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Diversity of Rogue Wave Solutions to the (1+1)-Dimensional Boussinesq Equation

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Abstract

A periodically homoclinic solution and some rogue wave solutions of (1+1)-dimensional Boussinesq equation are obtained via the limit behavior of parameters and different polynomial functions. Besides, the mathematics reasons for different spatiotemporal structures of rogue waves are analyzed using the extreme value theory of the two-variables function. The diversity of spatiotemporal structures not only depends on the disturbance parameter u_0 but also has a relationship with the other parameters c_0, α, β.

Keywords

Boussinesq Equation, Rogue wave, Periodically Homoclinic Solution, Spatiotemporal Structure

1. Introduction

The (1+1)-dimensional Boussinesq (Bq) equation

$$u_{tt} - c_0^2 u_{xx} - \alpha \left(u^2\right)_{xx} - \beta u_{xxxx} = 0, \tag{1}$$

where $u = u(x,t)$ denotes an average longitudinal wave velocity in fluids, c_0 is the velocity of the linear wave, α is the nonlinear term coefficient, and β is the dispersion coefficient. Bq equation can be used to describe many real-world processes such as the oscillations of nonlinear elastic strings and the propagation of long waves in shallow water. Then $\beta < 0$, Equation (1) is called "good" Bq equation, which describes the two-dimensional irrotational flow of a nonviscous fluid in a uniform rectangular channel. Then $\beta > 0$, Equation (1) is called the "bad" Bq equation, which is used to describe the two-dimensional flow of small amplitude shallow water waves.

Recently, many methods have been proposed to study the solitary wave solution of the nonlinear partial differential equation (NLPDEs), such as the inverse scattering method [1], Darboux transformation method [2], extended homoclinic test technique [3], Parameter limit method [4] and Hirota's direct method [5] [6] [7]. As a typical NLPDE, Equation (1) is studied by using different methods. Dai *et al.* reported its homoclinic orbit, two-wave solutions, homoclinic breather solution and rational homoclinic wave solution, respectively [8] [9] [10]. Rao *et al.* studied its rogue waves and hybrid solutions via applying Hirota bilinear method [11]. Zha *et al.* discussed its soliton interactions on nonzero backgrounds and resonant interactions by using Darboux transformation method [12]. Clarkson *et al.* obtained its algebraically decaying rational solutions of different orders by choosing a special polynomial function [13]. However, there are many properties and exact solutions of Equation (1) worth further study. In this paper, a periodically homoclinic solution that is different from [10] is studied via selecting a new test function, a rogue wave solution has emerged via the parameter p_1 limit behavior, and the mathematics reasons for the formation of different structures are given. Besides, one-rogue wave and two-rogue wave solutions different from [13] are studied by choosing different polynomial functions.

2. Emergence and Spatiotemporal Theory of Rogue Wave

By the following Cole-Hopf transformation,

$$u(x,t)=u_0+\frac{6\beta}{\alpha}(\ln f)_{xx}, \tag{2}$$

where u_0 is an arbitrary constant, and $f(x,t)$ is an unknown function of x and t. The following bilinear form is obtained by substituting Equation (2) into Equation (1),

$$\left(D_t^2-\left(c_0^2+2\alpha u_0\right)D_x^2-\beta D_x^4\right)f\cdot f=0, \tag{3}$$

while $D\_$ is the bilinear differential operators defined by

$$D_x^m D_t^n(g\cdot h)=\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^m\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^n g(x,t)h(x',t')\Big|_{x'=x,t'=t}, \tag{4}$$

where g and f are functions of x and t, and x' and t' are both the formal variables, m and n are non-negative integers [14]. When $c_0^2=1$, $\alpha=3$ and $\beta=1$ in Equation (1), a pair of periodic homoclinic breather waves is obtained by choosing a special kind of test function, and the following form of rogue-wave solution is studied using homoclinic limit method [10],

$$U_{\text{rogue-wave}}=u_0+\frac{8\left(-\frac{6}{1+6u_0}-2\left(x+\frac{1+6u_0}{\beta}t\right)(x+\beta t)\right)}{\left(\left(x+\frac{1+6u_0}{\beta}t\right)^2+(x+\beta t)^2-\frac{6}{1+6u_0}\right)^2}, \tag{5}$$

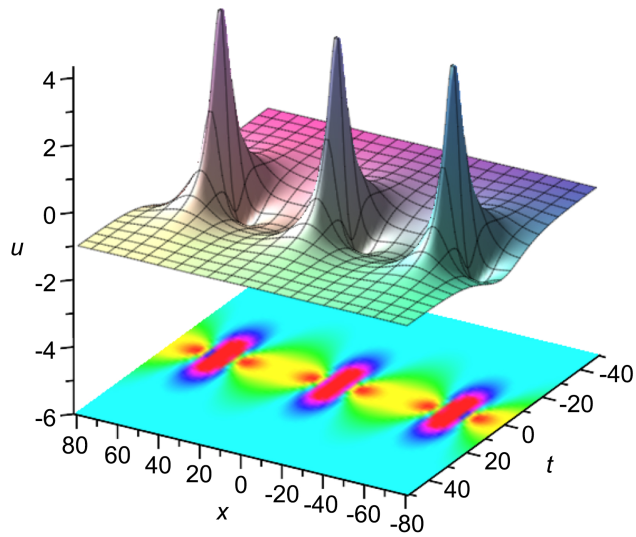
where $\beta^2=-(1+6u_0)$. Here, we choose the following test function which is different from [10],

$$f(x,t)=1+b_1\left(e^{ip_1x}+e^{-ip_1x}\right)e^{p_2t+\gamma}+b_2e^{2(p_2t+\gamma)}, \tag{6}$$

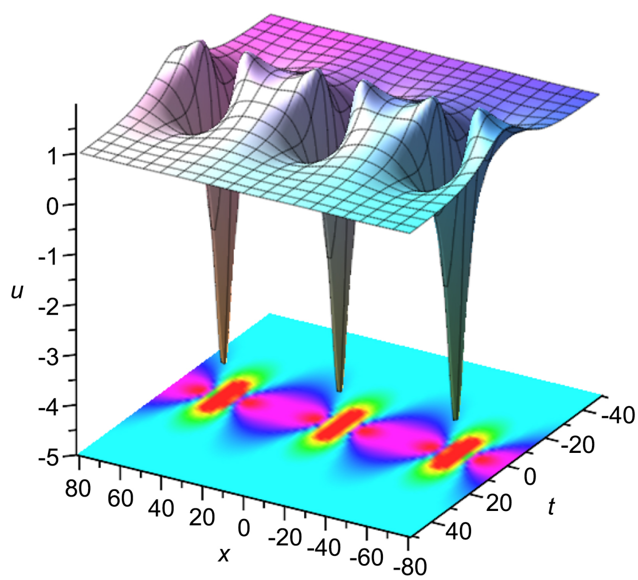
where i is the imaginary unit, $b_j, p_j\ (j=1,2)$ and γ are some real constants. Substituting the Equation (6) into bilinear form (3) with Maple 2023, we can get the following relations:

$$p_2=p_1\sqrt{\beta p_1^2-\left(c_0^2+2\alpha u_0\right)}, b_1=-\sqrt{b_2\Delta}, \Delta\triangleq\frac{\beta p_1^2-\left(c_0^2+2\alpha u_0\right)}{4\beta p_1^2-\left(c_0^2+2\alpha u_0\right)}. \tag{7}$$

A periodically homoclinic solution as follows is obtained by inserting Equation (6) with Equations (7) into Equation (2), (see **Figure 1**).



(a)



(b)

Figure 1. Spatiotemporal structure of Equation (8) as $p_1=\frac{1}{9}$, $b_2=1$, $\lambda=0$, $\beta=20$, (a) $u_0=c_0=-1$, $\alpha=1$; (b) $u_0=1$, $c_0=\alpha=-1$.

$$u(x,t) = u_0 + \frac{6\beta\sqrt{b_2\Delta}p_1^2\left(\cos(p_1x)\cosh\left(p_1\sqrt{\beta p_1^2-(c_0^2+2\alpha u_0)}t+\gamma+\ln\sqrt{b_2}\right)-\sqrt{\Delta}\right)}{\alpha\left(\cosh\left(p_1\sqrt{\beta p_1^2-(c_0^2+2\alpha u_0)}t+\gamma+\ln\sqrt{b_2}\right)-\sqrt{\Delta}\cos(p_1x)\right)^2}. \quad (8)$$

Taking $\gamma=0$ and $b_2=1$ in Equation (8), we get a rogue wave solution that is different from [10] by the limit behavior of the parameter p_1, namely,

$$\begin{aligned} u(x,t)_{\text{rogue-wave}} &= \lim_{p_1\to 0}\left(u_0+\frac{6\beta\sqrt{b_2\Delta}p_1^2\left(\cos(p_1x)\cosh\left(p_1\sqrt{\beta p_1^2-(c_0^2+2\alpha u_0)}t\right)-\sqrt{\Delta}\right)}{\alpha\left(\cosh\left(p_1\sqrt{\beta p_1^2-(c_0^2+2\alpha u_0)}t\right)-\sqrt{\Delta}\cos(p_1x)\right)^2}\right) \\ &= u_0-\frac{12\beta}{\alpha}\frac{x^2+(c_0^2+2\alpha u_0)t^2+\frac{3\beta}{c_0^2+2\alpha u_0}}{\left(x^2-(c_0^2+2\alpha u_0)t^2-\frac{3\beta}{c_0^2+2\alpha u_0}\right)^2}. \end{aligned} \quad (9)$$

We get two different forms of spatiotemporal structure of the rogue wave Equation (9) when the free parameter (c_0,u_0,α,β) takes different values (see **Figure 2**). The periodic solution Equation (8) has similar phenomena (see **Figure 1**). Now, we discuss the mathematical reason for the structure change of rogue wave Equation (9) via using the extreme value theory of the element function. The critical point of the rogue wave solution Equation (9):

$$p_1(x,t)=(0,0),\ p_2(x,t)=\left(\pm\sqrt{\frac{9\beta}{c_0^2+2\alpha u_0}},0\right). \quad (10)$$

Setting

$$\Delta=\frac{\partial^2}{\partial x^2}u(x,t), H(u)=\frac{\partial^2u(x,t)}{\partial x^2}\frac{\partial^2u(x,t)}{\partial t^2}-\left(\frac{\partial^2u(x,t)}{\partial x\partial t}\right)^2, \quad (11)$$

then

$$\Delta\big|_{p_1}=\frac{8(c_0^2+2\alpha u_0)^2}{-\alpha\beta}, H(u)\big|_{p_1}=-\frac{64(c_0^2+2\alpha u_0)^5}{3\alpha^2\beta^2}, u(x,t)\big|_{p_1}=-7u_0-\frac{4c_0^2}{\alpha}. \quad (12)$$

$$\Delta\big|_{p_2}=\frac{(c_0^2+2\alpha u_0)^2}{4\alpha\beta}, H(u)\big|_{p_2}=-\frac{(c_0^2+2\alpha u_0)^5}{12\alpha^2\beta^2}, u(x,t)\big|_{p_2}=2u_0+\frac{c_0^2}{2\alpha}. \quad (13)$$

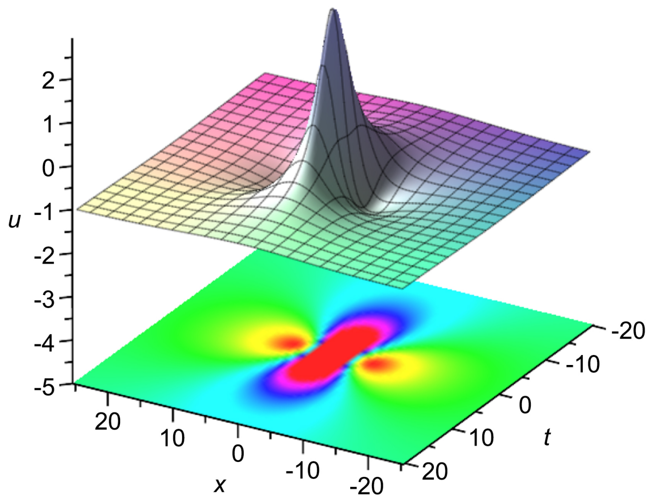
According to the discussions of Equation (12) and Equation (13), we can get the following results:

1) When $\alpha\beta>0$ and $c_0^2+2\alpha u_0<0$, namely, the symbols of the parameters (c_0,u_0,α,β) must be satisfied: $(\pm,+,-,-)$ or $(\pm,-,+,+)$. There are $\beta u_0<0$, $\Delta\big|_{p_1}<0$, $\Delta\big|_{p_2}>0$, $H(u)\big|_{p_1}>0$ and $H(u)\big|_{p_2}>0$, the critical point p_1 is a global maximum point and p_2 are two global minimum points, $u(x,t)$ shows an upward peak and two downward bulges. This form of rogue wave structure is also called bright rogue solution [15] and lump-type solution [16] [17] (see **Fig-**

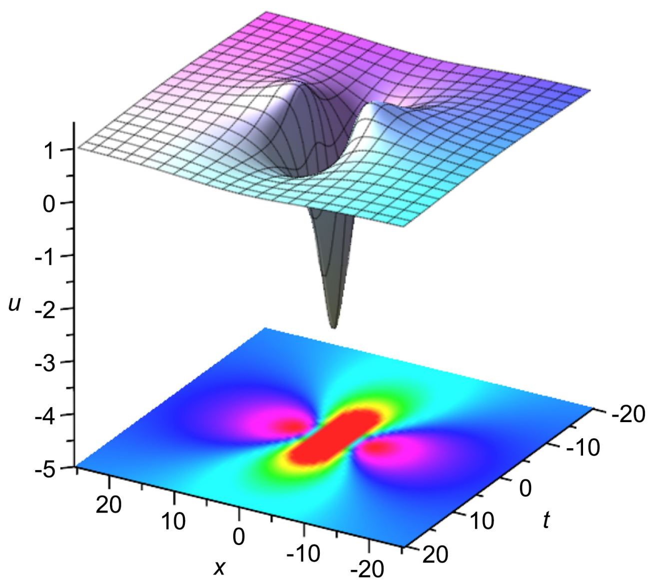
ure 2(a)).

2) When $\alpha\beta < 0$ and $c_0^2 + 2\alpha u_0 < 0$, namely, the symbols of the parameters $(c_0, u_0, \alpha, \beta)$ must be satisfied: $(\pm, +, -, +)$ or $(\pm, -, +, -)$. There are $\beta u_0 > 0$, $\Delta|_{p_1} > 0$, $\Delta|_{p_2} < 0$, $H(u)|_{p_1} > 0$ and $H(u)|_{p_2} > 0$, the critical point p_2 are two global maximum points and p_1 is a global maximum point, $u(x,t)$ shows two small upward bulges and a downward deep hole. This form of rogue wave structure is also called dark rogue solution [16] and lump-type solution (see Figure 2(b)).

3) When $c_0^2 + 2\alpha u_0 > 0$ and $\alpha\beta \neq 0$, there is $H(u) < 0$; or the symbols of the parameters $(c_0, u_0, \alpha, \beta)$ are not satisfied with 1) and 2), the critical point p is not a global extremum point, there is no corresponding rogue wave structure.



(a)



(b)

Figure 2. Spatiotemporal structure of Equation (9), (a) $u_0 = -1$, $c_0 = \alpha = 1$, $\beta = 4$; (b) $u_0 = c_0 = 1$, $\alpha = -1$, $\beta = 4$.

3. Different Forms of Rogue Wave

3.1. One-Rogue Wave Solution

Inspired by the rational form of rogue-wave solution Equation (5) and Equation (9), and on the foundation of Refs. [18] [19], we choose the following $f(x,t)$ in Equation (3)

$$f(x,t)=a_1+(a_2x+a_3t+a_4)^2+(a_5x+a_6t+a_7)^2, \tag{14}$$

where $a_i, i=1,\cdots,7$ are some free real numbers to be determined. Substituting Equation (14) into bilinear form Equation (3) with Maple, through extensive computations, we can obtain the following relations:

$$a_1=\frac{3\beta\left(a_2^2+a_5^2\right)}{-\left(c_0^2+2\alpha u_0\right)}, a_3=a_5\sqrt{-c_0^2-2\alpha u_0}, a_6=-a_2\sqrt{-c_0^2-2\alpha u_0}, \tag{15}$$

and

$$a_1=\frac{3\beta\left(a_6^2-\left(c_0^2+2\alpha u_0\right)a_5^2\right)}{\left(c_0^2+2\alpha u_0\right)^2}, a_2=\frac{-a_6}{\sqrt{-c_0^2-2\alpha u_0}}, a_3=a_5\sqrt{-c_0^2-2\alpha u_0}. \tag{16}$$

One-rogue wave solution is obtained via inserting Equations (15) with Equation (14) into Equation (2) (see **Figure 3**),

$$u(x,t)=u_0+\frac{6\beta}{\alpha}\left(\frac{2a_2^2+2a_5^2}{\left(a_2x+a_5\sqrt{-c_0^2-2\alpha u_0}t+a_4\right)^2+\left(a_5x-a_2\sqrt{-c_0^2-2\alpha u_0}t+a_7\right)^2-\frac{3\beta\left(a_2^2+a_5^2\right)}{c_0^2+2\alpha u_0}}\right.$$
$$\left.-\frac{\left(2\left(a_2x+a_5\sqrt{-c_0^2-2\alpha u_0}t+a_4\right)a_2+2\left(a_5x-a_2\sqrt{-c_0^2-2\alpha u_0}t+a_7\right)a_5\right)^2}{\left(\left(a_2x+a_5\sqrt{-c_0^2-2\alpha u_0}t+a_4\right)^2+\left(a_5x-a_2\sqrt{-c_0^2-2\alpha u_0}t+a_7\right)^2-\frac{3\beta\left(a_2^2+a_5^2\right)}{c_0^2+2\alpha u_0}\right)^2}\right). \tag{17}$$

A similar result is obtained via inserting Equations (16) with Equation (14) into Equation (2).

3.2. Two-Rogue Wave Solution

We choose a new function $f(x,t)$ different from [13] in Equation (3)

$$f(x,t)=a_1+\left(a_2x^2+a_3t^2\right)+\left(a_4x^4+a_5t^4+a_6x^2t^2\right)+\left(a_7x^2+a_8t^2\right)^3, \tag{18}$$

where $a_i, i=1,\cdots,8$ are some free real numbers. Two-rogue wave solution is obtained via substituting Equation (18) into Equation (2),

$$u(x,t)=u_0+\frac{6\beta}{\alpha}\left(\frac{2a_2+12a_4x^2+2a_6t^2+8a_7^2x^2+4a_7\left(a_7x^2+a_8t^2\right)}{a_1+a_2x^2+a_3t^2+a_4x^4+a_5t^4+a_6x^2t^2+\left(a_7x^2+a_8t^2\right)^2}\right.$$
$$\left.-\frac{\left(2a_2x+4a_4x^3+2a_6xt^2+4a_7\left(a_7x^2+a_8t^2\right)x\right)^2}{\left(a_1+a_2x^2+a_3t^2+a_4x^4+a_5t^4+a_6x^2t^2+\left(a_7x^2+a_8t^2\right)^2\right)^2}\right). \tag{19}$$

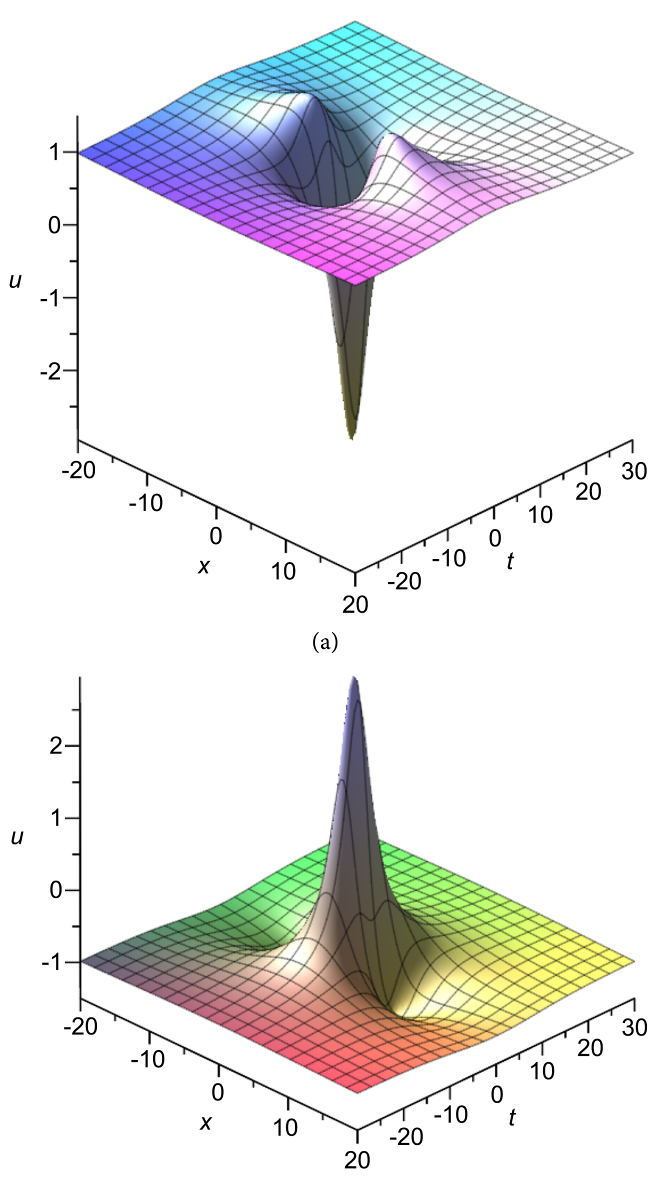
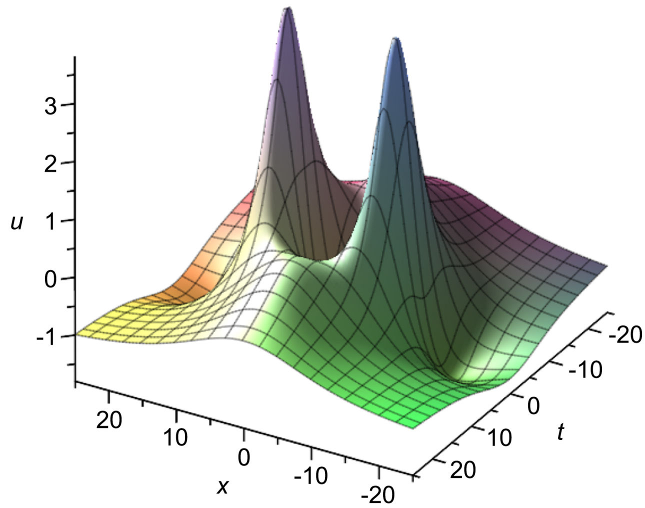


Figure 3. Spatiotemporal structure of Equation (17) as $a_2 = a_5 = 1$, $a_4 = a_7 = 2$: (a) $u_0 = -1$, $\alpha = c_0 = 1$, $\beta = 4$; (b) $u_0 = c_0 = 1$, $\alpha = -1$, $\beta = 4$.

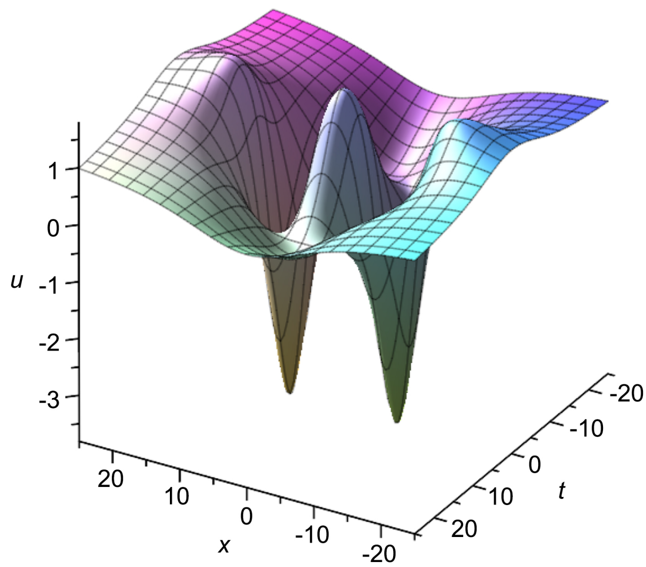
The parameters $a_i, i = 1, \cdots, 6$ and a_8 must satisfy the following relationships by calculation similar to section 3.1,

$$\begin{aligned} & a_1 = -\frac{1875\beta^3 a_7^3}{\left(c_0^2 + 2\alpha u_0\right)^3},\ a_2 = -\frac{125\beta^2 a_7^3}{\left(c_0^2 + 2\alpha u_0\right)^2},\ a_3 = -\frac{475\beta^2 a_7^3}{c_0^2 + 2\alpha u_0}, \\ & a_4 = \frac{-25\beta a_7^2}{c_0^2 + 2\alpha u_0},\ a_5 = -17\beta\left(c_0^2 + 2\alpha u_0\right)a_7^3,\ a_6 = 90\beta a_7^3, \\ & a_8 = -a_7\left(c_0^2 + 2\alpha u_0\right). \end{aligned} \tag{20}$$

By selecting the parameters $(c_0, u_0, \alpha, \beta)$, two different spatial structures of a two-rogue wave solution can be obtained (see **Figure 4**).



(a)



(b)

Figure 4. Spatiotemporal structure of Equation (19) as $a_7 = 2$, $\beta = 12$, (a) $u_0 = -1$, $\alpha = c_0 = 1$; (b) $u_0 = c_0 = 1$, $\alpha = -1$.

4. Conclusions

In summary, applying Hirota's bilinear form and different polynomial functions to the (1+1)-dimensional Bq equation, we obtained a periodically homoclinic solution and some rogue wave solutions. Besides, we also discussed that the deflection of rogue wave structures not only depends on the perturbation parameter u_0 but also has a relationship with the α, β, c_0. We got different forms of spatiotemporal structures of rogue wave solution: bright rogue wave structure, dark rogue wave structure and two-rogue wave structure. In fact, many nonlinear systems can get similar results, such as (1+1)-dimensional Nonlinear Schrödinger equation, Kadomtsev-Petviashvili equation, Benjamin-Ono equation, Yu-Toda-Sasa-Fukuyama equation. We will study the multiple-rogue wave solution of NLPDEs in the future.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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