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On Existence of Entropy Solution for a Doubly Nonlinear Differential Equation with L^1-Data

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Abstract

We consider a class of doubly nonlinear history-dependent problems having a convection term and a pseudomonotone nonlinear diffusion operator associated an equation of the type $\partial_t\left(k*\left(b(v)-b(v_0)\right)\right)-div\left(a(x,Dv)+F(v)\right)=f$ where the right hand side belongs to L^1. The kernel k belongs to the large class of $\mathcal{PC}$ kernels. In particular, the case of fractional time derivatives of order $\alpha\in(0,1)$ is included. Assuming b nondecreasing with L^1-data, we prove existence in the framework of entropy solutions. The approach adopted for the proof is based on a several step approximation method and by using a result in the case of a strictly increasing b.

Keywords

Fractional Time Derivative, Nonlinear Volterra Equation, Doubly Nonlinear, Entropy Solution

1. Introduction

This paper is devoted to the study of a class of doubly nonlinear history-dependent initial boundary value type probems of the form

$$(EP)^{b,F,k}_{f,v_0}\begin{cases}\partial_t\left(k*\left(b(v)-b(v_0)\right)\right)-div\left(a(x,Dv)+F(v)\right)=f & \text{in } Q_T,\\ b(v)(0,\cdot)=b(v_0) & \text{in } \Omega,\\ v=0 & \text{on } \Sigma_T.\end{cases}\quad(1)$$

Our aim is to prove existence of entropy solutions to the problem $(EP)^{b,F,k}_{f,v_0}$. In our problem, the framework is the following: Ω is boundary domain of $\mathbb{R}^N$ $(N\geq 2)$, T is positive number, $Q_T:=(0,T)\times\Omega$ is the space-time cylinder, $\Sigma_T:=(0,T)\times\partial\Omega$ where $\partial\Omega$ denotes the boundary of Ω, Dv stands for the gradient of v with respect to the spatial, $p>1$ is a real number, $p'=\frac{p}{p-1}$, $k\in L^1_{loc}([0,\infty))$ is a singular kernel which is type $\mathcal{PC}$, *i.e.* $k\in L^1_{loc}(\mathbb{R}^+)$, non-negative, non-increasing, such that there exists a function $l\in L^1_{loc}(\mathbb{R}^+)$ satisfying $(k*l)(t)=1$ for every $t>0$ and the expression $(k*u)$ represents the convolution operation over the positive half-line in relation to the time variable

$$(k*u)(t)=\int_0^t k(t-s)u(s)\mathrm{d}s, t>0.$$

We further assume that the kernel k satisfies additional assumptions which are introduced in Section 2.3. Under these assumptions on k, our work cover the case of a time-fractional derivative of order $0<\alpha<1$, *i.e.* $k(t)=t^{-\alpha}/\Gamma(1-\alpha)$, $t\in(0,\infty)$ where Γ denotes the Gamma function.

Here, the partial derivative with respect to time of the product of the functions k and u, denoted as $\partial_t(k*u)$, can be expressed as a distributed order derivative. This type of derivative is employed to characterize ultraslow diffusion scenarios, where the mean square displacement exhibits logarithmic growth. Such behaviour of the mean square displacement has been observed in various systems, including polymer physics and signal processing, as documented in references such as [1] and others cited therein. The diffusion term, $Av=-div(a(\cdot,Dv))$ is a Leray-Lions operator which is coercive, monotone and which grows like $|Dv|^{p-1}$ with respect to Dv. The $\mathbb{R}^N$-valued function F, representing the convection flux term is assumed to be defined and locally Lipschitz continuous on the whole $\mathbb{R}$. Let us stress that because the convection flux F is assumed merely Lipschitz continuous, existence techniques for $(EP)^{b,F,k}_{f,v_0}$ are those of entropy solutions. The lack of regularity of F and b are the only reasons why the doubling of variables in space can be needed for existence solution of type problems $(EP)^{b,F,k}_{f,v_0}$. The $\mathbb{R}-$valued function b is assumed to be a C^0-function defined on the whole $\mathbb{R}$, which is non-decreasing and satisfies the renormalization condition $b(0)=0$. Finally, f represents a source term. The data v_0 and f are such that $f\in L^1(Q_T)$, and $v_0:\Omega\to\mathbb{R}$ is measurable function with $b(v_0)\in L^1(\Omega)$.

We should note that equations of the form $(EP)^{b,F,k}_{f,v_0}$ finds application in describing nonlinear heat flow in certain dielectric materials at extremely low temperatures. Experimental observations have revealed a finite speed of propagation for thermal disturbances in this situation. Various models have been proposed to explain this phenomenon, with [2] presenting a model in which the constitutive relations for internal energy and heat flux, unlike Fourier's law, depend on the history of temperature and temperature gradient, respectively. As demonstrated in [3], this formulation leads to a problem in the form of $(EP)^{b,F,k}_{f,v_0}$ under certain assumptions on the relaxation functions of internal energy and heat flux.

It is worth noting that the assumptions on k are driven by the need to ensure the positivity of solutions, which is a crucial physical requirement in several applications. When modeling nonlinear heat flow in materials with memory, the function $v(t,x)$ in problem $(EP)_{f,v_0}^{b,F,k}$ is considered to represent the absolute temperature at the location x in the domain Ω at time t. Such assumptions were initially introduced in [4], giving rise to the concept of complete positivity, as discussed in [3] and [5].

Under all of these assumptions and for $1 < p \le 2 - \frac{1}{N}$, the above problem does not admit in general a weak solution for L^1-data, since that the fields $a(\cdot, Dv)$ do not belong $\left(L^1_{loc}(Q_T)\right)^N$ in general, see e.g. ([6], Appendix I). As it has been in [7] and [8]. When the problem $(EP)_{f,v_0}^{b,F,k}$, in the case $F \equiv 0$, it has been shown in ([9], section 3.1) that the problems of nonexistence and non-uniqueness of weak solutions to the elliptic problem carry over to the time-fractional case with $k(t) = t^{-\alpha}/\Gamma(1-\alpha)$, $\alpha \in (0,1)$, $t \in (0,\infty)$.

To address the challenges associated with the nonexistence and nonuniqueness of weak solutions, two novel solution concepts have been introduced. In [10] [11], the existence and uniqueness of renormalized solutions are demonstrated for elliptic and parabolic problems, respectively. Additionally, [12] explores the uniqueness of renormalized solutions for elliptic-parabolic problems without history dependence.

The second concept, known as entropy solutions, is equivalent to the notion of renormalized solutions for problems without history dependence and was initially proposed in [6] for an elliptic problem. For the parabolic problem, refer to [13].

These innovative concepts share the characteristic that a solution is not expected to be found as an element of a Sobolev space. Instead, the objective is to identify a measurable function, denoted as v, such that all truncations $T_K(v)$ of v belong to a specific Sobolev space. In the case problems of type $(EP)_{f,v_0}^{b,F,k}$, this notion was introduced by Jakubowski and Wittbold in [14].

In particular, if $b \equiv 0$, then the problem $(EP)_{f,v_0}^{b,F,k}$ is a purely elliptic problem and the existence of entropy solution has been shown in this case in [15]. Note that for $F \equiv 0$, the problem $(EP)_{f,v_0}^{b,F,k}$ is a special case of the problems considered in [14]. The authors prove existence of entropy solution in the particular case $b = id$, but the uniqueness is only shown in the general case where b is increasing. When $(a(x,Dv) + F(v))$ is replaced by $a(x,Dv)$, this problem has been studied by M. Scholtes and P. Wittbold in [16] and by N. Sapountzoglou in [17]. In [16], the authors show the existence of entropy solutions for a strictly increasing function b and in [17] the author proves the existence for any non-decreasing function b.

The main novelty of the work that the present here comes from the fact that we make an extension of [17], adding a convection term. We will combine the techniques of [17] and the approach developed in [16].

In this article, we will show the existence of entropy solution to initial boundary value problem $(EP)_{f,v_0}^{b,F,k}$. Our definition of entropy solution for above problem is similar to the definition in [16].

The organization of the paper is the following. In the next section, we prepare assumptions, some tools, namely the adaptation of the regularization method of R-Landes (see [18]) and some fundamental equality and inequality, which play a crucial role in our proofs. Finally, in the third section, we will state the Definition of our solution, main existence result and its proof. More precisely, we prove that under Leray Lions assumptions on the vector field a, F locally Lipschitz-continuous and b increasing that the generalized solution of an associated Volterra equation is an entropy solution.

Numerous references are provided at the conclusion of the paper. This list is by no means exhaustive, and additional pertinent references can be found in the cited works.

2. Preliminaries

2.1. Assumptions

Throughout the paper, we assume that the following assumptions hold true:

Ω is boundary domain of $\mathbb{R}^N\,(N \geq 2)$ with boundary $\partial\Omega$, $T > 0$ is given and we set $Q_T := (0,T)\times\Omega$ is the space-time cylinder, $\Sigma_T := (0,T)\times\partial\Omega$, $p > 1$ is a real number, $1 < p < +\infty$ and $p' = \dfrac{p}{p-1}$.

$$a : \Omega\times\mathbb{R}^N \to \mathbb{R}^N \text{ is Carathéodory function,} \tag{H1}$$

i.e. $a(\cdot,\xi):\Omega\to\mathbb{R}^N$ is measurable for all $\xi\in\mathbb{R}^N$, and $a(x,\cdot):\mathbb{R}^N\to\mathbb{R}^N$ is continuous vector field a.e. $x\in\Omega$. Moreover, we assume that a satisfies the classical Leray-Lions conditions, *i.e.*, for some real number $1 < p < +\infty$, we assume that a is monotone

$$\forall \xi,\zeta\in\mathbb{R}^N \text{ and a.e } x\in\Omega : \big(a(x,\xi)-a(x,\zeta)\big)\cdot(\xi-\zeta)\geq 0, \tag{H2}$$

coercive

$$\exists\lambda>0,\ \forall\xi\in\mathbb{R}^N \text{ and a.e } x\in\Omega : a(x,\xi)\cdot\xi \geq \lambda|\xi|^p, \tag{H3}$$

and satisfies a growth condition

$$\exists\Lambda>0,\ g\in L^{p'}(\Omega),\ \forall\xi\in\mathbb{R}^N \text{ and a.e } x\in\Omega : |a(x,\xi)| \leq \Lambda\Big(g(x)+|\xi|^{p-1}\Big). \tag{H4}$$

Thus, assumptions on a are rather general.

Next, we assume that

$$\text{The scalar kernel } k:(0,\infty) \text{ is of type } \mathcal{PC}, \tag{H5}$$

i.e. $k\in L^1_{loc}([0,\infty))$, nonnegative, nonincreasing and such that there exists a function $l\in L^1_{loc}([0,\infty))$ satisfying $(k*l)(t)=1$ for every $t>0$.

The function $b:\mathbb{R}\to\mathbb{R}$ is continuous, nondecreasing and

$$\text{satifying the normalization condition } b(0)=0. \tag{H6}$$

F is locally Lipschitz-continuous function defined on $\mathbb{R}$ with value in $\mathbb{R}^N$, (H7)

i.e. $F=(F_1,\cdots,F_N)$ with F_i is continuous on $\mathbb{R}$ for $i=1,\cdots,N$.

v_0 is a measurable function defined on Ω such that $u_0 := b(v_0)$ belongs to $L^1(\Omega)$. (H8)

$$f : Q_T \to \mathbb{R}\text{, is an element of } L^1(Q_T). \tag{H9}$$

In the following subsection we give some of the notation, functions, definitions and the basic results which will be used later.

2.2. Notations and Functions

If $A \subset \Omega$ is a Lebesgue measurable set, we will denote its Lebesgue measure by $|A|$ and by χ_A its characteristic function.

For any real number $K \geq 0$, we denote by $T_K : \mathbb{R} \to \mathbb{R}$, the truncation function at the level *K*, defined by

$$T_K(r) = \min(K, \max(r, -K)).$$

More precisely, for $1 \leq p < \infty$, the functional space $\mathcal{T}_0^{1,p}(\Omega)$ can be defined by

$$\mathcal{T}_0^{1,p}(\Omega) := \left\{ v : \Omega \to \mathbb{R} : v \text{ is a measurable and } T_K(v) \in W_0^{1,p}(\Omega) \text{ for every } K > 0 \right\}.$$

For more details about the class of functions $v : \Omega \to \mathbb{R}$ whose truncations $T_K(v)$ belong, for every $K > 0$, to some Sobolev space see, e.g., [6] [19] and [20]. We will frequently use the notation $T_{K,L} := T_L - T_K$, for $L > K$. For $r \in \mathbb{R}$, $r \mapsto sign_0^+(r)$ is the function defined by $sign_0^+(r) = 1$ if $r > 0$ and $sign_0^+(r) = 0$ if $r \geq 0$.

Throughout the paper, for the sake of simplicity for any $u : Q_T \to \mathbb{R}$ and for K a positive real number, we write $\{|u| \leq (<, >, \geq, =) K\}$ for the set $\{(t,x) \in Q_T : |u(t,x)| \leq (<, >, \geq, =) K\}$. In addition, we set

$$\mathscr{P} := \left\{ S \in C^1(\mathbb{R}) : S'(t) \geq 0 \text{ for every } t > 0, \text{ Supp } S' \text{ is compact}, S(0) = 0 \right\};$$

$AC([0,T])$: Absolutely continuous functions on $[0,T]$

and

$$W^{1,q,0}(0,T;X) := \left\{ u \in W^{1,q}(0,T;X) : u(0) = 0 \right\}.$$

In the sequel, C denotes a constant that may change from line to line.

2.3. Approximating the Kernel *k*

In this subsection, we adapt the regularization method of R-Landes [18] to kernels of type $\mathcal{PC}$. This regularization will be a fundamental tool for the proof of our existence result. Note that those type of kernels have been introduced by [21]. The readers can also see [16] [22] [23] for the same expositions.

Definition 2.1 A kernel $k \in L^1_{loc}([0,\infty))$ is called to be of type $\mathcal{PC}$ if it is nonnegative, nonincreasing and there exists a kernel $l \in L^1_{loc}([0,\infty))$ such that

$$(k * l)(t) = 1 \quad \text{for all } t \in [0,\infty).$$

In this case, we say that (k,l) is a $\mathcal{PC}$ pair and write $(k,l) \in \mathcal{PC}$.

From $(k,l) \in \mathcal{PC}$ it follows that l is completely positive (see Theorem 2.2 in

[23]). We next discuss an important method regularizing kernels of type $\mathcal{PC}$.

To do this, for $(k,l)\in\mathcal{PC}$, $\lambda>0$ and from ([24]; p37, p44), we shall denote by s_λ (resp r_λ) the unique solution in $AC([0,T])$ (resp in $L^1([0,T])$) of the scalar Volterra equations:

$$s_\lambda(t)+\frac{1}{\lambda}(l*s_\lambda)(t)=1,\quad t\geq 0 \tag{2}$$

$$r_\lambda(t)+\frac{1}{\lambda}(l*r_\lambda)(t)=\frac{1}{\lambda}l(t),\quad t\geq 0. \tag{3}$$

Note that the function r_λ is called the resolvent of $\frac{1}{\lambda}l$. Recall that, according to ([24], part 1, chapter 2, Theorem 3.5) the solution of the equation (2) is given by the variation of constants formula:

$$s_\lambda(t)=1-\int_0^t r_\lambda(\tau)\mathrm{d}\tau,\quad t\geq 0,\quad \lambda\geq 0.$$

Next, for $1\leq q<\infty, T>0$ and a real Banach space *X*, we consider the operator *L* defined by:

$$D(L)=\left\{u\in L^q(0,T;X):k*u\in W^{1,q,0}(0,T;X)\right\} \tag{4}$$

and for $u\in D(L)$

$$L(u)(t):=\partial_t(k*u)(t), \tag{2.4}$$

with $(k,l)\in\mathcal{PC}$. According to [([5], Theorem 3.1) and [25], it is know that this operator is m-accretive in $L^q(0,T;X)$. Its resolvent $J_\lambda^L=(I+\lambda L)^{-1}$, $\lambda>0$, which is of the form $J_\lambda^L u=r_\lambda*u$ where r_λ is the solution the Equation (3) (see Theorem 2.1 in [23]). Therefore, $L_\lambda=LJ_\lambda^L$, $\lambda>0$, the Yosida approximation of *L* can be written in the form

$$L_\lambda(u)=L(r_\lambda*u)=\partial_t(k*r_\lambda*u)=\partial_t(k_\lambda*u),$$

where $k_\lambda:=k*r_\lambda$. By the Equation (3), we get

$$k_\lambda:=k*r_\lambda=\frac{1}{\lambda}-\frac{1}{\lambda}\int_0^t r_\lambda(\tau)\mathrm{d}\tau=\lambda^{-1}s_\lambda. \tag{5}$$

Note that, since *l* is completely positive on $(0,T)$, then according to ([23], Proposition 2.1), s_λ is nonnegative and nonincreasing on $(0,T)$. Since s_λ belongs to $AC([0,T])$, then by the equality (5), we have

$$k_\lambda\in W^{1,1}(0,T),\quad k_\lambda(0)=\frac{1}{\lambda}\quad\text{and}\quad \|r_\lambda\|_{L^1(0,T)}\leq 1.$$

For $u\in L^p(0,T;X)$, we obtain that

$$(k*l*u)(t)=(1*u)(t)=\int_0^t u(\tau)\mathrm{d}\tau,\quad t\geq 0.$$

Then, $k*l*u\in W^{1,q,0}(0,T;X)$ which entails that $l*u\in D(L)$ for all $u\in L^p(0,T;X)$. Thus, it follows that

$$r_\lambda*u=\partial_t(k*r_\lambda*l*u)=L_\lambda(l*u)\to L(l*u)=u\ \text{ in } L^p(0,T;X),$$

for any $u\in L^p(0,T;X)$ as $\lambda\to 0$, which proof that $r_\lambda*u\to u$ in $L^p(0,T;X)$. In particular,

$$k_\lambda := k * r_\lambda \to k \quad \text{in } L^1([0,T]), \tag{6}$$

as $\lambda \to 0$ (see [16]). We can now to introduce a modification of the regularization in time by R-Landes (see e.g. Definition 2.2 in [16]).

Definition 2.2. Let X be a real Banach space, X^* its dual.

For $v \in L^{p'}(0,T;X^*)$ we define $v_\mu \in L^{p'}(0,T;X^*)$ by

$$v_\mu(t) = \int_t^T r_\mu(\tau - t) v(\tau) \mathrm{d}\tau, \quad t \in (0,T), \quad \mu > 0.$$

In the sequel the letter μ is used in this meaning only. Note that $v_\mu = J_\mu^{L^*} v$, where $L^* : D(L^*) \subset L^{p'}(0,T;X^*) \to L^{p'}(0,T;X^*)$ is the adjoint operator of L. Consequently, we have for any $v \in L^{p'}(0,T;X^*)$:

$$v_\mu \to v \quad \text{in } L^{p'}(0,T;X^*) \text{ as } \mu \to 0.$$

To be able to proof to the existence of an entropy solution to $(EP)_{k,f}^{b,F}(v_0)$, let us make same further assumptions on k and k_λ:

$$\begin{aligned}&\text{There exist constants } C_1; C_2 > 0 \text{ such that}\\ &0 \le k_\lambda(t) \le C_1 k(t) + C_2, \quad \text{for any } \lambda > 0 \text{ and almost } t \in (0,T)\end{aligned} \tag{K1}$$

$$\begin{aligned}&k \in AC_{loc}((0,T]) \text{ and there exist constants } C_1; C_2 > 0 \text{ such that}\\ &0 \le -k_\lambda'(t) \le -C_1 k'(t) + C_2, \quad \text{for all } l > 0 \text{ and almost } t \in (0,T)\end{aligned} \tag{K2}$$

and

$$k_\lambda'(t) \to k'(t) \quad \text{a.e.}\, t \in (0,T) \text{ as } \lambda \to 0. \tag{K3}$$

The most important example of a kernel which satisfies conditions (K1)-(K3) is the kernel corresponding to the case of fractional derivatives, *i.e.*; $k(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}$, $\alpha \in (0,1)$. Moreover, also the kernel corresponding to exponential weighted fractional derivatives, *i.e.*, $k(t) = \frac{t^{-\alpha} \mathrm{e}^{-\mu t}}{\Gamma(1-\alpha)}$, $\alpha \in (0,1)$, $\mu > 0$, satisfies these assumptions. For more details and another example on these kernel types see [16].

We next recall a fundamental identity for integro-differential operators of the form $\partial_t(k * u)$ which will be needed for the energy estimate.

Lemma 2.3. **([16], *Lemma* 2.4).** *Let $T > 0$ and U be an open subset of $\mathbb{R}$. Let further $k \in W^{1,1}(0,T)$, $H \in C^1(U)$ and $u \in L^1(0,T)$ with $u(t) \in U$ for almost every $t \in (0,T)$. Suppose that $H(u), H'(u), H'(u)(k' * u) \in L^1(0,T)$. Then, we have for almost every $t \in (0,T)$,*

$$\begin{aligned}&H'(u(t))\partial_t(k * u)(t) = \partial_t(k * H(u))(t) + \big(H'(u(t))u(t) - H(u(t))\big)k(t)\\ &+ \int_0^t \Big(H(u(t-s)) - H(u(t)) - H'(u(t))\big[u(t-s) - u(t)\big]\Big)\big[-k'(s)\big]\mathrm{d}s\end{aligned}$$

An integrated version of (2.3) can fund in ([24], p574, Lemma 18.4.1). Equality (2.3) is highly important for deriving a priori estimates for problems of the form $(EP)_{k,f}^{b,F}(v_0)$.

2.4. Approach to the Abstract Volterra Equations

The proof of our an entropy solution existence result presented in this article will be based on the theory of G. Gripenberg (see [26]) for abstract nonlinear Volterra integro-differentials equations of the form

$$\frac{\partial}{\partial t}\left(\gamma\left(u(t)-u_0\right)+\int_0^t k(t-s)\left(u(s)-u_0\right)\mathrm{d}s\right)+A\left(u(t)\right)\ni f(t), \quad t\in[0,T) \tag{7}$$

in a real Banach space *X*. Here γ is a nonnegative constant, *k* is a scalar kernel that is assumed to be locally integrable, nonnegative and nonincreasing function on $\mathbb{R}^+$, A is an m-acrretive, possibly multivalued operator in *X*, $u_0\in X$ and $f\in L^1(0,T;X)$.

In this subsection, we recall the definitions and the main results of the abstract theory. We limit ourselves to the case which will be treated in our purpose, *i.e.* $\gamma=0$ and *k* of type $\mathcal{PC}$. The abstract problem (7) then takes form

$$\partial_t\left[k*(u-u_0)\right](t)+A\left(u(t)\right)\ni f(t), \quad t\in[0,T) \tag{8}$$

The theory of G. Gripenberg is to consider for $\lambda>0$ the following approximating problem

$$\partial_t\left[k_\lambda*(u-u_0)\right](t)+A\left(u(t)\right)\ni f(t), \quad t\in[0,T). \tag{9}$$

Here, $k_\lambda,\lambda>0$ are the kernels associated to the Yosida approximations of the operator given by (4).

Definition 2.4. A measurable function $u:[0,T)\to X$ is called strong solution to the approximating Equation (9), if $u\in L^1(0,T;X)$ and there exists $w\in L^1(0,T;X)$ such that $w(t)\in A\left(u(t)\right)$ and

$$\partial_t\left[k_\lambda*(u-u_0)\right](t)+w(t)=f(t), \tag{10}$$

for almost every $t\in[0,T)$.

The abstract approximating problem (9) admits a unique strong solution u_λ, for every $\lambda>0$ in the sense of Definition 2.4 (see [26]; Theorem 1]).

The generalized solution to (8) is defined as follows (see [26]):

Definition 2.5. Let $(u_\lambda)_\lambda>0$ be the strong solutions to the approximating problem (9).

If there exists a functions $u\in L^1(0,T;X)$ such that $u_\lambda\to u$ in $L^1(0,T;X)$ as λ tends to 0, then *u* is called the generalized solution to (8).

By definition, the generalized solution is unique.

The following theorem is the main existence result of the abstract theory (for the proof, see ([26]; Theorem 1]).

Theorem 2.6. *Let X be a real Banach space. Assume that A is an m-acrretive operator in* $X, u_0\in\overline{D(A)}$ *and* $f\in L^1(0,T;X)$, *then there exists a generalized solution to* (8).

3. Definition of Entropy Solution and Main Result

The definition of a entropy solutions for problem $(EP)_{f,v_0}^{b,F,k}$ can be stated as follows:

Definition 3.1. A measurable function $v:Q_T\to\overline{\mathbb{R}}$ is called an entropy solution of (1) if

(P1) $b(v)\in L^1(Q_T)$ and $T_K(v)\in L^p(0,T;W_0^{1,p}(\Omega))$ for any $K>0$, (3.1) and for any functions $\varphi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, $\xi\in\mathscr{D}([0,T))$ with $\xi\ge 0$, $S\in\mathscr{P}$, for all, we have

(P2)

$$\begin{aligned}
&-\int_{Q_T}\left(\int_0^t k_{1,j}(t-\tau)\int_{v_0(x)}^{v(\tau,x)}S(\sigma-\varphi(x))\mathrm{d}b(\sigma)\right)\mathrm{d}\tau\,\xi_t(t)\mathrm{d}x\mathrm{d}t\\
&+\int_{Q_T}k_{2,j}(0^+)\big(b(v(t,x))-b(v_0(x))\big)S(v(t,x)-\varphi(x))\xi(t)\mathrm{d}x\mathrm{d}t\\
&+\int_{Q_T}\left(\int_0^t\big(b(v(t,x))-b(v_0(x))\big)\mathrm{d}k_{2,j}(\tau)\right)S(v(t,x)-\varphi(x))\xi(t)\mathrm{d}x\mathrm{d}t \qquad (11)\\
&+\int_{Q_T}\big(a(x,Dv(t,x))+F(v(t,x))\big)\cdot DS(v(t,x)-\varphi(x))\xi((t)\mathrm{d}x\mathrm{d}t\\
&\le\int_{Q_T}f(t,x)S(v(t,x)-\varphi(x))\xi(t)\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

where $k_{1,j}:=(k-j)^+$ and $k_{2,j}:=k-k_{1,j}$.

The following remarks are concerned with a few comments on Definition 3.1.

Remark 3.2. Note that in Definition 3.1, $a(\cdot,Dv)$ and $F(v)$ does note general make sense in the first equation of problem (1), but that to (3.1) each term in inequality (11) has meaning in $\mathcal{D}'(Q_T)$.

We can now formulate our main existence result of a entropy solution of $(EP)_{f,v_0}^{b,F,k}$ which is given by the following Theorem:

Theorem 3.3. *Assume that b satisfies* $(H6)$, *the vector fields a and F satisfy* $(H1)$-$(H4)$ *and* $(H7)$ *and that the scalar kernel k satisfy* $(K1)$-$(K3)$. *Let* $f\in L^1(Q_T)$ *and* $v_0:\Omega\to\mathbb{R}$ *a measurable function. Then there exists at least one entropy solution v of problem* $(EP)_{f,v_0}^{b,F,k}$.

Proof of Theorem 3.3

To prove Theorem 3.3, we will use several techniques and approximation procedures. First, we will construct the abstract problem corresponding to our problem $(EP)_{f,v_0}^{b,F,k}$.

3.1. Abstract Problem Corresponding to $(EP)_{f,v_0}^{b,F,k}$

Since our objective is to apply the abstract theory of G.Gripenberg, let then the graph of the possibly multivalued operator $A_b:L^1(\Omega)\to 2^{L^1(\Omega)}$ be defined by

$$\begin{aligned}
&A_b\subset L^1(\Omega)\times L^1(\Omega),(b(v),w)\in A_b\Leftrightarrow b(v),\,w\in L^1(\Omega),\,v\in\mathcal{T}_0^{1,p}(\Omega),\\
&\text{and }\int_\Omega\big(a(x,Dv)+F(v)\big)\cdot DT_K(v-\phi)\mathrm{d}x\le\int_\Omega wT_K(v-\phi)\mathrm{d}x \qquad (12)\\
&\text{for any }\phi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega).
\end{aligned}$$

This characterization of the operator A_b is based on the results that are shown in [15].

Thus, using this characterization of the operator A_b and by the same arguments as in [[27], Lemma 3.3.4], we can establish the following Lemma:

Lemma 3.4. *Let* A_b *the operator defined* (12). *Then* $(b(v),w)\in A_b$ *implies* *that*

$$\int_\Omega \left(a(x,Dv)+F(v)\right)\cdot D\left(h(v)\xi\right)\mathrm{d}x=\int_\Omega wh(v)\xi\mathrm{d}x \tag{13}$$

for all $h\in\mathcal{C}_c^1(\mathbb{R})$ and all $\xi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$.

Proof. Let $(b(v),w)\in A_b$. From characterizations of the operator A_b in (12), we have $v\in\mathcal{T}_0^{1,p}(\Omega)$. Since $h\in\mathcal{C}_c^1(\mathbb{R})$, it follows by Lemma 2.3 of [28] that $h(v)\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$. We may therefore take $\varphi=T_L(v)\pm h(v)\xi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, $L>0$ as an admissible test function in (12) which yields

$$\int_\Omega \left(a(x,Dv)+F(v)\right)\cdot DT_K\left(v-T_L(v)\pm h(v)\xi\right)\mathrm{d}x\le\int_\Omega wT_K\left(v-T_L(v)\pm h(v)\xi\right)\mathrm{d}x \tag{14}$$

for all $L>K>0$. By Lebesgue's theorem, we get

$$\int_\Omega wT_K\left(v-T_L(v)\pm h(v)\xi\right)\mathrm{d}x\xrightarrow{L\to\infty}\pm\int_\Omega wT_K\left(h(v)\xi\right)\mathrm{d}x$$

where we used that $T_K(v)\to v$ almost everywhere on Ω. Since $h(v)\xi\in L^\infty(\Omega)$, we obtain that

$$\pm\int_\Omega wT_K\left(h(v)\xi\right)\mathrm{d}x=\pm\int_\Omega wh(v)\xi\mathrm{d}x$$

for any K sufficiently large which shows that

$$\lim_{K\to\infty}\lim_{L\to\infty}\int_\Omega wT_K\left(v-T_L(v)\pm h(v)\xi\right)\mathrm{d}x=\pm\int_\Omega wh(v)\xi\mathrm{d}x.$$

Next, we have

$$\begin{aligned}&\int_\Omega a(x,Dv)\cdot DT_K\left(v-T_L(v)\pm h(v)\xi\right)\mathrm{d}x\\&=\int_\Omega \chi_{\{|v-T_L(v)\pm h(v)\xi|<K\}}a(x,Dv)\cdot D\left(v-T_L(v)\right)\mathrm{d}x\\&\quad\pm\int_\Omega \chi_{\{|v-T_L(v)\pm h(v)\xi|<K\}}a(x,Dv)\cdot D\left(h(v)\xi\right)\mathrm{d}x.\end{aligned}$$

The coercivity condition (H3) entails that

$$\begin{aligned}&\int_\Omega \chi_{\{|v-T_L(v)\pm h(v)\xi|<K\}}a(x,Dv)\cdot D\left(v-T_L(v)\right)\mathrm{d}x\\&=\int_\Omega \chi_{\{|v-T_L(v)\pm h(v)\xi|<K\}}a(x,Dv)\cdot Dv\chi_{\{|v|>L\}}\mathrm{d}x\ge 0\end{aligned}$$

for all $L>K>0$. On the over hand, we may conclude by Lebesgue's theorem that

$$\lim_{K\to\infty}\lim_{L\to\infty}\int_\Omega \chi_{\{|v-T_L(v)\pm h(v)\xi|<K\}}a(x,Dv)\cdot D\left(h(v)\xi\right)\mathrm{d}x=\int_\Omega a(x,Dv)\cdot D\left(h(v)\xi\right)\mathrm{d}x.$$

In order to pass to the limit in the left-hand side of (14), observe that

$$\begin{aligned}&\int_\Omega F(v)\cdot DT_K\left(v-T_L(v)\pm h(v)\xi\right)\mathrm{d}x\\&=\int_\Omega \chi_{\{|v-T_L(v)\pm h(v)\xi|<K\}}F(v)\cdot D\left(v-T_L(v)\right)\mathrm{d}x\\&\quad\pm\int_\Omega \chi_{\{|v-T_L(v)\pm h(v)\xi|<K\}}F(v)\cdot D\left(h(v)\xi\right)\mathrm{d}x.\end{aligned}$$

To the first integral on the right-hand side, since $|v|<\infty$ a.e. on Ω, then we have

$$\begin{aligned}&\int_\Omega \chi_{\{|v-T_L(v)\pm h(v)\xi|<K\}}F(v)\cdot D\left(v-T_L(v)\right)\mathrm{d}x\\&=\int_\Omega \chi_{\{|v-T_L(v)\pm h(v)\xi|<K\}}F(v)\cdot Dv\chi_{\{|v|>L\}}\mathrm{d}x=0\end{aligned}$$

for any L sufficiently large which shows that

$$\lim_{K\to\infty}\lim_{L\to\infty}\int_\Omega \chi_{\{|v-T_L(v)\pm h(v)\xi|<K\}}F(v)\cdot D(v-T_L(v))\mathrm{d}x=0.$$

For the second integral, Lebesgue's theorem yields

$$\lim_{K\to\infty}\lim_{L\to\infty}\int_\Omega \chi_{\{|v-T_L(v)\pm h(v)\xi|<K\}}F(v)\cdot D(h(v)\xi)\mathrm{d}x=\int_\Omega F(v)\cdot D(h(v)\xi)\mathrm{d}x.$$

Thus, we get that

$$\pm\int_\Omega (a(x,Dv)+F(v))\cdot D(h(v)\xi)\mathrm{d}x\le\pm\int_\Omega wh(v)\xi\mathrm{d}x$$

which show the equality (13). □

Using the result of Lemma 3.4 we can prove the following result:

Corollary 3.5 *Let* A_b *the operator defined* (12). *Then* $(b(v),w)\in A_b$ *implies that*

$$\int_\Omega (a(x,Dv)+F(v))\cdot DS(v-\varphi)\mathrm{d}x=\int_\Omega wS(v-\varphi)\mathrm{d}x \tag{15}$$

for all $S\in\mathscr{P}$ and all $\varphi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$.

We state some properties of the operator A_b in the following Proposition (for the proof see [29] pp44 and Proposition 4.1.1).

Proposition 3.6 *The operator* A_b *satisfies the following properties*:

1) A_b is m-accretive in $L^1(\Omega)$,
2) $\overline{D(A_b)}=\{u\in L^1(\Omega):u(x)\in\overline{ran(b)}$ for almost every $x\in\Omega\}$.

Now, taking the Banach space $X=L^1(\Omega)$ and using the operator L defined in (4), since the operator A_b is m-accretive, then Theorem 2.6 entails that the abstract Volterra equation

$$L(u-u_0)(t)+A_b(u(t))\ni f(t),\quad \text{in } L^1(\Omega) \tag{16}$$

admit for almost all $t\in(0,T)$, all $u_0=b(v_0)\in\overline{D(A_b)}$ and $f\in L^1(0,T;L^1(\Omega))\equiv L^1(Q_T)$ a unique generalized solution $u\in L^1(Q_T)$. But, it is a priori not clear in which sense the generalized solution satisfies $(EP)_{f,v_0}^{b,F,k}$.

In order to show that $u=b(v)$ where v satisfies $(EP)_{f,v_0}^{b,F,k}$, we will define approximate and perturbed problems associated to $(EP)_{f,v_0}^{b,F,k}$ in the next subsection.

3.2. Regularised and Perturbed Problem Corresponding to $(EP)_{f,v_0}^{b,F,k}$

Note that, for general b, we can not expect to find a strong solution which solves the inclusion problem (16).

In order to overcome this difficulty, let us introduce the following regularizations:

1) $b_l:=b+\frac{1}{l}\cdot id_{\mathbb{R}}$, for $l>0$.
2) $f^{m,n}:=\max(\min(f,m),-n)$, for $m,n\in\mathbb{N}^*$.
3) $v_0^{m,n}:=\max(\min(v_0,m),-n)$, for $m,n\in\mathbb{N}^*$.

4) $\psi^{m,n}(r) := \frac{1}{m}r^{+} - \frac{1}{n}r^{-}$, for all $r \in \mathbb{R}$.

5) $A_b^{\psi^{m,n}}$ the perturbed operator defined as:

$$\begin{aligned}&A_b^{\psi^{m,n}} \subset L^1(\Omega)\times L^1(\Omega), (b(v), w) \in A_b^{\psi^{m,n}} \Leftrightarrow b(v), w \in L^1(\Omega), v \in \mathcal{T}_0^{1,p}(\Omega),\\ &\text{and } \int_\Omega \left(a(x,Dv)+F(v)\right)\cdot DT_K(v-\phi)\mathrm{d}x\\ &+\int_\Omega \psi^{m,n}(v)T_K(v-\phi)\mathrm{d}x \le \int_\Omega wT_K(v-\phi)\mathrm{d}x\\ &\text{for any } \phi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega).\end{aligned} \tag{17}$$

Furthermore, we set $u_{0,l}^{m,n} = b_l\left(v_0^{m,n}\right)$.

By [30] (see also [29] [31]), we affirm that operator $A_b^{\psi^{m,n}}$ is m-accretive in $L^1(\Omega)$ and we have

$$\overline{D\left(A_b^{\psi^{m,n}}\right)}^{\|\cdot\|_{L^1(\Omega)}} = \left\{u \in L^1(\Omega) : u(x) \in \overline{ran(b)} \text{ for almost every } x \in \Omega\right\} = \overline{D(A_b)}^{\|\cdot\|_{L^1(\Omega)}}.$$

Note that, the function b_l is a strictly increasing approximation of b, $f^{m,n} \in L^\infty(Q_T)$ for each $m,n \in \mathbb{N}$, $\left|f^{m,n}(t,x)\right| \le \left|f(t,x)\right|$ a.e. in Q_T, hence $\lim_{n\to\infty}\lim_{m\to\infty} f^{m,n} = f$ in $L^1(Q_T)$ and almost everywhere in Q_T. Similarly, we have $v_0^{m,n} \in L^\infty(\Omega)$, $v_0^{m,n} \to v_0$ a.e. in Ω, $b_l\left(v_0^{m,n}\right) \to b(v_0)$ in $L^1(\Omega)$ and a.e. in Ω as m and n tend to infinity and l tends to infinity.

Let us now consider the following regularized problem:

$$(EP)_{f^{m,n},v_0^{m,n}}^{b_l,F,k,\psi^{m,n}} \begin{cases} \partial_t\left(k*\left(b_l(v)-b_l\left(v_0^{m,n}\right)\right)\right) - div\left(a(x,Dv)+F(v)\right)+\psi^{m,n(v)} = f^{m,n} & \text{in } Q_T,\\ b_l\left(v^{m,n}\right)(0,\cdot) = b_l\left(v_0^{m,n}\right) & \text{in } \Omega,\\ v = 0 & \text{on } \Sigma.\end{cases}$$

We cannot expect to have a strong solution to the abstract problem corresponding to $(EP)_{f^{m,n},v_0^{m,n}}^{b_l,F,k,\psi^{m,n}}$

$$L\left(u-u_{0,l}^{m,n}\right)(t) + A_{b_l}^{\psi^{m,n}}\left(u(t)\right) \ni f^{m,n}(t), \quad \text{in } L^1(\Omega) \tag{18}$$

for almost everywhere $t \in (0,T)$. However, we know by [[26], Theorem 1] that the corresponding approximating abstract problem with respect to the Yosida approximating of L defined by:

$$L_\lambda\left(u-u_{0,l}^{m,n}\right)(t) + A_{b_l}^{\psi^{m,n}}\left(u(t)\right) \ni f^{m,n}(t), \quad \text{in } L^1(\Omega) \tag{19}$$

for almost everywhere $t \in (0,T)$, admits a unique strong solution $u_{\lambda,l}^{m,n} = b_l(v_{\lambda,l}^{m,n})$ in $L^1(Q_T)$ in the sense of Definition 2.4 and through the Theorem 2.6, there exists a measurable function $u_l^{m,n}$ in $L^1(Q_T)$ such that $u_{\lambda,l}^{m,n} \to u_l^{m,n}$ in $L^1(Q_T)$ as λ tends to 0, where $u_l^{m,n}$ is the generalized solution to (18) in the sense of Definition 2.5.

As the function b_l is bijective, then there exists a unique measurable function $v_{\lambda,l}^{m,n}$ such that $b_l\left(v_{\lambda,l}^{m,n}\right) = u_{\lambda,l}^{m,n}$.

3.3. Entropy Solution to Approximate and Perturbed Problem with L^∞-Data

In this subsection, our plan is to show existence of entropy solution to

$(EP)^{b_l,F,k,\psi^{m,n}}_{f^{m,n},v_0^{m,n}}$. This is done via the study to approximate problem $(EP)^{b_l,F,k_\lambda,\psi^{m,n}}_{f^{m,n},v_0^{m,n}}$. The next proposition will give us existence of entropy solutions $v^{m,n}$ of $(EP)^{b_l,F,k,\psi^{m,n}}_{f^{m,n},v_0^{m,n}}$ for each $m,n\in\mathbb{N}^*$.

Proposition 3.7 *For* $m,n\in\mathbb{N}^*$, *there exists a function* $v^{m,n}\in L^1(Q_T)$ *which is a entropy solution of* $(EP)^{b_l,F,k,\psi^{m,n}}_{f^{m,n},v_0^{m,n}}$ *in the sense of Definition* 2.5.

***Proof of Proposition* 3.7.** The proof will be divided into five steps.

***Step* 1:** In this step, we will show existence of entropy solution to $(EP)^{b_l,F,k,\psi^{m,n}}_{f^{m,n},v_0^{m,n}}$.

Corollary 3.8. *The generalized solution* $u_l^{m,n}$ *of* (18) *is of the form* $u_l^{m,n}=b_l(v_l^{m,n})$ *where* $v_l^{m,n}$ *is an entropy solution to* $(EP)^{b_l,F,k,\psi^{m,n}}_{f^{m,n},v_0^{m,n}}$.

Proof of Corollary 3.8. We use the following result which gives a few basic a priori estimates of the sequence $(v_{\lambda,l}^{m,n})_\lambda$ (for m,n and l fixed) which are going through standard method of L^1-theory.

Lemma 3.9. *Let* $\{u_{\lambda,l}^{m,n}=b_l(v_{\lambda,l}^{m,n})\}$ *be the sequence of strong solutions of* (19). *Then, the sequence* $\{v_{\lambda,l}^{m,n}\}$ *satisfies,*

1) $\left|\left\{\left|v_{\lambda,l}^{m,n}\right|\ge K\right\}\right|\le\int_0^T\int_\Omega\left|DT_K(v_{\lambda,l}^{m,n})\right|\mathrm{d}x\mathrm{d}t\le CK$ for every $K>0$, where $C=C(f,k,v_0,b)>0$ is a constant independant of λ,l,m and n.

2) $\left\|v_{\lambda,l}^{m,n}\right\|_{L^\infty(Q_T)}\le\max(m^2,n^2)$.

Proof. The proof is classical (see [16] and [17]). For the sake of completness, let us recall the arguments.

(i): For K fixed, we choose $T_K(v_{\lambda,l}^{m,n}(t))\in L^\infty(\Omega)\cap W_0^{1,p}(\Omega), t\in(0,T)$ as a test function into (19). Using the characterization (17) of operator $A_{b_l}^{\psi^{m,n}}$ and the representation (4) of the Yosida approximation, we find

$$\begin{aligned}&\int_\Omega\partial_t\left[k_\lambda*\left(b_l(v_{\lambda,l}^{m,n})-b_l(v_0^{m,n})\right)\right](t)T_K(v_{\lambda,l}^{m,n}(t))\mathrm{d}x\\&+\int_\Omega a(x,Dv_{\lambda,l}^{m,n}(t))\cdot DT_K(v_{\lambda,l}^{m,n}(t))+\int_\Omega F(v_{\lambda,l}^{m,n}(t))\cdot DT_K(v_{\lambda,l}^{m,n}(t))\mathrm{d}x\\&+\int_\Omega\psi^{m,n}(v_{\lambda,l}^{m,n}(t))T_K(v_{\lambda,l}^{m,n}(t))\mathrm{d}x\le\int_\Omega f^{m,n}(t)T_K(v_{\lambda,l}^{m,n}(t))\mathrm{d}x,\end{aligned}\tag{20}$$

for almost everywhere $t\in(0,T)$.

Note that applying Gauss-Green Theorem and boundary condition

$$\int_\Omega F\left(T_K(v_{\lambda,l}^{m,n}(t))\right)\cdot DT_K(v_{\lambda,l}^{m,n}(t))\mathrm{d}x=0$$

for almost all $t\in(0,T)$. Moreover, by the monotonicity of $\psi^{m,n}$, we have

$$\int_\Omega\psi^{m,n}(v_{\lambda,l}^{m,n}(t))T_K(v_{\lambda,l}^{m,n}(t))\mathrm{d}x\ge0$$

for almost all $t\in(0,T)$. Thus, applying Lemma 2.5 of [16] to the first term in (20), integrate in time over $(0,T)$, taking into account the assumption of coercivity (H3) of vector field *a*, we have

$$\begin{aligned}&\int_\Omega\left[k_\lambda*\int_0^{v_{\lambda,l}^{m,n}}T_K(\sigma)\mathrm{d}b_l(\sigma)\right](T)\mathrm{d}x\\&+\int_0^T\int_\Omega\left[T_K(v_{\lambda,l}^{m,n}(t))b_l(v_{\lambda,l}^{m,n}(t))-\int_0^{v_{\lambda,l}^{m,n}(t)}T_K(\sigma)\mathrm{d}b_l(\sigma)\right]k_\lambda(t)x\mathrm{d}t\\&+\int_0^T\int_\Omega\int_0^t\left[\int_{v_{\lambda,l}^{m,n}(t)}^{v_{\lambda,l}^{m,n}(t-s)}T_K(\sigma)\mathrm{d}b_l(\sigma)\right.\end{aligned}\tag{21}$$

$$-T_K\left(v_{\lambda,l}^{m,n}(t)\right)\left(b_l\left(v_{\lambda,l}^{m,n}(t-s)\right)-b_l\left(v_{\lambda,l}^{m,n}(t)\right)\right)\Big]\left[-k_\lambda'(s)\right]\mathrm{d}s\mathrm{d}x\mathrm{d}t$$

$$+\int_0^T\int_\Omega\left|DT_K\left(v_{\lambda,l}^{m,n}(t)\right)\right|^p\mathrm{d}x\mathrm{d}t$$

$$\le K\|f\|_{L^1(Q_T)}+K\int_0^T\int_\Omega k_\lambda(t)\left|b(v_0)\right|\mathrm{d}x\mathrm{d}t\le KC$$

for all $K>0$, all $m,n\in\mathbb{N}^*$ and all $l>0$ where $C:=C(f,k,v_0,b)>0$ is a positive constant independent of λ,l,m and *n*. Then, defining a function *H* as

$$H(r):=\int_0^r T_K\circ b_l^{-1}(\sigma)\mathrm{d}\sigma$$

for every $K>0$ and every $r\in ran(b)$, we deduce that

$$H\left(b_l\left(v_{\lambda,l}^{m,n}(t)\right)\right)=\int_0^{v_{\lambda,l}^{m,n}(t)}T_K(\sigma)\mathrm{d}b_l(\sigma)$$

for almost all $t\in(0,T)$.

By the convexity of the function *H*, we obtain that each term on the left-hand side in (21) is nonnegative. So, we have

$$\int_0^T\int_\Omega\left|DT_K\left(v_{\lambda,l}^{m,n}(t)\right)\right|^p\mathrm{d}x\mathrm{d}t\le KC$$

for some constant *C* independent of λ,l,m and *n*. By applying Poincaré's inequality, this involves (i). The proof of estimate (ii) follows the same lines as the proof of ([17], Lemma 3.3). Indeed, taking $S\left(v_{\lambda,l}^{m,n}(t)\right)$, $t\in(0,T)$ in (19) as test function where $S\in\mathscr{P}$, taking into account the boundary condition and the Lipschitz character of *F*, then by divergence theorem, we obtain

$$\int_\Omega F\left(v_{\lambda,l}^{m,n}(t)\right)\cdot DS\left(v_{\lambda,l}^{m,n}(t)\right)\mathrm{d}x=0$$

for almost all $t\in(0,T)$. □

The following result states useful convergences result (see [16] [17]).

Lemma 3.10. *As* $\lambda\to0$, *we have* (*up to subsequences*):

1) $u_{\lambda,l}^{m,n}\to u_l^{m,n}$ almost everywhere in Q_T.

2) $v_{\lambda,l}^{m,n}\to v_l^{m,n}$ almost everywhere in Q_T, where $v_l^{m,n}:Q_T\to\mathbb{R}$ is a measurable function satisfying $u_l^{m,n}=b_l\left(v_l^{m,n}\right)$ a.e. in Q_T.

3) $T_K\left(v_{\lambda,l}^{m,n}\right)\to T_K\left(v_l^{m,n}\right)$ weakly in $L^p\left(0,T;W_0^{1,p}(\Omega)\right)$ and almost everywhere in Q_T.

4) $a\left(\cdot,DT_K\left(v_{\lambda,l}^{m,n}\right)\right)\rightharpoonup a\left(\cdot,DT_K\left(v_l^{m,n}\right)\right)$, weakly in $\left(L^{p'}(Q_T)\right)^N$, for every $K>0$.

Remark 3.11. *Note that, by* (ii) *of Lemma* 3.10, *a immediate consequence of Lemma* 3.9 *is the following result*:

$$\left\|v_l^{m,n}\right\|_{L^\infty(Q_T)}\le\max\left(m^2,n^2\right). \tag{22}$$

Hence, by the coercivity condition (H3) and Lemma 3.9, we find following result:

$$\left\|v_l^{m,n}\right\|_{L^p\left(0,T;W_0^{1,p}(\Omega)\right)}\le C(m,n) \tag{23}$$

where $C(m,n)>0$ is a constant independent of *l*.

Now, thanks to Lemma 3.9 and 3.10, we will show that $v_l^{m,n}$ satisfies the inequality of type (11).

Let $S\in\mathscr{P}$, $\varphi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ and $\xi\in\mathcal{D}([0,T))$, $\xi\ge 0$. Let R be a positive real number such that $Supp(S')\subset[-R,R]$ and define $K:=R+\|\varphi\|_{L^\infty(\Omega)}$.

Pointwise multiplication of the approximate abstract problem (19) by $S(v_{\lambda,l}^{m,n}(t,\cdot)-\varphi)\xi(t)$ and integration over $(0,T)$ leads to

$$\begin{aligned}&\int_0^T\int_\Omega\partial_t\left[k_\lambda*\left(b_l\left(v_{\lambda,l}^{m,n}\right)-b_l\left(v_0^{m,n}\right)\right)\right](t)S\left(v_{\lambda,l}^{m,n}(t,\cdot)-\varphi\right)\xi(t)\mathrm{d}x\mathrm{d}t\\&+\int_0^T\int_\Omega a\left(x,Dv_{\lambda,l}^{m,n}\right)\cdot DS\left(v_{\lambda,l}^{m,n}(t,\cdot)-\varphi\right)\xi(t)\mathrm{d}x\mathrm{d}t\\&+\int_0^T\int_\Omega F\left(v_{\lambda,l}^{m,n}\right)\cdot DS\left(v_{\lambda,l}^{m,n}(t,\cdot)-\varphi\right)\xi(t)\mathrm{d}x\mathrm{d}t\\&+\int_0^T\int_\Omega\psi^{m,n}\left(v_{\lambda,l}^{m,n}\right)S\left(v_{\lambda,l}^{m,n}(t,\cdot)-\varphi\right)\xi(t)\mathrm{d}x\mathrm{d}t\\&=\int_0^T\int_\Omega f^{m,n}S\left(v_{\lambda,l}^{m,n}(t,\cdot)-\varphi\right)\xi(t)\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{24}$$

Here, we used Corollary 3.5. Next, for $j\in\mathbb{N}$, we define $k_{1,\lambda,j}:=(k_\lambda-j)^+$ and $k_{2,\lambda,j}:=k_\lambda-k_{1,\lambda,j}$. It is obvious to see that $k_{1,\lambda,j}+k_{2,\lambda,j}=k_\lambda$ and since $k_\lambda\to k$ in $L^1(0,T)$ then $k_{2,\lambda,j}\to k_{1,j}$ and $k_{1,\lambda,j}\to k_{2,j}$ in $L^1(0,T)$ where $k_{1,j},k_{2,j}$ are as in the definition of the entropy solution. Applying Lemma 2.6 of the reference [16] to $k_{1,\lambda,j}$, we obtain

$$\begin{aligned}&-\int_0^T\int_\Omega\left[k_{1,\lambda,j}*\int_{v_0^{m,n}}^{v_{\lambda,l}^{m,n}}S(\sigma-\varphi)\mathrm{d}b_l(\sigma)\right]\xi_t(t)\mathrm{d}x\mathrm{d}t\\&+\int_0^T\int_\Omega\partial_t\left[k_{2,\lambda,j}*\left(b_l\left(v_{\lambda,l}^{m,n}\right)-b_l\left(v_0^{m,n}\right)\right)\right](t)S\left(v_{\lambda,l}^{m,n}(t,\cdot)-\varphi\right)\xi(t)\mathrm{d}x\mathrm{d}t\\&+\int_0^T\int_\Omega a\left(x,Dv_{\lambda,l}^{m,n}\right)\cdot DS\left(v_{\lambda,l}^{m,n}(t,\cdot)-\varphi\right)\xi(t)\mathrm{d}x\mathrm{d}t\\&+\int_0^T\int_\Omega F\left(v_{\lambda,l}^{m,n}\right)\cdot DS\left(v_{\lambda,l}^{m,n}(t,\cdot)-\varphi\right)\xi(t)\mathrm{d}x\mathrm{d}t\\&\le\int_0^T\int_\Omega\left(f^{m,n}-\psi^{m,n}\left(v_{\lambda,l}^{m,n}\right)\right)S\left(v_{\lambda,l}^{m,n}(t,\cdot)-\varphi\right)\xi(t)\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{25}$$

In whats follows we pass to the limit-inf as λ tends to 0 in (25).

- Limit of $-\int_0^T\int_\Omega\left[k_{1,\lambda,j}*\int_{v_0^{m,n}}^{v_{\lambda,l}^{m,n}}S(\sigma-\varphi)\mathrm{d}b_l(\sigma)\right]\xi_t(t)\mathrm{d}x\mathrm{d}t$.

We know that the convergence $v_{\lambda,l}^{m,n}\to v_l^{m,n}$ a.e. in Q_T entails

$$\int_{v_0^{m,n}}^{v_{\lambda,l}^{m,n}}S(\sigma-\varphi)\mathrm{d}b_l(\sigma)\to\int_{v_0^{m,n}}^{v_l^{m,n}}S(\sigma-\varphi)\mathrm{d}b_l(\sigma),\quad\text{a.e. in } Q_T.$$

Since $S\in\mathscr{P}$ is bounded and continuous function, then we have

$$\left|\int_{v_0^{m,n}}^{v_{\lambda,l}^{m,n}}S(\sigma-\varphi)\mathrm{d}b_l(\sigma)\right|\le\|S\|_\infty\left|b_l\left(v_{\lambda,l}^{m,n}\right)-b_l\left(v_0^{m,n}\right)\right|.$$

As $b_l\left(v_{\lambda,l}^{m,n}\right)\to b_l\left(v_l^{m,n}\right)$ in $L^1(Q_T)$, there exists a function in $L^1(Q_T)$ independent of λ which dominates $\left|b_l\left(v_{\lambda,l}^{m,n}\right)-b_l\left(v_0^{m,n}\right)\right|$. Thus, by Lebesgue's convergence theorem, it follows

$$\int_{v_0^{m,n}}^{v_{\lambda,l}^{m,n}}S(\sigma-\varphi)\mathrm{d}b_l(\sigma)\xrightarrow{\lambda\to 0}\int_{v_0^{m,n}}^{v_l^{m,n}}S(\sigma-\varphi)\mathrm{d}b_l(\sigma)\quad\text{in } L^1(Q_T),$$

and

$$\int_\Omega\int_{v_0^{m,n}}^{v_{\lambda,l}^{m,n}}S(\sigma-\varphi)\mathrm{d}b_l(\sigma)\xrightarrow{\lambda\to 0}\int_\Omega\int_{v_0^{m,n}}^{v_l^{m,n}}S(\sigma-\varphi)\mathrm{d}b_l(\sigma)\quad\text{in } L^1([0,T)).$$

As $k_{1,\lambda,j} \to k_{1,j}$ in $L^1([0,T))$, it follows by Young's inequality that

$$\int_\Omega \left[k_{1,\lambda,j} * \int_{v_0^{m,n}}^{v_{\lambda,l}^{m,n}} S(\sigma-\varphi) \mathrm{d}b_l(\sigma) \right](\cdot) \mathrm{d}x$$

$$\xrightarrow{\lambda\to 0} \int_\Omega \left[k_{1,j} * \int_{v_0^{m,n}}^{v_\varepsilon^{m,n}} S(\sigma-\varphi) \mathrm{d}b_l(\sigma) \right](\cdot) \mathrm{d}x \quad \text{in } L^1([0,T)),$$

and for a subsequence if necessary a.e. in $(0,T)$. Hence,

$$-\int_0^T \int_\Omega \left[k_{1,\lambda,j} * \int_{v_0^{m,n}}^{v_{\lambda,l}^{m,n}} S(\sigma-\varphi) \mathrm{d}b_l(\sigma) \right](t) \xi_t \mathrm{d}x$$

$$\xrightarrow{\lambda\to 0} -\int_0^T \int_\Omega \left[k_{1,j} * \int_{v_0^{m,n}}^{v_l^{m,n}} S(\sigma-\varphi) \mathrm{d}b_l(\sigma) \right](t) \xi_t \mathrm{d}x\mathrm{d}t$$

in $L^1(Q_T)$.

• Limit of

$$I_{2,\lambda,l}^{m,n} := \int_0^T \int_\Omega \partial_t \left[k_{2,\lambda,j} * \left(b_l\left(v_{\lambda,l}^{m,n}\right) - b_l\left(v_0^{m,n}\right)\right)\right](t) S\left(v_{\lambda,l}^{m,n}(t,\cdot) - \varphi\right) \xi(t) \mathrm{d}x\mathrm{d}t$$

By the triangle ineguality, we have the following estimate:

$$\left\| \partial_t \left[k_{2,\lambda,j} * \left(b_l\left(v_{\lambda,l}^{m,n}\right) - b_l\left(v_0^{m,n}\right)\right)\right] - \partial_t \left[k_{2,j} * \left(b_l\left(v_l^{m,n}\right) - b_l\left(v_0^{m,n}\right)\right)\right] \right\|_{L^1(Q_T)}$$

$$\le \left\| \partial_t \left[k_{2,\lambda,j} * \left(b_l\left(v_{\lambda,l}^{m,n}\right) - b_l\left(v_0^{m,n}\right)\right) - \left(b_l\left(v_l^{m,n}\right) - b_l\left(v_0^{m,n}\right)\right)\right] \right\|_{L^1(Q_T)}$$

$$+ \left\| \partial_t \left[k_{2,\lambda,j} * \left(b_l\left(v_l^{m,n}\right) - b_l\left(v_0^{m,n}\right)\right)\right] - \partial_t \left[k_{2,j} * \left(b_l\left(v_l^{m,n}\right) - b_l\left(v_0^{m,n}\right)\right)\right] \right\|_{L^1(Q_T)}$$

$$:= A_{l,\lambda,j}^{m,n} + B_{l,\lambda,j}^{m,n}.$$

Since $k_{2,\lambda,j} \in W^{1,1}(0,T)$ verifies $0 \le k_{2,\lambda,j}(0) \le j$ for any $\lambda > 0$ and $j \in \mathbb{N}$. So, from Young's inequality, we have

$$A_{l,\lambda,j}^{m,n} \le j \left\| \left(b_l\left(v_{\lambda,l}^{m,n}\right) - b_l\left(v_0^{m,n}\right)\right) - \left(b_l\left(v_l^{m,n}\right) - b_l\left(v_0^{m,n}\right)\right) \right\|_{L^1(Q_T)}$$

$$+ \left\| k'_{2,\lambda,j} \right\|_{L^1(0,T)} \left\| \left(b_l\left(v_{\lambda,l}^{m,n}\right) - b_l\left(v_0^{m,n}\right)\right) - \left(b_l\left(v_l^{m,n}\right) - b_l\left(v_0^{m,n}\right)\right) \right\|_{L^1(Q_T)}$$

for any $\lambda > 0$ and $j \in \mathbb{N}$. As $k'_{2,\lambda,j}$ is non-negative and non-increasing, then

$$Var_{[0,T]} k_{2,\lambda,j} = \left\| k'_{2,\lambda,j} \right\|_{L^1(0,T)} \le k_{2,\lambda,j}(0) \le j$$

for any $\lambda > 0$ and $j \in \mathbb{N}$, where $Var_{[0,T]}$ denotes the variation on the interval $[0,T]$. Thus, we have just shown that $A_{l,\lambda,j}^{m,n} \to 0$ as $\lambda \to 0$.

Moreover, we may conclude by [[26], Lemma 3.4] that $B_{l,\lambda,j}^{m,n} \to 0$ as $\lambda \to 0$. Its follows that

$$\partial_t \left[k_{2,\lambda,j} * \left(b_l\left(v_{\lambda,l}^{m,n}\right) - b_l\left(v_0^{m,n}\right)\right)\right] \to \partial_t \left[k_{2,j} * \left(b_l\left(v_l^{m,n}\right) - b_l\left(v_0^{m,n}\right)\right)\right]$$

in $L^1(Q_T)$ as $\lambda \to 0$. Since, S is bounded and continuous, the convergence $v_{\lambda,l}^{m,n} \to v_l^{m,n}$ a.e. in Q_T implies that $S\left(v_{\lambda,l}^{m,n} - \varphi\right)$ converges to $S\left(v_l^{m,n} - \varphi\right)$ a.e. in Q_T and $L^\infty(Q_T)$ weak-$\star$. Hence,

$$I_{2,\lambda,l}^{m,n} \xrightarrow{\lambda\to 0} \int_0^T \int_\Omega \partial_t \left[k_{2,j} * \left(b_l\left(v_l^{m,n}\right) - b_l\left(v_0^{m,n}\right)\right)\right](t) S\left(v_l^{m,n}(t,\cdot) - \varphi\right) \xi(t) \mathrm{d}x\mathrm{d}t.$$

• Limit of $\int_0^T \int_\Omega a\left(x, Dv_{\lambda,l}^{m,n}\right) \cdot DS\left(v_{\lambda,l}^{m,n}(t,\cdot) - \varphi\right) \xi(t) \mathrm{d}x\mathrm{d}t$.

Here, we will try to make an estimate of $\varliminf_{\lambda\to 0}\int_0^T\int_\Omega a\left(x,Dv_{\lambda,l}^{m,n}\right)\cdot DS\left(v_{\lambda,l}^{m,n}(t,\cdot)-\varphi\right)\xi(t)\,\mathrm{d}x\mathrm{d}t$. Recall that it has been assumed that $Supp(S)\subset[-R,R]$ and $K:=R+\|\varphi\|_{L^\infty(\Omega)}$ where $R>0$. Furthermore,

$$S'\left(v_{\lambda,l}^{m,n}(t,\cdot)-\varphi\right)a\left(\cdot,Dv_{\lambda,l}^{m,n}\right)\cdot D\left(v_{\lambda,l}^{m,n}(t,\cdot)-\varphi\right)$$

is identified with the term

$$S'\left(T_K\left(v_{\lambda,l}^{m,n}(t,\cdot)\right)-\varphi\right)a\left(\cdot,DT_K\left(v_{\lambda,l}^{m,n}\right)\right)\cdot D\left(T_K\left(v_{\lambda,l}^{m,n}(t,\cdot)\right)-\varphi\right).$$

Thus, from the monotonicity assumption (H2), weak convergences in Lemma 3.10 and the same argument as in ([16], p. 494-495), we obtain

$$\varliminf_{\lambda\to 0}\int_0^T\int_\Omega a\left(x,Dv_{\lambda,l}^{m,n}\right)\cdot DS\left(v_{\lambda,l}^{m,n}(t,\cdot)-\varphi\right)\xi(t)\,\mathrm{d}x\mathrm{d}t$$
$$\geq\int_0^T\int_\Omega a\left(x,Dv_l^{m,n}\right)\cdot DS\left(v_l^{m,n}(t,\cdot)-\varphi\right)\xi(t)\,\mathrm{d}x\mathrm{d}t$$

for any $\varphi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, $S\in\mathscr{P}$ and $\xi\in\mathcal{D}$, $\xi\geq 0$.

• Limit of $\int_0^T\int_\Omega F\left(v_{\lambda,l}^{m,n}\right)\cdot DS\left(v_{\lambda,l}^{m,n}(t,\cdot)-\varphi\right)\xi(t)\,\mathrm{d}x\mathrm{d}t$.

We have

$$F\left(v_{\lambda,l}^{m,n}\right)\cdot DS\left(v_{\lambda,l}^{m,n}-\varphi\right)=S'\left(T_K\left(v_{\lambda,l}^{m,n}\right)-\varphi\right)F\left(T_K\left(v_{\lambda,l}^{m,n}\right)\right)\cdot D\left(T_K\left(v_{\lambda,l}^{m,n}\right)-\varphi\right),\quad \text{a.e. in } Q_T.$$

Due to $\varphi\in W_0^{1,p}(\Omega)$, the weak convergence $T_K\left(v_{\lambda,l}^{m,n}\right)\rightharpoonup T_K\left(v_l^{m,n}\right)$ in $L^p\left(0,T;W_0^{1,p}(\Omega)\right)$ and a.e. in Q_T, we deduce that

$$D\left(T_K\left(v_{\lambda,l}^{m,n}\right)-\varphi\right)\rightharpoonup D\left(T_K\left(v_l^{m,n}\right)-\varphi\right)\quad\text{weakly in }\left(L^p(Q_T)\right)^N$$

as λ tends to 0, while $S'\left(T_K\left(v_{\lambda,l}^{m,n}\right)-\varphi\right)F\left(T_K\left(v_{\lambda,l}^{m,n}\right)\right)$ is uniformly bounded with respect to λ and converges a.e. in Q_T to $S'\left(T_K\left(v_{\lambda,l}^{m,n}\right)-\varphi\right)F\left(T_K\left(v_{\lambda,l}^{m,n}\right)\right)$. As a consequence, it follows that for $1\leq q<\infty$

$$S'\left(T_K\left(v_{\lambda,l}^{m,n}\right)-\varphi\right)F\left(T_K\left(v_{\lambda,l}^{m,n}\right)\right)\to S'\left(T_K\left(v_l^{m,n}\right)-\varphi\right)F\left(T_K\left(v_l^{m,n}\right)\right)$$

strongly in $L^q(Q_T)$ and

$$F\left(T_K\left(v_{\lambda,l}^{m,n}\right)\right)\cdot DS\left(T_K\left(v_{\lambda,l}^{m,n}\right)-\varphi\right)\xrightarrow{\lambda\to 0}F\left(T_K\left(v_l^{m,n}\right)\right)\cdot DS\left(T_K\left(v_l^{m,n}\right)-\varphi\right)$$

weakly in $L^1(Q_T)$. So,

$$\int_0^T\int_\Omega F\left(v_{\lambda,l}^{m,n}\right)\cdot DS\left(v_{\lambda,l}^{m,n}-\varphi\right)\xi\,\mathrm{d}x\mathrm{d}t\xrightarrow{\lambda\to 0}\int_0^T\int_\Omega F\left(v_l^{m,n}\right)\cdot DS\left(v_l^{m,n}-\varphi\right)\xi\,\mathrm{d}x\mathrm{d}t.$$

• Limit of $\int_0^T\int_\Omega\left(f^{m,n}-\psi^{m,n}\left(v_{\lambda,l}^{m,n}\right)\right)S\left(v_{\lambda,l}^{m,n}(t,\cdot)-\varphi\right)\xi(t)\,\mathrm{d}x\mathrm{d}t$.

Recalling that $f^{m,n}$ belongs to $L^1(Q_T)$, $v_{\lambda,l}^{m,n}\to v_l^{m,n}$ a.e. in Q_T, $\left\|v_{\lambda,l}^{m,n}\right\|_{L^\infty(Q_T)}\leq C$ where is a positive constant independent of λ (see Lemma 3.9 and 3.10) and that $S\left(v_{\lambda,l}^{m,n}-\varphi\right)$ is bounded and convergences to $S\left(v_l^{m,n}-\varphi\right)$ a.e. Q_T. as λ tends to 0. Then, it possible to obtain

$$f^{m,n}S\left(v_{\lambda,l}^{m,n}-\varphi\right)\xi\to f^{m,n}S\left(v_l^{m,n}-\varphi\right)\xi\quad\text{in }L^1(Q_T)$$

as λ tends 0. Moreover, by definition of the function $\psi^{m,n}$, we also have $\psi^{m,n}\left(v_{\lambda,l}^{m,n}\right)$ is uniformly bounded with respect to λ and converges a.e. in Q_T

to $\psi^{m,n}\left(v_l^{m,n}\right)$ as λ tends to 0. So, we also have

$$\psi^{m,n}\left(v_{\lambda,l}^{m,n}\right)S\left(v_{\lambda,l}^{m,n}-\varphi\right)\xi \to \psi^{m,n}\left(v_l^{m,n}\right)S\left(v_l^{m,n}-\varphi\right)\xi \quad \text{in } L^1\left(Q_T\right)$$

as λ tends 0.

As a consequence of the above convergence results, we can say $v_l^{m,n}$ satisfies

$$\begin{aligned}
&-\int_{Q_T}\left[k_{1,j} * \int_{v_0^{m,n}}^{v_l^{m,n}} S(\sigma-\varphi)\mathrm{d}b_l(\sigma)\right]\xi(t)\mathrm{d}x\mathrm{d}t \\
&+\int_{Q_T}\partial_t\left[k_{2,j} * \left(b_l\left(v_l^{m,n}\right)-b_l\left(v_0^{m,n}\right)\right)\right](t)S\left(v_l^{m,n}(t,\cdot)-\varphi\right)\xi(t)\mathrm{d}x\mathrm{d}t \\
&+\int_{Q_T}\left[a\left(x,Dv_l^{m,n}\right)+F\left(v_l^{m,n}\right)\right]\cdot DS\left(v_l^{m,n}(t,\cdot)-\varphi\right)\xi(t)\mathrm{d}x\mathrm{d}t \\
&+\int_{Q_T}\psi^{m,n}\left(v_l^{m,n}\right)S\left(v_l^{m,n}(t,\cdot)-\varphi\right)\xi(t)\mathrm{d}x\mathrm{d}t \\
&\le \int_{Q_T} f^{m,n} S\left(v_l^{m,n}(t,\cdot)-\varphi\right)\xi(t)\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{26}$$

Moreover, we know that

$$b_l\left(v_l^{m,n}\right)\in L^1\left(0,T;L^1(\Omega)\right)$$

and

$$T_K\left(v_l^{m,n}\right)\in L^p\left(0,T;W_0^{1,p}(\Omega)\right).$$

Hence, $v_l^{m,n}$ is entropy solution to $(EP)_{k,f^{m,n}}^{b_l,F^{m,n}}\left(v_0^{m,n},\psi^{m,n}\right)$. This ends the proof of Corollary 3.8. □

Using the existence result with L^∞-data of the Corollary 3.8, we get a strong convergence in the following step.

***Step* 2:** In this step, we will show that a subsequence of $\left(v_l^{m,n}\right)_l$ (still denoted by $\left(v_l^{m,n}\right)_l$, for simplicity) converges in $L^1\left(Q_T\right)$ as $l\to\infty$.

In order to pass at the limit-inf with $l\to\infty$ in inequality (26), we need some type of strong convergence of an subsequence of entropy solution $v_l^{m,n}$ to $v^{m,n}$ in $L^1\left(Q_T\right)$. It is the perturbation term that allows us to prove this result by comparing two different entropy solutions $v_l^{m,n}$ and $v_i^{m,n}$. To this end, we use Kruzhkov's method of doubling variables to show the strong convergence of $v_l^{m,n}$ in the following Lemma. Before, stating the lemma, we should note that by ([26], Theorem 5) we have $u_l^{m,n}=b_l\left(v_l^{m,n}\right)\to u^{m,n}$ in $L^1\left(Q_T\right)$ where $u^{m,n}$ is the generalized solution to abstract problem

$$L\left(u-u_0^{m,n}\right)(t)+A_b^{\psi^{m,n}}\left(u(t)\right)\ni f^{m,n}(t) \quad L^1(\Omega)$$

for almost all $t\in(0,T)$. Then, by Remark 3.11, we can deduce that

$$\left\|u^{m,n}\right\|_{L^\infty(Q_T)}\le C(m,n,b)$$

where $C(m,n,b)>0$ is a constant which depends on m,n and *b*.

Lemma 3.12. *Let* $\left(v_l^{m,n}\right)_l$ *be the sequence of entropy solution* $(EP)_{f^{m,n},v_0^{m,n}}^{b_l,F,k,\psi^{m,n}}$. *Then, there exist a measurable function* $v^{m,n}$ *such that (up to subsequences):*

$v_l^{m,n}\to v^{m,n}$ *in* $L^1\left(Q_T\right)$ *and a.e. in* Q_T

as $l\to\infty$.

Proof. The proof follows the same lines as the proof of Lemma 3.5 in [17].

First of all let us note that by an approximation argument an entropy solution of $(EP)^{b_l,F,k,\psi^{m,n}}_{f^{m,n},v_0^{m,n}}$ additionally satisfies inequality (26) for all $S\in\{T_K;T_{K,L}\}$ with $L>K>0$.

We apply Kruzskov's method of doubling variables in time. Let s,t two variables in $(0,T)$ and consider $v_l^{m,n}$ the entropy solution to $(EP)^{b_l,F,k,\psi^{m,n}}_{f^{m,n},v_0^{m,n}}$ as a function of (t,x) and $v_l^{m,n}$ the entropy solution to $(EP)^{b_l,F,k,\psi^{m,n}}_{f^{m,n},v_0^{m,n}}$ as a function of (s,x). Moreover, let $\xi\in\mathcal{D}([0,T)\times[0,T))$ with $\xi\ge 0$, we take $\varphi=v_i^{m,n}(s)\in W_0^{1,p}(\Omega)$ as a test function in the of entropy solution for $v_l^{m,n}$ and $\varphi=v_l^{m,n}(t)\in W_0^{1,p}(\Omega)$ as a test function in the of entropy solution for $v_i^{m,n}$ with for $L>0$ $S=T_L$. Adding the two variational inequalities, we obtain,

$$I_1^L+I_2^L+I_3^L+I_4^L+I_5^L+I_6^L+I_7^L\le I_8^L. \tag{27}$$

Here, for simplicity, we set $Q_{2,T}:=(0,T)\times(0,T)\times\Omega$ and use the abbreviations $u_l^{m,n}:=b_l(v_l^{m,n})$, $u_i^{m,n}:=b_i(v_i^{m,n})$, $u_{0,l}^{m,n}:=b_l(v_0^{m,n})$, $u_{0,i}^{m,n}:=b_i(v_0^{m,n})$ and furthermore

$$I_1^L:=-\int_{Q_{2,T}}\xi_t(t,s)\left[\int_0^t k_{1,j}(t-\sigma)\int_{v_0^{m,n}}^{v_l^{m,n}(\sigma)}T_L\left(r-v_i^{m,n}(s)\right)\mathrm{d}b_l(r)\,\mathrm{d}\sigma\right]\mathrm{d}x\mathrm{d}t\mathrm{d}s$$

$$I_2^L:=-\int_{Q_{2,T}}\xi_s(t,s)\left[\int_0^s k_{1,j}(s-\tau)\int_{v_0^{m,n}}^{v_i^{m,n}(\tau)}T_L\left(r-v_l^{m,n}(t)\right)\mathrm{d}b_i(r)\,\mathrm{d}\tau\right]\mathrm{d}x\mathrm{d}t\mathrm{d}s$$

$$I_3^L:=\int_{Q_{2,T}}\xi(t,s)\left[\partial_t\left[k_{2,j}*\left(u_l^{m,n}-u_{0,l}^{m,n}\right)\right](t)\right]T_L\left[v_l^{m,n}(t)-v_i^{m,n}(s)\right]\mathrm{d}x\mathrm{d}t\mathrm{d}s$$

$$I_4^{K,L}:=\int_{Q_{2,T}}\left[\xi(t,s)\partial_s\left[k_{2,j}*\left(u_i^{m,n}-u_{0,i}^{m,n}\right)\right](s)\right]T_L\left[v_i^{m,n}(s)-v_l^{m,n}(t)\right]\mathrm{d}x\mathrm{d}t\mathrm{d}s$$

$$I_5^L:=\int_{Q_{2,T}}\xi(t,s)\left[a\left(x,Dv_l^{m,n}(t)\right)-a\left(x,Dv_i^{m,n}(s)\right)\right]\cdot DT_L\left[v_l^{m,n}(t)-v_i^{m,n}(s)\right]\mathrm{d}x\mathrm{d}t\mathrm{d}s$$

$$I_6^L:=\int_{Q_{2,T}}\xi(t,s)\left[F\left(v_l^{m,n}(t)\right)-F\left(v_i^{m,n}(s)\right)\right]\cdot DT_L\left[v_l^{m,n}(t)-v_i^{m,n}(s)\right]\mathrm{d}x\mathrm{d}t\mathrm{d}s$$

$$I_7^L:=\int_{Q_{2,T}}\xi(t,s)\left[\psi^{m,n}\left(v_l^{m,n}(t)\right)-\psi^{m,n}\left(v_i^{m,n}(s)\right)\right]T_L\left[v_l^{m,n}(t)-v_i^{m,n}(s)\right]\mathrm{d}x\mathrm{d}t\mathrm{d}s$$

$$I_8^L:=\int_{Q_{2,T}}\xi(t,s)\left[f^{m,n}(t)-f^{m,n}(s)\right]T_L\left[v_l^{m,n}(t)-v_i^{m,n}(s)\right]\mathrm{d}x\mathrm{d}t\mathrm{d}s$$

Dividing inequality (27) by *L*, we get

$$\frac{1}{L}I_1^L+\frac{1}{L}I_2^L+\frac{1}{L}I_3^L+\frac{1}{L}I_4^L+\frac{1}{L}I_5^L+\frac{1}{L}I_6^L+\frac{1}{L}I_7^L\le\frac{1}{L}I_8^L. \tag{28}$$

Now, we will pass to the limit with $L\to\infty$ in (28).

Note that the term I_5^L is nonnegative by the monotonicity assumption (H2). Moreover, as F is locally Lipschitz continuous, let $C_F>0$ be the Lipschitz constant of *F*. Then, we find follows

$$\frac{1}{L}\left|I_6^L\right|\le C_F\|\xi\|_\infty\frac{1}{L}\int_{Q_{2,T}}\chi_{\left\{0<v_l^{m,n}(t)-v_i^{m,n}(s)<L\right\}}T_L\left[v_l^{m,n}(t)-v_i^{m,n}(s)\right]\left\|D\left[v_l^{m,n}(t)-v_i^{m,n}(s)\right]\right\|\mathrm{d}x\mathrm{d}t\mathrm{d}s.$$

Since,

$$\chi_{\left\{0<v_l^{m,n}(t)-v_i^{m,n}(s)<L\right\}}\frac{1}{L}T_L\left[v_l^{m,n}(t)-v_i^{m,n}(s)\right]\to\chi_{\left\{v_l^{m,n}(t)=v_i^{m,n}(s)\right\}}sign_0^+\left(v_l^{m,n}(t)-v_i^{m,n}(s)\right)=0$$

almost everywhere in $Q_{2,T}$ as $L\to 0$, it follows that

$$\lim_{L\to 0}\frac{1}{L}I_6^L=0.$$

Therefore, using the same arguments as in [[17]; p9-12], we get

$$\lim_{l,i\to\infty}\int_\tau^\theta\int_\Omega\left|v_l^{m,n}-v_i^{m,n}\right|\mathrm{d}x\mathrm{d}t=0 \quad (29)$$

for almost every $0\le\tau<\theta<T$. So, $\left(v_l^{m,n}\right)_l$ is a Cauchy sequence in $L^1\left((\tau,\theta)\times\Omega\right)$. Since, $\left(v_l^{m,n}\right)_l$ is uniformly bounded and by (23), we have (up to subsequence)

$$v_l^{m,n}\rightharpoonup v^{m,n} \quad \text{in } L^p\left(0,T;W_0^{1,p}(\Omega)\right),$$

then from (29), we deduce that

$$v_l^{m,n}\to v^{m,n} \quad \text{in } L^1\left(Q_T\right) \text{ and a.e. in } Q_T$$

for a subsequence. □

Remark 3.13. *First, as a consequence of this Lemma, we have*

$$\left\|v^{m,n}\right\|_{L^\infty(Q_T)}\le\max\left(m^2,n^2\right). \quad (30)$$

Indeed, since $v_l^{m,n}\to v^{m,n}$ a.e. in Q_T, then by (22), we deduce (30). Moreover, recall that for all $\lambda>0$ the function $u_{\lambda,l}^{m,n}$ is the unique strong solution to (19) and if we set $u_{\lambda,l}^{m,n}:=b_l\left(v_{\lambda,l}^{m,n}\right)$, then we have $v_{\lambda,l}^{m,n}=b_l^{-1}\left(u_{\lambda,l}^{m,n}\right)$. From the diagonal principle, there exists a subsequence $\left(\lambda(l)\right)_l$ with $\lambda(l)\to 0$ as $l\to\infty$ such that setting $\tilde{v}_l^{m,n}:=v_{\lambda(l),l}^{m,n}$, we have the following convergence results for $l\to\infty$

$$\tilde{v}_l^{m,n}\to v^{m,n} \quad \text{in } L^1\left(Q_T\right) \text{ and a.e. in } Q_T \quad (31)$$

$$b_l\left(v_0^{m,n}\right)\to b\left(v_0^{m,n}\right) \quad \text{in } L^1\left(Q_T\right) \quad (32)$$

$$b_l\left(\tilde{v}_l^{m,n}\right)\to b\left(v^{m,n}\right) \quad \text{in } L^1\left(Q_T\right) \quad (33)$$

$$k_l\to k \quad \text{in } L^1\left(Q_T\right). \quad (34)$$

In addition, note that, by Lemma 3.9 and the growth condition (H4), the sequence $\left(a\left(\cdot,DT_k\left(\tilde{v}_l^{m,n}\right)\right)\right)_{l>0}$ is bounded in $\left(L^{p'}\left(Q_T\right)\right)^N$, So, there exist $\Phi_K^{m,n}\in\left(L^{p'}\left(Q_T\right)\right)^N$ and subsequence, still denotes by $\left(a\left(\cdot,DT_K\left(\tilde{v}_l^{m,n}\right)\right)\right)_{l>0}$, such that

$$a\left(\cdot,DT_K\left(\tilde{v}_l^{m,n}\right)\right)\rightharpoonup\Phi_K^{m,n} \quad \text{weakly in } \left(L^{p'}\left(Q_T\right)\right)^N. \quad (35)$$

In the next steps we will show that $div\left(\chi_K^{m,n}\right)=div\left(a\left(\cdot,DT_K\left(v^{m,n}\right)\right)\right)$ in $\mathcal{D}'\left(Q_T\right)$ for all $K>0$.

***Step* 3:** In this step, we prove the following lemma, which is the main estimate in the arguments that will be developed in step 4. The idea of the proof is the same as in ([17], p. 13-19). Let $h_i:\mathbb{R}\to\mathbb{R}$ be defined by $h_i(r):=\min\left(\left(i+1-|r|\right)^+,1\right)$ for each $r\in\mathbb{R}$.

Lemma 3.14. *For any* $K\ge 0$, $m,n>0$, *the subsequence* $\left(\tilde{v}_l^{m,n}\right)_l$ *defined above in Step* 2 *satisfies*

$$\liminf_{i\to\infty}\liminf_{\mu\to\infty}\liminf_{l\to\infty}\int_{Q_\tau}\partial_t\left(k_l*\left[b_l\left(\tilde{v}_l^{m,n}\right)-b_l\left(v_0^{m,n}\right)\right]\right)\times\left[T_K\left(\tilde{v}_l^{m,n}\right)-h_i\left(\tilde{v}_l^{m,n}\right)T_K\left(v^{m,n}\right)_\mu\right]\mathrm{d}x\mathrm{d}t\geq 0 \tag{36}$$

for almost any $\tau\in(0,T)$ with $Q_\tau:=(0,\tau)\times\Omega$.

Here, $T_K\left(v^{m,n}\right)_\mu\in L^p\left(0,T;W_0^{1,p}(\Omega)\right)$ is the time regularization of $T_K\left(v^{m,n}\right)$ introduced in Definition 2.2.

Proof. We know that $T_K\left(\tilde{v}_l^{m,n}\right)-h_i\left(\tilde{v}_l^{m,n}\right)T_K\left(v^{m,n}\right)_\mu$ converges *a.e.* on Q_T towards $T_K\left(v^{m,n}\right)-h_i\left(v^{m,n}\right)T_K\left(v^{m,n}\right)_\mu$ as $l\to\infty$ and from Lemma 2.3 in [16], it is uniformly bounded by $2K$. Since convergences (31), (32) and (34) holds true, then using the properties of $\left(T_K\left(v^{m,n}\right)\right)_\mu$, h_i and Lebesgue's theorem, we get that

$$\lim_{i\to\infty}\lim_{\mu\to 0}\lim_{l\to\infty}\int_{Q_\tau}k_l b_l\left(v_0^{m,n}\right)\left[T_K\left(\tilde{v}_l^{m,n}\right)-h_i\left(\tilde{v}_l^{m,n}\right)T_K\left(v^{m,n}\right)_\mu\right]\mathrm{d}x\mathrm{d}t=0$$

for any $\tau\in(0,T)$. So, it remains to show that

$$\liminf_{i\to\infty}\liminf_{\mu\to\infty}\liminf_{l\to\infty}\int_{Q_\tau}\partial_t\left(k_l*b_l\left(\tilde{v}_l^{m,n}\right)\right)\left[T_K\left(\tilde{v}_l^{m,n}\right)-h_i\left(\tilde{v}_l^{m,n}\right)T_K\left(v^{m,n}\right)_\mu\right]\mathrm{d}x\mathrm{d}t\geq 0$$

for almost any $\tau\in(0,T)$.

If we apply Lemma 2.3 on $\partial_t\left(k_l*b_l\left(\tilde{v}_l^{m,n}\right)\right)h_i\left(\tilde{v}_l^{m,n}\right)\left(T_K\left(v^{m,n}\right)\right)_\mu$, we obtain

$$\begin{aligned}
&-\int_{Q_\tau}\partial_t\left(k_l*b_l\left(\tilde{v}_l^{m,n}\right)\right)h_i\left(\tilde{v}_l^{m,n}\right)\left(T_K\left(v^{m,n}\right)\right)_\mu\\
&=-\int_{Q_\tau}\partial_t\left(k_l*\int_0^{\tilde{v}_l^{m,n}}h_i(\sigma)\mathrm{d}b_l(\sigma)\right)\left(T_K\left(v^{m,n}\right)\right)_\mu\mathrm{d}x\mathrm{d}t\\
&-\int_{Q_\tau}\left[h_i\left(\tilde{v}_l^{m,n}\right)b_l\left(\tilde{v}_l^{m,n}\right)-\int_0^{\tilde{v}_l^{m,n}}h_i(\sigma)\mathrm{d}b_l(\sigma)\right]k_l(t)\left(T_K\left(v^{m,n}\right)\right)_\mu\mathrm{d}x\mathrm{d}t\\
&-\int_{Q_\tau}\int_0^t\left[\int_{\tilde{v}_l^{m,n}(t)}^{\tilde{v}_l^{m,n}(t-s)}h_i(\sigma)\mathrm{d}b_l(\sigma)-h_i\left(\tilde{v}_l^{m,n}(t)\right)\left[b_l\left(\tilde{v}_l^{m,n}(t-s)-b_l\left(\tilde{v}_l^{m,n}(t)\right)\right)\right]\right]\\
&\times\left[-k_l'(s)\right]\mathrm{d}s\left(T_K\left(v^{m,n}\right)\right)_\mu\mathrm{d}x\mathrm{d}t\\
&:=-I_{l,\mu,i}^1-I_{l,\mu,i}^2-I_{l,\mu,i}^3.
\end{aligned}$$

For, $i\in\mathbb{N}$, we define the following functions

$$T_{i,i+1}^+(r):=\left(T_{i,i+1}(r)\right)^+\ \text{ and }\ T_{i,i+1}^-(r):=-\left(T_{i,i+1}(r)\right)^-$$

for every $r\in\mathbb{R}$. As, we observe that $h_i(r)=T_{i,i+1}^-(r)-T_{i,i+1}^+(r)+1$, then it follows that

$$\begin{aligned}
I_{l,\mu,i}^2&=\int_{Q_\tau}\left[T_{i,i+1}^-\left(\tilde{v}_l^{m,n}\right)b_l\left(\tilde{v}_l^{m,n}\right)-\int_0^{\tilde{v}_l^{m,n}}T_{i,i+1}^-(\sigma)\mathrm{d}b_l(\sigma)\right]k_l(t)\left(T_K\left(v^{m,n}\right)\right)_\mu\mathrm{d}x\mathrm{d}t\\
&\quad-\int_{Q_\tau}\left[T_{i,i+1}^+\left(\tilde{v}_l^{m,n}\right)b_l\left(\tilde{v}_l^{m,n}\right)+\int_0^{\tilde{v}_l^{m,n}}T_{i,i+1}^+(\sigma)\mathrm{d}b_l(\sigma)\right]k_l(t)\left(T_K\left(v^{m,n}\right)\right)_\mu\mathrm{d}x\mathrm{d}t
\end{aligned}$$

and

$$\begin{aligned}
I_{l,\mu,i}^3&=\int_{Q_\tau}\int_0^t\left[\int_{\tilde{v}_l^{m,n}(t)}^{\tilde{v}_l^{m,n}(t-s)}T_{i,i+1}^-(\sigma)\mathrm{d}b_l(\sigma)-T_{i,i+1}^-\left(\tilde{v}_l^{m,n}(t)\right)\left[b_l\left(\tilde{v}_l^{m,n}(t-s)\right)-b_l\left(\tilde{v}_l^{m,n}(t)\right)\right]\right]\\
&\times\left[-k_l'(s)\right]\mathrm{d}s\left(T_K\left(v^{m,n}\right)\right)_\mu\mathrm{d}x\mathrm{d}t
\end{aligned}$$

$$-\int_{Q_\tau}\int_0^t\left[\int_{\tilde{v}_l^{m,n}(t)}^{\tilde{v}_l^{m,n}(t-s)} T_{i,i+1}^+(\sigma)\mathrm{d}b_l(\sigma)-T_{i,i+1}^+\left(\tilde{v}_l^{m,n}(t)\right)\left[b_l\left(\tilde{v}_l^{m,n}(t-s)\right)-b_l\left(\tilde{v}_l^{m,n}(t)\right)\right]\right]$$
$$\times\left[-k_l'(s)\right]\mathrm{d}s\left(T_K\left(v^{m,n}\right)\right)_\mu \mathrm{d}x\mathrm{d}t.$$

Now, our objective is to show that

$$\limsup_{i\to\infty}\limsup_{\mu\to\infty}\limsup_{l\to\infty}\left(I_{l,\mu,i}^2+I_{l,\mu,i}^3\right)=0.$$

To do this end, we take $T_{i,i+1}\left(\tilde{v}_l^{m,n}\right)$ as a test function in (19).

We know that taking into account of $\tilde{v}_l^{m,n}=0$ on $\partial\Omega$, the divergence theorem gives

$$\int_0^\tau\int_\Omega F\left(\tilde{v}_l^{m,n}\right)\cdot DT_{i,i+1}\left(\tilde{v}_l^{m,n}\right)\mathrm{d}x\mathrm{d}s=0$$

Then, using coercivity condition (H3) and Lemma 2.5 in [16], we get that

$$\begin{aligned}&C\int_{Q_\tau}\left|DT_{i,i+1}\left(\tilde{v}_l^{m,n}\right)\right|^p\mathrm{d}x\mathrm{d}t+\int_{Q_\tau}\left[T_{i,i+1}\left(\tilde{v}_l^{m,n}\right)b_l\left(\tilde{v}_l^{m,n}\right)-\int_0^{\tilde{v}_l^{m,n}}T_{i,i+1}(\sigma)\mathrm{d}b_l(\sigma)\right]k_l(t)\mathrm{d}x\mathrm{d}t\\&+\int_{Q_\tau}\int_0^t\left[\int_{\tilde{v}_l^{m,n}(t)}^{\tilde{v}_l^{m,n}(t-s)}T_{i,i+1}(\sigma)\mathrm{d}b_l(\sigma)-T_{i,i+1}\left(\tilde{v}_l^{m,n}(t)\right)\left[b_l\left(\tilde{v}_l^{m,n}(t-s)\right)-b_l\left(\tilde{v}_l^{m,n}(t)\right)\right]\right]\\&\times\left[-k_l'(s)\right]\mathrm{d}s\mathrm{d}x\mathrm{d}t+\int_{Q_\tau}\psi^{m,n}\left(\tilde{v}_l^{m,n}\right)T_{i,i+1}\left(\tilde{v}_l^{m,n}\right)\mathrm{d}x\mathrm{d}t\\&\le\int_{Q_\tau\cap\left\{\left|\tilde{v}_l^{m,n}\right|>i\right\}}\left|f^{m,n}\right|\mathrm{d}x\mathrm{d}t+\int_{Q_\tau\cap\left\{\left|\tilde{v}_l^{m,n}\right|>i\right\}}k_l\left|b_l\left(v_0^{m,n}\right)\right|\mathrm{d}x\mathrm{d}t\end{aligned}\tag{37}$$

for any $\tau\in(0,T)$ where $C>0$ is a constant. As $\tilde{v}_l^{m,n}\to v^{m,n}$ a.e. in Q_T and by Lemma 3.9 $\left|v^{m,n}\right|<\infty$ a.e. in Q_T, then we get that

$$\limsup_{i\to\infty}\limsup_{l\to\infty}\left(\int_{Q_\tau\cap\left\{\left|\tilde{v}_l^{m,n}\right|>i\right\}}\left|f^{m,n}\right|\mathrm{d}x\mathrm{d}t+\int_{Q_\tau\cap\left\{\left|\tilde{v}_l^{m,n}\right|>i\right\}}k_l\left|b_l\left(v_0^{m,n}\right)\right|\mathrm{d}x\mathrm{d}t\right)=0.$$

Since all terms on the left-hand side in (37) are nonnegative, then it follows that

$$\limsup_{i\to\infty}\limsup_{l\to\infty}\int_{Q_\tau}\left|DT_{i,i+1}\left(\tilde{v}_l^{m,n}\right)\right|^p\mathrm{d}x\mathrm{d}t=0\tag{38}$$

$$\limsup_{i\to\infty}\limsup_{l\to\infty}\int_{Q_\tau}\left[T_{i,i+1}\left(\tilde{v}_l^{m,n}\right)b_l\left(\tilde{v}_l^{m,n}\right)-\int_0^{\tilde{v}_l^{m,n}}T_{i,i+1}(\sigma)\mathrm{d}b_l(\sigma)\right]k_l(t)\mathrm{d}x\mathrm{d}t=0\tag{39}$$

$$\begin{aligned}&\limsup_{i\to\infty}\limsup_{l\to\infty}\int_{Q_\tau}\int_0^t\left[\int_{\tilde{v}_l^{m,n}(t)}^{\tilde{v}_l^{m,n}(t-s)}T_{i,i+1}(\sigma)\mathrm{d}b_l(\sigma)\right.\\&\left.-T_{i,i+1}\left(\tilde{v}_l^{m,n}(t)\right)\left[b_l\left(\tilde{v}_l^{m,n}(t-s)\right)-b_l\left(\tilde{v}_l^{m,n}(t)\right)\right]\right]\left[-k_l'(s)\right]\mathrm{d}s\mathrm{d}x\mathrm{d}t=0\end{aligned}\tag{40}$$

for any $\tau\in(0,T)$. Noting that the above results hold true in case $T_{i,i+1}$ replaced by $T_{i,i+1}^+$ or $T_{i,i+1}^-$, we deduce that

$$\limsup_{i\to\infty}\limsup_{\mu\to\infty}\limsup_{l\to\infty}\left(I_{l,\mu,i}^2+I_{l,\mu,i}^3\right)=0$$

for almost every $\tau\in(0,T)$. Thus, it remains to show that

$$\liminf_{i\to\infty}\liminf_{\mu\to\infty}\liminf_{l\to\infty}\left(\int_{Q_\tau}\partial_t\left(k_l*b_l\left(\tilde{v}_l^{m,n}\right)\right)T_K\left(\tilde{v}_l^{m,n}\right)\mathrm{d}x\mathrm{d}t-I_{l,\mu,i}^1\right)\ge0$$

for almost every $\tau\in(0,T)$. Applying Lemma 2.5 in [16] and integrating over

Q_τ for some τ, we get

$$\int_{Q_\tau} \partial_t \left(k_l * b_l\left(\tilde{v}_l^{m,n}\right)\right) T_K\left(\tilde{v}_l^{m,n}\right) \mathrm{d}x\mathrm{d}t = \int_\Omega \left(k_l * \int_0^{\tilde{v}_l^{m,n}} T_K(\sigma)\mathrm{d}b_l(\sigma)\right)(\tau)\mathrm{d}x$$

$$\int_{Q_\tau}\left[T_K\left(\tilde{v}_l^{m,n}(t)\right)b_l\left(\tilde{v}_l^{m,n}(t)\right) - \int_0^{\tilde{v}_l^{m,n}(t)} T_K(\sigma)\mathrm{d}b_l(\sigma)\right]k_l(t)\mathrm{d}x\mathrm{d}t$$

$$+\int_{Q_\tau}\int_0^t\left[\int_{\tilde{v}_l^{m,n}(t)}^{\tilde{v}_l^{m,n}(t-s)} T_K(\sigma)\mathrm{d}b_l(\sigma) \tag{41}$$

$$-T_K\left(\tilde{v}_l^{m,n}(t)\right)\left[b_l\left(\tilde{v}_l^{m,n}(t-s)\right) - b_l\left(\tilde{v}_l^{m,n}(t)\right)\right]\right]\left[-k_l'(s)\right]\mathrm{d}s\mathrm{d}x\mathrm{d}t.$$

Now, our objective is to estimate limit-inf of (41) with $l \to \infty$. Since, $\tilde{v}_l^{m,n} \to v^{m,n}$ a.e. in Q_T, b is continuous on $\mathbb{R}$ and $d\left(b_l(\sigma) - b(\sigma)\right) = \frac{1}{l}d\sigma$, then it comes that

$$\int_0^{\tilde{v}_l^{m,n}} T_K(\sigma)\mathrm{d}b_l(\sigma) \to \int_0^{v^{m,n}} T_K(\sigma)\mathrm{d}b(\sigma) \quad \text{a.e. in } Q_T.$$

Moreover, as

$$b_l\left(\tilde{v}_l^{m,n}\right) \to b\left(v^{m,n}\right) \text{ in } L^1(Q_T) \text{ and } \left|\int_0^{\tilde{v}_l^{m,n}} T_K(\sigma)\mathrm{d}b_l(\sigma)\right| \le K\left|b_l\left(\tilde{v}_l^{m,n}\right)\right|,$$

then there exist at least a subsequence of $\left(b_l\left(\tilde{v}_l^{m,n}\right)\right)_l$, still denoted same way that $\left(b_l\left(\tilde{v}_l^{m,n}\right)\right)_l$, which is dominated by an $L^1(Q_T)$-function. So, from Lebesgue's theorem, we get

$$\int_0^{\tilde{v}_l^{m,n}} T_K(\sigma)\mathrm{d}b_l(\sigma)\mathrm{d}x \to \int_0^{v^{m,n}} T_K(\sigma)\mathrm{d}b(\sigma)\mathrm{d}x, \quad \text{in } L^1(Q_T)$$

and

$$\int_\Omega\int_0^{\tilde{v}_l^{m,n}} T_K(\sigma)\mathrm{d}b_l(\sigma)\mathrm{d}x \to \int_\Omega\int_0^{v^{m,n}} T_K(\sigma)\mathrm{d}b(\sigma)\mathrm{d}x, \quad \text{in } L^1(0,T).$$

Since $k_l \to k$ in $L^1(0,T)$, then using the Young inequality for convolution, it follows that (up to subsequence)

$$\int_\Omega\left(k_l * \int_0^{\tilde{v}_l^{m,n}} T_K(\sigma)\mathrm{d}b_l(\sigma)\right)\mathrm{d}x \to \int_\Omega\left(k * \int_0^{v^{m,n}} T_K(\sigma)\mathrm{d}b(\sigma)\right)\mathrm{d}x \tag{42}$$

in $L^1(0,T)$ and a.e. in $(0,T)$. Next, note that the nonnegativity of all terms in (21) implies that

$$\int_{Q_\tau}\left[T_K\left(\tilde{v}_l^{m,n}(t)\right)b_l\left(\tilde{v}_l^{m,n}(t)\right) - \int_0^{\tilde{v}_l^{m,n}(t)} T_K(\sigma)\mathrm{d}b_l(\sigma)\right]k_l(t)\mathrm{d}x\mathrm{d}t$$

$$+\int_{Q_\tau}\int_0^t\left[\int_{\tilde{v}_l^{m,n}(t)}^{\tilde{v}_l^{m,n}(t-s)} T_K(\sigma)\mathrm{d}b_l(\sigma)\right.$$

$$\left.-T_K\left(\tilde{v}_l^{m,n}(t)\right)\left(b_l\left(\tilde{v}_l^{m,n}(t-s)\right) - b_l\left(\tilde{v}_l^{m,n}(t)\right)\right)\right]\left[-k_l'(s)\right]\mathrm{d}s\mathrm{d}x\mathrm{d}t \le C$$

for any $\tau \in (0,T)$ and some constant $C = C(K, f, k, v_0, b) > 0$ independent of l, m and n. Since the kernel verifies assumptions (K1) and (K2), then the Lemma of Fatou gives

$$\left[T_K\left(v^{m,n}(t)\right)b\left(v^{m,n}(t)\right) - \int_0^{v^{m,n}(t)} T_K(\sigma)\mathrm{d}b(\sigma)\right]k \in L^1(Q_T) \tag{43}$$

and

$$\chi_{(0,t)}\left[\int_{v^{m,n}(t)}^{v^{m,n}(t-s)} T_K(\sigma)\mathrm{d}b(\sigma) - T_K\left(v^{m,n}(t)\right)\left[b\left(v^{m,n}(t-s)\right) - b\left(v^{m,n}(t)\right)\right]\right]\left[-k_l'(s)\right] \in L^1\left((0,T)\times Q_t\right) \quad (44)$$

for any $t\in(0,T)$. By convergences (42), (43) and (44), we can estimate lim-inf of (41) with $l\to\infty$. Thus, passing to the limit in (41) with $l\to\infty$, we obtain

$$\begin{aligned}
&\liminf_{l\to\infty}\int_{Q_\tau}\partial_t\left(k_l * b_l\left(\tilde{v}_l^{m,n}\right)\right)T_K\left(\tilde{v}_l^{m,n}\right)\mathrm{d}x\mathrm{d}t \geq \int_\Omega\left(k*\int_0^{v^{m,n}} T_K(\sigma)\mathrm{d}b(\sigma)\right)(\tau)\mathrm{d}x \\
&+\int_{Q_\tau}\left[T_K\left(v^{m,n}(t)\right)b\left(v^{m,n}(t)\right) - \int_0^{v^{m,n}(t)} T_K(\sigma)\mathrm{d}b(\sigma)\right]k(t)\mathrm{d}x\mathrm{d}t \\
&+\int_{Q_\tau}\int_0^t\left[\int_{v^{m,n}(t)}^{v^{m,n}(t-s)} T_K(\sigma)\mathrm{d}b(\sigma)\right. \\
&\left.-T_K\left(v^{m,n}(t)\right)\left[b\left(v^{m,n}(t-s)\right) - b\left(v^{m,n}(t)\right)\right]\right]\left[-k_l'(s)\right]\mathrm{d}s\mathrm{d}x\mathrm{d}t
\end{aligned} \quad (45)$$

for almost any $\tau\in(0,T)$. Next, we want to pass to the limit with $I^1_{l,\mu,i}$ as $l\to\infty$. As h_i is bounded and compact support, then we observe that

$$\int_0^{\tilde{v}_l^{m,n}} h_i(\sigma)\mathrm{d}b_l(\sigma) \in L^{p'}(Q_T)$$

and

$$\partial_t\left(k_l * \int_0^{\tilde{v}_l^{m,n}} h_i(\sigma)\mathrm{d}b_l(\sigma)\right) \in L^{p'}\left(0,T;W^{-1,p'}(\Omega)\right).$$

So, taking into account that $\left(T_K\left(v^{m,n}\right)\right)_\mu = J_\mu^{L^*}\left(T_K\left(v^{m,n}\right)\right)$, we obtain that

$$\begin{aligned}
I^1_{l,\mu,i} &= \int_{Q_\tau}\partial_t\left(gk_l * \int_0^{\tilde{v}_l^{m,n}} h_i(\sigma)\mathrm{d}b_l(\sigma)\right)\left(T_K\left(v^{m,n}\right)\right)_\mu \mathrm{d}x\mathrm{d}t \\
&= \left\langle L_{\lambda(l)}\int_0^{\tilde{v}_l^{m,n}} h_i(\sigma)\mathrm{d}b_l(\sigma),\, J_\mu^{L^*}\left(T_K\left(v^{m,n}\right)\right)\right\rangle_{L^{p'}\left(0,\tau;W^{-1,p'}(\Omega)\right)\times L^p\left(0,\tau;W^{1,p}(\Omega)\right)} \\
&= \left\langle L_\mu J_{\lambda(l)}^L\int_0^{\tilde{v}_l^{m,n}} h_i(\sigma)\mathrm{d}b_l(\sigma), T_K\left(v^{m,n}\right)\right\rangle_{L^{p'}\left(0,\tau;W^{-1,p'}(\Omega)\right)\times L^p\left(0,\tau;W^{1,p}(\Omega)\right)}
\end{aligned}$$

for all $\tau\in(0,T)$ where $L_{\lambda(l)}$ denotes the Yosida approximation of operator L as defined in subsection 2.3. Here, we used that $v\in D(L)$ involves $J_\mu^L Lv = LJ_\mu^L$ for $\mu>0$. In particular,

$$J_\mu^L L_{\lambda(l)}v = LJ_\mu^L J_{\lambda(l)}^L v = L_\mu J_{\lambda(l)}^L v$$

for any $v\in L^{p'}\left(0,\tau;W^{-1,p'}(\Omega)\right)$ and any $\mu,\lambda(l)>0$. Since, h_i is compactly supported, then we get that la sequence $\left(\int_0^{\tilde{v}_l^{m,n}} h_i(\sigma)\mathrm{d}b_l(\sigma)\right)_l$ is uniformly bounded. As,

$$\int_0^{\tilde{v}_l^{m,n}} h_i(\sigma)\mathrm{d}b_l(\sigma) \to \int_0^{v^{m,n}} h_i(\sigma)\mathrm{d}b(\sigma), \quad \text{e.a. in } Q_T,$$

then, from Lebesgue's theorem, it comes that

$$\int_0^{\tilde{v}_l^{m,n}} h_i(\sigma)\mathrm{d}b_l(\sigma) \to \int_0^{v^{m,n}} h_i(\sigma)\mathrm{d}b(\sigma), \quad \text{in } L^{p'}(Q_T)$$

as $l \to \infty$. $J^L_{\lambda(l)}$ being a bounded operator verifying $J^L_{\lambda(l)} w \to w$ for any $w \in L^{p'}\left(0,\tau; W^{-1,p'}(\Omega)\right)$ as $l \to \infty$, then by an application of the triangle inequality, it follows that

$$J^L_{\lambda(l)} \int_0^{\tilde{v}_l^{m,n}} h_i(\sigma) \mathrm{d}b_l(\sigma) \to \int_0^{v^{m,n}} h_i(\sigma) \mathrm{d}b(\sigma), \quad \text{in } L^{p'}\left(0,\tau; W^{-1,p'}(\Omega)\right)$$

as $l \to \infty$. Hence, the continuity of the Yosida approximation L_μ gives

$$\begin{aligned} \lim_{l\to\infty} I^1_{i,\mu,l} &= \left\langle L_\mu \int_0^{v^{m,n}} h_i(\sigma) \mathrm{d}b(\sigma), T_K\left(v^{m,n}\right) \right\rangle_{L^{p'}\left(0,\tau;W^{-1,p'}(\Omega)\right) \times L^p\left(0,\tau;W^{1,p}(\Omega)\right)} \\ &= \int_{Q_\tau} \partial_t \left(k_\mu * \int_0^{v^{m,n}} h_i(\sigma) \mathrm{d}b(\sigma) \right) T_K\left(v^{m,n}\right) \mathrm{d}x\mathrm{d}t \end{aligned}$$

for any $\tau \in (0,T)$. Now, using Lemma A5 in ([17], Appendix), we get that

$$\begin{aligned} &\int_{Q_\tau} \partial_t \left(k_\mu * \int_0^{v^{m,n}} h_i(\sigma) \mathrm{d}b(\sigma) \right) T_K\left(v^{m,n}\right) \mathrm{d}x\mathrm{d}t = \int_\Omega \left(k_\mu * \int_0^{v^{m,n}} T_K(\sigma) h_i(\sigma) \mathrm{d}b(\sigma) \right)(\tau) \mathrm{d}x \\ &+ \int_{Q_\tau} \left[T_K\left(v^{m,n}(t)\right) \int_0^{v^{m,n}(t)} h_i(\sigma) \mathrm{d}b(\sigma) - \int_0^{v^{m,n}} T_K(\sigma) h_i(\sigma) \mathrm{d}b(\sigma) \right] k_\mu(t) \mathrm{d}x\mathrm{d}t \\ &+ \int_{Q_\tau} \int_0^t \left[\int_{v^{m,n}(t)}^{v^{m,n}(t-s)} T_K(\sigma) h_i(\sigma) \mathrm{d}b(\sigma) - T_K\left(v^{m,n}(t)\right) \int_{v^{m,n}(t)}^{v^{m,n}(t-s)} h_i(\sigma) \mathrm{d}b(\sigma) \right] \left[-k'_\mu\right] \mathrm{d}s\mathrm{d}x\mathrm{d}t \end{aligned} \tag{46}$$

for almost every $\tau \in (0,T)$. As $k_\mu \to k$ in $L^1(0,T)$, then by Young's inequality, it follows that

$$\begin{aligned} &\int_\Omega \left(k_\mu * \int_0^{v^{m,n}} T_K(\sigma) h_i(\sigma) \mathrm{d}b(\sigma) \right)(\tau) \mathrm{d}x \\ &\xrightarrow{\mu\to 0} \int_\Omega \left(k * \int_0^{v^{m,n}} T_K(\sigma) h_i(\sigma) \mathrm{d}b(\sigma) \right)(\tau) \mathrm{d}x \end{aligned} \tag{47}$$

in $L^1(0,T)$ and passing to a subsequence if necessary, a.e. in $(0,T)$. In addition, by the properties of h_i, we have

$$\int_0^{v^{m,n}} T_K(\sigma) h_i(\sigma) \mathrm{d}b(\sigma) \to \int_0^{v^{m,n}} T_K(\sigma) \mathrm{d}b(\sigma)$$

a.e. in Q_T as $i \to \infty$. The function b being monotone, it follows that

$$0 \le \int_0^{v^{m,n}} T_K(\sigma) h_i(\sigma) \mathrm{d}b(\sigma) \le \int_0^{v^{m,n}} T_K(\sigma) \mathrm{d}b(\sigma),$$

a.e. in Q_T. As, the kernel k is nonnegative, then we have

$$0 \le k * \int_0^{v^{m,n}} T_K(\sigma) h_i(\sigma) \mathrm{d}b(\sigma) \le k * \int_0^{v^{m,n}} T_K(\sigma) \mathrm{d}b(\sigma) \in L^1(Q_T).$$

Hence, Lebesgue's theorem involves

$$\int_\Omega \left[k * \int_0^{v^{m,n}} T_K(\sigma) h_i(\sigma) \mathrm{d}b(\sigma) \right](\tau) \mathrm{d}x \to \int_\Omega \left[k * \int_0^{v^{m,n}} T_K(\sigma) \mathrm{d}b(\sigma) \right](\tau) \mathrm{d}x \tag{48}$$

in $L^1(0,T)$, and, extracting subsequence if necessary, a.e. in $(0,T)$. Next, note that the monotonicity of b, the condition $b(0) = 0$ and $0 \le h_i \le 1$ involves that

$$\begin{aligned} 0 &\le T_K\left(v^{m,n}(t)\right) \int_0^{v^{m,n}(t)} h_i(\sigma) \mathrm{d}b(\sigma) - \int_0^{v^{m,n}(t)} T_K(\sigma) h_i(\sigma) \mathrm{d}b(\sigma) \\ &\le T_K\left(v^{m,n}(t)\right) b\left(v^{m,n}(t)\right) - \int_0^{v^{m,n}(t)} T_K(\sigma) \mathrm{d}b(\sigma) \quad \text{a.e. in } Q_T. \end{aligned}$$

Moreover, the same arguments entails that

$$0 \le \int_{v^{m,n}(t)}^{v^{m,n}(t-s)} T_K(\sigma) h_i(\sigma) \mathrm{d}b(\sigma) - T_K\left(v^{m,n}(t)\right) \int_{v^{m,n}(t)}^{v^{m,n}(t-s)} h_i(\sigma) \mathrm{d}b(\sigma)$$
$$\le \int_{v^{m,n}(t)}^{v^{m,n}(t-s)} T_K(\sigma) \mathrm{d}b(\sigma) - T_K\left(v^{m,n}(t)\right)\left[b\left(v^{m,n}(t)\right) - b\left(v^{m,n}(t-s)\right)\right] \text{ a.e. in } (0,T)\times Q_T.$$

From, (43) and (44), we see that

$$\left[T_K\left(v^{m,n}(t)\right) b\left(v^{m,n}(t)\right) - \int_0^{v^{m,n}(t)} T_K(\sigma) \mathrm{d}b(\sigma)\right] k \in L^1(Q_T)$$

and

$$\chi_{(0,t)}\left[\int_{v^{m,n}(t)}^{v^{m,n}(t-s)} T_K(\sigma) \mathrm{d}b(\sigma) - T_K\left(v^{m,n}(t)\right)\left[b\left(v^{m,n}(t-s)\right) - b\left(v^{m,n}(t)\right)\right]\right]\left[-k_l'(s)\right]$$
$$\in L^1\left((0,T)\times Q_t\right)$$

for any $t \in (0,T)$ with $Q_t := (0,t)\times\Omega$. As, k_μ is non-negative, non-increasing and verifies assumptions (K1)-(K3), then by application of Lebesgue's theorem, we get that

$$\begin{aligned} &\lim_{i\to\infty}\lim_{\mu\to 0}\int_{Q_\tau}\left[T_K\left(v^{m,n}(t)\right)\int_0^{v^{m,n}(t)} h_i(\sigma)\mathrm{d}b(\sigma) - \int_0^{v^{m,n}(t)} T_K(\sigma) h_i(\sigma)\mathrm{d}b(\sigma)\right] k_\mu(t)\mathrm{d}x\mathrm{d}t \\ &= \int_{Q_\tau}\left[T_K\left(v^{m,n}(t)\right) b\left(v^{m,n}(t)\right) - \int_0^{v^{m,n}(t)} T_K(\sigma)\mathrm{d}b(\sigma)\right] k(t)\mathrm{d}x\mathrm{d}t \end{aligned} \tag{49}$$

for any $\tau \in (0,T)$ and using the same arguments as above we find

$$\begin{aligned} &\lim_{i\to\infty}\lim_{\mu\to 0}\int_{Q_\tau}\int_0^t\left[\int_{v^{m,n}(t)}^{v^{m,n}(t-s)} T_K(\sigma) h_i(\sigma)\mathrm{d}b(\sigma)\right. \\ &\left.-T_K\left(v^{m,n}(t)\right)\int_{v^{m,n}(t)}^{v^{m,n}(t-s)} h_i(\sigma)\mathrm{d}b(\sigma)\right]\left[-k_\mu'(s)\right]\mathrm{d}s\mathrm{d}x\mathrm{d}t \\ &= \int_{Q_\tau}\int_0^t\left[\int_{v^{m,n}(t)}^{v^{m,n}(t-s)} T_K(\sigma)\mathrm{d}b(\sigma)\right. \\ &\left.-T_K\left(v^{m,n}(t)\right)\left[b\left(v^{m,n}(t)\right) - b\left(v^{m,n}(t-s)\right)\right]\right]\left[-k'(s)\right]\mathrm{d}s\mathrm{d}x\mathrm{d}t \end{aligned} \tag{50}$$

for any $\tau \in (0,T)$. Hence, using the convergence results (48), (49), (50) and passing to the limit in equality (46), we deduce

$$\begin{aligned} &\lim_{i\to\infty}\lim_{\mu\to 0}\lim_{l\to\infty} I_{l,\mu,i}^1 = \int_\Omega\left[k * \int_0^{v^{m,n}} T_K(\sigma)\mathrm{d}b(\sigma)\right](\tau)\mathrm{d}x \\ &+\int_{Q_\tau}\left[T_K\left(v^{m,n}(t)\right) b\left(v^{m,n}(t)\right) - \int_0^{v^{m,n}(t)} T_K(\sigma)\mathrm{d}b(\sigma)\right] k(t)\mathrm{d}x\mathrm{d}t \\ &+\int_{Q_\tau}\int_0^t\left[\int_{v^{m,n}(t)}^{v^{m,n}(t-s)} T_K(\sigma)\mathrm{d}b(\sigma)\right. \\ &\left.-T_K\left(v^{m,n}(t)\right)\left[b\left(v^{m,n}(t)\right) - b\left(v^{m,n}(t-s)\right)\right]\right]\left[-k'(s)\right]\mathrm{d}s\mathrm{d}x\mathrm{d}t \end{aligned} \tag{51}$$

for almost every $\tau \in (0,T)$. Thus, from results (45) and (51), we get that

$$\liminf_{i\to\infty}\ \liminf_{\mu\to\infty}\ \liminf_{l\to\infty}\left(\int_{Q_\tau}\partial_t\left(k_l * b_l\left(\tilde{v}_l^{m,n}\right)\right) T_K\left(\tilde{v}_l^{m,n}\right)\mathrm{d}x\mathrm{d}t - I_{l,\mu,i}^1\right) \ge 0$$

for almost every $\tau \in (0,T)$ and the proof of the lemma 3.14 is completed.

square

***Step* 4:** In this step we identify the weak limit $\Phi_K^{m,n}$ in (35).

Lemma 3.15. *For fixed* $K \geq 0$, *we have*

$$\Phi_K^{m,n} = a\left(\cdot, DT_K\left(v^{m,n}\right)\right) \quad \text{a.e. in } Q_T. \tag{52}$$

But to prove this lemma, we need the following assertion:

Assertion 3.16. *Let* $\left(\tilde{v}_l^{m,n}\right)_l$ *be the sequence defined in Remark* 3.13 *and* $v^{m,n}$ *given by Lemma* 3.12. *Let* $\tau > 0$ *be fixed such that the estimate* (36) *hold. Then, we have* (*up to subsequence*) *that for every* $K \geq 0$

$$\limsup_{\mu\to 0} \limsup_{l\to\infty} \int_{Q_\tau} a\left(x, DT_K\left(\tilde{v}_l^{m,n}\right)\right) \cdot D\left[T_K\left(\tilde{v}_l^{m,n}\right) - T_K\left(v^{m,n}\right)_\mu\right] \mathrm{d}x\mathrm{d}t \leq 0.$$

***Proof of Assertion* 3.16:** We choose, $T_K\left(\tilde{v}_l^{m,n}\right) - h_i\left(\tilde{v}_l^{m,n}\right) T_K\left(v^{m,n}\right)_\mu$ as a test function in (19) and integrate over Q_τ. It comes that

$$\begin{aligned}
&\int_{Q_\tau} \partial_t\left(k_l * \left[b_l\left(\tilde{v}_l^{m,n}\right) - b_l\left(v_0^{m,n}\right)\right]\right) \times \left[T_K\left(\tilde{v}_l^{m,n}\right) - h_i\left(\tilde{v}_l^{m,n}\right) T_K\left(v^{m,n}\right)_\mu\right] \mathrm{d}x\mathrm{d}t \\
&+ \int_{Q_\tau} a\left(x, D\tilde{v}_l^{m,n}\right) \cdot D\left[T_K\left(\tilde{v}_l^{m,n}\right) - h_i\left(\tilde{v}_l^{m,n}\right) T_K\left(v^{m,n}\right)_\mu\right] \mathrm{d}x\mathrm{d}t \\
&+ \int_{Q_\tau} F\left(\tilde{v}_l^{m,n}\right) \cdot D\left[T_K\left(\tilde{v}_l^{m,n}\right) - h_i\left(\tilde{v}_l^{m,n}\right) T_K\left(v^{m,n}\right)_\mu\right] \mathrm{d}x\mathrm{d}t \\
&+ \int_{Q_\tau} \psi^{m,n}\left(\tilde{v}_l^{m,n}\right)\left[T_K\left(\tilde{v}_l^{m,n}\right) - h_i\left(\tilde{v}_l^{m,n}\right) T_K\left(v^{m,n}\right)_\mu\right] \mathrm{d}x\mathrm{d}t \\
&\leq \int_{Q_\tau} f^{m,n}\left[T_K\left(\tilde{v}_l^{m,n}\right) - h_i\left(\tilde{v}_l^{m,n}\right) T_K\left(v^{m,n}\right)_\mu\right] \mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{53}$$

Here, we used the Lemma 3.4. The boundary condition and divergence theorem yield

$$\int_{Q_\tau} F\left(\tilde{v}_l^{m,n}\right) \cdot D\left[T_K\left(\tilde{v}_l^{m,n}\right) - h_i\left(\tilde{v}_l^{m,n}\right) T_K\left(v^{m,n}\right)_\mu\right] \mathrm{d}x\mathrm{d}t = 0.$$

Moreover, from Lebesgue's dominated convergence theorem, we have

$$\lim_{i\to\infty} \lim_{\mu\to\infty} \lim_{l\to\infty} \int_{Q_\tau} f^{m,n}\left[T_K\left(\tilde{v}_l^{m,n}\right) - h_i\left(\tilde{v}_l^{m,n}\right) T_K\left(v^{m,n}\right)_\mu\right] \mathrm{d}x\mathrm{d}t = 0.$$

Next, note that if $\left|\tilde{v}_l^{m,n}\right| \geq i+1$, then $h_i\left(\tilde{v}_l^{m,n}\right) = 0$ and this case we have

$$\psi^{m,n}\left(\tilde{v}_l^{m,n}\right)\left[T_K\left(\tilde{v}_l^{m,n}\right) - h_i\left(\tilde{v}_l^{m,n}\right) T_K\left(v^{m,n}\right)_\mu\right] = \psi^{m,n}\left(\tilde{v}_l^{m,n}\right) T_K\left(\tilde{v}_l^{m,n}\right) \geq 0.$$

If, $\left|\tilde{v}_l^{m,n}\right| < i+1$, then $\left|\psi^{m,n}\left(\tilde{v}_l^{m,n}\right)\left[T_K\left(\tilde{v}_l^{m,n}\right) - h_i\left(\tilde{v}_l^{m,n}\right) T_K\left(v^{m,n}\right)_\mu\right]\right| \leq 2K(i+1)$. Moreover, since $h_i \to 1$ as $i \to \infty$, then we can apply Fatou's Lemma three times and we get

$$\begin{aligned}
&\liminf_{i\to\infty} \liminf_{\mu\to 0} \liminf_{l\to\infty} \int_{Q_\tau} \psi^{m,n}\left(\tilde{v}_l^{m,n}\right)\left[T_K\left(\tilde{v}_l^{m,n}\right) - h_i\left(\tilde{v}_l^{m,n}\right) T_K\left(v^{m,n}\right)_\mu\right] \mathrm{d}x\mathrm{d}t \\
&\geq \int_{Q_\tau} \liminf_{i\to\infty} \liminf_{\mu\to 0} \liminf_{l\to\infty} \left(\psi^{m,n}\left(\tilde{v}_l^{m,n}\right)\left[T_K\left(\tilde{v}_l^{m,n}\right) - h_i\left(\tilde{v}_l^{m,n}\right) T_K\left(v^{m,n}\right)_\mu\right]\right) \mathrm{d}x\mathrm{d}t = 0.
\end{aligned}$$

Hence, inequality (53) and (36) involves that

$$\limsup_{i\to\infty} \limsup_{\mu\to 0} \limsup_{l\to\infty} \int_{Q_\tau} a\left(x, D\tilde{v}_l^{m,n}\right) \cdot D\left[T_K\left(\tilde{v}_l^{m,n}\right) - h_i\left(\tilde{v}_l^{m,n}\right) T_K\left(v^{m,n}\right)_\mu\right] \mathrm{d}x\mathrm{d}t \leq 0. \tag{54}$$

Since, $|h'(r)|=1$ if $|r|\in(i,i+1)$ and $h_i'(r)=0$ if $|r|>i+1$ or $|r|<i$, it comes that

$$\begin{aligned}&\int_{Q_\tau} a\left(x,D\tilde{v}_l^{m,n}\right)\cdot D\left[\left(h_i\left(\tilde{v}_l^{m,n}\right)-1\right)T_K\left(\tilde{v}_l^{m,n}\right)\right]\mathrm{d}x\mathrm{d}t\\ &\le\int_{Q_\tau}\left(h_i\left(\tilde{v}_l^{m,n}\right)-1\right)a\left(x,D\tilde{v}_l^{m,n}\right)\cdot DT_K\left(\tilde{v}_l^{m,n}\right)\mathrm{d}x\mathrm{d}t\\ &\quad+\int_{Q_\tau\cap\left\{i<\left|\tilde{v}_l^{m,n}\right|<i+1\right\}}\left|a\left(x,D\tilde{v}_l^{m,n}\right)\cdot DT_K\left(\tilde{v}_l^{m,n}\right)\right|\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Observe that, if $i>k$ then, almost everywhere on Q_τ, either $h_i\left(\tilde{v}_l^{m,n}\right)-1=0$ or $DT_K\left(\tilde{v}_l^{m,n}\right)=0$, so the first integral on the right-hand side equals zero in this case. Applying the growth condition (H4), Höder's inequality and Young's inequality, we get that

$$\int_{Q_\tau\cap\left\{i<\left|\tilde{v}_l^{m,n}\right|<i+1\right\}}\left|a\left(x,D\tilde{v}_l^{m,n}\right)\cdot DT_K\left(\tilde{v}_l^{m,n}\right)\right|\mathrm{d}x\mathrm{d}t\le C\int_{Q_\tau\cap\left\{i<\left|\tilde{v}_l^{m,n}\right|<i+1\right\}}\left|DT_K\left(\tilde{v}_l^{m,n}\right)\right|^p\mathrm{d}x\mathrm{d}t$$

with $C=C(g,K)>0$. As a consequence, (38) entails that

$$\limsup_{i\to\infty}\limsup_{l\to\infty}\int_{Q_\tau} a\left(x,D\tilde{v}_l^{m,n}\right)\cdot D\left[\left(h_i\left(\tilde{v}_l^{m,n}\right)-1\right)T_K\left(\tilde{v}_l^{m,n}\right)\right]\mathrm{d}x\mathrm{d}t\le 0. \tag{55}$$

From, results (54) and (55), we obtain

$$\limsup_{i\to\infty}\limsup_{\mu\to 0}\limsup_{l\to\infty}\int_{Q_\tau} a\left(x,D\tilde{v}_l^{m,n}\right)\cdot D\left[h_i\left(\tilde{v}_l^{m,n}\right)\left(T_K\left(\tilde{v}_l^{m,n}\right)-T_K\left(v^{m,n}\right)_\mu\right)\right]\mathrm{d}x\mathrm{d}t\le 0. \tag{56}$$

The growth condition (H4), Lemma 2.3 of [16] and Höder's inequality involve the existence of a constant $C=C(g,k)$ such that

$$\begin{aligned}&-\int_{Q_\tau} h_i'\left(\tilde{v}_l^{m,n}\right)a\left(x,D\tilde{v}_l^{m,n}\right)\cdot D\tilde{v}_l^{m,n}\left(T_K\left(\tilde{v}_l^{m,n}\right)-T_K\left(v^{m,n}\right)_\mu\right)\mathrm{d}x\mathrm{d}t\\ &\le\int_{Q_\tau\cap\left\{i<\left|\tilde{v}_l^{m,n}\right|<i+1\right\}}\left|D\tilde{v}_l^{m,n}\right|^p\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Therefore, (38) involves that

$$\limsup_{i\to\infty}\limsup_{\mu\to 0}\limsup_{l\to\infty}\left(-\int_{Q_\tau} h_i'\left(\tilde{v}_l^{m,n}\right)a\left(x,D\tilde{v}_l^{m,n}\right)\cdot D\tilde{v}_l^{m,n}\left(T_K\left(\tilde{v}_l^{m,n}\right)-T_K\left(v^{m,n}\right)_\mu\right)\mathrm{d}x\mathrm{d}t\right)\le 0.$$

So, by (56), it follows that

$$\limsup_{i\to\infty}\limsup_{\mu\to 0}\limsup_{l\to\infty}\int_{Q_\tau} h_i\left(\tilde{v}_l^{m,n}\right)a\left(x,D\tilde{v}_l^{m,n}\right)\cdot D\left[T_K\left(\tilde{v}_l^{m,n}\right)-T_K\left(v^{m,n}\right)_\mu\right]\mathrm{d}x\mathrm{d}t\le 0. \tag{57}$$

As, $h_i\left(\tilde{v}_l^{m,n}\right)=0$, if $\left|\tilde{v}_l^{m,n}\right|>i+1$, then we have

$$\begin{aligned}&\int_{Q_\tau\cap\left\{\left|\tilde{v}_l^{m,n}\right|>K\right\}} h_i\left(\tilde{v}_l^{m,n}\right)a\left(x,D\tilde{v}_l^{m,n}\right)\cdot DT_K\left(v^{m,n}\right)_\mu\mathrm{d}x\mathrm{d}t\\ &=\int_{Q_\tau\cap\left\{\left|\tilde{v}_l^{m,n}\right|>K\right\}} h_i\left(\tilde{v}_l^{m,n}\right)a\left(x,DT_{i+1}\left(\tilde{v}_l^{m,n}\right)\right)\cdot DT_K\left(v^{m,n}\right)_\mu\mathrm{d}x\mathrm{d}t\\ &=\int_{Q_\tau\cap\left\{\left|\tilde{v}_l^{m,n}\right|>K\right\}\cap\left\{\left|v^{m,n}\right|\neq K\right\}} h_i\left(\tilde{v}_l^{m,n}\right)a\left(x,DT_{i+1}\left(\tilde{v}_l^{m,n}\right)\right)\cdot DT_K\left(v^{m,n}\right)_\mu\mathrm{d}x\mathrm{d}t\\ &\quad+\int_{Q_\tau\cap\left\{\left|\tilde{v}_l^{m,n}\right|>K\right\}\cap\left\{\left|v^{m,n}\right|=K\right\}} h_i\left(\tilde{v}_l^{m,n}\right)a\left(x,DT_{i+1}\left(\tilde{v}_l^{m,n}\right)\right)\cdot DT_K\left(v^{m,n}\right)_\mu\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{58}$$

Since $\chi_{\left\{\left|\tilde{v}_l^{m,n}\right|>K\right\}\cap\left\{\left|v^{m,n}\right|\neq K\right\}}\to\chi_{\left\{\left|v^{m,n}\right|>K\right\}\cap\left\{\left|v^{m,n}\right|\neq K\right\}}=\chi_{\left\{\left|v^{m,n}\right|>K\right\}}$ a.e. in Q_τ, so applying Lebesgue's theorem obtain that

$$\chi_{\{|\tilde{v}_l^{m,n}|>K\}\cap\{|v^{m,n}|\neq K\}} h_i\left(\tilde{v}_l^{m,n}\right) DT_K\left(v^{m,n}\right)_\mu \to \chi_{\{|v^{m,n}|>K\}} h_i\left(v^{m,n}\right) DT_K\left(v^{m,n}\right)_\mu \text{ in } \left(L^p\left(Q_\tau\right)\right)^N$$

as $l\to\infty$. We see also that passing to a subsequence of l if necessary, we find that

$a\left(x, DT_{i+1}\left(\tilde{v}_l^{m,n}\right)\right) \rightharpoonup \Phi_{i+1}^{m,n}$ in $\left(L^{p'}\left(Q_\tau\right)\right)^N$ for some $\Phi_{i+1}^{m,n} \in \left(L^{p'}\left(Q_\tau\right)\right)^N$. Thus, it follows that the first integral on the right hand side of (58) converges to

$$\int_{Q_\tau\cap\{|v^{m,n}|>K\}} h_i\left(v^{m,n}\right)\Phi_{i+1}^{m,n}\cdot DT_K\left(v^{m,n}\right)_\mu \mathrm{d}x\mathrm{d}t$$

as $l\to\infty$. We see also that $\left(\chi_{\{|\tilde{v}_l^{m,n}|>K\}} a\left(x, DT_{i+1}\left(\tilde{v}_l^{m,n}\right)\right)\right)_{l}$ is bounded in $\left(L^{p'}\left(Q_\tau\cap\{|v^{m,n}|=K\}\right)\right)^N$. So, there exists subsequence of l, still denoted the same away, such that in

$$\chi_{\{|\tilde{v}_l^{m,n}|>K\}} a\left(x, DT_{i+1}\left(\tilde{v}_l^{m,n}\right)\right) \rightharpoonup \tilde{\Phi}_{i+1}^{m,n} \quad \text{in } \left(L^{p'}\left(Q_\tau\cap\{|v^{m,n}|=K\}\right)\right)^N$$

for some $\tilde{\Phi}_{i+1}^{m,n} \in \left(L^{p'}\left(Q_\tau\cap\{|v^{m,n}|=K\}\right)\right)^N$. Thus, we get also that the second integral on the right hand side of (58) converges to

$$\int_{Q_\tau\cap\{|v^{m,n}|=K\}} h_i\left(v^{m,n}\right)\tilde{\Phi}_{i+1}^{m,n}\cdot DT_K\left(v^{m,n}\right)_\mu \mathrm{d}x\mathrm{d}t$$

as $l\to\infty$. Moreover, since $T_K\left(v^{m,n}\right)_\mu \to T_K\left(v^{m,n}\right)$ in $L^p\left(0,T;W_0^{1,p}(\Omega)\right)$ as $\mu\to\infty$, then taking into account that $DT_K\left(v^{m,n}\right)=0$ on $\{|v^{m,n}|>K\}$, it follows that

$$\begin{aligned}
&\lim_{\mu\to 0}\lim_{l\to\infty}\int_{Q_\tau\cap\{|\tilde{v}_l^{m,n}|>K\}} h_i\left(\tilde{v}_l^{m,n}\right) a\left(x, D\tilde{v}_l^{m,n}\right)\cdot DT_K\left(v^{m,n}\right)_\mu \mathrm{d}x\mathrm{d}t\\
&=\int_{Q_\tau\cap\{|v^{m,n}|>K\}} h_i\left(v^{m,n}\right)\Phi_{i+1}^{m,n}\cdot DT_K\left(v^{m,n}\right)\mathrm{d}x\mathrm{d}t\\
&\quad+\int_{Q_\tau\cap\{|v^{m,n}|=K\}} h_i\left(v^{m,n}\right)\tilde{\Phi}_{i+1}^{m,n}\cdot DT_K\left(v^{m,n}\right)\mathrm{d}x\mathrm{d}t\\
&=0.
\end{aligned}$$

Hence, the inequality (57) is equivalent to

$$\limsup_{i\to\infty}\limsup_{\mu\to 0}\limsup_{l\to\infty}\int_{Q_\tau} h_i\left(\tilde{v}_l^{m,n}\right) a\left(x, DT_K\left(\tilde{v}_l^{m,n}\right)\right)\cdot D\left[T_K\left(\tilde{v}_l^{m,n}\right)-T_K\left(v^{m,n}\right)_\mu\right]\mathrm{d}x\mathrm{d}t \le 0.$$

As, if $i>K$, we have $h_i\left(\tilde{v}_l^{m,n}\right)=0$ on $\{|\tilde{v}_l^{m,n}|\le K\}$, then it follows that

$$\limsup_{\mu\to 0}\limsup_{l\to\infty}\int_{Q_\tau} a\left(x, DT_K\left(\tilde{v}_l^{m,n}\right)\right)\cdot D\left[T_K\left(\tilde{v}_l^{m,n}\right)-T_K\left(v^{m,n}\right)_\mu\right]\mathrm{d}x\mathrm{d}t \le 0$$

which conclude the proof of Assertion refmonotonie. □

***Proof of Lemma* 3.15**: Since $DT_K\left(v^{m,n}\right)_\mu \to DT_K\left(v^{m,n}\right)$ in $\left(L^P\left(Q_T\right)\right)^N$ as $\mu\to 0$ and $a\left(\cdot, DT_K\left(\tilde{v}_l^{m,n}\right)\right) \rightharpoonup \Phi_K^{m,n}$ in $\left(L^{P'}\left(Q_T\right)\right)^N$ as $l\to\infty$, then by Assertion 3.16, we obtain that

$$\limsup_{l\to\infty}\int_{Q_\tau} a\left(x, DT_K\left(\tilde{v}_l^{m,n}\right)\right)\cdot DT_K\left(\tilde{v}_l^{m,n}\right)\mathrm{d}x\mathrm{d}t \le \int_{Q_\tau}\Phi_K^{m,n}\cdot DT_K\left(v^{m,n}\right)\mathrm{d}x\mathrm{d}t. \quad (59)$$

As, $T_K\left(\tilde{v}_l^{m,n}\right)$ weakly converges to $T_K\left(v^{m,n}\right)$ in $L^p\left(0,T;W^{1,p}(\Omega)\right)$, we deduce that

$$\limsup_{l\to\infty}\int_{Q_\tau}\left[a\left(x,DT_K\left(\tilde{v}_l^{m,n}\right)\right)-a\left(x,DT_K\left(v^{m,n}\right)\right)\right]\cdot D\left(T_K\left(\tilde{v}_l^{m,n}\right)-T_K\left(v^{m,n}\right)\right)\mathrm{d}x\mathrm{d}t\le 0. \quad (60)$$

Thus, using the monotony assumption (H2) and Minty's monotonicity argument, we get

$$\Phi_K^{m,n}=a\left(\cdot,DT_K\left(v^{m,n}\right)\right)\quad \text{in } \left(L^{P'}\left(Q_\tau\right)\right)^N.$$

Since, this is true for almost every $\tau\in(0,T)$, then we have shown that

$$\Phi_K^{m,n}=a\left(\cdot,DT_K\left(v^{m,n}\right)\right)\quad \text{in } \left(L^{P'}\left(Q_T\right)\right)^N. \quad \square$$

-

***Step* 5:** In this step, $v^{m,n}$ is shown to satisfy (11). The idea of the proof is the same in [[17], p. 22-23] and [[16], p. 493-495]. Let $S\in\mathscr{P}$, $\xi\ge 0$ and $\varphi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$. Using $S\left(\tilde{v}_l^{m,n}-\varphi\right)\xi$ as a test function in (19), integrating over Q_T and applying the Lemma 2.6 of [16], we obtain

$$\begin{aligned}&-\int_{Q_T}\left[k_{1,l,j}*\int_{v_0^{m,n}}^{\tilde{v}_l^{m,n}}S(\sigma-\varphi)\mathrm{d}b_l(\sigma)\right]\xi_t(t)\mathrm{d}x\mathrm{d}t\\&+\int_{Q_T}\partial_t\left[k_{2,l,j}*\left(b_l\left(\tilde{v}_l^{m,n}\right)-b_l\left(v_0^{m,n}\right)\right)\right](t)S\left(\tilde{v}_l^{m,n}(t,\cdot)-\varphi\right)\xi(t)\mathrm{d}x\mathrm{d}t\\&+\int_{Q_T}\left(a\left(x,D\tilde{v}_l^{m,n}\right)+F\left(\tilde{v}_l^{m,n}\right)\right)\cdot DS\left(\tilde{v}_l^{m,n}-\varphi\right)\xi\mathrm{d}x\mathrm{d}t\\&+\int_{Q_T}\psi^{m,n}\left(\tilde{v}_l^{m,n}\right)S\left(\tilde{v}_l^{m,n}-\varphi\right)\xi\mathrm{d}x\mathrm{d}t\le\int_{Q_T}f^{m,n}S\left(\tilde{v}_l^{m,n}-\varphi\right)\xi\mathrm{d}x\mathrm{d}t.\end{aligned} \quad (61)$$

Note that

$$\begin{aligned}&F\left(\tilde{v}_l^{m,n}\right)\cdot DS\left(\tilde{v}_l^{m,n}-\varphi\right)\\&=S'\left(T_K\left(\tilde{v}_l^{m,n}\right)-\varphi\right)F\left(T_K\left(\tilde{v}_l^{m,n}\right)\right)\cdot D\left(T_K\left(\tilde{v}_l^{m,n}\right)-\varphi\right),\quad \text{a.e. in } Q_T.\end{aligned}$$

Due to $\varphi\in W_0^{1,p}(\Omega)$, the weak convergence $T_K\left(\tilde{v}_l^{m,n}\right)\rightharpoonup T_K\left(v^{m,n}\right)$ in $L^p\left(0,T;W_0^{1,p}(\Omega)\right)$ and a.e. in Q_T, we deduce that

$$D\left(T_K\left(\tilde{v}_l^{m,n}\right)-\varphi\right)\rightharpoonup D\left(T_K\left(v^{m,n}\right)-\varphi\right)\quad \text{weakly in } \left(L^p\left(Q_T\right)\right)^N$$

as $l\to\infty$, while $S'\left(T_K\left(\tilde{v}_l^{m,n}\right)-\varphi\right)F\left(T_K\left(\tilde{v}_l^{m,n}\right)\right)$ is uniformly bounded with respect to l and converges a.e. in Q_T to $S'\left(T_K\left(v^{m,n}\right)-\varphi\right)F\left(T_K\left(v^{m,n}\right)\right)$. As a consequence, it follows that for $1\le q<\infty$

$$S'\left(T_K\left(\tilde{v}_l^{m,n}\right)-\varphi\right)F\left(T_K\left(\tilde{v}_l^{m,n}\right)\right)\to S'\left(T_K\left(v^{m,n}\right)-\varphi\right)F\left(T_K\left(v^{m,n}\right)\right)$$

strongly in $L^q\left(Q_T\right)$ and

$$F\left(T_K\left(\tilde{v}_l^{m,n}\right)\right)\cdot DS\left(T_K\left(\tilde{v}_l^{m,n}\right)-\varphi\right)\xrightarrow{l\to\infty}F\left(T_K\left(v^{m,n}\right)\right)\cdot DS\left(T_K\left(v^{m,n}\right)-\varphi\right)$$

weakly in $L^1\left(Q_T\right)$. So,

$$\int_0^T\int_\Omega F\left(\tilde{v}_l^{m,n}\right)\cdot DS\left(\tilde{v}_l^{m,n}-\varphi\right)\xi\mathrm{d}x\mathrm{d}t\xrightarrow{l\to\infty}\int_0^T\int_\Omega F\left(v^{m,n}\right)\cdot DS\left(v^{m,n}-\varphi\right)\xi\mathrm{d}x\mathrm{d}t.$$

Thus, using the same argument as in the proof of Lemma 4.3. of [17], we can pass to the limit inferior with $l\to\infty$ on both sides of the inequality (61) and we find that

$$
\begin{aligned}
&-\int_{Q_T}\left[k_{1,j}*\int_{v_0^{m,n}}^{v^{m,n}} S(\sigma-\varphi)\mathrm{d}b(\sigma)\right]\xi_t(t)\mathrm{d}x\mathrm{d}t\\
&+\int_{Q_T}\partial_t\left[k_{2,j}*\left(b\left(v^{m,n}\right)-b\left(v_0^{m,n}\right)\right)\right](t)S\left(v^{m,n}(t,\cdot)-\varphi\right)\xi(t)\mathrm{d}x\mathrm{d}t\\
&+\int_{Q_T}\left(a\left(x,Dv^{m,n}\right)+F\left(v^{m,n}\right)\right)\cdot DS\left(v^{m,n}-\varphi\right)\xi\mathrm{d}x\mathrm{d}t\\
&+\int_{Q_T}\psi^{m,n}\left(v^{m,n}\right)S\left(v^{m,n}-\varphi\right)\xi\mathrm{d}x\mathrm{d}t\le\int_{Q_T}f^{m,n}S\left(v^{m,n}-\varphi\right)\xi\mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{62}
$$

Hence, $v^{m,n}$ is an entropy solution of $(EP)^{b,F,k,\psi^{m,n}}_{f^{m,n},v_0^{m,n}}$. With this last step the proof of Proposition 3.7 is achieved. □

In the next subsection, we will show that the perturbed term $\psi^{m,n}\to 0$ and there exists some measurable function $v:Q_T\to\mathbb{R}$ such that $v^{m,n}\to v$ a.e. in Q_T as $m,n\to\infty$. To this end, in a first step, we state a comparison result which is main tool for the conclusion of the proof of Theorem 3.3.

3.4. A Comparison Principle and Conclusion of the Proof of Theorem 3.3

***Step* 1:** A comparison principle

Lemma 3.17. *Let* $v_0,\tilde{v}_0\in L^\infty(\Omega)$, $f,\tilde{f}\in L^\infty(Q_T)$, $\psi,\tilde{\psi}:\mathbb{R}\to\mathbb{R}$ *be continuous, strictly increasing functions with* $\psi(0)=\tilde{\psi}(0)=0$ *and let* $v,\tilde{v}\in L^\infty(Q_T)$ *be entropy solution of* $(EP)^{b,f,k,\psi}_{f,v_0}$ *and* $(EP)^{b,F,k,\tilde{\psi}}_{\tilde{f},\tilde{v}_0}$ *respectively. If,*

$v_0\le\tilde{v}_0$ a.e. in Ω

$f\le\tilde{f}$ a.e. in Q_T

and

$\tilde{\psi}(r)\le\psi(r)$ for all $r\in\mathbb{R}$

then

$v\le\tilde{v}$ a.e. in Q_T.

Proof. The proof of this Lemma follows the same lines as the proof of the Theorem 5.1 of [17] and is omitted here in detail. It is based on Kruzhkov's doubling of variable technique. In our particular case, we only have to double the time variables: Let t,s denote two variables in $[0,T)$. We write the t variable in the weak formulation of $(EP)^{b,f,k,\psi}_{f,v_0}$ and the s variable in the weak formulation of $(EP)^{b,F,k,\tilde{\psi}}_{\tilde{f},\tilde{v}_0}$. Moreover, for $\phi\in D([0,T))$, $\phi\ge 0$ and $(\rho_p)_p$ a sequence of mollifiers in $\mathbb{R}$ with Supp $\rho_p\subset\left]-\frac{1}{p};\frac{1}{p}\right[$. Next, we set $\xi_p(t,s):=\phi(t)\rho_p(t-s)$ and for $L>0$, we still pose on the one hand $S=T_L\left(v(t,x)-\tilde{v}(t,x)\right)^+$ and the other hand $S=-T_L\left(\tilde{v}(s,x)-v(t,x)\right)^-$. Now, we use $\frac{1}{L}\xi_pT_L\left(v(t,x)-\tilde{v}(t,x)\right)^+$ and $-\frac{1}{L}\xi_pT_L\left(\tilde{v}(s,x)-v(t,x)\right)^-$, respectively as test functions in the definition of entropy solutions. There is no problem to pass to the limit with $L\to 0$ in the diffusion and the convection term because F is assumed to be locally Lipschitz continuous. Thus, proceeding with the arguments similar to those of Theorem 5.1 of [17], we get that

$$\left(\tilde{\psi}(v)-\tilde{\psi}(\tilde{v})\right)^{+}=0 \quad \text{a.e. in } Q_T.$$

It comes that $\tilde{\psi}(v)\le\tilde{\psi}(\tilde{v})$ a.e. in Q_T. As $\tilde{\psi}$ is strictly increasing, then it follows that

$$v\le\tilde{v} \quad \text{a.e. in } Q_T. \quad \square$$

***Step* 2:** Approximation procedure.

By Proposition 3.7, the problem $(EP)^{b,F,k,\psi^{m,n}}_{f^{m,n},v_0^{m,n}}$ has a entropy solution for all $m,n\in\mathbb{N}^*$. We have $\psi^{m,n}\ge\psi^{m+1,n}$ for all $n\in\mathbb{N}^*$ and $\psi^{m,n}\le\psi^{m,n+1}$ for all $m\in\mathbb{N}^*$ on $\mathbb{R}$. From Lemma 3.17, it follows that $v^{m,n}\le v^{m+1,n}$ and $v^{m,n+1}\le v^{m,n}$ almost everywhere in Q_T for all $m,n\in\mathbb{N}^*$. So there exist measurable functions $\tilde{v}^n: Q_T\to\mathbb{R}\cup\{+\infty\}$, $v: Q_T\to\overline{\mathbb{R}}$ such that $v^{m,n}\uparrow\tilde{v}^n$ as $m\to\infty$ and $\tilde{v}^n\downarrow v$ as $n\to\infty$ almost everywhere in Q_T. Note that we have also $\psi^{m,n}(r)\downarrow\psi^n=-\frac{1}{n}\max(-r;0)$ as $m\to\infty$ and $\psi^n\uparrow 0$ as $n\to\infty$ for all $r\in\mathbb{R}$. Therefore, the operator $A_b^{\psi^n}\subset\liminf_{m\to\infty}A_b^{\psi^{m,n}}$ and $A_b\subset\liminf_{n\to\infty}A_b^{\psi^n}$.

Remark 3.18. *Lemma* 3.9 *shows that* $\tilde{v}^n$ *and v are finite almost everywhere in* Q_T, *more precisely there exists a constant* $C>0$, *such that*

$$\left|\left\{\left|\tilde{v}^n\right|\ge K\right\}\right|\le CK, \tag{63}$$

and from (63), it follows that

$$\lim_{K\to 0}\left|\left\{|v|\ge K\right\}\right|=0. \tag{64}$$

In the next step, we will prove that v is a entropy solution of $(EP)^{b,F,k}_{f,v_0}$. Therefore, we need the following convergence results.

Convergence results

Now, applying the diagonal principle and Lemma 3.9, we get the following convergence:

Lemma 3.19. *For* $m,n\in\mathbb{N}^*$, $f\in L^1(Q_T)$ *and* $b(v_0)\in\overline{D(A_b)}^{\|\cdot\|}_{L^1(\Omega)}$, *let* $v^{m,n}$ *be the entropy solution to* $(EP)^{b,F,k,\psi^{m,n}}_{f^{m,n},v_0^{m,n}}$. *Then, there exists subsequences* $(m(n))_n,(l(n))_n$ *and* $(\lambda(l(n)))_n$ *such that setting* $\psi^n:=\psi^{m(n),n}$; $f^n:=f^{m(n),n}$; $b_n:=b_{l(n)}$; $v_0^n:=v_0^{m(n),n}$; $\tilde{v}^n:=v^{m(n),n}$ *and* $v^n:=v^{m(n),n}_{\lambda(l(n)),l(n)}$, *we have the following convergence results for any* $K>0$ *and* $n\to\infty$

(i) $f^n\to f$ in $L^1(Q_T)$,

(ii) $k_n\to k$ in $L^1([0,T))$,

(iii) $v_0^n\to v_0$ a.e. in Ω,

(iv) $\tilde{v}^n\to v$ a.e. in Q_T,

(v) $v^n\to v$ a.e. in Q_T,

(vi) $b_n(v_0^n)\to b(v_0)$ in $L^1(\Omega)$,

(vii) $b(\tilde{v}^n)\to b(v)$ in $L^1(Q_T)$,

(viii) $b_n(v^n)\to b(v)$ in $L^1(Q_T)$,

(ix) $T_K(v^n)\rightharpoonup T_K(v)$ in $L^p(0,T;W_0^{1,p}(\Omega))$,

(x) $a\left(\cdot, DT_K\left(v^n\right)\right) \rightharpoonup a\left(\cdot, DT_K(v)\right)$ in $\left(L^{p'}\left(Q_T\right)\right)^N$.

Proof. (i)-(v) *are direct consequences of approximation procedure*, *Lemma* 3.9 *and Remark* 3.18.

(vi) Choosing $l(n)$ such that $l(n) \geq m(n)$ for all $n \in \mathbb{N}^*$, we obtain

$$\left|b_n\left(v_0^n\right)-b\left(v_0\right)\right| \leq\left|b_n\left(v_0^n\right)-b\left(v_0^n\right)\right|+\left|b\left(v_0^n\right)-b\left(v_0\right)\right|=\frac{1}{l(n)}\left|v_0^n\right|+\left|b\left(v_0^n\right)-b\left(v_0\right)\right|$$

Since $\left|v_0^n\right| \leq m(n) \leq l(n)$ almost everywhere in Ω and for all $n \in \mathbb{N}$, then we get that $\left|b_n\left(v_0^n\right)-b\left(v_0\right)\right| \rightarrow 0$ a.e. in Ω as $n \rightarrow \infty$. Moreover, we have $\left|b_n\left(v_0^n\right)-b\left(v_0\right)\right| \leq 1+2\left|b\left(v_0\right)\right|$. Thus, by Lebesgue's theorem, we conclude that $b_n\left(v_0^n\right) \rightarrow b\left(v_0\right)$ in $L^1(\Omega)$ as $n \rightarrow \infty$.

(vii) We know that by theorem 5 of [26], the abstract problem

$$L\left(u-b\left(v_0^{m,n}\right)\right)(t)+A_b^{\psi^{m,n}}(u(t)) \ni f^{m,n}(t), \quad t \in[0, T)$$

admits a unique generalized solution $u^{m,n}=b\left(v^{m,n}\right)$ belongs to $L^1\left(Q_T\right)$ and by diagonal principle, there exists some function $u \in L^1\left(Q_T\right)$ such that $b\left(v^{m(n),n}\right) \rightarrow u$ in $L^1\left(Q_T\right)$ as $n \rightarrow \infty$. Since $v^{m(n),n} \rightarrow v$ a.e. in Q_T and b is continuous on $\mathbb{R}$, then it follows that $u=b(v)$. (viii) In the same way, since $v^n \rightarrow v$ a.e. in Q_T and $|v|<\infty$ a.e. on Q_T, it also comes that $b_n\left(v^n\right) \rightarrow b(v)$ in $L^1\left(Q_T\right)$.

From Lemma 3.9 and (v) we deduce (ix). It is left to show that (x). From Lemma 3.9 and growth condition (H4), it follows that for any $K>0$, exists $\tilde{\Phi}_K \in\left(L^{p'}\left(Q_T\right)\right)^N$ such that

$$a\left(\cdot, DT_K\left(v^n\right)\right) \rightharpoonup \tilde{\Phi}_K \quad \text{weakly in } \left(L^{p'}\left(Q_T\right)\right)^N$$

as $n \rightarrow \infty$. To prove that $\tilde{\Phi}_K=a\left(\cdot, DT_K(v)\right)$, we proceed as in Lemma 3.15. Simply replace $\tilde{v}_l^{m,n}$ by v^n; $v^{m,n}$ by v; b_l by b_n; k_l by k_n; $v_0^{m,n}$ by v_0 and $\psi^{m,n}$ by ψ^n. Note that, as $v^n \rightarrow v$ a.e. in Q_T, $\psi^n \rightarrow 0$ uniformly on compact sets, then by Fatou's lemma it follows that

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \int_{Q_\tau} \psi^n\left(v^n\right)\left(T_K\left(v^n\right)-h_i\left(v^n\right) T_K(v)_\mu\right) \mathrm{d}x \mathrm{d}t \\ & \geq \int_{Q_\tau} \liminf_{n \rightarrow \infty} \psi^n\left(v^n\right)\left(T_K\left(v^n\right)-h_i\left(v^n\right) T_K(v)_\mu\right) \mathrm{d}x \mathrm{d}t=0 .\end{aligned}$$

Thus, we obtain that

$$\liminf_{i \rightarrow \infty} \liminf_{\mu \rightarrow 0} \liminf_{n \rightarrow \infty} \int_{Q_\tau} \psi^n\left(v^n\right)\left(T_K\left(v^n\right)-h_i\left(v^n\right) T_K(v)_\mu\right) \mathrm{d}x \mathrm{d}t \geq 0 .$$

So, by the same argument as in Lemma 3.15, we can deduce that

$$a\left(\cdot, DT_K\left(v^n\right)\right) \rightharpoonup a\left(\cdot, DT_K(v)\right) \quad \text{weakly in } \left(L^{p'}\left(Q_T\right)\right)^N$$

for al $K>0$ and as $n \rightarrow \infty$. □

***Step* 3: *Conclusion of the proof of Theorem* 3.3.**

Now, we are able to conclude the proof of Theorem 3.3. By Remark 3.18 and Lemma 3.19 it follows immediately that (P1) hold for all $K>0$. Now, observe that for any $\varphi \in W_0^{1,p}(\Omega) \cap L^{\infty}(\Omega)$, $S \in \mathscr{P}$ and $\xi \in D([0, T))$ with $\xi \geq 0$, there exists a constant $C>0$ such that

$$\psi^n\left(v^n\right)S\left(v^n-\varphi\right)\xi \geq -C.$$

From Fatou's Lemma, we have

$$\liminf_{n\to\infty}\int_{Q_T}\psi^n\left(v^n\right)S\left(v^n-\varphi\right)\xi \mathrm{d}x\mathrm{d}t \geq 0. \tag{65}$$

Thus, proceeding as in the proof of Proposition 3.7 and as in [16] pp494-495, by monotonicity condition (H2), we have

$$\liminf_{n\to\infty}\int_{Q_T}a\left(x,Dv^n\right)\cdot S\left(v^n-\varphi\right)\xi \mathrm{d}x\mathrm{d}t \geq \int_{Q_T}a\left(x,Dv\right)\cdot S\left(v-\varphi\right)\xi \mathrm{d}x\mathrm{d}t.$$

Taking $S\left(v^n-\varphi\right)\xi$ as a test function in (19), we get

$$\begin{aligned}
&-\int_{Q_T}\left[k_{1,n,j}*\int_{v_0^n}^{v^n}S\left(\sigma-\varphi\right)\mathrm{d}b_n\left(\sigma\right)\right]\xi_t\left(t\right)\mathrm{d}x\mathrm{d}t \\
&+\int_{Q_T}\partial_t\left[k_{2,n,j}*\left(b_n\left(v^n\right)-b_n\left(v_0^n\right)\right)\right]\left(t\right)S\left(v^n\left(t,\cdot\right)-\varphi\right)\xi\left(t\right)\mathrm{d}x\mathrm{d}t \\
&+\int_{Q_T}\left(a\left(x,Dv^n\right)+F\left(v^n\right)\right)\cdot DS\left(v^n-\varphi\right)\xi \mathrm{d}x\mathrm{d}t \\
&+\int_{Q_T}\psi^n\left(v^n\right)S\left(v^n-\varphi\right)\xi \mathrm{d}x\mathrm{d}t \leq \int_{Q_T}f^nS\left(v^n-\varphi\right)\xi \mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Next, using the same argument as in step 5 of the proof of Proposition 3.7 and the inequality (65), we obtain that v satisfies the entropy formulation (P2). Hence, v is a entropy solution of $(EP)_{f,v_0}^{b,F,k}$. □

3.5. Conclusion

We note that the condition Lipschitz continuous on F allowed us to show strong convergence of a sub-sequence of the sequence $\left(v_l^{m,n}\right)_l$ in the Lemma 3.12, which was a crucial result for the proof of our main result Theorem 3.3.

In this paper we do not deal with uniqueness of solutions. For the moment, this remains an open problem.

Finally, in the literature no result on the existence can be found for the degenerated case of $(EP)_{f,v_0}^{b,F,k}$ with F merely continuous. In particular, this problem is still open for the history dependent problem $(EP)_{f,v_0}^{b,F,k}$.

Furthermore, it is important to note that the existence of weak solutions for bounded data remains an open problem in the case of the degenerate equation with history dependence.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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