{H}^{-1}(\mathbb{R}^2)$ is defined as

$$\dot{H}^{-1}(\mathbb{R}^2) = \left\{u; (-\Delta)^{-\frac{1}{2}} u \in L^2(\mathbb{R}^2)\right\},$$

with norm

$$\|u\|_{\dot{H}^{-1}}\left(\mathbb{R}^2\right)=\left\|(-\Delta)^{-\frac{1}{2}}u\right\|_{L^2\left(\mathbb{R}^2\right)}=\left\||\xi|^{-1}\hat{u}(\xi)\right\|_{L^2\left(\mathbb{R}^2\right)},$$

where $\hat{u}$ is the Fourier transform of *u*.

We denote the product space X_M as

$$X_M:=H^{M-1}\left(\mathbb{R}^2\right)\times H^M\left(\mathbb{R}^2\right)\times H^{M-1}\left(\mathbb{R}^2\right)\times\left(H^{M-3}\left(\mathbb{R}^2\right)\cap\dot{H}^{-1}\left(\mathbb{R}^2\right)\right),$$

and

$$\left\|(u_1,u_2,u_3,u_4)\right\|_{X_M}:=\|u_1\|_{H^{M-1}\left(\mathbb{R}^2\right)}+\|u_2\|_{H^M\left(\mathbb{R}^2\right)}+\|u_3\|_{H^{M-1}\left(\mathbb{R}^2\right)}+\|u_4\|_{H^{M-3}\left(\mathbb{R}^2\right)\cap\dot{H}^{-1}\left(\mathbb{R}^2\right)}.$$

The main result is stated in the following theorem.

Theorem 1 Let $\left(F_{1,0},F_{2,0},n_0,n_1\right)\in X_M$ and $M\geq 4$ is a positive integer. Then, the system (5) has a unique global solution $\left(F_1,F_2,n,n_t\right)$ satisfying

$$\left(F_1,F_2,n,n_t\right)\in C\left(\mathbb{R}^+;X_M\right).$$

Theorem 1 gives the global existence result without any size restriction under the quantum effect. This is quite different from the classical Zakharov system where a global solution exists with small initial data.

2. Preliminaries

In this section, we give the conserved quantities and a basic L^∞ type estimate.

Lemma 2 (Young inequality) [16] Let $1\leq p$, $q\leq\infty$, $\frac{1}{p}+\frac{1}{q}=1$, then

$$ab\leq\frac{a^p}{p}+\frac{b^q}{q}.$$

Lemma 3 (Hölder inequality) [16] Let $1\leq p$, $q\leq\infty$, $\frac{1}{p}+\frac{1}{q}=1$, if there has $u\in L^p(U)$, $v\in L^q(U)$, then

$$\int_U|uv|\mathrm{d}x\leq\|u\|_{L^p}\|v\|_{L^q}.$$

Lemma 4 (Gagliardo-Nirenberg inequality) [16] Let $u\in L^q\left(\mathbb{R}^n\right)$, $D^m u\in L^r\left(\mathbb{R}^n\right)$, $1\leq p$, $r\leq\infty$, $0\leq j\leq m$, Then, there are $C>0$, satisfying

$$\left\|D^j u\right\|_{L^p\left(\mathbb{R}^n\right)}\leq C\left\|D^m u\right\|_{L^r\left(\mathbb{R}^n\right)}^{\alpha}\|u\|_{L^q\left(\mathbb{R}^n\right)}^{1-\alpha},$$

with $0\leq\frac{j}{m}\leq\alpha\leq 1$,

$$\frac{1}{p}=\frac{j}{n}+\alpha\left(\frac{1}{r}-\frac{m}{n}\right)+(1-\alpha)\frac{1}{q}.$$

Lemma 5 Let $u\in W^{k,p}\left(\mathbb{R}^d\right)\cap W^{s,q}\left(\mathbb{R}^d\right)$, $k,s>0$, $p>1$, $q\geq 1$, $kp=d<sq$, Then, there holds

$$\|u\|_{L^\infty}\leq C\|u\|_{W^{k,p}}\left(1+\ln\left(\frac{\|u\|_{W^{s,q}}}{\|u\|_{W^{k,p}}}\right)\right)^{1-\frac{1}{p}}$$

with C depending on k,s,p,q,d.

The proof of this lemma can be found in [17] [18]. When $d=2$, $k=1$, $s=2$, $p=q=2$, we have

$$\|u\|_{L^\infty} \le C\left(1+\ln\left(1+\|u\|_{H^2}\right)\right)^{\frac{1}{2}} \tag{6}$$

for $u \in H^2\left(\mathbb{R}^2\right)$ and $\|u\|_{H^1} \le K$, where C is a constant depending only on K.

Proposition 6 *For smooth solutions of (5), there hold two conserved quantities:*

$$\|F\|_{L^2} = \|F_0\|_{L^2}, \mathcal{H}(t) = \mathcal{H}(0),$$

where $F=(F_1,F_2)$ and $\mathcal{H}(t)=\mathcal{H}\left(F_1(t),F_2(t),n(t),n_t(t)\right)$ with

$$\begin{aligned} &\mathcal{H}\left(F_1,F_2,n,n_t\right) \\ &:= \|\nabla F_1\|_{L^2}^2 + \|\nabla F_2\|_{L^2}^2 + \|\Delta F_2\|_{L^2}^2 + \frac{1}{2}\left(\|n_t\|_{\dot{H}^{-1}}^2 + \|n\|_{L^2}^2 + \|\nabla n\|_{L^2}^2\right) + \int_{\mathbb{R}^2} n|F|^2 \,\mathrm{d}x. \end{aligned} \tag{7}$$

Proof. We first derive the L^2 bound of F_1. Taking the imaginary part of the inner product in L^2 between the first equation of (5) and F_1, we have $\frac{\mathrm{d}}{\mathrm{d}t}\|F_1\|_{L^2}^2 = 0$. Therefore,

$$\|F_1(t)\|_{L^2}^2 = \|F_{1,0}\|_{L^2}^2.$$

Similarly, we have

$$\|F_2(t)\|_{L^2}^2 = \|F_{2,0}\|_{L^2}^2.$$

Thus, we get

$$\|F(t)\|_{L^2} = \|F(0)\|_{L^2}.$$

Next, we multiply the first equation of (5) by $\overline{F_{1t}}$ and consider the real part. This leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla F_1\|_{L^2}^2 = -\int_{\mathbb{R}^2} n\partial_t |F_1|^2 \,\mathrm{d}x. \tag{8}$$

Similarly, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla F_2\|_{L^2}^2 + \|\Delta F_2\|_{L^2}^2\right) = -\int_{\mathbb{R}^2} n\partial_t |F_2|^2 \,\mathrm{d}x. \tag{9}$$

On the other hand, we take the inner product of the third equation of (5) with $(-\Delta)^{-1} n_t$, then we can obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|n_t\|_{\dot{H}^{-1}}^2 + \|n\|_{L^2}^2 + \|\nabla n\|_{L^2}^2\right) = -\int_{\mathbb{R}^2} \partial_t n|F|^2 \,\mathrm{d}x. \tag{10}$$

From (8)-(10), we obtain

$$\mathcal{H}(t) = \mathcal{H}(0).$$

3. Proof of Theorem 1

Now we are going to prove Theorem 1.

Proof of Theorem 1. According to the local existence theory, it is sufficient to show the *a-priori* bound of the solution. From Proposition 6, we know

$\mathcal{H}(F,n)(t)=\mathcal{H}(F,n)(0)$ which implies

$$\begin{aligned}&\|\nabla F_1\|_{L^2}^2+\|\nabla F_2\|_{L^2}^2+\|\Delta F_2\|_{L^2}^2+\frac{1}{2}\|n_t\|_{\dot{H}^{-1}}^2+\frac{1}{2}\|n\|_{L^2}^2+\frac{1}{2}\|\nabla n\|_{L^2}^2\\&\le C+\left|\int_{\mathbb{R}^2}n|F|^2\,\mathrm{d}x\right|\\&\le C+\frac{1}{4}\|\nabla n\|_{L^2}^2+\frac{1}{2}\|\nabla F\|_{L^2}^2\\&\le C.\end{aligned}\tag{11}$$

Here, for the nonlinear integral term, we have used the Gagliardo-Nirenberg inequality and Young's inequality to obtain

$$\int_{\mathbb{R}^2}n|F|^2\,\mathrm{d}x\le C\|n\|_{L^6}\|F\|_{L^{\frac{12}{5}}}^2\le\frac{1}{4}\|\nabla n\|_{L^2}^2+\frac{1}{2}\|\nabla F\|_{L^2}^2+C.\tag{12}$$

Therefore, we get

$$\|F_1\|_{H^1}+\|F_2\|_{H^2}+\|n_t\|_{\dot{H}^{-1}}+\|n\|_{H^1}\le C,\ \forall t>0.\tag{13}$$

and the inequality (6) implies

$$\begin{aligned}&\|F_1\|_{L^\infty}\le C\left(1+\ln\left(1+\|\Delta F_1\|_{L^2}\right)\right)^{\frac{1}{2}},\\&\|n\|_{L^\infty}\le C\left(1+\ln\left(1+\|\Delta n\|_{L^2}\right)\right)^{\frac{1}{2}},\\&\|\nabla F_2\|_{L^\infty}\le C\left(1+\ln\left(1+\|\nabla\Delta F_2\|_{L^2}\right)\right)^{\frac{1}{2}}.\end{aligned}\tag{14}$$

Next, we estimate the higher-order norms for F_1, F_2 and n. We perform $\dot{H}^2$ energy estimate for F_1, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta F_1\|_{L^2}^2=-2\mathrm{Re}\int_{\mathbb{R}^2}\nabla(nF_1)\nabla\overline{F_{1t}}\mathrm{d}x.\tag{15}$$

Recalling (5), we see $F_{1t}=\mathrm{i}\Delta F_1-\mathrm{i}nF_1$. Hence, we deduce

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta F_1\|_{L^2}^2&=-2\,\mathrm{Im}\int_{\mathbb{R}^2}\nabla(nF_1)\nabla\Delta\overline{F_1}\mathrm{d}x\\&=2\,\mathrm{Im}\int_{\mathbb{R}^2}\Delta(nF_1)\Delta\overline{F_1}\mathrm{d}x\\&\le C\|\Delta F_1\|_{L^2}\|\Delta n\|_{L^2}\|F_1\|_{L^\infty}+C\|\Delta F_1\|_{L^2}^2\|n\|_{L^\infty}.\end{aligned}\tag{16}$$

Similarly, we can obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\Delta F_2\|_{L^2}^2+\|\nabla\Delta F_2\|_{L^2}^2\right)&=-2\,\mathrm{Re}\int_{\mathbb{R}^2}\nabla(nF_2)\nabla\overline{F_{2t}}\mathrm{d}x\\&=-2\,\mathrm{Im}\int_{\mathbb{R}^2}\nabla(nF_2)\nabla\Delta\overline{F_2}\mathrm{d}x+2\,\mathrm{Im}\int_{\mathbb{R}^2}\nabla(nF_2)\nabla\Delta^2\overline{F_2}\mathrm{d}x\\&:=I_{21}+I_{22}.\end{aligned}\tag{17}$$

For I_{21}, it is easy to see (by (13))

$$\begin{aligned}I_{21}&=-2\,\mathrm{Im}\int_{\mathbb{R}^2}\nabla(nF_2)\nabla\Delta\overline{F_2}\mathrm{d}x\\&=2\,\mathrm{Im}\int_{\mathbb{R}^2}\Delta(nF_2)\Delta\overline{F_2}\mathrm{d}x\\&\le C\left(1+\|n\|_{H^2}^2\right).\end{aligned}\tag{18}$$

As to I_{22}, we have

$$
\begin{aligned}
I_{22} &= -2\operatorname{Im}\int_{\mathbb{R}^2} \Delta n F_2 \Delta^2 \overline{F_2}\mathrm{d}x - 4\operatorname{Im}\int_{\mathbb{R}^2} \nabla n \nabla F_2 \Delta^2 \overline{F_2}\mathrm{d}x \\
&\quad - \operatorname{Im}\int_{\mathbb{R}^2} n\Delta F_2 \Delta^2 \overline{F_2}\mathrm{d}x - \operatorname{Im}\int_{\mathbb{R}^2} n\Delta F_2 \Delta^2 \overline{F_2}\mathrm{d}x \\
&\le -2\operatorname{Im}\int_{\mathbb{R}^2} \Delta n F_2 \Delta^2 \overline{F_2}\mathrm{d}x + C\left\|\nabla\Delta F_2\right\|_{L^2}\left\|\Delta n\right\|_{L^2}\left\|\nabla F_2\right\|_{L^\infty} \\
&\quad + C\left(\left\|\Delta n\right\|_{L^2}^2 + \left\|\nabla \Delta F_2\right\|_{L^2}^2\right) + C\left\|n\right\|_{L^\infty}\left\|\nabla\Delta F_2\right\|_{L^2}^2 .
\end{aligned} \tag{19}
$$

Taking inner product of both sides of the third equation of (5) with n_t, there is

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\left\|n_t\right\|_{L^2}^2 + \frac{1}{2}\left\|\nabla n\right\|_{L^2}^2 + \frac{1}{2}\left\|\Delta n\right\|_{L^2}^2\right) \\
&= \int_{\mathbb{R}^2} \Delta\left|F\right|^2 n_t \mathrm{d}x = -\int_{\mathbb{R}^2} \nabla\left|F\right|^2 \nabla n_t \mathrm{d}x \\
&= -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2} \nabla\left|F\right|^2 \nabla n \mathrm{d}x + \int_{\mathbb{R}^2} \nabla\left|F\right|_t^2 \nabla n \mathrm{d}x.
\end{aligned}
$$

Thus, we have

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\left\|n_t\right\|_{L^2}^2 + \frac{1}{2}\left\|\nabla n\right\|_{L^2}^2 + \frac{1}{2}\left\|\Delta n\right\|_{L^2}^2 + \int_{\mathbb{R}^2} \nabla\left|F\right|^2 \nabla n \mathrm{d}x\right) = \int_{\mathbb{R}^2} \nabla\left|F\right|_t^2 \nabla n \mathrm{d}x. \tag{20}
$$

The Equation (5) indicates that

$$
\left|F_1\right|_t^2 = -2\operatorname{Im}\left(\Delta F_1 \overline{F_1}\right), \tag{21}
$$

$$
\left|F_2\right|_t^2 = -2\operatorname{Im}\left(\Delta F_2 \overline{F_2}\right) + 2\operatorname{Im}\left(\Delta^2 F_2 \overline{F_2}\right). \tag{22}
$$

Now using (21)-(22), the integral term $\int_{\mathbb{R}^2} \nabla\left|F\right|_t^2 \nabla n \mathrm{d}x$ can be estimated as

$$
\begin{aligned}
&\int_{\mathbb{R}^2} \nabla\left|F\right|_t^2 \nabla n \mathrm{d}x \\
&= -2\operatorname{Im}\int_{\mathbb{R}^2} \nabla\left(\Delta F_1 \overline{F_1} + \Delta F_2 \overline{F_2}\right)\nabla n \mathrm{d}x + 2\operatorname{Im}\int_{\mathbb{R}^2} \nabla\left(\Delta^2 F_2 \overline{F_2}\right)\nabla n \mathrm{d}x \\
&:= I_{31} + I_{32}.
\end{aligned} \tag{23}
$$

For I_{31}, it can be estimated by

$$
\begin{aligned}
I_{31} &= 2\operatorname{Im}\int_{\mathbb{R}^2} \left(\Delta F_1 \overline{F_1} + \Delta F_2 \overline{F_2}\right)\Delta n \mathrm{d}x \\
&\le C\left\|\Delta n\right\|_{L^2}\left\|\Delta F_1\right\|_{L^2}\left\|F_1\right\|_{L^\infty} + C\left\|\Delta n\right\|_{L^2} .
\end{aligned} \tag{24}
$$

For I_{32}, we have

$$
I_{32} = -2\operatorname{Im}\int_{\mathbb{R}^2} \Delta^2 F_2 \overline{F_2}\Delta n \mathrm{d}x. \tag{25}
$$

Then it follows from (20) and (23)-(25) that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\left\|n_t\right\|_{L^2}^2 + \frac{1}{2}\left\|\nabla n\right\|_{L^2}^2 + \frac{1}{2}\left\|\Delta n\right\|_{L^2}^2 + \int_{\mathbb{R}^2} \nabla\left|F\right|^2 \nabla n \mathrm{d}x\right) \\
&\le C\left\|\Delta n\right\|_{L^2}\left\|\Delta F_1\right\|_{L^2}\left\|F_1\right\|_{L^\infty} + C\left\|\Delta n\right\|_{L^2} - 2\operatorname{Im}\int_{\mathbb{R}^2} \Delta^2 F_2 \overline{F_2}\Delta n \mathrm{d}x.
\end{aligned} \tag{26}
$$

Now, collecting the estimates (16)-(19) and (26) yield

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\Delta F_1\right\|_{L^2}^2 + \left\|\Delta F_2\right\|_{L^2}^2 + \left\|\nabla\Delta F_2\right\|_{L^2}^2 + \frac{1}{2}\left\|n_t\right\|_{L^2}^2 + \frac{1}{2}\left\|\nabla n\right\|_{L^2}^2 + \frac{1}{2}\left\|\Delta n\right\|_{L^2}^2 + \int_{\mathbb{R}^2} \nabla\left|F\right|^2 \nabla n \mathrm{d}x\right) \\
&\le C\left\|\Delta F_1\right\|_{L^2}\left\|\Delta n\right\|_{L^2}\left\|F_1\right\|_{L^\infty} + C\left\|\Delta F_1\right\|_{L^2}^2\left\|n\right\|_{L^\infty} + C\left(1 + \left\|\Delta n\right\|_{L^2}^2 + \left\|\nabla\Delta F_2\right\|_{L^2}^2\right) \\
&\quad + C\left\|\nabla\Delta F_2\right\|_{L^2}^2\left\|n\right\|_{L^\infty} + C\left\|\nabla\Delta F_2\right\|_{L^2}\left\|\Delta n\right\|_{L^2}\left\|\nabla F_2\right\|_{L^\infty} .
\end{aligned} \tag{27}
$$

The nonlinear part in the left-hand side of (27) can be estimated by

$$\left|\int_{\mathbb{R}^2} \nabla |F|^2 \nabla n \mathrm{d}x\right| = \left|\int_{\mathbb{R}^2} |F|^2 \Delta n \mathrm{d}x\right| \le \frac{1}{4}\|\Delta n\|_{L^2}^2 + \left\||F|^2\right\|_{L^2}^2 . \tag{28}$$

And using inequality (21)-(22), we get

$$\begin{aligned}\left\||F|^2\right\|_{L^2}^2 &= 2\int_0^t\int_{\mathbb{R}^2}\left(|F|^2\right)\left(|F|_t^2\right)\mathrm{d}x\mathrm{d}s \\ &\le C\int_0^t\left(1+\|\Delta F_1\|_{L^2}^2+\|\nabla\Delta F_2\|_{L^2}^2\right)\mathrm{d}s.\end{aligned} \tag{29}$$

For $t \in [0,T]$, we define

$$\psi(t) := 1+\|\Delta F_1\|_{L^2}^2+\|\Delta F_2\|_{L^2}^2+\|\nabla\Delta F_2\|_{L^2}^2+\|n_t\|_{L^2}^2+\|\nabla n\|_{L^2}^2+\|\Delta n\|_{L^2}^2 ,$$

then the estimates (27)-(29) give

$$\begin{aligned}\psi(t) &\le C\int_0^t \psi(s)\left(1+\|F_1\|_{L^\infty}^2+\|\nabla F_2\|_{L^\infty}^2+\|n\|_{L^\infty}^2\right)\mathrm{d}s \\ &\le C\int_0^t \psi(s)\left(1+\ln\left(1+\|\Delta F_1\|_{L^2}\right)+\ln\left(1+\|\nabla\Delta F_2\|_{L^2}\right)+\ln\left(1+\|\Delta n\|_{L^2}\right)\right)\mathrm{d}s \\ &\le C\int_0^t \psi(s)\left(1+\ln\psi(s)\right)\mathrm{d}s.\end{aligned}$$

By Gronwall's inequality and (13), one deduces from the above inequality that

$$\|F_1\|_{H^2}+\|F_2\|_{H^3}+\|n_t\|_{L^2}+\|n\|_{H^2} \le C$$

for any $t \in [0,T]$. Following the same argument as above, we can obtain

$$\|F_1\|_{H^{M-1}}+\|F_2\|_{H^M}+\|n\|_{H^{M-1}}+\|n_t\|_{H^{M-3}} \le C, \forall t \in [0,T].$$

Since the proof is similar, we omit further details. The proof of Theorem 1 is then completed.

4. Appendix

According to the equality $\nabla\times(\nabla\times E)=\nabla(\nabla\cdot E)-\Delta E$, the linear system (1) is equivalent to

$$\mathrm{i}E_t+(1-\alpha)\nabla(\nabla\cdot E)+\alpha\Delta E-\Gamma\nabla\left(\Delta(\nabla\cdot E)\right)=0, \tag{30}$$

we take the Fourier transform for (30) to obtain

$$\mathrm{i}\hat{E}_t+(1-\alpha)(\mathrm{i}\xi)\left(\mathrm{i}\xi\cdot\hat{E}\right)-\alpha|\xi|^2\hat{E}-\Gamma(\mathrm{i}\xi)\left(-|\xi|^2\right)\left((\mathrm{i}\xi)\cdot\hat{E}\right)=0, \tag{31}$$

(31) can be rewritten as in the matrix form

$$\mathrm{i}\begin{pmatrix}\hat{E}_1\\ \hat{E}_2\end{pmatrix}_t-\begin{pmatrix}\beta\xi_1^2+\alpha|\xi|^2 & \beta\xi_1\xi_2\\ \beta\xi_1\xi_2 & \beta\xi_2^2+\alpha|\xi|^2\end{pmatrix}\begin{pmatrix}\hat{E}_1\\ \hat{E}_2\end{pmatrix}=0,$$

where $\beta=(1-\alpha)+\Gamma|\xi|^2$.

Let

$$A=\begin{pmatrix}\beta\xi_1^2+\alpha|\xi|^2 & \beta\xi_1\xi_2\\ \beta\xi_1\xi_2 & \beta\xi_2^2+\alpha|\xi|^2\end{pmatrix},$$

then there is

$$|\lambda E - A| = \begin{vmatrix} \lambda - \beta\xi_1^2 - \alpha|\xi|^2 & -\beta\xi_1\xi_2 \\ -\beta\xi_1\xi_2 & \lambda - \beta\xi_2^2 - \alpha|\xi|^2 \end{vmatrix} = \left(\lambda - \alpha|\xi|^2\right)\left(\lambda - |\xi|^2\left(1+\Gamma|\xi|^2\right)\right). \tag{32}$$

Therefore, the determinant (32) implies

$$\lambda_1 = \alpha|\xi|^2, \lambda_2 = |\xi|^2\left(1+\Gamma|\xi|^2\right).$$

For $\lambda_1 = \alpha|\xi|^2$, the corresponding eigenvector is

$$x_1 = (-\xi_2, \xi_1)^{\mathrm{T}}.$$

For $\lambda_2 = |\xi|^2\left(1+\Gamma|\xi|^2\right)$, the corresponding eigenvector is

$$x_2 = (\xi_1, \xi_2)^{\mathrm{T}}.$$

After unitization, we attain

$$\begin{cases} \eta_1 = \left(-\frac{\xi_2}{\sqrt{\xi_1^2+\xi_2^2}}, \frac{\xi_1}{\sqrt{\xi_1^2+\xi_2^2}}\right)^{\mathrm{T}}, \\ \eta_2 = \left(\frac{\xi_1}{\sqrt{\xi_1^2+\xi_2^2}}, \frac{\xi_2}{\sqrt{\xi_1^2+\xi_2^2}}\right)^{\mathrm{T}}. \end{cases} \tag{33}$$

Then from (33), we can obtain the orthogonal matrix

$$Q = \begin{pmatrix} -\frac{\xi_2}{\sqrt{\xi_1^2+\xi_2^2}} & \frac{\xi_1}{\sqrt{\xi_1^2+\xi_2^2}} \\ \frac{\xi_1}{\sqrt{\xi_1^2+\xi_2^2}} & \frac{\xi_2}{\sqrt{\xi_1^2+\xi_2^2}} \end{pmatrix}.$$

Since Q satisfies $Q^{\mathrm{T}}Q = E$ and

$$Q^{\mathrm{T}}AQ = \begin{pmatrix} \alpha|\xi|^2 & 0 \\ 0 & |\xi|^2\left(1+\Gamma|\xi|^2\right) \end{pmatrix}.$$

If we set

$$Q^{\mathrm{T}}\begin{pmatrix} \hat{E}_1 \\ \hat{E}_2 \end{pmatrix} = \begin{pmatrix} \hat{F}_1 \\ \hat{F}_2 \end{pmatrix},$$

then

$$\mathrm{i}\begin{pmatrix} \hat{F}_1 \\ \hat{F}_2 \end{pmatrix}_t - \begin{pmatrix} \alpha|\xi|^2 & 0 \\ 0 & |\xi|^2\left(1+\Gamma|\xi|^2\right) \end{pmatrix}\begin{pmatrix} \hat{F}_1 \\ \hat{F}_2 \end{pmatrix} = 0. \tag{34}$$

Now we take the inverse Fourier transform for (34) to derive an equivalent form of the linear part of (1)

$$\begin{cases} \mathrm{i}F_{1t} + \alpha\Delta F_1 = 0, \\ \mathrm{i}F_{2t} + \Delta F_2 - \Gamma\Delta^2 F_2 = 0. \end{cases} \tag{35}$$

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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