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Variational Approach to Heat Conduction Modeling

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Abstract

It is known that Fourier's heat equation, which is parabolic, implies an infinite velocity propagation, or, in other words, that the mechanism of heat conduction is established instantaneously under all conditions. This is unacceptable on physical grounds in spite of the fact that Fourier's law agrees well with experiment. However, discrepancies are likely to occur when extremely short distances or extremely short time intervals are considered, as they must in some modern problems of aero-thermodynamics. Cattaneo and independently Vernotte proved that such process can be described by Heaviside's telegraph equation. This paper shows that this fact can be derived using calculus of variations, by application of the Euler-Lagrange equation. So, we proved that the equation of heat conduction with finite velocity propagation of the thermal disturbance can be obtained as a solution to one variational problem.

Keywords

Telegraph Equation, Heat Equation, Heat Conduction, Calculus of Variations

1. Introduction

Partial differential equations play an important role in various branches of mathematics, and mathematical physics as well as in a large number of engineering applications. Differential equations describe various physical phenomena whose solutions are often quite complex and difficult to apply in practice. Practical problems, which are mostly obtained experimentally or by measurement at an industrial plant, require the simplest possible solutions that satisfy the set conditions.

Heat conduction in solid environments (solid materials) is widely used in various branches of process engineering and thermotechnics. The equations that describe heat conduction are partial differential equations. The Fourier's heat equation, proposed by Joseph Fourier in 1822 for the purpose of modeling how heat diffuses through a given region, describes well the phenomenon of heat conduction in solids for most practical problems [1] [2] [3] [4] [5].

It predicts an infinite rate of expansion of the thermal disturbance, which is physically unacceptable and contradicts the existing recent theories that treat the phenomenon of heat transfer [6]-[11]. Very fast material heating processes, e.g. during the absorption of energy coming from ultra-short laser pulses, cannot be satisfactorily explained by the Fourier's heat equation [12] [13]. The proposed theory includes a time delay (relaxation time) and introduces the temperature gradient rate as an additional term in the classical heat conduction equation.

Cattaneo [14] [15] and independently Vernotte [16] proved that such process can be described by Heaviside's telegraph equation. This equation comes from Oliver Heaviside who, in 1876, developed the transmission line model, which demonstrates that the electromagnetic waves can be reflected on the wire, and that wave patterns can form along the line. Based on these assumptions, thermal disturbance in a solid medium behaves like a wave (the so-called "second sound").

In this paper, it has been shown that this fact can be derived using calculus of variations, by application of the Euler-Lagrange equation. So, we proved that the equation of heat conduction with finite velocity propagation of the thermal disturbance can be obtained as a solution to one variational problem.

Further, the characteristic equation of the telegraph equation will be derived. In general, information about the speed of physical quantities $T(x,\tau)$, which is the subject of the telegraphic equation, extends along the characteristics, *i.e.* along characteristic lines. Will be shown that the rate of disturbance can be determined using the expression $v=\sqrt{C/A}$ ($A\neq 0$, A,C-same sign) where A and C are coefficients in the telegraph equation:

$$A\frac{\partial^2 T}{\partial \tau^2}+2B\frac{\partial T}{\partial \tau}=C\frac{\partial^2 T}{\partial x^2}.$$

The paper describes an expression for the approximate solution of the telegraph equation using calculus of variations. It shows that in limit, when the relaxation time is obtained, an approximate solution of the classical (Fourier) equation of heat conduction is obtained.

2. Mathematical Formulation

2.1. The Telegraph Equation

Let's consider the partial differential equation:

$$A\frac{\partial^2 T}{\partial \tau^2}+2B\frac{\partial T}{\partial \tau}=C\frac{\partial^2 T}{\partial x^2}\quad (A,B,C\in R). \tag{1}$$

If in the set of real numbers:

1) $\Delta = AC > 0$, we say that Equation (1) is of hyperbolic type;
2) $\Delta = AC = 0$, we say that Equation (1) is of parabolic type;
3) $\Delta = AC < 0$, we say that Equation (1) is of elliptic type.

Mathematical expression $\Delta = AC$ can vary from point to point in a given area and therefore the type of Equation (1) can vary. The telegraph Equation (1) is of hyperbolic type, which means that in the given area it is $\Delta = AC > 0$.

Dividing Equation (1) by $2B$, $B \neq 0$ Equation (1) is equivalent to the equation:

$$\frac{A}{2B}\frac{\partial^2 T}{\partial \tau^2} + \frac{\partial T}{\partial \tau} = \frac{C}{2B}\frac{\partial^2 T}{\partial x^2}. \tag{2}$$

The differential Equation (1), *i.e.* (2) can be obtained as the Euler-Lagrange equation from the Lagrangrian:

$$L\left(x, \varsigma, T, \frac{\partial T}{\partial \tau}, \frac{\partial T}{\partial x}\right) = \left[\frac{A}{2B}\left(\frac{\partial T}{\partial \tau}\right)^2 - \frac{C}{2B}\left(\frac{\partial T}{\partial x}\right)^2\right] \mathrm{e}^{\frac{\tau}{A/2B}}. \tag{3}$$

Euler-Lagrange equation for a function of two variables $T(x,\tau)$ is given by the expression [17]:

$$\frac{\partial L}{\partial T} - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial T_x}\right) - \frac{\partial}{\partial \tau}\left(\frac{\partial L}{\partial T_\tau}\right) = 0. \tag{4}$$

From Equation (3), we get:

$$\frac{\partial L}{\partial T} = 0,\ \frac{\partial L}{\partial T_x} = -\frac{C}{2B}\cdot\frac{\partial T}{\partial x}\cdot \mathrm{e}^{\frac{\tau}{A/2B}},\ \frac{\partial L}{\partial T_\tau} = \frac{A}{2B}\cdot\frac{\partial T}{\partial \tau}\cdot \mathrm{e}^{\frac{\tau}{A/2B}}.$$

By changing the value $\frac{\partial L}{\partial T}, \frac{\partial L}{\partial T_x}, \frac{\partial L}{\partial T_\tau}$ into Equation (4) and after simple calculations, we come to the equation:

$$-\frac{A}{2B}\frac{\partial^2 T}{\partial \tau^2} - \frac{\partial T}{\partial \tau} + \frac{C}{2B}\frac{\partial^2 T}{\partial x^2} = 0,$$

whence follows:

$$A\frac{\partial^2 T}{\partial \tau^2} + 2B\frac{\partial T}{\partial \tau} = C\frac{\partial^2 T}{\partial x^2},$$

which had to be proved.

Same vice-versa—Equation (1), *i.e.* (2) represents the variational task of finding the stationary value of the integral:

$$F = \int_0^\tau \int_0^L L \mathrm{d}x \mathrm{d}\tau, \tag{5}$$

where L is the Lagrangian given at (3).

Determining the Characteristics of the Telegraph Equation

Equation (1) can be written in the form:

$$C\frac{\partial^2 T}{\partial x^2} - A\frac{\partial^2 T}{\partial \tau^2} = 2B\frac{\partial T}{\partial \tau}. \tag{6}$$

The characteristic equations will be determined if the equations of total differentials for partial derivatives are added to equation T_x and T_τ like:

$$\mathrm{d}\left(T_x\right) = T_{xx}\mathrm{d}x + T_{x\tau}\mathrm{d}\tau, \tag{7}$$

$$\mathrm{d}\left(T_\tau\right) = T_{\tau x}\mathrm{d}x + T_{\tau\tau}\mathrm{d}\tau. \tag{8}$$

In matrix form, the system of Equations (6)-(8) can be written as:

$$\begin{pmatrix} C & 0 & -A \\ \mathrm{d}x & \mathrm{d}\tau & 0 \\ 0 & \mathrm{d}x & \mathrm{d}\tau \end{pmatrix} \cdot \begin{pmatrix} T_{xx} \\ T_{x\tau} \\ T_{\tau\tau} \end{pmatrix} = \begin{pmatrix} 2BT_\tau \\ \mathrm{d}\left(T_x\right) \\ \mathrm{d}\left(T_\tau\right) \end{pmatrix}. \tag{9}$$

Equation (9) has infinitely many solutions or no solution if the determinant of the system is equal to zero. From this condition, the characteristics of Equation (6) are provided in the form:

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \pm\sqrt{\frac{C}{A}}, \quad \left(A \neq 0, A, C\right) \text{ of the same sign.} \tag{10}$$

Integrating the last equation, the characteristics of equations are:

$$x - \sqrt{\frac{C}{A}} \cdot \tau = c_1, \tag{11}$$

$$x + \sqrt{\frac{C}{A}} \cdot \tau = c_2. \tag{12}$$

The characteristics of Equation (6) are two systems of parallel directions $x - \sqrt{C/A} \cdot \tau = c_1$ and $x + \sqrt{C/A} \cdot \tau = c_2$ where c_1 and c_2 are arbitrary constants. Through every point of the plane $xO\tau$ passes one characteristic from the mentioned system. Therefore, the general integral of the partial Equation (1) is:

$$T\left(x,\tau\right) = f\left(x - \sqrt{C/A} \cdot \tau\right) + g\left(x + \sqrt{C/A} \cdot \tau\right), \tag{13}$$

where are they f and g are arbitrary functions.

The physical meaning of the telegraph Equation (1) is that the size $T\left(x,\tau\right)$ extends (transports) along the characteristic curve. Velocity of disturbance (transport) of magnitude $T\left(x,\tau\right)$ is given by expression (10), *i.e.*:

$$v = \frac{\mathrm{d}x}{\mathrm{d}\tau} = \sqrt{\frac{C}{A}}, \quad \left(A \neq 0,\ A, C - \text{same symbol}\right).$$

2.2. Telegraph Equation and Heat Equation

For $A = 0$ because of $\Delta = 0$, the telegraph Equation (1) becomes a parabolic equation in the considered area:

$$\frac{\partial T}{\partial \tau} = \frac{C}{2B}\frac{\partial^2 T}{\partial x^2}. \tag{14}$$

Equation (14) is equivalent to the classic one-dimensional heat equation:

$$\frac{\partial T}{\partial \tau} = a\frac{\partial^2 T}{\partial x^2}, \tag{15}$$

where $a = C/2B$ is thermal diffusivity of solid material.

According to (10), the rate of thermal disturbance in a solid material is infinitely high $v=\infty$, which is physically inadmissible and this result is in accordance with data from the literature [6]-[11]. The characteristics' equations of Equation (15) based on (10) are $\mathrm{d}\tau=0$ that is $\tau=c=const$. Equation (14) or Equation (15) has a system of parallel lines as a feature in the plane $xO\tau$. It can be demonstrated that Equation (15) does not have an exact Lagrange function and cannot be formulated as a variational problem. By shift $A/2B=\tau^*$ and $C/2B=a$ in Equation (2), Equation (2) becomes:

$$\tau^*\frac{\partial^2 T}{\partial \tau^2}+\frac{\partial T}{\partial \tau}=a\frac{\partial^2 T}{\partial x^2}, \tag{16}$$

where τ^* is relaxation time and $a=\lambda/(\rho\cdot c_p)$ thermal (heat) diffusivity of solid material, λ,ρ,c_p heat conduction coefficient, density and specific heat capacity of solid material. The relaxation time is the time required to establish the stationary temperature state of the volume element of the solid material due to the effect of the temperature gradient.

Equation (16) represents the generalized heat conduction equation. In Reference [14], the generalized heat conduction equation was derived for the first time. It is a hyperbolic partial differential equation that can be derived as shown by the application of the calculus of variations and predicts the wave nature of heat conduction with a finite rate of thermal disturbance $v=\sqrt{a/\tau^*}$. When $\tau^*\to 0$, Equation (16) turns into the classic heat conduction equation (Equation (15)). The physical difference between Equation (15) and Equation (16) is that the rate of thermal disturbance according to Equation (15) is infinitely large, while the rate of thermal disturbance according to Equation (16) is finite.

2.2.1. Determining the Exact Solution of the Telegraph Equation

In References [18] [19], there are certain analytical methods for solving hyperbolic partial differential equations, and solving hyperbolic equations dates back to B.Rimann. In recent times, in engineering practice, the Laplace transform is used more and more when solving partial differential equations. Difficulties can arise when determining the inverse Laplace transform, so more and more authors [20] [21] use approximate numerical methods for determining the inverse Laplace transform. In this paper, the Fourier method of separation of variables will be used to solve the considered partial differential equation.

Now, let's solve the differential Equation (1), *i.e.* the equation:

$$\tau^*\frac{\partial^2 T}{\partial \tau^2}+\frac{\partial T}{\partial \tau}=a\frac{\partial^2 T}{\partial x^2}, \tag{17}$$

where $A/2B=\tau^*$ and $C/2B=a$. The solution will be sought for the case when the initial condition has the form:

$$T(x,0)=f(x)=x\cdot(L-x),\quad T_\tau(x,0)=0,\quad 0\le x\le L,\quad \tau>0, \tag{18}$$

and the boundary condition reads:

$$T(0,\tau)=T(L,\tau)=0. \tag{19}$$

We will look for a non-trivial solution to problems (17)-(19) in the form:

$$T(x,\tau)=X(x)\cdot Y(\tau)\neq 0,\quad 0\leq x\leq L,\quad \tau>0. \tag{20}$$

Equation (17) becomes:

$$\tau^{*}X(x)\cdot Y''(\tau)+X(x)\cdot Y'(\tau)=aX''(x)Y(\tau),$$

i.e.:

$$\frac{X''(x)}{X(x)}=\frac{\tau^{*}}{a}\cdot\frac{Y''(\tau)}{Y(\tau)}+\frac{1}{a}\cdot\frac{Y'(\tau)}{Y(\tau)}=-K,\quad K=const, \tag{21}$$

in which the variables x and τ are separated. Equation (21) now breaks down into a system of equations:

$$X''(x)+K\cdot X(x)=0, \tag{22}$$

$$Y''+\frac{1}{\tau^{*}}\cdot Y'+\frac{K\cdot a}{\tau^{*}}\cdot Y=0. \tag{23}$$

Boundary condition (19) reduces to:

$$X(0)=0,\quad X(L)=0. \tag{24}$$

The solution of Equation (22) has the form:

$$X(x)=C_1\cdot\cos\left(\sqrt{K}x\right)+C_2\cdot\sin\left(\sqrt{K}x\right). \tag{25}$$

It follows from the boundary conditions (24):

$$C_1=0,\quad C_2\cdot\sin\left(\sqrt{K}L\right)=0,\quad K_n=\frac{n^2\pi^2}{L^2},\quad (n=1,2,\cdots). \tag{26}$$

Now, the solution of Equation (22) based on (26) can be written in the form:

$$X_n(x)=C_2\cdot\sin\left(\frac{n\pi}{L}x\right),\quad (n=1,2,\cdots). \tag{27}$$

The solution of Equation (23) has the form:

$$Y_n(\tau)=A_n\cdot e^{r_{1,n}\cdot\tau}+B_n\cdot e^{r_{2,n}\cdot\tau}, \tag{28}$$

where:

$$r_{1,n}=\frac{-1+\sqrt{1-4K_n\tau^{*}a}}{2\tau^{*}},\quad r_{2,n}=\frac{-1-\sqrt{1-4K_n\tau^{*}a}}{2\tau^{*}}, \tag{29}$$

and the relaxation time belongs to the interval $\tau^{*}\in\left(0,1/4K_n a\right]$.

From (27) and (28), a particular solution is reached:

$$T_n(x,\tau)=X_n(x)\cdot Y_n(\tau)=C_2\cdot\sin\left(\frac{n\pi}{L}x\right)\cdot\left(A_n\cdot e^{r_{1,n}\cdot\tau}+B_n\cdot e^{r_{2,n}\cdot\tau}\right), \tag{30}$$

so the initial conditions entail $B_n=-A_n\cdot\frac{r_{1,n}}{r_{2,n}}$.

Now, the particular solution is:

$$Y_n(\tau)=C_n\cdot\left(r_{2,n}\cdot e^{r_{1,n}\cdot\tau}-r_{1,n}\cdot e^{r_{2,n}\cdot\tau}\right), \tag{31}$$

where it is $C_n=\frac{A_n}{r_{2,n}},(n=1,2,\cdots)$.

Now, based on (20) functions:

$$T_n(x,\tau) = X_n(x) \cdot Y_n(\tau) = D_n \cdot \left(r_{2,n} \cdot e^{r_{1,n} \cdot \tau} - r_{1,n} \cdot e^{r_{2,n} \cdot \tau}\right) \cdot \sin\left(\frac{n\pi}{L} \cdot x\right). \quad (32)$$

It satisfies Equation (17) and conditions (18) and (19) and based on the principle of linear superposition and functions, $T(x,\tau)$ defined by:

$$T(x,\tau) = \sum_{n=1}^{\infty} D_n \cdot \left(r_{2,n} \cdot e^{r_{1,n} \cdot \tau} - r_{1,n} \cdot e^{r_{2,n} \cdot \tau}\right) \cdot \sin\left(\frac{n\pi}{L} \cdot x\right). \quad (33)$$

It represents the solution to problems (17)-(19).

A constant D_n is determined from the conditions $T(x,0) = f(x)$, *i.e.*:

$$T(x,0) = \sum_{n=1}^{\infty} D_n \cdot \left(r_{2,n} - r_{1,n}\right) \cdot \sin\left(\frac{n\pi}{L} \cdot x\right) = f(x). \quad (34)$$

If it is stated that:

$$D_n \cdot \left(r_{2,n} - r_{1,n}\right) = E_n, \quad (35)$$

then it is:

$$\begin{aligned} E_n &= \frac{2}{L} \cdot \int_0^L f(x) \cdot \sin\left(\frac{n\pi}{L} x\right) \mathrm{d}x = \frac{2}{L} \cdot \int_0^L x(L-x) \cdot \sin\left(\frac{n\pi}{L} x\right) \mathrm{d}x \\ &= \frac{8L^2}{(2n-1)^3 \pi^3}, \quad (n = 1, 2, \cdots) \end{aligned} \quad (36)$$

so the solution to the considered problem is (17)-(19):

$$T(x,\tau) = \sum_{n=1}^{\infty} \frac{E_n}{r_{2,n} - r_{1,n}} \cdot \left(r_{2,n} \cdot e^{r_{1,n} \cdot \tau} - r_{1,n} \cdot e^{r_{2,n} \cdot \tau}\right) \cdot \sin\left(\frac{n\pi}{L} \cdot x\right), \quad (37)$$

and based on (36), the solution to problems (17)-(19) is given by:

$$T(x,\tau) = \frac{8L^2}{\pi^3} \sum_{n=1}^{\infty} \frac{r_{2,2n-1} \cdot e^{r_{1,2n-1} \cdot \tau} - r_{1,2n-1} \cdot e^{r_{2,2n-1} \cdot \tau}}{(2n-1)^3 \cdot \left(r_{2,2n-1} - r_{1,2n-1}\right)} \cdot \sin\left(\frac{(2n-1)\pi}{L} \cdot x\right). \quad (38)$$

2.2.2. Determining the Approximate Solution of the Telegraph Equation

Now, let's find an approximate solution to the equation:

$$\tau^* \frac{\partial^2 T}{\partial \tau^2} + \frac{\partial T}{\partial \tau} = a \frac{\partial^2 T}{\partial x^2}. \quad (39)$$

An approximate solution to the problem will be assumed in the form:

$$T(x,\tau) = x(L-x) \cdot f(\tau), \quad (40)$$

where $f(\tau)$ is an unknown function that satisfies the initial condition:

$$f(0) = 1. \quad (41)$$

If we start from the functional:

$$\begin{aligned} F\left(T(x,\tau)\right) &= \iint_D F\left(x,\tau,T(x,\tau),T'_x,T'_\tau\right) \mathrm{d}x\mathrm{d}\tau \\ &= \int_0^L \int_0^\tau \left[\frac{\tau^*}{2}\left(\frac{\partial T}{\partial \tau}\right)^2 - \frac{a}{2}\left(\frac{\partial T}{\partial x}\right)^2\right] \cdot e^{\frac{\tau}{\tau^*}} \mathrm{d}x\mathrm{d}\tau, \end{aligned} \quad (42)$$

and by inserting the trial solution (40) into Equation (42), we get:

$$F = \int_0^{\tau} \left(\frac{\tau^* L^5}{60} \cdot f'^2(\tau) - \frac{aL^3}{6} \cdot f^2(\tau) \right) \cdot e^{\tau/\tau^*} \mathrm{d}\tau. \tag{43}$$

If we now introduce the Lagrangian function:

$$L(f, f') = \left(\frac{\tau^* L^5}{60} \cdot f'^2(\tau) - \frac{aL^3}{6} \cdot f^2(\tau) \right) \cdot e^{\tau/\tau^*}, \tag{44}$$

and using the Euler-Lagrange equation:

$$\frac{\partial L}{\partial f} - \frac{\partial}{\partial \tau} \left(\frac{\partial L}{\partial f_\tau} \right) = 0, \tag{45}$$

and dividing with $\exp\left(\tau/\tau^*\right)$ the following differential equation is obtained:

$$\frac{\tau^* L^5}{30} \cdot f''(\tau) + \frac{L^5}{30} \cdot f'(\tau) + \frac{aL^3}{3} \cdot f(\tau) = 0. \tag{46}$$

After a short calculation, the solution to Equation (46) is arrived:

$$\begin{aligned} f(\tau) = {} & C_1 \cdot \exp\left(\frac{\left(-L^2 + \sqrt{L^4 - 40a\tau^* L^2} \right) \cdot \tau}{2\tau^* L^2} \right) \\ & + C_2 \cdot \exp\left(\frac{\left(-L^2 - \sqrt{L^4 - 40a\tau^* L^2} \right) \cdot \tau}{2\tau^* L^2} \right) \end{aligned} \tag{47}$$

at which they are C_1 and C_2 constants that are determined from condition (41) and from the condition that $T_\tau(x, 0) = 0$.

The first condition gives:

$$C_1 + C_2 = 1, \tag{48}$$

and the second condition gives:

$$\begin{aligned} & C_1 \cdot e^{\frac{\left(-L^2 + \sqrt{L^4 - 40a\tau^* L^2} \right) \cdot \tau}{2\tau^* L^2}} \cdot \frac{-L^2 + \sqrt{L^4 - 40a\tau^* L^2}}{2\tau^* L^2} \\ & + C_2 \cdot e^{\frac{\left(-L^2 - \sqrt{L^4 - 40a\tau^* L^2} \right) \cdot \tau}{2\tau^* L^2}} \cdot \frac{-L^2 - \sqrt{L^4 - 40a\tau^* L^2}}{2\tau^* L^2} = 0. \end{aligned} \tag{49}$$

The constants are determined from the system of Equations (48) and (49). C_1 and C_2:

$$C_1 = \frac{1}{2} \cdot \left(1 + \frac{L^2}{\sqrt{L^4 - 40a\tau^* L^2}} \right), \tag{50}$$

$$C_2 = \frac{1}{2} \cdot \left(1 - \frac{L^2}{\sqrt{L^4 - 40a\tau^* L^2}} \right). \tag{51}$$

Now, the final solution of Equation (47) is:

$$f(\tau)=\frac{1}{2}\cdot\left(1+\frac{L^2}{\sqrt{L^4-40a\tau^*L^2}}\right)\cdot \mathrm{e}^{\frac{\left(-L^2+\sqrt{L^4-40a\tau^*L^2}\right)\cdot\tau}{2\tau^*L^2}} + \frac{1}{2}\cdot\left(1-\frac{L^2}{\sqrt{L^4-40a\tau^*L^2}}\right)\cdot \mathrm{e}^{\frac{\left(-L^2-\sqrt{L^4-40a\tau^*L^2}\right)\cdot\tau}{2\tau^*L^2}}, \tag{52}$$

so the approximate solution of Equation (39) based on (40) is given by:

$$T(x,\tau)=\frac{x}{2}\cdot(L-x)\cdot\left(\left(1+\frac{L}{\sqrt{L^2-40a\tau^*}}\right)\cdot \mathrm{e}^{\frac{\left(-L+\sqrt{L^2-40a\tau^*}\right)\cdot\tau}{2\tau^*L}} + \left(1-\frac{L}{\sqrt{L^2-40a\tau^*}}\right)\cdot \mathrm{e}^{\frac{\left(-L-\sqrt{L^2-40a\tau^*}\right)\cdot\tau}{2\tau^*L}}\right), \tag{53}$$

where the relaxation time belongs to the interval $\tau^*\in\left(0, L^2/40a\right)$.

By shift $\tau^*=A/2B$ and $a=C/2B, B\neq 0$ the solution (53) is also the solution of the telegraph Equation (1). If the boundary condition $\tau^*\to 0$ is applied to Equations (38) and (53), the exact and approximate solution of the classic heat conduction equation (Equation (15)) is obtained, which is given:

$$T(x,\tau)=\frac{8L^2}{\pi^3}\cdot\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\cdot \mathrm{e}^{-\frac{a\tau}{L^2}\cdot(2n-1)^2\cdot\pi^2}\cdot\sin\frac{(2n-1)\pi}{L}x, \tag{54}$$

$$T(x,\tau)=x\cdot(L-x)\cdot \mathrm{e}^{-\frac{10a\tau}{L^2}}. \tag{55}$$

3. Numerical Example

In the previous section, the analytical and approximate solution of the telegraph equation was presented. When relaxation time tends to zero, an approximate solution of the classical (Fourier) heat conduction equation is obtained. In order to show the relevant physical effects of the obtained results, numerical results will be presented.

As an example, let us consider heat propagation through an aluminum rod of limited length $0\le x\le 10\ \text{cm}$. In order for the temperature at all points of the cross-section of the rod to be the same, it will be assumed that the rod is sufficiently narrow and adiabatically insulated so that heat spreads only along x axis. The temperature field is then described by the telegraph Equation (17). The thermal diffusivity of aluminum is $a=0.8418\ \text{cm}^2\cdot\text{s}^{-1}$ [22] and relaxation time $\tau^*=10^{-9}\ \text{s}$ [23] [24]. **Table 1** shows the comparative results of non-stationary heat conduction calculations using Equation (38) and Equation (53) with the same initial and boundary conditions. Obviously, these two solutions agree well. Taking more terms in Equation (38) would give better results.

The behavior of the temperature field in the considered aluminum rod is shown in **Figure 1** and **Figure 2**. It can be observed that the temperature field gradually decreases as the heat in the rod decreases over time. During engineering

Table 1. Comparison of the calculation results of the determination of the temperature field $T(x,\tau)$ using the exact solution (Equation (38)) and the approximate solution of heat conduction obtained by variational calculus (Equation (53)) ($L = 10\text{ cm}$, $a = 0.8418\text{ cm}^2\cdot\text{s}^{-1}$, $\tau^* = 10^{-9}\text{ s}$).

$\tau(\text{s})$	1		10		20		50	
$x(\text{cm})$	$T(x,\tau)$ (°C) Correct solution Equation (38)	$T(x,\tau)$ (°C) Approximate solution Equation (53)	$T(x,\tau)$ (°C) Correct solution Equation (38)	$T(x,\tau)$ (°C) Approximate solution Equation (53)	$T(x,\tau)$ (°C) Correct solution Equation (38)	$T(x,\tau)$ (°C) Approximate solution Equation (53)	$T(x,\tau)$ (°C) Correct solution Equation (38)	$T(x,\tau)$ (°C) Approximate solution Equation (53)
0.0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
2.0	13.97	14.71	6.62	6.89	2.89	2.97	0.24	0.24
4.0	22.61	22.06	10.71	10.34	4.67	4.46	0.34	0.36
6.0	22.62	22.06	10.72	10.34	4.67	4.46	0.39	0.36
8.0	14.00	14.71	6.63	6.89	2.89	2.97	0.24	0.24
10.0	0.04	0.00	0.02	0.00	0.01	0.00	0.00	0.00

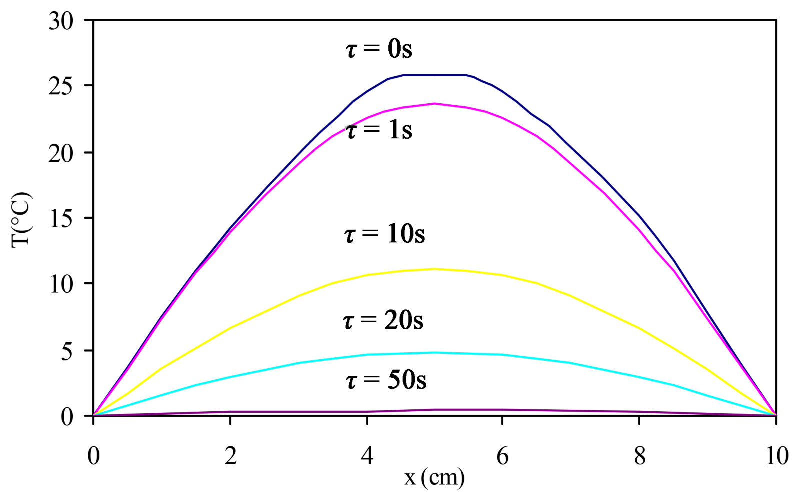

Figure 1. Temperature field of heat conduction in an aluminum rod depending on the depth of the rod x at a fixed time τ.

calculations of heat conduction through solid materials (aluminum rod), the practical question of determining time τ arises, whereas the entire rod will cool down to a certain temperature τ^*. The hottest spot is obviously on the spot $L/2$, so time τ is determined based on the equation $T(L/2,\tau) = T^*$.

For the considered example, the time required for the aluminium rod to cool, for example, to 1˚C, requires solving the equation $T(5,\tau) = 1$ and based on Equation (38) or Equation (53) $\tau \approx 38.23\text{ s}$ is obtained.

Figure 3 shows the temporal behavior of the temperature field depending on the fixed relaxation time τ^*. It is observed that the temperature field decreases slightly as the relaxation time τ^* decreases.

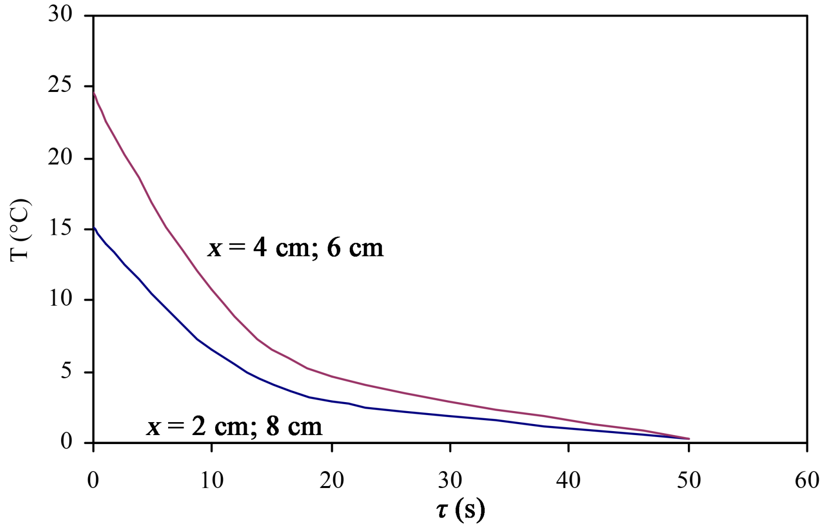

Figure 2. Temperature field of heat conduction in an aluminum rod depending on the weather τ at fixed depths *x*.

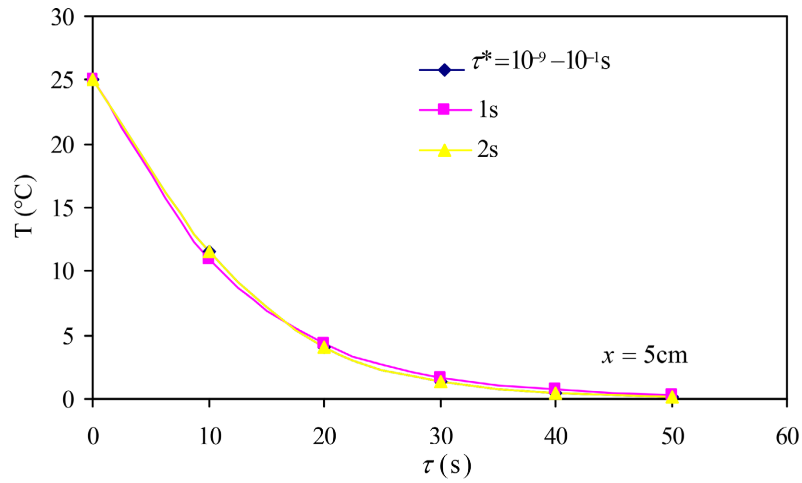

Figure 3. Distribution of the temperature field of heat conduction in an aluminum rod depending on the weather τ and fixed relaxation time τ^* at depth $x = 5\text{ cm}$.

Figure 4 shows the two-dimensional temperature field in the aluminum rod. It can be seen that the diagrams shown in **Figure 1** and **Figure 2** are obtained by sections of the surface $T(x,\tau)$ with planes $x = \text{const.}$ and $\tau = \text{const.}$

4. Conclusions

The telegraph equation was considered in the paper $A\frac{\partial^2 T}{\partial \tau^2} + 2B\frac{\partial T}{\partial \tau} = C\frac{\partial^2 T}{\partial x^2}\ (A, B, C \in R)$ and it was equivalent to the heat conduction equation. The conditions for switching to the classic heat conduction equation (parabolic equation) $\frac{\partial T}{\partial \tau} = a\frac{\partial^2 T}{\partial x^2}$ are shown and the generalized heat conduction equation (hyperbolic equation) $\tau^*\frac{\partial^2 T}{\partial \tau^2} + \frac{\partial T}{\partial \tau} = a\frac{\partial^2 T}{\partial x^2}$.

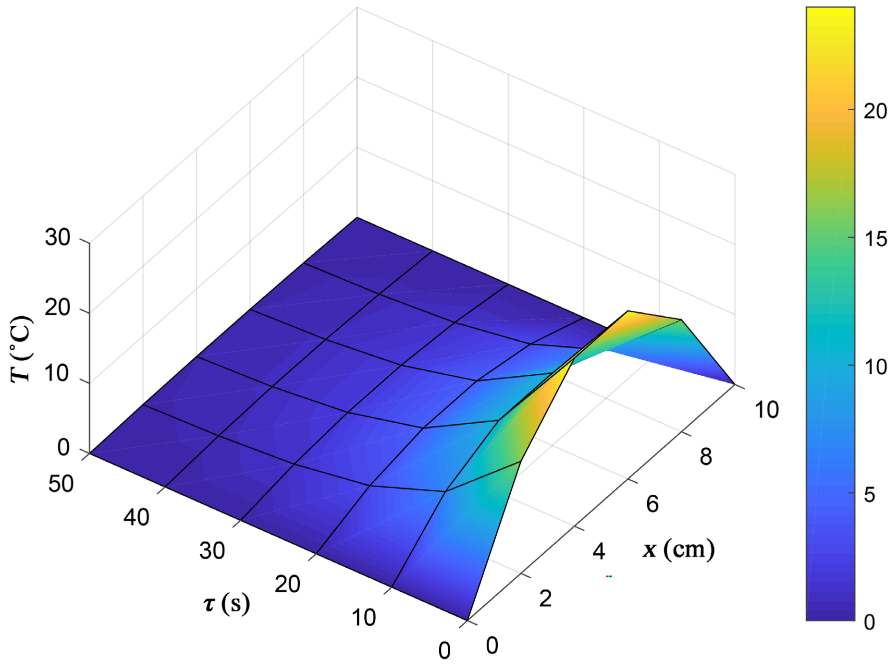

Figure 4. Two-dimensional temperature field of an aluminum rod.

It is demonstrated that the considered telegraph equation can be obtained by applying the Euler-Lagrange equation from the given Lagrangian (Equation (3)).

In the paper, the propagation speed of disturbances (thermal disturbance $T(x,\tau)$) $v = \frac{\mathrm{d}x}{\mathrm{d}\tau} = \sqrt{\frac{C}{A}}$, ($A \neq 0$, A, C -same symbol) using the characteristic equations of the telegraph equation is determined.

An exact solution of the telegraph equation was found using the Fourier method of separation of variables (Equation (38)) and an approximate solution using calculus of variations (Equation (53)). A numerical example showed the effectiveness of the proposed method (exact and approximate solution). It can be seen that the exact solution of the telegraph equation agrees well with the approximate solution involving the same initial and boundary conditions.

The diagrams in Figure 1 and Figure 2 show that the temperature field in the aluminum rod gradually decreases as the heat in the rod decreases over time, so the speed of heat propagation is higher for a smaller τ. Finally, the effect of the relaxation time τ^* at a fixed length of aluminum rod of 10 cm is shown in Figure 3. It is observed that the temperature field changes very little as the relaxation time τ^* becomes smaller.

In addition, it should be noted that the proposed approximate solution (Equation (53) and (55)) of heat conduction, which is used very easily and flexibly, is much simpler than the solutions offered in the cited literature.

The results obtained in this work also provide a theoretical basis for further analysis of heat conduction, especially for the further study of laser heating, and are comparable to the results in the cited literature. For engineering practice, the approximate solutions of the hyperbolic equation (Equation (53)) and parabolic equation (Equation (55)) are of special interest due to their simplicity, in contrast to the exact solutions and some approximate solutions that are not simple and describe heat transport.

Finally, the main results presented in this paper can be summarized as:

- It has been proven that the heat conduction equation (telegraph Equation (1)) can be obtained as a solution to a variational problem;
- It has been proven that the speed of propagation of thermal disturbances $v = \sqrt{C/A} = \sqrt{a/\tau^*}$, ($A \neq 0$, A, C -same sign) is final and that it is in agreement with the data in the cited literature;
- An approximate solution to the telegraph equation (Equation (53)) was determined using calculus of variations;
- An approximate solution of the classical parabolic equation (Equation (55)) was determined as the limiting case of Equation (53) when the relaxation time $\tau^* \to 0$;
- The performed numerical example of heat conduction through a 10 cm aluminum rod shows a good agreement between the results using the exact solution (Equation (38)) and the approximate solution (Equation (53)) using the calculus of variations. It should be emphasized that by taking more terms in Equation (38), better results would be obtained.

Therefore, this research provides some new possibilities for applying the calculus of variations to the equation of heat conduction and applying it in practice.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Jannot, Y., Moyne, C. and Degiovanni, A. (2023) Heat Transfer, Volume 1: Conduction and Convection. John Wiley & Sons, Inc., Hoboken. https://doi.org/10.1002/9781394228102

[2] Lienhard IV, J.H. and Lienhard V, J.L. (2019) A Heat Transfer Textbook. Dover Books on Engineering, Dover Publications, Inc., Mineola.

[3] Mahaffy, J.M. (2020) Heat Conduction in Higher Dimensions. State University, San Diego.

[4] Ouedraogo, D., Igo, S.W., Sawadogo, G.L., Compaore, A., Zeghmati, B. and Chesneau, X. (2020) Modeling and Numerical Simulation of Heat Transfer in a Metallic Pressure Cooker Isolated with Kapok Wool. *Modeling and Numerical Simulation of Material Science*, **10**, 15-30. https://doi.org/10.4236/mnsms.2020.102002

[5] Maturi, D.A., Alsulami, N.M. and Alaidarous, E.S (2020) Finite Difference Approxi-

mation for Solving Transiet Heat Conduction Equation for Copper. *Advances in Pure Mathematics*, **10**, 350-358. https://doi.org/10.4236/apm.2020.105021

[6] Zhang, Z.W., *et al.* (2022) Heat Conduction Theory Including Phonon Coherence. *Physical Review Letters*, **128**, Article ID: 015901. https://doi.org/10.1103/PhysRevLett.128.015901

[7] Lu, Y.K., Zhang, S., Zhang, Y. and Zhou, Q.B. (2001) Depth Profile Reconstruction of Thermal Relaxation Time. *Analytical Science*, **17**, 9390-9393.

[8] Shakeriaski, F., Ghodrat, M., Escobedo-Diaz, J. and Behnia, M. (2021) Recent Advances in Generalized Thermoelasticity Theory and the Modified Models: A Review. *Journal of Computational Design and Engineering,* **8**, 15-35. https://doi.org/10.1093/jcde/qwaa082

[9] Povstenko, Y. (2019) Generalized Theory of Diffusive Stresses Associated with the Time-Fractional Diffusion Equation and Nonlocal Constitutive Equations for the Stress Tensor. *Computers & Mathematics with Applications,* **78**, 1819-1825. https://doi.org/10.1016/j.camwa.2016.02.034

[10] Vasileiadis, T., Reparaz, J.S. and Graczykowski, B. (2022) Phonon Transport in the Gigahertz to Terahertz Range: Confinement, Topology and Second Sound. *Journal of Applied Physics*, **131**, Article ID: 180901. https://doi.org/10.1063/5.0073508

[11] Christov, C.I. and Jordan, P.M. (2005) Heat Conduction Pardox Involving Second-Sound Propagation in Moving Media. *Physical Review Letters*, **94**, Article ID: 154301. https://doi.org/10.1103/PhysRevLett.94.154301

[12] Zarei, A. and Pilla, S. (2023) An Improved Theory of Thermoelasticity for Ultrafast Heating of Materials Using Short and Ultrashort Laser Pulses. *International Journal of Heat and Mass Transfer*, **215**, Article ID: 124510. https://doi.org/10.1016/j.ijheatmasstransfer.2023.124510

[13] Hussein, E.M. (2023) Fractional Model in the Theory of Generalized Thermoelastic Diffusion. *Special Topics & Reviews in Porous Media*: *An International Journal,* **14**, 1-16. https://doi.org/10.1615/SpecialTopicsRevPorousMedia.2022044574

[14] Cattaneo, C. (1949) Sul calcolo di alcuni potenziali e sul loro intervento nella risoluzione di particolari problemi Armonici. *Atti del Seminario Matematico e Fisico dell' Universita di Modena e Reggio Emilia*, **3**, 29-45

[15] Cattaneo, C. (1958) Sur uneforme de l'équation de la chaleuréliminant le paradoxe d'une propagation instantanée. *Comptes Rendus de l' Académie des Sciences*, **247**, 431-433.

[16] Vernotte, P. (1958) Les paradoxes de la théorie continue de l'équation de la chaleur. *Comptes Rendus de l' Académie des Sciences*, **246**, 3154-3155.

[17] Olver, J.P. (2021) The Calculus of Variations. University of Minnesota, Minneapolis. https://www-users.cse.umn.edu/~olver/ln_/cv.pdf

[18] Duffy, D.G. (2015) Green's Functions with Applications. CRC Press, Boca Raton.

[19] Pinchover, Y. and Rubinstein, J. (2005) An Introduction to Partial Differential Equations. Cambridge University Press, Cambridge. https://doi.org/10.1017/CBO9780511801228

[20] Abbate, J. and Whitt, W. (2006) A Unified Framework for Numerically Inverting Laplace Transforms. *INFORMS Journal on Computing*, **18**, 408-421. https://doi.org/10.1287/ijoc.1050.0137

[21] Cohen, A.A. (2007) Numerical Methods for Laplace Transforms Inversion. Springer, New York.

[22] David, W., David, H. and Ozisik, M.N. (2012) Heat Conduction. 3rd Edition, John

Wiley & Sons, Inc., Hoboken.

[23] Muluken Tilahum, B.S. (1998) Experimental Investigation of Hyperbolic Heat Transfer in Heterogeneous Materijals. Master's Thesis, Virginia Polytechnic Institute and State University, Blacksburg.

[24] Smilth, M.J. (2012) Femtosecond-Laser Irradiation as a Platform for Tailoring the Optoelectronic Properties of Silicon. Ph.D. Thesis, Massachusetts Institute of Technology, Cambridge, MA.

Nomenclatures

A: coefficient in the telegraph equation

a: thermal diffusivity of the material (cm^2/s)

B: coefficient in the telegraph equation

C: coefficient in the telegraphic equation

c_p: specific heat capacity of the material (kJ/kg·K)

F: functional

L: length (cm)

L: Lagrangian

T: temperature (˚C)

v: velocity (m/s)

x: length (cm)

τ: time (s)

τ^*: relaxation time (s)

ρ: material density (kg/m^3)

λ: coefficient of thermal conductivity of the material (W/m·K)