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Weak External Bisection of Some Graphs

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Abstract

Let G be a graph. A bipartition of G is a bipartition of $V(G)$ with $V(G)=V_1\cup V_2$ and $V_1\cap V_2=\varnothing$. If a bipartition satisfies $\left||V_1|-|V_2|\right|\le 1$, we call it a bisection. The research in this paper is mainly based on a conjecture proposed by Bollobás and Scott. The conjecture is that every graph G has a bisection (V_1,V_2) such that $\forall v\in V_1$, at least half minuses one of the neighbors of v are in the V_2; $\forall v\in V_2$, at least half minuses one of the neighbors of v are in the V_1. In this paper, we confirm this conjecture for some bipartite graphs, crown graphs and windmill graphs.

Keywords

Weak External Bisection, Bipartite Graph, Windmill Graph

1. Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. An external bipartition of G is $V(G)=V_1\cup V_2$ and $V_1\cap V_2=\varnothing$ and requires that at least half of the neighbors of each vertex are in the other part. If a bipartition satisfies $\left||V_1|-|V_2|\right|\le 1$, we call it a bisection and denote it by (V_1,V_2). Ban and Linial [1] showed that every class 1, 3 or 4 regular graph G has an external bisection. Bollobás and Scott [2] observed that not every graph has an external bisection. In the same paper, they gave a counterexample that $K_{2l+1,m}$, where $m\ge 2l+3$ doesn't have an external bisection. Esperet, Mazzuoccolo and Tarsi [3] found a set of cubic graphs without external bisection and containing at least 2 bridges.

For vertices $u,v\in V(G)$, if $uv\in E(G)$, then edge uv is said to be associated with u and v. The degree of v is the number of edges in G associated with v, denoted as $d(v)$. Let (V_1,V_2) be a bisection of G. For a vertex v in V_1, the internal degree of v is the number of the edges associated with v and other endpoints of these edges are in V_1 too; and the external degree of v is the number of the edges associated with v and other endpoints of these edges are in V_2. For a vertex v in V_2, the internal degree of v is the number of the edges associated with v and other endpoints of these edges are in V_2; and the external degree of v is the number of the edges associated with v and other endpoints of these edges are in V_1. The internal degree of v is denoted as $d_{in}(v)$ and the external degree of v is denoted as $d_{ex}(v)$.

A graph is said to be a bipartite graph, denoted by $G[X,Y]$, if the set of vertices of the graph can be partitioned into two non-empty subsets X and Y such that no two vertices in X are connected to each other with an edge and no two vertices in Y are connected with an edge.

The crown graph of [4] $G_{n,m}$ satisfies the condition:

$$V(G_{n,m}) = \{u_i \mid i = 1,2,\cdots,n\} \cup \{v_i \mid i = 1,2,\cdots,n\} \bigcup_{i=1}^{n} \{u_{ij} \mid j = 1,2,\cdots,m\};$$

$$E(G_{n,m}) = \{u_1u_2, u_2u_3, \cdots, u_nu_1\} \cup \{v_1v_2, v_2v_3, \cdots, v_nv_1\} \cup \{v_1u_1, v_2u_2, \cdots, v_nu_n\} \bigcup_{i=1}^{n} \{u_iu_{ij} \mid j = 1,2,\cdots,m\} \bigcup_{i=1}^{n} \{u_{ij}u_{i(j+1)} \mid j = 1,2,\cdots,m-1\}, \ (n \geq 3, m \geq 1).$$

For example, **Figure 1** shows $G_{3,3}$.

A windmill graph $K_m^{(n)}$ is a graph consisting of n m-order complete graphs K_m with a common vertex.

For example, **Figure 2** shows $K_4^{(n)}$.

Conjecture 1.1. (Bollobás and Scott [2]). *Every graph G has a weak external bisection that is G has a bisection* (V_1, V_2), *such that*:

$$d_{ex}(v) \geq d_{in}(v) - 1 \quad \textit{for all} \quad v \in V(G).$$

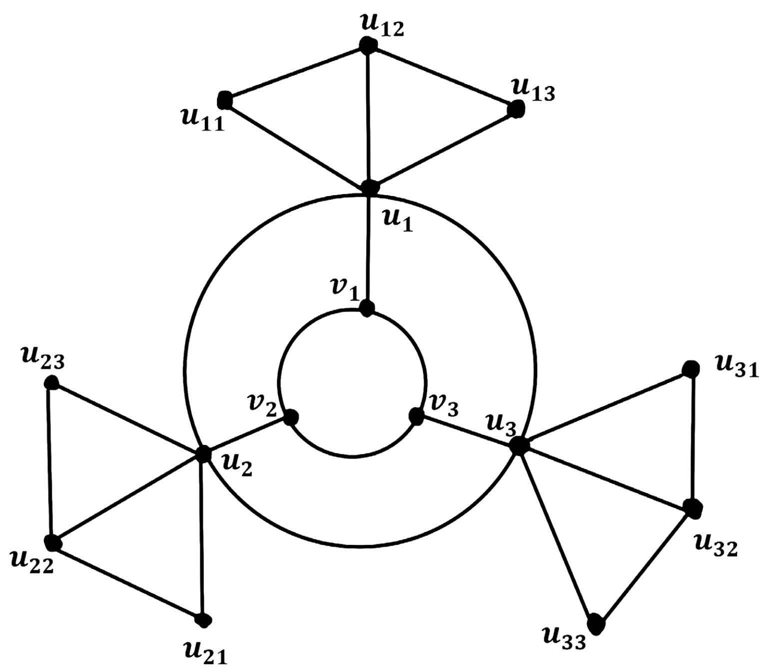

Figure 1. $G_{3,3}$.

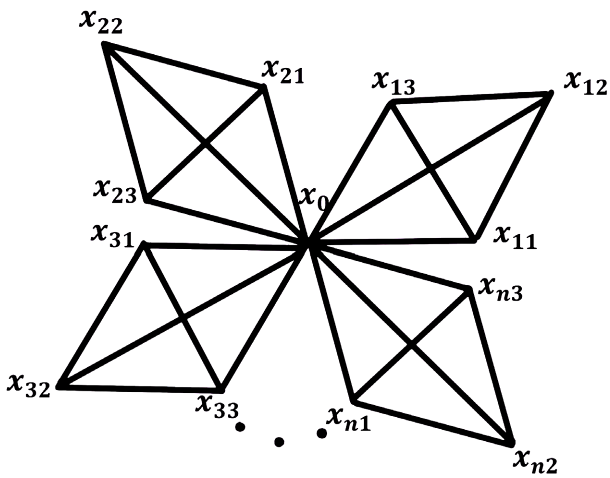

Figure 2. $K_4^{(n)}$.

Ji, Ma, Yan and Yu [5] showed that every graphic sequence has a realization for which Conjecture 1.1 holds. In the same paper, they gave an infinite family of counterexamples to Conjecture 1.1.

In this paper, we confirm this conjecture for some graphs by showing the following three theorems.

Theorem 1.1. *Let* $G[X,Y]$ *be a bipartite graph with* $|X|=3$, *then* $G[X,Y]$ *admits a weak external bisection.*

Theorem 1.2. *Every crown graph* $G_{n,m}$ *admits a weak external bisection.*

Theorem 1.3. *Every windmill graph* $K_m^{(n)}$ *admits a weak external bisection.*

In fact, by the proof Theorem 1.2, $G_{n,m}$ admits an external bisection.

2. Weak External Bisection

In this section, we prove Theorem 1.1, Theorem 1.2 and Theorem 1.3.

Proof of Theorem 1.1. Let $G[X,Y]$ be a bipartite graph with two parts X and Y, we define $Y_S=\{y\in Y \mid N(y)=S, S\subseteq X\}$.

For a set S, we define a function:

$$f(S)=\begin{cases}1 & \text{if } |S| \text{ is odd;}\\ 0 & \text{if } |S| \text{ is even.}\end{cases} \quad (1)$$

Let $X=\{v_1,v_2,v_3\}$, and assume, without loss of generality, that:

$$\left|Y_{\{v_1,v_3\}}\right| \ge \left|Y_{\{v_1,v_2\}}\right| \ge \left|Y_{\{v_2,v_3\}}\right|.$$

Otherwise, we can re-label the three vertices in X. We give a weak external bisection (V_1,V_2) of $V(G)$ by three steps.

First, let $\{v_1,v_2\}\subseteq V_1$, $\{v_3\}\subseteq V_2$, $Y_{\{v_1,v_2\}}\subseteq V_2$ and $Y^*_{\{v_1,v_3\}}\subseteq V_1$ where $Y^*_{\{v_1,v_3\}}\subseteq Y_{\{v_1,v_3\}}$ and $\left|Y^*_{\{v_1,v_3\}}\right|=\left|Y_{\{v_1,v_2\}}\right|$. Because $\left|Y_{\{v_1,v_3\}}\right|\ge\left|Y_{\{v_1,v_2\}}\right|$, such $Y^*_{\{v_1,v_3\}}$ exists. Let $Y^{**}_{\{v_1,v_3\}}=Y_{\{v_1,v_3\}}\setminus Y^*_{\{v_1,v_3\}}$.

Then, we partition the odd sets of $Y_{\{v_2\}}$, $Y_{\{v_1\}}$, $Y^{**}_{\{v_1,v_3\}}$, $Y_{\{v_1,v_2,v_3\}}$, $Y_{\{v_2,v_3\}}$, $Y_{\{v_3\}}$

one after another. Denote by $S_1, S_2, \cdots, S_m$, the odd sets of $Y_{\{v_2\}}$, $Y_{\{v_1\}}$, $Y^{**}_{\{v_1,v_3\}}$, $Y_{\{v_1,v_2,v_3\}}$, $Y_{\{v_2,v_3\}}$, $Y_{\{v_3\}}$, with the same order. For each $i \in \{1, \cdots, m\}$, put $\left\lceil \frac{|S_i|}{2} \right\rceil$ vertices of S_i into V_2 and $\left\lfloor \frac{|S_i|}{2} \right\rfloor$ vertices of S_i into V_1 if i is odd; and put $\left\lceil \frac{|S_i|}{2} \right\rceil$ vertices of S_i into V_1 and $\left\lfloor \frac{|S_i|}{2} \right\rfloor$ vertices of S_i into V_2 if i is even.

Finally, denote by $T_1, T_2, \cdots, T_k$ the even sets of $Y_{\{v_2\}}$, $Y_{\{v_1\}}$, $Y^{**}_{\{v_1,v_3\}}$, $Y_{\{v_1,v_2,v_3\}}$, $Y_{\{v_2,v_3\}}$, $Y_{\{v_3\}}$. For each $i = 1, \cdots, k$, put $\frac{|T_i|}{2}$ vertices of T_i into V_1 and $\frac{|T_i|}{2}$ vertices of T_i into V_2.

Now, we show that V_1 and V_2 form a weak external bisection of $G[X, Y]$. Clearly, $|V_1| - |V_2| = 0$ if m is odd, and $|V_1| - |V_2| = 1$ if m is even. So (V_1, V_2) is a bisection. Since $\{v_1, v_2\} \subseteq V_1$ and $Y_{\{v_1,v_2\}} \subseteq V_2$, then $d_{ex}(v) - d_{in}(v) = 2$ for each vertex $v \in Y_{\{v_1,v_2\}}$. Moreover, $|d_{ex}(v) - d_{in}(v)| = 1$ for each vertex $v \in Y_S$ if $S \subseteq \{v_1, v_2, v_3\}$ and $|S| = 1$ or 3; and $d_{ex}(v) = d_{in}(v)$ for each vertex $v \in Y_S$ if $S \subseteq \{v_1, v_2, v_3\}$, $S \neq \{v_1, v_2\}$ and $|S| = 2$. So, for each $v \in Y$, $d_{ex}(v) \geq d_{in}(v) - 1$.

Let:

$$\mathcal{S}_1 = \left\{Y_{\{v_1\}}, Y^{**}_{\{v_1,v_3\}}, Y_{\{v_1,v_2,v_3\}}\right\},$$
$$\mathcal{S}_2 = \left\{Y_{\{v_1,v_2,v_3\}}, Y_{\{v_2,v_3\}}\right\},$$
$$\mathcal{S}_3 = \left\{Y^{**}_{\{v_1,v_3\}}, Y_{\{v_1,v_2,v_3\}}, Y_{\{v_2,v_3\}}, Y_{\{v_3\}}\right\}.$$

Let $\mathcal{S}'_i = \{S_1, S_2, \cdots, S_m\} \cap \mathcal{S}_i$, for each $i = 1, 2, 3$. Note that $S_1, S_2, \cdots, S_m$ are the odd sets of $Y_{\{v_2\}}$, $Y_{\{v_1\}}$, $Y^{**}_{\{v_1,v_3\}}$, $Y_{\{v_1,v_2,v_3\}}$, $Y_{\{v_2,v_3\}}$, $Y_{\{v_3\}}$, with the same order. Then, each $\mathcal{S}'_i$ contains several continuous set of $S_1, S_2, \cdots, S_m$. Let $\mathcal{T}_i = \{T_1, T_2, \cdots, T_k\} \cap \mathcal{S}_i$, for each $i = 1, 2, 3$. We have:

$$d_{ex}(v_1) = \left|Y_{\{v_1,v_2\}}\right| + \sum_{\substack{S_i \subseteq \mathcal{S}'_1 \\ i \text{ is odd}}} \left\lceil \frac{|S_i|}{2} \right\rceil + \sum_{\substack{S_i \subseteq \mathcal{S}'_1 \\ i \text{ is even}}} \left\lfloor \frac{|S_i|}{2} \right\rfloor + \sum_{T_i \subseteq \mathcal{T}_1} \frac{|T_i|}{2};$$

$$d_{in}(v_1) = \left|Y^{*}_{\{v_1,v_3\}}\right| + \sum_{\substack{S_i \subseteq \mathcal{S}'_1 \\ i \text{ is odd}}} \left\lfloor \frac{|S_i|}{2} \right\rfloor + \sum_{\substack{S_i \subseteq \mathcal{S}'_1 \\ i \text{ is even}}} \left\lceil \frac{|S_i|}{2} \right\rceil + \sum_{T_i \subseteq \mathcal{T}_1} \frac{|T_i|}{2}.$$

Since $\mathcal{S}'_1$ contains continuous set of $S_1, S_2, \cdots, S_m$, then we see that $|d_{ex}(v_1) - d_{in}(v_1)| = f(\mathcal{S}'_1)$, where $f(\mathcal{S}'_1)$ is defined as (1). It is easy to see that:

$$d_{ex}(v_2) = \left|Y_{\{v_1,v_2\}}\right| + \left\lceil \frac{\left|Y_{\{v_2\}}\right|}{2} \right\rceil + \sum_{\substack{S_i \subseteq \mathcal{S}'_2 \\ i \text{ is odd}}} \left\lceil \frac{|S_i|}{2} \right\rceil + \sum_{\substack{S_i \subseteq \mathcal{S}'_2 \\ i \text{ is even}}} \left\lfloor \frac{|S_i|}{2} \right\rfloor + \sum_{T_i \subseteq \mathcal{T}_2} \frac{|T_i|}{2};$$

$$d_{in}(v_2) = \left\lfloor \frac{\left|Y_{\{v_2\}}\right|}{2} \right\rfloor + \sum_{\substack{S_i \subseteq \mathcal{S}'_2 \\ i \text{ is odd}}} \left\lfloor \frac{|S_i|}{2} \right\rfloor + \sum_{\substack{S_i \subseteq \mathcal{S}'_2 \\ i \text{ is even}}} \left\lceil \frac{|S_i|}{2} \right\rceil + \sum_{T_i \subseteq \mathcal{T}_2} \frac{|T_i|}{2}.$$

Clearly, if $\left|Y_{\{v_2\}}\right|$ is odd, then $S_1 = Y_{\{v_2\}}$. Thus, we have $d_{ex}(v_2) - d_{in}(v_2) \ge \left|Y_{\{v_1,v_2\}}\right| + f\left(Y_{\{v_2\}}\right) - f\left(\mathcal{S}_2'\right) \ge -1$. Similarly, we have:

$$d_{ex}(v_3) = \left|Y^*_{\{v_1,v_3\}}\right| + \sum_{\substack{S_i \subseteq \mathcal{S}_3' \\ i \text{ is odd}}} \left\lfloor \frac{|S_i|}{2} \right\rfloor + \sum_{\substack{S_i \subseteq \mathcal{S}_3' \\ i \text{ is even}}} \left\lceil \frac{|S_i|}{2} \right\rceil + \sum_{T_i \subseteq \mathcal{T}_3} \frac{|T_i|}{2};$$

$$d_{in}(v_3) = \sum_{\substack{S_i \subseteq \mathcal{S}_3' \\ i \text{ is odd}}} \left\lceil \frac{|S_i|}{2} \right\rceil + \sum_{\substack{S_i \subseteq \mathcal{S}_3' \\ i \text{ is even}}} \left\lfloor \frac{|S_i|}{2} \right\rfloor + \sum_{T_i \subseteq \mathcal{T}_3} \frac{|T_i|}{2}.$$

So, we have $\left|d_{ex}(v_3) - d_{in}(v_3)\right| \ge \left|Y^*_{\{v_1,v_3\}}\right| - f\left(\mathcal{S}_2'\right)$. Then, for each $v_i \in X$, $d_{ex}(v_i) \ge d_{in}(v_i) - 1$. Thus, (V_1, V_2) is a weak external bisection of $G[X,Y]$.■

Proof of Theorem 1.2. Let $V(G_{n,m}) = \{u_i \mid i = 1,\cdots,n\} \cup \{v_i \mid i = 1,\cdots,n\} \bigcup_{i=1}^{n} \{u_{ij} \mid j = 1,\cdots,m\}$. We give a weak external bisection (V_1, V_2) of $V(G_{n,m})$ by two steps.

First, let $\{v_i \mid i \in \{1,2,\cdots,n\} \text{ and } i \text{ is odd}\} \subset V_2$ and $\{v_i \mid i \in \{1,2,\cdots,n\} \text{ and } i \text{ is even}\} \subset V_1$. Let $\{u_i \mid i \in \{1,2,\cdots,n\} \text{ and } i \text{ is odd}\} \subset V_1$ and $\{u_i \mid \{i \in 1,2,\cdots,n\} \text{ and } i \text{ is even}\} \subset V_2$.

Then, we partition the set $\bigcup_{i=1}^{n} \{u_{ij} \mid j = 1,2,\cdots,m\}$. The partition of $\bigcup_{i=1}^{n} \{u_{ij} \mid j = 1,2,\cdots,m\}$ is determined by $\bigcup_{i=1}^{n} \{u_i\}$. For a given *k*, if $u_k \in V_1$, let $\{u_{kj} \mid j \in \{1,2,\cdots,m\} \text{ and } j \text{ is odd}\} \subset V_2$ and $\{u_{kj} \mid j \in \{1,2,\cdots,m\} \text{ and } j \text{ is even}\} \subset V_1$; if $u_k \in V_2$, let $\{u_{kj} \mid j \in \{1,2,\ldots,m\} \text{ and } j \text{ is odd}\} \subset V_1$ and $\{u_{kj} \mid j \in \{1,2,\cdots,m\} \text{ and } j \text{ is even}\} \subset V_2$.

Now, we show that V_1 and V_2 form a weak external bisection of $G_{n,m}$. Clearly, $|V_1| - |V_2| = 1$ if both *n* and *m* are odd. Otherwise, $|V_1| - |V_2| = 0$. So, (V_1, V_2) is a bisection. If *n* is even, then $d_{ex}(v) - d_{in}(v) = 3$ for each $v \in \bigcup_{i=1}^{n} \{v_i\}$. If *n* is odd, then $d_{ex}(v_1) - d_{in}(v_1) = 1$, $d_{ex}(v_n) - d_{in}(v_n) = 1$ and $d_{ex}(v) - d_{in}(v) = 3$ for each $v \in \bigcup_{i=2}^{n-1} \{v_i\}$. So, in any case, $d_{ex}(v_i) \ge d_{in}(v_i)$ for $i = 1,2,\cdots,n$.

If *m* is odd, then $d_{ex}(v) - d_{in}(v) = 2$ for each $v \in \bigcup_{i=1}^{n} \{u_{i1}, u_{im}\}$, $d_{ex}(v) - d_{in}(v) = 1$ for each $v \in \bigcup_{i=1}^{n} \{u_{ij} \mid j \in \{2,\cdots,m-1\} \text{ and } j \text{ is even}\}$ and $d_{ex}(v) - d_{in}(v) = 3$ for each $v \in \bigcup_{i=1}^{n} \{u_{ij} \mid j \in \{2,\cdots,m-1\} \text{ and } j \text{ is odd}\}$. If *m* is even, then $d_{ex}(u_{i1}) - d_{in}(u_{i1}) = 2$, $d_{ex}(u_{im}) = d_{in}(u_{im})$, $d_{ex}(v) - d_{in}(v) = 1$ for each $v \in \bigcup_{i=1}^{n} \{u_{ij} \mid j \in \{2,\cdots,m-1\} \text{ and } j \text{ is even}\}$ and $d_{ex}(v) - d_{in}(v) = 3$ for each $v \in \bigcup_{i=1}^{n} \{u_{ij} \mid j \in \{2,\cdots,m-1\} \text{ and } j \text{ is odd}\}$. So, in any case, $d_{ex}(u_{ij}) \ge d_{in}(u_{ij})$ for

$i=1,2,\cdots,n$; $j=1,2,\cdots,m$.

If both n and m are odd, then $d_{ex}(u_1)-d_{in}(u_1)=2$, $d_{ex}(u_n)-d_{in}(u_n)=2$ and $d_{ex}(v)-d_{in}(v)=4$ for each $v\in\bigcup_{i=2}^{n-1}\{u_i\}$. If n is odd and m is even, then $d_{ex}(u_1)-d_{in}(u_1)=1$, $d_{ex}(u_n)-d_{in}(u_n)=1$ and $d_{ex}(v)-d_{in}(v)=3$ for each $v\in\bigcup_{i=2}^{n-1}\{u_i\}$. If n is even and m is odd, then $d_{ex}(v)-d_{in}(v)=4$ for each $v\in\bigcup_{i=1}^{n}\{u_i\}$. If both n and m are even, then $d_{ex}(v)-d_{in}(v)=3$ for each $v\in\bigcup_{i=1}^{n}\{u_i\}$. So, in any case, $d_{ex}(u_i)\ge d_{in}(v_i)$ for $i=1,2,\cdots,n$. Thus, (V_1,V_2) is a weak external bisection of $G_{n,m}$.■

Proof of Theorem 1.3. We labeled the vertices of graph $K_m^{(n)}$ as in Figure 2. $V\left(K_m^{(n)}\right)=\{x_0\}\bigcup_{i=1}^{n}\{x_{i1},x_{i2},\cdots,x_{i,m-1}\}$. Let $\{x_0\}\subset V_1$.

We consider the following two cases.

Case 1. m is odd.

Let $\bigcup_{i=1}^{n}\{x_{ij}\mid j\in\{1,2,\cdots,m-1\}\text{ and } j \text{ is even}\}\subset V_1$ and $\bigcup_{i=1}^{n}\{x_{ij}\mid j\in\{1,2,\cdots,m-1\}\text{ and } j \text{ is odd}\}\subset V_2$. Now $|V_1|-|V_2|=1$. So (V_1,V_2) is a bisection. $d_{ev}(x_0)-d_{in}(x_0)=1$. For each $v\in\bigcup_{i=1}^{n}\{x_{ij}\mid j\in\{1,2,\cdots,m-1\}\text{ and } j \text{ is even}\}$, $d_{ev}(v)=d_{in}(v)$. For each $v\in\bigcup_{i=1}^{n}\{x_{ij}\mid j\in\{1,2,\cdots,m-1\}\text{ and } j \text{ is odd}\}$, $d_{ev}(v)-d_{in}(v)=2$. Thus, (V_1,V_2) is a weak external bisection of $K_m^{(n)}$.

Case 2. m is even.

If i is odd, let $\bigcup_{i=1}^{n}\{x_{ij}\mid j\in\{1,2,\cdots,m-1\}\text{ and } j \text{ is odd}\}\subset V_2$ and let $\bigcup_{i=1}^{n}\{x_{ij}\mid j\in\{1,2,\cdots,m-1\}\text{ and } j \text{ is even}\}\subset V_1$. If i is even, let $\bigcup_{i=1}^{n}\{x_{ij}\mid j\in\{1,2,\cdots,m-1\}\text{ and } j \text{ is odd}\}\subset V_1$ and let $\bigcup_{i=1}^{n}\{x_{ij}\mid j\in\{1,2,\cdots,m-1\}\text{ and } j \text{ is even}\}\subset V_2$.

Now, we show that V_1 and V_2 form a weak external bisection of $K_m^{(n)}$. Clearly, $|V_1|-|V_2|=0$ if n is odd and $|V_1|-|V_2|=1$ if n is even. So (V_1,V_2) is a bisection. If i is odd, $d_{ev}(v)-d_{in}(v)=1$ for each $v\in\{x_{ij}\mid j\in\{1,\cdots,m-1\}\text{ and } j \text{ is odd}\}$; and $d_{ev}(v)-d_{in}(v)=1$ for each $v\in\{x_{ij}\mid j\in\{1,\cdots,m-1\}\text{ and } j \text{ is even}\}$. If i is even, then $d_{ev}(v)-d_{in}(v)=-1$ for each $v\in\{x_{ij}\mid j\in\{1,\cdots,m-1\}\text{ and } j \text{ is odd}\}$; and $d_{ev}(v)-d_{in}(v)=3$ for each $v\in\{x_{ij}\mid j\in\{1,\cdots,m-1\}\text{ and } j \text{ is even}\}$. If n is even, then $d_{ev}(x_0)=d_{in}(x_0)$. If n is odd, then $d_{ev}(x_0)-d_{in}(x_0)=1$. Thus, (V_1,V_2) is a weak external bisection of $K_m^{(n)}$.■

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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