Scientific Research Publishing

Journal of Applied Mathematics and Physics, 2023, 11, 4016-4026
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Kernel Thesis Reproduction and Truncated Toeplitz Operators

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How to cite this paper: Ali, Z.A.M.A., Bakhit, A.M.A., Ali, I.A.H., Juma, M.Y.A., Mohamed, T.M.A. and Abdelmajeed, S.I.A. (2023) Kernel Thesis Reproduction and Truncated Toeplitz Operators. *Journal of Applied Mathematics and Physics*, **11**, 4016-4026.
https://doi.org/10.4236/jamp.2023.1112257

Received: November 14, 2023
Accepted: December 24, 2023
Published: December 27, 2023

Abstract

Important operator characteristics (such boundedness or compactness) for particular classes of operators on particular reproducing kernel Hilbert spaces may be impacted by the behaviour of the operators on the reproducing kernels. These results have been shown for Toeplitz operators on the Paley-Wiener space, a reproducing kernel Hilbert space over C. Furthermore, we show how the norm of such an operator has no relation to the supremum of the norms of the pictures of the normalization reproducing kernels of the space. As a result, if this supremum is finite, the operator is implicitly bounded. To further demonstrate that the operator norm is not the same as the supremum of the norms of the pictures of the real normalized reproducing kernels, another example is also provided. We also set out a necessary and sufficient condition for the operators' compactness in terms of their limiting function on the reproducing kernels.

Keywords

Toeplitz Operators, Compact Operator, The Reproducing Kernel

1. Introduction and Notation

The "Bonsall's Theorem" implies that the supremum of the norms of the images of the normalization reproducing kernels of the Hardy space in [1] and [2] corresponds to the norm of a Hankel operator on the Hardy space of the disc. This statement may be used to the "Reproducing Kernel Thesis" (in [3]), it implies that the operation of the operators on the kernels defines important characteristics of certain classes of operators on replicating kernel Hilbert spaces. Huge

Hankel operators on the Hardy space of the bidisc and Hankel operators on the Bergman space of the disc both produced results that were equivalent in [4] and [5], respectively. The replication of kernels for Hankel operators on an extensive group of function spaces are shown to determine boundedness, but there doesn't seem to be a single theorem that proves this. Actually, it has been shown in [6] that the comparable claim for tiny Hankel operators on the Hardy space of the bidisc is not true. We will prove an equivalent of Bonsall's Theorem for Toeplitz operators acting on the function space Paley-Wiener.

For more details on the value of this space in signal processing, in [7]. It will be established later. Let P_E is orthogonal projection onto *E*, $E^{\perp}$ is orthogonal complement of *E* for *E*, a closed subspace of a Hilbert space. Let $L^2(J)$ be the closed subspace of $L^2(\mathbb{R})$ containing those functions that vanish, or are outside of *J*, for *J*, a measurable subset of $\mathbb{R}$, the real line, and χ_J convey the characteristic function of *J*. We define in $L^2(\mathbb{R})$) inner product by $\langle,\rangle$, as well as the $L^2(\mathbb{R})$ function and operator norms by $\|\,\|$. Other $L^2(\mathbb{R})$ norms will be identified by the symbol $\|\,\|_p$. The (unitary) Fourier transform and its inverse will be denoted by the following definitions and notations:

$$\mathcal{F}f(x)=\hat{f}(x)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(t)\mathrm{e}^{-ixt}\mathrm{d}t,$$

$$\mathcal{F}^{-1}f(t)=\tilde{f}(t)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)\mathrm{e}^{itx}\mathrm{d}x.$$

2. Hankel Operators in the Hardy Space

Suppose $\mathcal{H}_-$ and $\mathcal{H}_+$ are lower and half upper planes of $\mathbb{C}$. We define $H^2(\mathcal{H}_+)$ is the Hardy space of the upper half plane. By the theorem of Paley–Wiener, $H^2(\mathcal{H}_+)=\mathcal{F}^{-1}(L^2[0,\infty))$, in [8]. Therefore, for $f\in L^2(\mathbb{R})$,

$$P_{H^2(\mathcal{H}_+)}f=\mathcal{F}^{-1}\left(\chi_{[0,\infty)}\hat{f}\right),\ P_{H^2(\mathcal{H}_+)^{\perp}}f=\mathcal{F}^{-1}\left(\chi_{(-\infty,0]}\hat{f}\right). \quad (1)$$

If $W(t)=\sqrt{\pi}(t+i)$. For $\phi\in WL^2(\mathbb{R})$, let Γ_ϕ is a Hankel operator on $H^2(\mathcal{H}_+)$ with symbol ϕ so:

$$\Gamma_\phi: H^2(\mathcal{H}_+)\to H^2(\mathcal{H}_+)^{\perp},\ \Gamma_\phi f=P_{H^2(\mathcal{H}_+)^{\perp}}(\phi f).$$

See [9] for more information on these operators. It is fundamental that $\phi\in WL^2(\mathbb{R})$ for Γ_ϕ to be defined, since $\frac{1}{W}\in H^2(\mathcal{H}_+)$. Note that under this condition, Γ_ϕ is at least defined on $\frac{1}{W}H^\infty(\mathcal{H}_+)$, a dense subspace of $H^2(\mathcal{H}_+)$, also:

$$L^2(\mathbb{R})\cup L^\infty(\mathbb{R})\subseteq WL^2(\mathbb{R}).$$

The following theorem provides both essential and enough criteria for Γ_ϕ to be bounded and closed (also known as "Theorem of Bonsall's").

Theorem (2.1): For $w\in\mathcal{H}_+$, let j_w be $H^2(\mathcal{H}_+)$ normalized Replication of kernel connected to *w*, that is

$$j_w(t)=\frac{i\sqrt{\operatorname{Im} w}}{\sqrt{\pi}(t-\overline{w})}.$$

Then, for $\phi\in WL^2(\mathbb{R})$, Γ_ϕ is limited if and only if it has limits on every $\{j_w : w\in\mathcal{H}_+\}$. Additionally, there is a universal constant M that exists such that

$$\|\Gamma_\phi\|\le M\sup\{\|\Gamma_\phi j_w\| : w\in\mathcal{H}_+\}.$$

In addition, Γ_ϕ is compact if and only if

$$\lim_{w\in\mathcal{H}_+,|w|\to\infty}\|\Gamma_\phi j_w\|=\lim_{w\in\mathcal{H}_+,\operatorname{Im} w\to 0}\|\Gamma_\phi j_w\|=0.$$

The boundedness result is proved in [1] for Hankel operators on the Hardy space of the disc by using Fefferman's duality theorem. The boundedness result for Hankel operators on the Hardy space of the disc is shown in [1] by using Fefferman's duality theorem. The compactness result is a simple corollary, which is quoted in [4], for instance. This version may then be obtained by a standard conformal mapping from the disc to the upper half plane and is essentially found in [10], for instance. Let PW be the Paley-Wiener space of those $L^2(\mathbb{R})$ functions supported on a compact subset of which Fourier transforms are possible, chosen to be $I=[-\pi,\pi]$ but similar conclusions are valid for any compact interval. PW is equal to $PW=\mathcal{F}^{-1}(L^2(I))$. These functions, as is well known, extend to complete functions over $\mathbb{C}$ of exponential type up to π in [11]. Recall that, given $f\in L^2(\mathbb{R})$,

$$P_{PW}f=\mathcal{F}^{-1}\left(\chi_I\hat{f}\right). \tag{2}$$

Let T_ϕ is Toeplitz operator on PW with respect to ϕ so

$$T_\phi : PW\to PW,\ T_\phi f=P_{PW}(\phi f).$$

Then,

$$\left(T_\phi f\right)^\wedge=\chi_I(\phi f)^\wedge. \tag{3}$$

Let $\operatorname{sinc}(t)=\frac{\sin\pi t}{\pi t}$ (for $t\ne 0$, $\operatorname{sinc}(0)=1$). It is well-known that

$$\operatorname{sinc}(t-w)=\mathcal{F}^{-1}\left(\frac{\chi_I(x)e^{-iwx}}{\sqrt{2\pi}}\right). \tag{4}$$

PW is a replicating kernel Hilbert space, as is also widely known [11], with kernels denoted by $r_w(t)=\operatorname{sinc}(t-\overline{w})$. Let any $f\in PW$, $w\in\mathbb{C}$,

$$f(w)=\langle f,r_w\rangle. \tag{5}$$

3. Dividing the Symbol

In [12], the Rochbergs technique is used in our analysis of Toeplitz operators T_ϕ on PW. Pick and change some $\psi_L\in C_0^\infty(\mathbb{R})$ ($\mathbb{R}$ -based, indefinitely differentiable functions which tend to 0 at $\pm\infty$) so that support is given of ψ_L lies in $\left[-4\pi,-\frac{\pi}{2}\right]$, $\psi_L\equiv 1$ on $[-3\pi,-\pi]$ and $0\le\psi(x)\le 1$. Let $\psi_R(x)=\psi_L(-x)$

and $\psi_C \equiv 0$ off *I*, $\psi_C(x) = 1 - \psi_L(x) - \psi_R(x)$ on *I*. Initially, we will assume that the symbols of our Toeplitz operators are in $WL^2(\mathbb{R})$, where W was previously defined. For this ϕ, consider:

$$\phi_L = \phi * \check{\psi}_L,\ \phi_C = \phi * \check{\psi}_C,\ \varphi_R = \phi * \check{\psi}_R,$$

where $*$ denotes convolution. Note that ψ_R, ψ_C and ψ_L all belong to $\mathcal{S}$, the functions that experience rapid descent at infinity in the Schwartz space, in [13] for instance. Since the Fourier transform is a bijection on $\mathcal{S}$, the Fourier transforms of these functions also belong to $\mathcal{S}$. The next lemma demonstrates that each of ϕ_L, ϕ_C and ϕ_R is also a member of $WL^2(\mathbb{R})$.

Lemma (3.1): Suppose $\phi \in WL^2(\mathbb{R})$ and $\Delta \in \mathcal{S}$ then $\phi * \Delta \in WL^2(\mathbb{R})$.

Proof: Let $f \in L^2(\mathbb{R})$. Then

$$\left\langle (\phi * \Delta) W^{-1}, f \right\rangle = \int_{\mathbb{R}} \Delta(s) \int_{\mathbb{R}} \phi(t-s) W^{-1}(t) \overline{f(t)} \mathrm{d}t\mathrm{d}s,$$

based on the Tonelli and Fubini Theorems. A straightforward justification using the Cauchy-Schwarz inequality demonstrates that

$$\left| \int_{\mathbb{R}} \phi(t-s) W^{-1}(t) \overline{f(t)} \mathrm{d}t \right| \le \| f \| \left\| \phi W^{-1} \right\| \sup_{t \in \mathbb{R}} \left| \frac{W(t)}{W(t+s)} \right|.$$

by way of a simple argument, $\left| W(t)/W(t+s) \right|^2 \le 2s^2 + 2$. then,

$$\left| \left\langle (\phi * \Delta) W^{-1}, f \right\rangle \right| \le \int_{\mathbb{R}} |\Delta(s)| \left(2s^2 + 2\right)^{1/2} \mathrm{d}s \| f \| \left\| \phi W^{-1} \right\|.$$

Given that $\Delta \in \mathcal{S}$, the integral in the aforementioned expression is unquestionably finite. Consequently, $(\phi * \Delta) W^{-1} \in L^2(\mathbb{R})$, according to the Riesz Representation Theorem [14]. Note that only a distributional perspective may be used to broadly consider the Fourier transforms of, ϕ, ϕ_R, ϕ_C and ϕ_L. Their supports also fit into this category. We definitely have

$$\operatorname{supp}\left(\hat{\phi}_L\right) \subseteq \left[-4\pi, -\frac{\pi}{2}\right],\ \operatorname{supp}\left(\hat{\phi}_C\right) \subseteq [-\pi, \pi],\ \operatorname{supp}\left(\hat{\phi}_R\right) \subseteq \left[\frac{\pi}{2}, 4\pi\right].$$

Recall that the Toeplitz operator with the sign depends simply on $\hat{\phi}$ being restricted to $[-2\pi, 2\pi]$. Consequently

$$T_\phi = T_{\phi_L} + T_{\phi_C} + T_{\phi_R}. \tag{6}$$

Additionally, the following, which is essentially illustrated in [12], demonstrates that the symbol's splitting is constant in the norm.

Lemma (3.2): Suppose ϕ_X be ϕ_L, ϕ_C or ϕ_R. Then K_X, a universal constant, exists in such a way that $\left\| T_{\phi_X} f \right\| \le K_X \left\| T_\phi f \right\|$ for all $f \in PW$.

Proof: For $y \in \mathbb{R}$, let U_y be the translator specified by $U_y f(t) = f(t-y)$. Then (in [12] p. 201), $T_{U_y}\phi = U_y T_\phi U_{-y}$, so for all *y*, $T_{U_y\phi}$ is unitarily equivalent to T_ϕ. Let $\Pi_X = \check{\Psi}_X$ so we know that $\phi_X = \phi * \Pi_X$ and $\Pi_X \in \mathcal{S} \subseteq L^1(\mathbb{R})$. For any $g \in L^2(\mathbb{R})$ and $t \in \mathbb{R}$, $P_{PW} g(t) = \langle g, r_t \rangle$ by (5). This fact, along with the translation operator already described, reveals that

$T_{\phi_X} f(t) = \int_{\mathbb{R}} T_{U_y\phi} f(t) \Pi_X(y) \mathrm{d}y$.

Therefore, by duality, we obtain

$$\left\|T_{\phi_X} f\right\| = \sup_{g\in PW,\|g\|\le 1}\left|\left\langle T_{\phi_X} f, g\right\rangle\right| \le \sup_{g\in PW,\|g\|\le 1}\sup_{y\in\mathbb{R}}\left|\left\langle T_{U_y\phi} f, g\right\rangle\right|\left\|\Pi_X\right\|_1 = \left\|\Pi_X\right\|_1\left\|T_\phi f\right\|,$$

since for all $y\in\mathbb{R}$, $T_{U_y\phi}$ is unitarily equivalent to T_ϕ. Therefore, for all $f\in PW$,

$$\left\|T_\phi f\right\| \approx \left\|T_{\phi_L} f\right\| + \left\|T_{\phi_C} f\right\| + \left\|T_{\phi_R} f\right\|, \tag{7}$$

Specifically, universal constants c,d exist such that

$$c\left(\left\|T_{\phi_L} f\right\| + \left\|T_{\phi_C} f\right\| + \left\|T_{\phi_R} f\right\|\right) \le \left\|T_\phi f\right\| \le d\left(\left\|T_{\phi_L} f\right\| + \left\|T_{\phi_C} f\right\| + \left\|T_{\phi_R} f\right\|\right).$$

We will first test the boundedness and compactness of each of the three components before testing the boundedness and compactness of a Toeplitz operator. The action of multiplying by θ on $L^2(\mathbb{R})$ is represented by M_θ, which M_θ is obviously a unitary operator if $\theta(t) = e^{i\pi t}$. The fact that, for example

$$f\in L^2(\mathbb{R}),\ (\theta f)^\wedge(x) = \hat{f}(x-\pi) \text{ and } (\bar{\theta} f)^\wedge(x) = \hat{f}(x+\pi).$$

The fact that follows holds for any $\psi\in WL^2(\mathbb{R})$ and $f\in L^2(\mathbb{R})$, where at least one has a Fourier transform with compact support, will be used frequently throughout this paper:

$$\operatorname{supp}\left((\psi f)^\wedge\right) = \operatorname{supp}\left(\hat{\psi} * \hat{f}\right) \subseteq \operatorname{supp}(\hat{\psi}) + \operatorname{supp}\left(\hat{f}\right), \tag{8}$$

In [15]. In essence, [12] also contains the following lemma.

Lemma (3.3): Let $\psi\in WL^2(\mathbb{R})$, $\operatorname{supp}(\hat{\psi}) \subseteq \left[\frac{\pi}{2}, 4\pi\right]$. Then

$$T_\psi = M_\theta \Gamma_{\bar{\theta}^2\psi} M_\theta \text{ and } \Gamma_{\bar{\theta}^2\psi} = M_{\bar{\theta}} T_\psi P_{PW} M_{\bar{\theta}}.$$

Therefore, $\left\|T_\psi\right\| = \left\|\Gamma_{\bar{\theta}^2\psi}\right\|$, provided these are finite, and T_ψ is limited and closed only if $\Gamma_{\bar{\theta}^2\psi}$ is closed and bounded.

Proof: By taking into account the support provided by the Fourier transforms of pertinent functions, the first two equality conditions are established. Given that M_θ and PPW are bounded operators of norm 1, the operators' norms are same, and their compactness is equivalent. For $w\in\mathbb{C}$, let h_w be the associated normalized reproducing kernel of *PW*.

$$h_w(t) = \sqrt{\frac{2\pi \operatorname{Im} w}{\sinh(2\pi \operatorname{Im} w)}}\operatorname{sinc}(t-\bar{w}), \tag{9}$$

when $\operatorname{Im} w = 0$, with a suitable interpretation.

The component T_{ϕ_R} boundedness will be established using the following statement.

Proposition (3.4): Let $\psi\in WL^2(\mathbb{R})$ and $\operatorname{supp}(\hat{\psi}) \subseteq \left[\frac{\pi}{2}, 4\pi\right]$. Then T_ψ is bounded if and only if it is bounded on $\{h_w : w\in\mathcal{H}_+\}$. Additionally, a universal constant M exists such that $\left\|T_\psi\right\| \le M \sup\left\{\left\|T_\psi h_w\right\| : w\in\mathcal{H}_+\right\}$.

Proof: Given that this is a collection of normalized functions, it is obvious that if T_ψ is bounded, it is limited on $\{h_w : w\in\mathcal{H}_+\}$. In contrast, T_ψ is bounded by Lemma (2.1.4) if and only if $\Gamma_{\bar{\theta}^2\psi}$ is limited. According to Theorem (2.1.1), if $\Gamma_{\bar{\theta}^2\psi}$ is restricted on $\{j_w : w \in \mathcal{H}_+\}$ and such a universal constant M exists.

$$\left\|\Gamma_{\bar{\theta}^2\psi}\right\| \le M \sup\left\{\Gamma_{\bar{\theta}^2\psi} j_w : w \in \mathcal{H}_+\right\}.$$

However, by Lemma (2.1.4), $\left\|\Gamma_{\bar{\theta}^2\psi} j_w\right\| = \left\|T_\psi P_{PW} M_{\bar{\theta}} j_w\right\|$. A straightforward calculation shows that

$$P_{PW} M_{\bar{\theta}} j_w(t) = \mathrm{e}^{-i\pi\bar{w}} \sqrt{2\pi \operatorname{Im} w} \operatorname{sinc}(t - \bar{w}),$$

and therefore

$$\left\|\Gamma_{\bar{\theta}^2}\psi j_w\right\| = \mathrm{e}^{-\pi \operatorname{Im} w} \sqrt{\sinh(2\pi \operatorname{Im} w)} \left\|T_\psi h_w\right\|. \tag{10}$$

If $\mathrm{e}^{-\pi \operatorname{Im} w} \sqrt{\sinh(2\pi \operatorname{Im} w)} < 1$ for $w \in \mathcal{H}_+$, where,

$$\sup\left\{\Gamma_{\bar{\theta}^2\psi} j_w : w \in \mathcal{H}_+\right\} < \infty \text{ and so } \Gamma_{\bar{\theta}^2\psi}$$

and hence T_ψ is limited. Moreover,

$$\left\|T_\psi\right\| = \left\|\Gamma_{\bar{\theta}^2\psi}\right\| \le M \sup\left\{\Gamma_{\bar{\theta}^2\psi} j_w : w \in \mathcal{H}_+\right\} \le M \sup\left\{\left\|T_\psi h_w\right\| : w \in \mathcal{H}_+\right\},$$

as required. We will now talk about Toeplitz operators with symbols whose Fourier transforms are supported on $\left[-4\pi, -\frac{\pi}{2}\right]$, such as the component T_{ϕ_L}.

We present the notation. $h*(t) = \overline{h(-t)}$. Then it is simple to demonstrate that $\left(\hat{h}\right)^* = \left(\bar{h}\right)^\wedge$ and that, for $g \in PW$,

$$\left(T_{\bar{\psi}} g\right)^\wedge = \left(\left(T_\psi \bar{g}\right)^\wedge\right)^*. \tag{11}$$

Corollary (3.5): Let $\psi \in WL^2(\mathbb{R})$, $\operatorname{supp}(\hat{\psi}) \subseteq \left[-4\pi, -\frac{\pi}{2}\right]$. T_ψ is then said to be limited if and only if it has limits on $\{h_w : w \in \mathcal{H}_-\}$. Additionally, An continuous constant M exists in a way that $\left\|T_\psi\right\| \le M \sup\left\{\left\|T_\psi h_w\right\| : w \in \mathcal{H}_-\right\}$.

Proof: Given that complex conjugation, the Fourier transform, and the operations on $*$ are all unitary, $\left\|T_\psi\right\| = \left\|T_{\bar{\psi}}\right\|$ by (11). But we may apply Proposition (3.4) to ψ. By (11) again, $\left\|T_{\bar{\psi}} h_w\right\| = \left\|T_\psi \overline{h_w}\right\|$. It is easily shown that $\overline{h_w} = h_{\bar{w}}$ and the result therefore stands. In order to study the component T_{ϕ_C}, we will now examine the scenario in which the symbol has support for the Fourier transform on $[-\pi, \pi]$.

Proposition (3.6): Let $\psi \in WL^2(\mathbb{R})$, $\hat{\psi}$ It is strengthened on $[-\pi, \pi]$ and $w \in \mathbb{R}$ then $\psi \in L^\infty(\mathbb{R})$ provided that $\sup\left\{T_\psi h_w : w \in \mathbb{R}\right\} < \infty$, Additionally, there is a universal constant M such that $\sup\|\psi\|_\infty \le M \sup\left\{T_\psi h_w : w \in \mathbb{R}\right\}$.

Proof: The inner product of $L^2(\mathbb{R})$ functions and, more broadly, the impact of a distribution on a function are both shown here using the notation $\langle , \rangle$. According to the Paley-Wiener-Schwartz Theorem in [7], ψ is the limit to $\mathbb{R}$ of an entire function of exponential type at most π, as it has a compactly supported Fourier transform.

It is simple to demonstrate $\left\langle T_\psi h_w, h_w\right\rangle = \left\langle \hat{\psi}, \hat{h}_w * \hat{h}_w\right\rangle$ and that

$$\hat{h}_w * h_w(x) = \frac{e^{-iwx}}{2\pi}(2\pi - |x|)\chi_{2I}(x),$$

using (4). Let $\Lambda(x) = \frac{2\pi - |x|}{\sqrt{2\pi}}\chi_{2I}(x)$.

Then $\langle T_\psi h_w, h_w \rangle = \langle \hat{\psi}, \Lambda \hat{h}_w \rangle = \langle \Lambda\hat{\psi}, \hat{h}_w \rangle$. However,

$$\begin{aligned}\check{\Lambda} * \psi(w) &= \langle \check{\Lambda} * \psi, \delta_w \rangle \quad \text{where } \delta_w \text{ is the point mass at } w,\\ &= \Lambda\hat{\psi}, \chi_I \hat{\delta}_w, \text{ as } \operatorname{supp}(\hat{\psi}) \subseteq I,\\ &= \Lambda\hat{\psi}, \hat{h}_w.\end{aligned}$$

Hence, $\check{\Lambda} * \psi(w) = \{T_\psi h_w, h_w\}$ and therefore,

$$\|\check{\Lambda} * \psi\|_\infty = \sup_{w\in\mathbb{R}} |\langle T_\psi h_w, h_w \rangle| \le \sup_{w\in\mathbb{R}} \|T_\psi h_w\|, \tag{12}$$

as $\|h_w\| = 1$ for all $w \in \mathbb{R}$.

We aim to demonstrate this $\|\check{\Lambda} * \psi\|_\infty \approx \|\psi\|_\infty$.

First off, we conclude that $\check{\Lambda} \in L^1(\mathbb{R})$ as Λ is an array of $\chi_I * \chi_I$, so $\check{\Lambda}$ is the square of an $L^2(\mathbb{R})$ function. Therefore, $\|\check{\Lambda} * \psi\|_\infty \le \|\check{\Lambda}\|_1 \|\psi\|_\infty$. For the opposite inference, we see that $\psi(x) = \frac{\sqrt{\pi}}{2\pi - |x|}\chi_{2I}(x)(\check{\Lambda} * \psi)^\wedge(x)$.

On I, however, $(\check{\Lambda} * \psi)^\wedge$ is supported. As a result, we create a function V that, for $x \in I$,

$$V(x) = \frac{\sqrt{2\pi}}{2\pi - |x|},$$

After that, and V is expanded to become even, supported by $2I$, and infinitely differentiable, save at 0. Therefore, since $\hat{\phi} = V(\check{\Lambda} * \phi)^\wedge$ so $\phi = \check{V} * (\hat{\Lambda} * \phi)$ The math below indicates that $\check{V} \in L^1(\mathbb{R})$.

$$\begin{aligned}\check{V}(t) &= \sqrt{\frac{2}{\pi}}\int_0^{2\pi} V(x)\cos(tx)\,dx, \text{ as } V \text{ is even}\\ &= -\sqrt{\frac{2}{\pi}}\frac{1}{t^2}\left(V_+'(0) + \int_0^{2\pi} V''(x)\cos(tx)\,dx\right),\end{aligned}$$

utilizing two integrations via sections. Therefore, $|\check{V}(t)| \le \frac{2}{\pi}\frac{1}{t^2}|V_+'(0)| + \frac{1}{2}\|V\|_1$.

Being continuous, $\check{V}$ is unquestionably locally integrable, hence $V \in L^1(\mathbb{R})$ follows.

Hence, $\|\psi\|_\infty \le \|\check{V}\|_1 \|\check{\Lambda} * \psi\|_\infty$, so that $\|\check{\Lambda} * \psi\|_\infty \approx \|\psi\|_\infty$. By combining this with (12), we arrive at the desired outcome.

4. Main Findings

The first fundamental theorem can now be stated. T_ϕ is divided into $T_{\phi_L} + T_{\phi_C} + T_{\phi_R}$, We shall see that the boundedness of T_ϕ on $\{h_w : w \in \mathcal{H}_+\}$. determines the boundedness of T_{ϕ_R}. The boundedness of T_ϕ on $\{h_w : w \in \mathcal{H}_-\}$ and $\{h_w : w \in \mathbb{R}\}$, respectively, determines the boundedness of T_{ϕ_L} and T_{ϕ_C}.

Theorem (4.1): Let $\phi \in WL^2(\mathbb{R})$. Subsequently, T_ϕ is limited if and only if it is limited on $\{h_w : w \in \mathbb{C}\}$. Additionally, a continuous constant M exists in a way that

$$\|T_\phi\| \le M \sup\{\|T_\phi h_w\| : w \in \mathbb{C}\}.$$

Proof: Clearly, if T_ϕ is bounded, it is bounded on $\{h_w : w \in \mathbb{C}\}$ since these are a collection of normalized functions. Conversely, we know by (7) that for all $f \in PW$,

$$\|T_\phi f\| \approx \|T_{\phi_L} f\| + \|T_{\phi_C} f\| + \|T_{\phi_R} f\|. \tag{13}$$

By Proposition (3.4), T_{ϕ_R} is limited provided that $\sup\{T_{\phi_R} h_w : w \in \mathcal{H}_+\} < \infty$ and this is undoubtedly accurate given that

$$\sup\{\|T_\phi h_w\| : w \in \mathcal{H}_+\} < \infty, \tag{14}$$

through Lemma (3.2). Similar to that, according to Corollary (3.5) and Lemma (3.2), T_{ϕ_L} is limited if

$$\sup\{\|T_\phi h_w\| : w \in \mathcal{H}_-\} < \infty, \tag{15}$$

Rochberg shows that $\|T_{\phi_C}\| \approx \|\phi_C\|_\infty$, in [12]. T_{ϕ_C} is therefore constrained by Proposition (3.6) and Lemma (3.2), provided that

$$\sup\{\|T_\phi h_w\| : w \in \mathbb{R}\} < \infty. \tag{16}$$

When (13), (14), (15), and (16) are combined, we discover that T_ϕ is bounded if $\sup\{\|T_\phi h_w\| : w \in \mathbb{C}\} < \infty$. The estimate for $\|T_\phi\|$ is similarly produced by estimating the norms of T_{ϕ_L}, T_{ϕ_R} and T_{ϕ_R} using Proposition (3.4), Corollary 2.5, and Proposition (3.6). By using a counterexample, we can demonstrate that the supremum of the norms of the pictures in the set of h_w for $w \in \mathbb{R}$ is not comparable to the norm of a Toeplitz operator.

Lemma (4.2): No universal constant M exists such that $\|T_\phi\| \le M \sup\{\|T_\phi h_w\| : w \in \mathbb{R}\}$, for all bounded T_ϕ.

Proof: For $\alpha \in [0, 2\pi)$, let $\phi_\alpha(t) = \mathrm{e}^{i\alpha t}$. Since $\|\phi_\alpha\|_\infty = 1$ it is clear that $\|T_{\phi_\alpha}\| \le 1$. Fourier transforms are used in an easy computation to demonstrate that

$$\|T_{\phi_\alpha} h_w\|^2 = \frac{\mathrm{e}^{v\alpha} \sinh(2\pi - \alpha)v}{\sinh(2\pi v)}.$$

where $v = \operatorname{Im} w$ (if v = 0, given a reasonable interpretation). In particular, if $w \in \mathbb{R}$ then

$$\|T_{\phi_\alpha} h_w\|^2 = \frac{2\pi - \alpha}{2\pi} \text{ so } \sup\{\|T_{\phi_\alpha} h_w\| : w \in \mathbb{R}\} = \sqrt{\frac{2\pi - \alpha}{2\pi}}. \tag{17}$$

However, $\lim_{v\to\infty} \dfrac{\mathrm{e}^{v\alpha} \sinh(2\pi - \alpha)v}{\sinh(2\pi v)} = 1$.

Therefore, $\sup\{T_{\phi_\alpha} h_w : w \in \mathbb{C}\} \ge 1$, so $\|T_{\phi_\alpha}\| = 1$. Therefore, any such M would need satisfy $M \geq \sqrt{\frac{2\pi}{2\pi - \alpha}}$, for all values of $\alpha \in [0, 2\pi)$, which is obviously not possible. We will start by demonstrating a statement that is true for any compact operator on *PW*.

Proposition (4.3): If T is any compact operator from *PW* to a Hilbert space *H*, assumable, $\lim_{|w|\to\infty} \|Th_w\| = 0$.

Proof: To do this, we must first demonstrate that h_w weakly converges to zero as

$$|w| \to \infty. \quad E = \bigcup_{\delta \in (0,\pi)} \mathcal{F}^{-1} L^2([-\pi + \delta, \pi - \delta]).$$

It is then simple to demonstrate that E is a dense subspace of *PW*. Let $f \in E$. Using the kernels' ability to reproduce,

$$\langle f, h_w \rangle = f(w) \sqrt{\frac{2\pi v}{\sinh(2\pi v)}}, \tag{18}$$

where $v = \operatorname{Im} w$, by (5) and (9). There is a constant K_f such that for some $\pi - \delta$, $\delta \in (0, \pi)$, *f* is a complete function of exponential type at most.

$|f(w)| \leq K_f e^{(\pi - \delta)|v|}$, in [11]. Therefore,

$$|f, h_w| \leq K_f e^{(\pi-\delta)|v|} \sqrt{\frac{2\pi v}{\sinh(2\pi v)}} = \frac{2K_f e^{-\delta|v|}\sqrt{\pi|v|}}{\sqrt{1 - e^{-4\pi|v|}}} \to 0 \text{ as } |v| \to \infty.$$

We can also show that, for any $R > 0$, $|f(u + iv)| \to 0$ as $|u| \to \infty$, this is a generalization of the Riemann-Lebesgue Lemma for $|v| \leq R^-$. Using (18) once more, it is evident that $\lim_{|w|\to\infty} \langle f, h_w \rangle = 0$.

However, a typical argument demonstrates that h_w converges weakly to zero as $|w| \to \infty$.| since E is a dense subspace of *PW* and $\{h_w : w \in \mathbb{C}\}$ is uniformly bounded. Compact operators transform weak convergence to norm convergence, so the outcome is as shown in [16]. If T is a Toeplitz operator, we can get the opposite of this conclusion.

Theorem (4.4): Let $\phi \in WL^2(\mathbb{R})$. Then T_ϕ is closed and bounded if and only if

$$\lim_{|w|\to\infty} \|T_\phi h_w\| = 0.$$

Proof: The prior assertion provides the forward implication. Note that by (7) T_ϕ is compact if and only if each of T_{ϕ_L}, T_{ϕ_C}, and T_{ϕ_R}) is, demonstrating the opposite conclusion. Let's start by thinking about T_{ϕ_R}. According to our theory and (7),

$$\lim_{w \in \mathcal{H}_+, |w| \to \infty} \|T_{\phi_R} h_w\| = 0.$$

By (10),

$$\|\Gamma_{\bar{\theta}^2} \phi_R j_w\| = e^{-\pi \operatorname{Im} w} \sqrt{\sinh(2\pi \operatorname{Im} w)} \|T_{\phi_R} h_w\|.$$

Therefore,

$$\lim_{w \in \mathcal{H}_+, |w| \to \infty} \|\Gamma_{\bar{\theta}^2 \phi_R} j_w\| = \lim_{w \in \mathcal{H}_+, \operatorname{Im} w \to 0} \|\Gamma_{\bar{\theta}^2 \phi_R} j_w\| = 0.$$

As a result, according to Theorem (2.1), $\Gamma_{\bar{\partial}^2 \phi_R}$ is compact, and by Lemma (3.3), T_{ϕ_R} is closed and bounded. The same argument (applied to $T_{\overline{\phi_L}}$ and using (11)) shows that $T_{\overline{\phi_L}}$ and hence T_{ϕ_L} is compact. T_{ϕ_C} is compact provided that $|\phi_C(w)| \to 0$ as $|w| \to \infty$, in [12]. T_{ϕ_C} is therefore implied to be closed and bounded by Lemma (3.2) and Proposition (3.6), according to our hypothesis. T_ϕ is hence compact.

5. Other Issues

It is possible to determine whether Toeplitz operators belong to a certain Schatten-von Neumann class by observing how the operators behave on the reproducing kernels. It would be interesting to see if similar results are obtained for Hankel-type operators on *PW*. For this space, Hankel-type operators in one form are considered in [12], and it is found that they are equivalent to the Toeplitz operators considered in this paper. However, Hankel-type operators defined on *PW* by

$$H_\phi : PW \to PW^\perp, \; H_\phi f = P_{PW^\perp}(\phi f),$$

do not appear to have been analyses.

Acknowledgements

This study is supported via funding from Prince Sattam bin Abdulaziz University Project number (PSAU/2023/R/144).

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Bonsall, F.F. (1984) Boundedness of Hankel Matrices. *Journal of the London Mathematical Society*, **29**, 289-300. https://doi.org/10.1112/jlms/s2-29.2.289

[2] Bonsall, F.F. (1994) Conditions for Boundedness of Hankel Matrices. *Bulletin of the London Mathematical Society*, **26**, 171-176. https://doi.org/10.1112/blms/26.2.171

[3] Havin, V.P. and Nikolski, N.K. (2000) Stanislav Aleksandrovich Vinogradov, His Life and Mathematics. In: Havin, V.P., Nikolski, N.K., Eds., *Complex Analysis, Operators, and Related Topics. Operator Theory: Advances and Applications*, Birkhäuser, Basel. https://doi.org/10.1007/978-3-0348-8378-8_1

[4] Axler, S. (1986) The Bergman Space, the Bloch Space, and Commutators of Multiplication Operators. *Duke Mathematical Journal*, **2**, 53315-53332. https://doi.org/10.1215/S0012-7094-86-05320-2

[5] Ferguson, S.H. and Sadosky, C. (2000) Characterizations of Bounded Mean Oscillation on the Polydisk in Terms of Hankel Operators and Carleson Measures. *Journal d'Analyse Mathématique*, **81**, 239-267. https://doi.org/10.1007/BF02788991

[6] Pott, S. and Sadosky, C. (2002) Bounded Mean Oscillation on the Bidisk and Operator BMO, *Journal of Functional Analysis*, **189**, 475-495. https://doi.org/10.1006/jfan.2001.3822

[7] Higgins, J.R. (1996) Sampling Theory in Fourier and Signal Analysis. Oxford University Press, Oxford. https://doi.org/10.1093/oso/9780198596998.001.0001

[8] Duren, P.L. (1970) Theory of H^p Spaces. Academic Press, New York.

[9] Partington, J.R. (1988) An Introduction to Hankel Operators. Cambridge University Press, Cambridge. https://doi.org/10.1017/CBO9780511623769

[10] Partington, J.R. and Weiss, G. (2000) Admissible Observation Operators for the Right-Shift Semigroup. *Mathematics of Control, Signals, and Systems*, **13**, 179-192. https://doi.org/10.1007/PL00009866

[11] Partington, J.R. (1997) Interpolation, Identification and Sampling. Oxford University Press, Oxford. https://doi.org/10.1093/oso/9780198500247.001.0001

[12] Rochberg, R. (1987) Toeplitz and Hankel Operators on the Paley Wiener Space. *Integral Equations Operator Theory*, **10**, 187-235. https://doi.org/10.1007/BF01199078

[13] Katznelson, Y. (1976) An Introduction to Harmonic Analysis. Dover Publications, New York.

[14] Young, N. (1988) An Introduction to Hilbert Space. Cambridge University Press, Cambridge. https://doi.org/10.1017/CBO9781139172011

[15] Donoghue, W.F. (1969) Distributions and Fourier Transforms. Academic Press, New York.

[16] Halmos, P.R. (1982) A Hilbert Space Problem Book. Springer-Verlag, New York. https://doi.org/10.1007/978-1-4684-9330-6