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On Prime Numbers between *kn* and (*k* + 1)*n*

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Abstract

In this paper along with the previous studies on analyzing the binomial coefficients, we will complete the proof of a theorem. The theorem states that for two positive integers n and k, when $n \ge k-1$, there always exists at least a prime number p such that $kn < p \le (k+1)n$. The Bertrand-Chebyshev's theorem is a special case of this theorem when $k = 1$. In the field of prime number distribution, just as the prime number theorem provides the approximate number of prime numbers relative to natural numbers, while the new theory indicates that prime numbers exist in the specific intervals between natural numbers, that is, the new theorem provides the approximate positions of prime numbers among natural numbers.

Keywords

Bertrand's Postulate-Chebyshev's Theorem, The Prime Number Theorem, Landau Problems, Legendre's Conjecture, Prime Number Distribution

1. Introduction

The Bertrand-Chebyshev's theorem states that for any positive integer n, there always exists a prime number p such that $n < p \le 2n$. Pafnuty Chebyshev proved this in 1850 [1]. In 2006, M. El Bachraoui [2] extended the theorem and proved that for any positive integer n, there exists a prime number p such that $2n < p \le 3n$. In 2011, Andy Loo [3] proved that when $n \ge 2$, there are prime numbers in the interval $(3n, 4n)$. In 2013, Vladimir Shevelev *et al.* [4] proved that when the integer $k \le 100{,}000{,}000$, only $k = 1, 2, 3, 5, 9, 14$, for all $n \ge 1$, the interval $(kn, (k+1)n)$ contains prime numbers. This raises the question: for all $k \ge 1$, under what conditions does the interval $(kn, (k+1)n)$ contain prime numbers? Previously, the author partially answered this question in the paper [5] by analyzing the binomial coefficients $\binom{\lambda n}{n}$ where λ is a positive integer. In that paper, the author proved that when $n \ge \lambda - 2 \ge 25$, *i.e.*, when $n \ge \lambda - 2$ and $\lambda \ge 27$, there exists at least a prime number p such that $(\lambda - 1)n < p \le \lambda n$. In this article, we will use the same method to complete the entire work on this problem. We will prove that when $n \ge \lambda - 2$ and $\lambda \ge 3$, there exists at least a prime number p such that $(\lambda - 1)n < p \le \lambda n$. Then, by converting λ to k, we prove that for two positive integers n and k, when $n \ge k - 1$, there is always at least a prime number p such that $kn < p \le (k+1)n$. The Bertrand-Chebyshev's theorem is a special case of this theorem when $k = 1$.

We will use the same definition and concepts from [5] in this section and in section 2. In section 3, we will prove that for λ from 3 to 26, when $n \ge \lambda - 2$, there exists at least a prime number p such that $(\lambda - 1)n < p \le \lambda n$. In section 4, we will convert λ to k, and complete this article.

Definition: $\Gamma_{a \ge p > b}\left\{\binom{\lambda n}{n}\right\}$ denotes the prime number factorization operator of the integer expression $\binom{\lambda n}{n}$. It is the product of the prime numbers in the decomposition of $\binom{\lambda n}{n}$ in the range of $a \ge p > b$. In this operator, p is a prime number, a and b are real numbers, and $\lambda n \ge a \ge p > b \ge 1$.

It has some properties:

It is always true that $\Gamma_{a \ge p > b}\left\{\binom{\lambda n}{n}\right\} \ge 1$ (1.1)

If there is no prime number in $\binom{\lambda n}{n}$ within the range of $a \ge p > b$, then $\Gamma_{a \ge p > b}\left\{\binom{\lambda n}{n}\right\} = 1$, or vice versa, if $\Gamma_{a \ge p > b}\left\{\binom{\lambda n}{n}\right\} = 1$, then there is no prime number in $\binom{\lambda n}{n}$ within the range of $a \ge p > b$. (1.2)

For example, when $\lambda = 5$ and $n = 4$, $\Gamma_{16 \ge p > 10}\left\{\binom{20}{4}\right\} = 13^0 \cdot 11^0 = 1$. No prime number 13 or 11 is in $\binom{20}{4}$ within the range of $16 \ge p > 10$.

If there is at least one prime number in $\binom{\lambda n}{n}$ in the range of $a \ge p > b$, then $\Gamma_{a \ge p > b}\left\{\binom{\lambda n}{n}\right\} > 1$, or vice versa, if $\Gamma_{a \ge p > b}\left\{\binom{\lambda n}{n}\right\} > 1$, then there is at least one prime number in $\binom{\lambda n}{n}$ within the range of $a \ge p > b$. (1.3)

For example, when $\lambda = 5$ and $n = 4$, $\Gamma_{18 \ge p > 16}\left\{\binom{20}{4}\right\} = 17 > 1$. A prime number 17 is in $\binom{20}{4}$ within the range of $18 \ge p > 16$.

Let $v_p(n)$ be the *p-adic valuation* of *n*, the exponent of the highest power of *p* that divides *n*.

We define $R(p)$ by the inequalities $p^{R(p)} \le \lambda n < p^{R(p)+1}$, and determine the *p-adic valuation* of $\binom{\lambda n}{n}$.

$$v_p\left(\binom{\lambda n}{n}\right) = v_p\left((\lambda n)!\right) - v_p\left(((\lambda-1)n)!\right) - v_p(n!)$$

$$= \sum_{i=1}^{R(p)} \left(\left\lfloor \frac{\lambda n}{p^i} \right\rfloor - \left\lfloor \frac{(\lambda-1)n}{p^i} \right\rfloor - \left\lfloor \frac{n}{p^i} \right\rfloor \right) \le R(p)$$

because for any real numbers a and b, the expression of $\lfloor a+b \rfloor - \lfloor a \rfloor - \lfloor b \rfloor$ is 0 or 1.

Thus, if *p* divides $\binom{\lambda n}{n}$, then $v_p\left(\binom{\lambda n}{n}\right) \le R(p) \le \log_p(\lambda n)$, or

$$p^{v_p\left(\binom{\lambda n}{n}\right)} \le p^{R(p)} \le \lambda n \tag{1.4}$$

If $n \ge p > \lfloor \sqrt{\lambda n} \rfloor$, then $0 \le v_p\left(\binom{\lambda n}{n}\right) \le R(p) \le 1$. (1.5)

Let $\pi(n)$ be the number of distinct prime numbers less than or equal to *n*. Among the first six consecutive natural numbers are three prime numbers 2, 3, and 5. Then, for each additional six consecutive natural numbers, at most one can add two prime numbers, $p \equiv 1(\text{MOD } 6)$ and $p \equiv 5(\text{MOD } 6)$. Thus,

$$\pi(n) \le \left\lfloor \frac{n}{3} \right\rfloor + 2 \le \frac{n}{3} + 2. \tag{1.6}$$

From the prime number decomposition, when $n > \lfloor \sqrt{\lambda n} \rfloor$,

$$\binom{\lambda n}{n} = \Gamma_{\lambda n \ge p > n}\left\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\right\} \cdot \Gamma_{n \ge p > \lfloor \sqrt{\lambda n} \rfloor}\left\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\right\} \cdot \Gamma_{\lfloor \sqrt{\lambda n} \rfloor \ge p}\left\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\right\}$$

When $n \le \lfloor \sqrt{\lambda n} \rfloor$, $\binom{\lambda n}{n} \le \Gamma_{\lambda n \ge p > n}\left\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\right\} \cdot \Gamma_{\lfloor \sqrt{\lambda n} \rfloor \ge p}\left\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\right\}$

Thus,

$$\binom{\lambda n}{n} \le \Gamma_{\lambda n \ge p > n}\left\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\right\} \cdot \Gamma_{n \ge p > \lfloor \sqrt{\lambda n} \rfloor}\left\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\right\} \cdot \Gamma_{\lfloor \sqrt{\lambda n} \rfloor \ge p}\left\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\right\}$$

$\Gamma_{\lambda n \ge p > n}\left\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\right\} = \Gamma_{\lambda n \ge p > n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\}$ since all prime numbers in $n!$ do not appear in the range of $\lambda n \ge p > n$.

If $n \ge \lambda - 2$, and there is a prime number p in $\Gamma_{\lambda n \ge p > n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\}$, then $p \ge n+1 = \sqrt{(n+2)n+1} > \sqrt{\lambda n}$. From (1.5),

$0 \le v_p\left(\Gamma_{\lambda n \ge p > n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\}\right) \le R(p) \le 1$. Thus, if $n \ge \lambda - 2$, every prime number in $\Gamma_{\lambda n \ge p > n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\}$ has a power of 0 or 1. (1.7)

Referring to (1.5), $\Gamma_{n \ge p > \lfloor\sqrt{\lambda n}\rfloor}\left\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\right\} \le \prod_{n \ge p} p$. It has been proven [6] that for $n \ge 3$, $\prod_{n \ge p} p < 2^{2n-3}$.

When $n = 2$, $\prod_{n \ge p} p = 2^{2n-3}$, then for $n \ge 2$,

$\Gamma_{n \ge p > \lfloor\sqrt{\lambda n}\rfloor}\left\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\right\} \le \prod_{n \ge p} p \le 2^{2n-3}$.

Referring to (1.4) and (1.6), $\Gamma_{\lfloor\sqrt{\lambda n}\rfloor \ge p}\left\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\right\} \le (\lambda n)^{\frac{\sqrt{\lambda n}}{3}+2}$.

Thus, for $\lambda \ge 3$ and $n \ge 2$, $\binom{\lambda n}{n} \le \Gamma_{\lambda n \ge p > n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\}\cdot 2^{2n-3}\cdot(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+2}$ (1.8)

2. Lemmas

Lemma 1: For $\lambda \ge 3$ and $n \ge 2$,

$$\Gamma_{\lambda n \ge p > n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\} > \frac{2\lambda^2\cdot\left(\frac{\lambda}{4}\cdot\left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{n-1}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+3}} = f_8(n,\lambda) \quad (2.1)$$

Proof:

Let a real number $x \ge 3$, and $f_1(x) = \frac{2(2x-1)}{x-1}$; then,

$$f_1'(x) = \frac{2(x-1)(2x-1)' - 2(2x-1)(x-1)'}{(x-1)^2} = \frac{-2}{(x-1)^2} < 0.$$

Thus, $f_1(x)$ is a strictly decreasing function for $x \ge 3$.

Since $f_1(3) = 5$, and $\lim_{x\to\infty} f_1(x) = 4$, for $x \ge 3$, we have

$$5 \ge f_1(x) = \frac{2(2x-1)}{x-1} \ge 4.$$

Let $f_2(x) = \left(\frac{x}{x-1}\right)^x$, then

$$f_2'(x) = \left(\left(\frac{x}{x-1}\right)^x\right)' = \left(e^{x\cdot\ln\frac{x}{x-1}}\right)' = e^{x\cdot\ln\frac{x}{x-1}}\cdot\left(x\cdot\ln\frac{x}{x-1}\right)'$$

$$f_2'(x)=\left(\frac{x}{x-1}\right)^x\cdot\left(\ln\frac{x}{x-1}+x\cdot\left(\ln\frac{x}{x-1}\right)'\right)$$

$$=\left(\frac{x}{x-1}\right)^x\cdot\left(\ln\frac{x}{x-1}+x\cdot\frac{x-1}{x}\cdot\frac{x-1-x}{(x-1)^2}\right)$$

$$f_2'(x)=\left(\frac{x}{x-1}\right)^x\cdot\left(\ln\frac{x}{x-1}-\frac{1}{x-1}\right)\tag{2.1.1}$$

In (2.1.1), for $x\geq 3$, $\frac{1}{x-1}=\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+\frac{1}{x^4}+\frac{1}{x^5}+\frac{1}{x^6}+\cdots$

Using the formula: $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\frac{x^6}{6}+\cdots$, we have

$$\ln\frac{x}{x-1}=\ln\frac{1}{1+\frac{-1}{x}}=-\ln\left(1+\frac{-1}{x}\right)=\frac{1}{x}+\frac{1}{2x^2}+\frac{1}{3x^3}+\frac{1}{4x^4}+\frac{1}{5x^5}+\frac{1}{6x^6}+\cdots$$

Thus, for $x\geq 3$, $\ln\frac{x}{x-1}-\frac{1}{x-1}<0$.

Since $\left(\frac{x}{x-1}\right)^x$ is a positive number for $x\geq 3$,

$$f_2'(x)=\left(\frac{x}{x-1}\right)^x\cdot\left(\ln\frac{x}{x-1}-\frac{1}{x-1}\right)<0.$$

Thus $f_2(x)$ is a strictly deceasing function for $x\geq 3$. Since $f_2(3)=3.375$ and $\lim_{x\to\infty}f_2(x)=e\approx 2.718$, for $x\geq 3$, $3.375\geq f_2(x)=\left(\frac{x}{x-1}\right)^x\geq e$ (2.1.2)

Since for $x\geq 3$, $f_1(x)$ has a lower bound of 4 and $f_2(x)$ has an upper bound of 3.375,

$$f_1(x)=\frac{2(2x-1)}{x-1}>f_2(x)=\left(\frac{x}{x-1}\right)^x$$

When $x=\lambda\geq 3$, we have $\frac{2(2\lambda-1)}{\lambda-1}>\left(\frac{\lambda}{\lambda-1}\right)^\lambda$ (2.1.3)

When $\lambda\geq 3$ and $n=2$, $\binom{\lambda n}{n}=\binom{2\lambda}{2}=\frac{2\lambda(2\lambda-1)(2\lambda-2)!}{2(2\lambda-2)!}=\lambda(2\lambda-1)$

(2.1.4)

$$\frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}}=\frac{\lambda^{2\lambda-\lambda+1}}{2(\lambda-1)^{2(\lambda-1)-\lambda+1}}=\frac{\lambda(\lambda-1)}{2}\cdot\left(\frac{\lambda}{\lambda-1}\right)^\lambda\tag{2.1.5}$$

Since $\frac{\lambda(\lambda-1)}{2}$ is a positive number for $\lambda\geq 3$, referring to (2.1.4) and (2.1.5), when $\frac{\lambda(\lambda-1)}{2}$ multiplies both sides of (2.1.3), we have

$$\left(\frac{\lambda(\lambda-1)}{2}\right)\left(\frac{2(2\lambda-1)}{\lambda-1}\right)=\lambda(2\lambda-1)=\binom{\lambda n}{n}>\left(\frac{\lambda(\lambda-1)}{2}\right)\left(\frac{\lambda}{\lambda-1}\right)^\lambda=\frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}}$$

Thus, $\binom{\lambda n}{n} > \frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}}$ when $\lambda \geq 3$ and $n = 2$. (2.1.6)

By induction on *n*, when $\lambda \geq 3$, if $\binom{\lambda n}{n} > \frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}}$ is true for *n*, then for $n+1$,

$$\binom{\lambda(n+1)}{n+1} = \binom{\lambda n+\lambda}{n+1} = \frac{(\lambda n+\lambda)(\lambda n+\lambda-1)\cdots(\lambda n+2)(\lambda n+1)}{(\lambda n+\lambda-n-1)(\lambda n+\lambda-n-2)\cdots(\lambda n-n+1)(n+1)}\cdot\binom{\lambda n}{n}$$

$$\binom{\lambda(n+1)}{n+1} > \frac{(\lambda n+\lambda)(\lambda n+\lambda-1)\cdots(\lambda n+2)(\lambda n+1)}{(\lambda n+\lambda-n-1)(\lambda n+\lambda-n-2)\cdots(\lambda n-n+1)(n+1)}\cdot\frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}}$$

$$\binom{\lambda(n+1)}{n+1} > \frac{(\lambda n+\lambda)(\lambda n+\lambda-1)\cdots(\lambda n+2)}{(\lambda n+\lambda-n-1)(\lambda n+\lambda-n-2)\cdots(\lambda n-n+1)}\cdot\frac{\lambda n+1}{n}\cdot\frac{1}{n+1}\cdot\frac{\lambda^{\lambda n-\lambda+1}}{(\lambda-1)^{(\lambda-1)n-\lambda+1}}$$

Notice $\frac{\lambda n+1}{n} > \lambda$, and

$\frac{(\lambda n+\lambda)(\lambda n+\lambda-1)\cdots(\lambda n+2)}{(\lambda n+\lambda-n-1)(\lambda n+\lambda-n-2)\cdots(\lambda n-n+1)} > \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}$ because

$\frac{\lambda n+\lambda}{\lambda n+\lambda-n-1} = \frac{\lambda}{\lambda-1}; \frac{\lambda n+\lambda-1}{\lambda n+\lambda-n-2} > \frac{\lambda}{\lambda-1}; \cdots; \frac{\lambda n+2}{\lambda n-n+1} > \frac{\lambda}{\lambda-1}$. Thus,

$$\binom{\lambda(n+1)}{n+1} > \frac{\lambda^{\lambda-1}}{(\lambda-1)^{\lambda-1}}\cdot\frac{\lambda}{1}\cdot\frac{1}{n+1}\cdot\frac{\lambda^{\lambda n-\lambda+1}}{(\lambda-1)^{(\lambda-1)n-\lambda+1}} = \frac{\lambda^{\lambda(n+1)-\lambda+1}}{(n+1)(\lambda-1)^{(\lambda-1)(n+1)-\lambda+1}} \quad (2.1.7)$$

From (2.1.6) and (2.1.7), we have for $\lambda \geq 3$ and $n \geq 2$,

$$\binom{\lambda n}{n} > \frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}}$$

Referring to (1.8), for $\lambda \geq 3$ and $n \geq 2$,

$$\binom{\lambda n}{n} \leq \Gamma_{\lambda n\geq p>n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\}\cdot 2^{2n-3}\cdot(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+2}.$$

Then, $\Gamma_{\lambda n\geq p>n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\}\cdot 2^{2n-3}\cdot(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+2} > \frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}}$.

Since when $\lambda \geq 3$ and $n \geq 2$, then $2^{2n-3} > 0$ and $(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+2} > 0$.

$$\Gamma_{\lambda n\geq p>n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\} > \frac{\lambda^{\lambda n-\lambda+1}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+2}\cdot 2^{2n-3}\cdot n(\lambda-1)^{(\lambda-1)n-\lambda+1}} = \frac{2\lambda^2\cdot\left(\frac{\lambda}{4}\cdot\left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{n-1}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+3}}$$

Thus, **Lemma 1** is proven.

Lemma 2: When $n \ge \lambda - 2 \ge 9$, $f_3(n,\lambda) = \dfrac{2\lambda^2 \cdot \left(\dfrac{\lambda-1}{4} \cdot e\right)^{n-1}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+3}}$ is an increasing function with respect to the product of λn and with respect to n. (2.2)

Proof:

Referring to (2.1.2), when $\lambda \ge 3$, $\left(\dfrac{\lambda}{\lambda-1}\right)^{\lambda} \ge e$. From (2.1), when $\lambda \ge 3$ and $n \ge 2$,

$$\Gamma_{\lambda n \ge p > n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\} > f_8(n,\lambda) = \frac{2\lambda^2 \cdot \left(\dfrac{\lambda-1}{4} \cdot \left(\dfrac{\lambda}{\lambda-1}\right)^{\lambda}\right)^{n-1}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+3}}, \text{ and}$$

$$f_8(n,\lambda) \ge \frac{2\lambda^2 \cdot \left(\dfrac{\lambda-1}{4} \cdot e\right)^{n-1}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+3}} = f_3(n,\lambda) > 0.$$

Thus, $\Gamma_{\lambda n \ge p > n}\left\{\dfrac{(\lambda n)!}{((\lambda-1)n)!}\right\} > f_8(n,\lambda) \ge f_3(n,\lambda) > 0$. (2.2.1)

Let $x \ge 2$ and $y \ge 4$ both be real numbers. When $x = y - 2$,

$$f_3(x,y) = \frac{2(x+2)^2 \cdot \left(\dfrac{x+1}{4} \cdot e\right)^{x-1}}{((x+2)\cdot x)^{\frac{\sqrt{x\cdot(x+2)}}{3}+3}} > f_4(x) = \frac{2(x+2)^2 \cdot \left(\dfrac{x+1}{4} \cdot e\right)^{x-1}}{((x+2)\cdot x)^{\frac{x+1}{3}+3}} > 0 \quad (2.2.2)$$

$$f_4'(x) = f_4(x) \cdot \left(\frac{2}{x+2} + \ln\left(\frac{x+1}{4}\right) + \frac{4}{3} - \frac{2}{x+1} - \frac{1}{3}\ln((x+2)\cdot x) - \frac{10}{3x} - \frac{8}{3(x+2)}\right)$$
$$= f_4(x) \cdot f_5(x)$$

where $f_5(x) = \dfrac{2}{x+2} + \ln\left(\dfrac{x+1}{4}\right) + \dfrac{4}{3} - \dfrac{2}{x+1} - \dfrac{1}{3}\ln((x+2)\cdot x) - \dfrac{10}{3x} - \dfrac{8}{3(x+2)}$

$f_5'(x) = \dfrac{4x+6}{(x+1)^2 \cdot (x+2)^2} + \dfrac{x^2+2x-2}{3x(x+1)(x+2)} + \dfrac{10}{3x^2} + \dfrac{8}{3(x+2)^2} > 0$ when $x \ge 2$.

Thus, $f_5(x)$ is a strictly increasing function for $x \ge 2$.

When $x = 9$,

$f_5(x) = \dfrac{2}{9+2} + \ln\left(\dfrac{9+1}{4}\right) + \dfrac{4}{3} - \dfrac{2}{9+1} - \dfrac{1}{3}\ln(9) - \dfrac{1}{3}\ln(9+2) - \dfrac{10}{27} - \dfrac{8}{33} > 0$. Thus, for $x \ge 9$, $f_5(x) > 0$. Then, $f_4'(x) = f_4(x) \cdot f_5(x) > 0$.

Thus, $f_4(x)$ is a strictly increasing function for $x \ge 9$.

From (2.2.2), when $x = y - 2$, $f_3(x,y) > f_4(x) > 0$. Thus, when $x = y - 2 \ge 9$ and $xy \ge 99$, then $f_3(x,y)$ is an increasing function respect to the product of xy. (2.2.3)

$$\frac{\partial f_3(x,y)}{\partial x} = f_3(x,y) \cdot \left(\ln\left(\frac{y-1}{4}\right) + 1 - \frac{\sqrt{y}}{6\sqrt{x}} \cdot \ln(yx) - \frac{\sqrt{y}}{3\sqrt{x}} - \frac{3}{x}\right) = f_3(x,y) \cdot f_6(x,y) \quad (2.2.4)$$

where $f_6(x,y)=\ln\left(\frac{y-1}{4}\right)+1-\frac{\sqrt{y}}{6\sqrt{x}}\cdot\ln(yx)-\frac{\sqrt{y}}{3\sqrt{x}}-\frac{3}{x}$

When $x=y-2$, then

$$f_6(x,y)=f_7(x)=\ln\left(\frac{x+1}{4}\right)+1-\frac{\sqrt{x+2}}{6\sqrt{x}}\cdot\left(\ln(x+2)+\ln(x)+2\right)-\frac{3}{x}$$

When $x\ge 2$, $f_7'(x)=\frac{1}{x+1}-\frac{\sqrt{x+2}}{6\sqrt{x}}\cdot\left(\frac{1}{x+2}+\frac{1}{x}\right)+\frac{\ln(x+2)+\ln(x)+2}{6x\sqrt{x(x+2)}}+\frac{3}{x^2}$

$$\begin{aligned} f_7'(x)&=\frac{1}{x+1}-\frac{\sqrt{x+2}}{6\sqrt{x}}\cdot\frac{x+x+2}{x(x+2)}+\frac{\ln(x+2)+\ln(x)+2}{6x\sqrt{x(x+2)}}+\frac{3}{x^2}\\ &=\frac{1}{x+1}-\frac{1}{3\sqrt{x}}\cdot\frac{x+1}{x\sqrt{x+2}}+\frac{2}{6x\sqrt{x(x+2)}}+\frac{\ln(x+2)+\ln(x)}{6x\sqrt{x(x+2)}}+\frac{3}{x^2} \quad (2.2.5)\\ &=\frac{1}{x+1}-\frac{x}{3x\sqrt{x(x+2)}}+\frac{\ln(x+2)+\ln(x)}{6x\sqrt{x(x+2)}}+\frac{3}{x^2}\end{aligned}$$

When $x\ge 2$, then $3\sqrt{x(x+2)}+(x+1)>0$

$$\begin{aligned}&\left(3\sqrt{x(x+2)}+(x+1)\right)\cdot\left(3\sqrt{x(x+2)}-(x+1)\right)\\ &=\left(3\sqrt{x(x+2)}\right)^2-(x+1)^2=8x^2+16x-1>0\end{aligned}$$

Thus, $\left(3\sqrt{x(x+2)}+(x+1)\right)\cdot\left(3\sqrt{x(x+2)}-(x+1)\right)>0$

$$3\sqrt{x(x+2)}-(x+1)>0$$

$$3\sqrt{x(x+2)}>x+1 \quad \text{then} \quad \frac{1}{x+1}>\frac{1}{3\sqrt{x(x+2)}}$$

When $x\ge 2$, $\frac{1}{x+1}-\frac{1}{3\sqrt{x(x+2)}}>0$, and from (2.2.5),

$\frac{\ln(x+2)+\ln(x)}{6x\sqrt{x(x+2)}}+\frac{3}{x^2}>0$.

Then $f_7'(x)=\left(\frac{1}{x+1}-\frac{1}{3\sqrt{x(x+2)}}\right)+\frac{\ln(x+2)+\ln(x)}{6x\sqrt{x(x+2)}}+\frac{3}{x^2}>0$.

Thus, when $x\ge 2$, $f_7(x)$ is a strictly increasing function.

When $x=y-2\ge 2$, since $f_6(x,y)=f_7(x)$, $f_6(x,y)$ is an increasing function respect to *xy*.

When $x=y-2=9$, $f_6(x,y)=\ln\left(\frac{11-1}{4}\right)+1-\frac{\sqrt{11}}{6\sqrt{9}}\cdot\ln(99)-\frac{\sqrt{11}}{3\sqrt{9}}-\frac{3}{9}>0$.

When $x\ge y-2\ge 2$,

$$\frac{\partial f_6(x,y)}{\partial x}=\frac{\sqrt{y}}{12x\sqrt{x}}\cdot\ln(y)+\frac{\sqrt{y}}{12x\sqrt{x}}\cdot\ln(x)+\frac{\sqrt{y}}{6x\sqrt{x}}+\frac{\sqrt{y}}{6x\sqrt{x}}+\frac{3}{x^2}>0.$$

Thus, when $x\ge y-2\ge 9$, $f_6(x,y)>0$, and it is an increasing function with respect to *x* and to the product of *xy*, then, from (2.2.4),

$$\frac{\partial f_3(x,y)}{\partial x}=f_3(x,y)\cdot f_6(x,y)>0.$$

Thus, when $x \ge y-2 \ge 9$, $f_3(x,y)$ is an increasing function with respect to x. (2.2.6)

Referring to (2.2.3) and (2.2.6), when $x \ge y-2 \ge 9$, then $xy \ge 99$, $f_3(x,y)$ is an increasing function with respect to the product of xy and with respect to x.

Let $x = n$ and $y = \lambda$. Then when $n \ge \lambda - 2 \ge 9$, $f_3(n,\lambda)$ is an increasing function with respect to the product of λn and with respect to n.

Thus, **Lemma 2** is proven.

Lemma 3: When $n \ge 24$ and $26 \ge \lambda \ge 3$,

$$f_8(n,\lambda) = \frac{2\lambda^2 \cdot \left(\frac{\lambda}{4} \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{n-1}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+3}}$$ is a strictly increasing function respect to n. (2.3)

Proof:

Referring to (2.1), when $\lambda \ge 3$ and $n \ge 2$,

$$f_8(n,\lambda) = \frac{2\lambda^2 \cdot \left(\frac{\lambda}{4} \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{n-1}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+3}} > 0$$

Let a real number $x \ge 2$. When $\lambda \ge 3$, then

$$f_8(x,\lambda) = \frac{2\lambda^2 \cdot \left(\frac{\lambda}{4} \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{x-1}}{(\lambda x)^{\frac{\sqrt{\lambda x}}{3}+3}} > 0 \quad (2.3.1)$$

When λ is an integer constant in the range of $10 \ge \lambda \ge 3$,

$$\frac{\partial f_8(x,\lambda)}{\partial x} = 2\lambda^2 \cdot \frac{\frac{\partial}{\partial x}\left(\left(\frac{\lambda}{4} \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{x-1}\right) \cdot \left((\lambda x)^{\frac{\sqrt{\lambda x}}{3}+3}\right) - \left(\frac{\lambda}{4} \cdot \left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{x-1} \cdot \frac{\partial}{\partial x}\left((\lambda x)^{\frac{\sqrt{\lambda x}}{3}+3}\right)}{\left((\lambda x)^{\frac{\sqrt{\lambda x}}{3}+3}\right)^2}$$

$$\begin{aligned}\frac{\partial f_8(x,\lambda)}{\partial x} &= f_8(x,\lambda) \cdot \left(\ln\left(\frac{\lambda}{4}\right) + \ln\left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1} - \frac{\sqrt{\lambda}\left(\ln(x)+\ln(\lambda)+2\right)}{6\sqrt{x}} - \frac{3}{x}\right) \\ &= f_8(x,\lambda) \cdot f_9(x,\lambda)\end{aligned}$$

where

$$f_9(x,\lambda) = \ln\left(\frac{\lambda}{4}\right) + \ln\left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1} - \frac{\sqrt{\lambda}\left(\ln(x)+\ln(\lambda)+2\right)}{6\sqrt{x}} - \frac{3}{x}$$

$\frac{\partial f_9(x,\lambda)}{\partial x} = \frac{\sqrt{\lambda}\ln(\lambda) + \sqrt{\lambda}\ln(x)}{12x\sqrt{x}} + \frac{3}{x^2} > 0$ for $x > 1$ and $\lambda > 2$; then, $f_9(x,\lambda)$ is a strictly increasing function respect to x.

When $x = 24$ and $10 \ge \lambda \ge 3$, $f_9(x,\lambda) > 0$. The calculations are below.

When $\lambda = 3$, $f_9(x,\lambda) = \ln\left(\frac{3}{4}\right) + \ln\left(\frac{3}{3-1}\right)^{3-1} - \frac{\sqrt{3}\left(\ln(24)+\ln(3)+2\right)}{6\sqrt{24}} - \frac{3}{24} \approx 0.0283 > 0$.

When $\lambda = 4$, $f_9(x,\lambda) = \ln\left(\frac{4}{4}\right) + \ln\left(\frac{4}{4-1}\right)^{4-1} - \frac{\sqrt{4}\left(\ln(24)+\ln(4)+2\right)}{6\sqrt{24}} - \frac{3}{24} \approx 0.2814 > 0$.

When $\lambda = 5$, $f_9(x,\lambda) = \ln\left(\frac{5}{4}\right) + \ln\left(\frac{5}{5-1}\right)^{5-1} - \frac{\sqrt{5}\left(\ln(24)+\ln(5)+2\right)}{6\sqrt{24}} - \frac{3}{24} \approx 0.4744 > 0$.

When $\lambda = 6$, $f_9(x,\lambda) = \ln\left(\frac{6}{4}\right) + \ln\left(\frac{6}{6-1}\right)^{6-1} - \frac{\sqrt{6}\left(\ln(24)+\ln(6)+2\right)}{6\sqrt{24}} - \frac{3}{24} \approx 0.6113 > 0$.

When $\lambda = 7$, $f_9(x,\lambda) = \ln\left(\frac{7}{4}\right) + \ln\left(\frac{7}{7-1}\right)^{7-1} - \frac{\sqrt{7}\left(\ln(24)+\ln(7)+2\right)}{6\sqrt{24}} - \frac{3}{24} \approx 0.7183 > 0$.

When $\lambda = 8$, $f_9(x,\lambda) = \ln\left(\frac{8}{4}\right) + \ln\left(\frac{8}{8-1}\right)^{8-1} - \frac{\sqrt{8}\left(\ln(24)+\ln(8)+2\right)}{6\sqrt{24}} - \frac{3}{24} \approx 0.8044 > 0$.

When $\lambda = 9$, $f_9(x,\lambda) = \ln\left(\frac{9}{4}\right) + \ln\left(\frac{9}{9-1}\right)^{9-1} - \frac{\sqrt{9}\left(\ln(24)+\ln(9)+2\right)}{6\sqrt{24}} - \frac{3}{24} \approx 0.8755 > 0$.

When $\lambda = 10$, $f_9(x,\lambda) = \ln\left(\frac{10}{4}\right) + \ln\left(\frac{10}{10-1}\right)^{10-1} - \frac{\sqrt{10}\left(\ln(24)+\ln(10)+2\right)}{6\sqrt{24}} - \frac{3}{24} \approx 0.9347 > 0$.

From (2.3.1), when $x \ge 2$ and $\lambda \ge 3$, $f_8(x,\lambda) > 0$.

Since $\frac{\partial f_8(x,\lambda)}{\partial x} = f_8(x,\lambda)\cdot f_9(x,\lambda)$, when $x = 24$ and $10 \ge \lambda \ge 3$, $\frac{\partial f_8(x,\lambda)}{\partial x} > 0$.

Thus, when $x \ge 24$ and $10 \ge \lambda \ge 3$, $f_8(x,\lambda)$ is a strictly increasing function respect to *x*.

Let $x = n \ge 24$ and $10 \ge \lambda \ge 3$, $f_8(n,\lambda)$ is a strictly increasing function respect to *n*. (2.3.2)

From (2.2.1), when $\lambda \ge 3$ and $n \ge 2$, $f_8(n,\lambda) \ge f_3(n,\lambda) > 0$. Referring to (2.2), when $\lambda \ge 11$ and $n \ge \lambda - 2$, $f_3(n,\lambda)$ is an increasing function with respect to the product of λn and with respect to *n*.

When $n \ge 24$ and $26 \ge \lambda \ge 11$, since $n \ge \lambda - 2$ and $f_8(n,\lambda) \ge f_3(n,\lambda)$, $f_8(n,\lambda)$ is an increasing function with respect to the product of λn and with respect to *n*. (2.3.3)

From (2.3.2) and (2.3.3), when $n \ge 24$ and $26 \ge \lambda \ge 3$, $f_8(n,\lambda)$ is an increasing function with respect to *n*.

Thus, **Lemma 3** is proven.

Lemma 4: When $n \ge n_0 \ge 24$ and $26 \ge \lambda \ge 3$, if

$$f_8(n,\lambda) = \frac{2\lambda^2 \cdot \left(\frac{\lambda}{4}\cdot\left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{n-1}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+3}} > 1,$$

then there exists at least a prime number *p* such that $(\lambda-1)n < p \le \lambda n$. (2.4)

Proof:

Let integers $m \ge n \ge n_0 \ge 24$.

From (2.3), when $n \ge n_0 \ge 24$ and $26 \ge \lambda \ge 3$, $f_8(n_0, \lambda)$ is an increasing function respect to *n*.

When $26 \ge \lambda \ge 3$, if $f_8(n_0, \lambda) > 1$, then $f_8(n, \lambda) > 1$, and thus, $f_8(m, \lambda) > 1$; then from (2.1), $\Gamma_{\lambda n \ge p > n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\} > f_8(n, \lambda) > 1$, and

$$\Gamma_{\lambda m \ge p > m}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\} > f_8(m, \lambda) > 1. \tag{2.4.1}$$

Note that $m \ge n \ge n_0 \ge 24 \ge \lambda - 2$ since $26 \ge \lambda \ge 3$.

From (1.7), when $n \ge \lambda - 2$, every prime number in $\Gamma_{\lambda n \ge p > n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\}$ has a power of 0 or 1; when $m \ge \lambda - 2$, every prime number in

$$\Gamma_{\lambda m \ge p > m}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\} \text{ has a power of 0 or 1.} \tag{2.4.2}$$

$$\Gamma_{\lambda m \ge p > m}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\} = \Gamma_{\lambda m \ge p > (\lambda-1)m}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}$$
$$\cdot \prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{(\lambda-1)m}{i} \ge p > \frac{\lambda m}{i+1}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\} \cdot \Gamma_{\frac{\lambda m}{i+1} \ge p > \frac{(\lambda-1)m}{i+1}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\right)$$

In $\prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{(\lambda-1)m}{i} \ge p > \frac{\lambda m}{i+1}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\right)$, for every distinct prime number *p* in these ranges, the numerator $(\lambda m)!$ has the product of $p \cdot 2p \cdot 3p \cdots ip = i! \cdot p^i$. The denominator $((\lambda-1)m)!$ also has the same product of $i! \cdot p^i$. Thus, they cancel each other in $\frac{(\lambda m)!}{((\lambda-1)m)!}$.

Referring to (1.2), $\prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{(\lambda-1)m}{i} \ge p > \frac{\lambda m}{i+1}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\right) = 1$.

Thus,

$$\Gamma_{\lambda m \ge p > m}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}$$
$$= \Gamma_{\lambda m \ge p > (\lambda-1)m}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\} \cdot \prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{\lambda m}{i+1} \ge p > \frac{(\lambda-1)m}{i+1}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\right)$$

$$\Gamma_{\lambda m \ge p > m}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\} = \prod_{i=1}^{i=\lambda-1}\left(\Gamma_{\frac{\lambda m}{i} \ge p > \frac{(\lambda-1)m}{i}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\right). \tag{2.4.3}$$

$\prod_{i=1}^{i=\lambda-1}\left(\Gamma_{\frac{\lambda m}{i} \ge p > \frac{(\lambda-1)m}{i}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\right)$ is the product of (λ − 1) sectors from $i = 1$ to $i = \lambda - 1$.

Each of these sectors is the prime number factorization of the product of the consecutive integers between $\frac{(\lambda-1)m}{i}$ and $\frac{\lambda m}{i}$.

Referring to (2.4.3), when $\Gamma_{\lambda m \ge p > m}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\} > 1$, then

$$\prod_{i=1}^{i=\lambda-1}\left(\Gamma_{\frac{\lambda m}{i} \ge p > \frac{(\lambda-1)m}{i}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\right) > 1.$$

Referring to (1.1), $\Gamma_{\frac{\lambda m}{i} \ge p > \frac{(\lambda-1)m}{i}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\} \ge 1$. Thus, when

$\Gamma_{\lambda m \ge p > m}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\} > 1$, at least one of the sectors is greater than one in

$$\prod_{i=1}^{i=\lambda-1}\left(\Gamma_{\frac{\lambda m}{i} \ge p > \frac{(\lambda-1)m}{i}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\right).$$

Let $\Gamma_{\frac{\lambda m}{i} \ge p > \frac{(\lambda-1)m}{i}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\} > 1$ be such a sector and let $m = ni$ where $\lambda - 1 \ge i \ge 1$ from (2.4.3). Thus, when $m = ni \ge n \ge \lambda - 2$,

$$\Gamma_{\frac{\lambda ni}{i} \ge p > \frac{(\lambda-1)ni}{i}}\left\{\frac{(\lambda ni)!}{((\lambda-1)ni)!}\right\} = \Gamma_{\lambda n \ge p > (\lambda-1)n}\left\{\frac{(\lambda ni)!}{((\lambda-1)ni)!}\right\} > 1. \tag{2.4.4}$$

$$\frac{(\lambda ni)!}{((\lambda-1)ni)!} = \frac{(\lambda ni)\cdot(\lambda ni-1)\cdots(\lambda ni-i)\cdots(\lambda ni-2i)\cdots(\lambda ni-(n-1)i)\cdots(\lambda ni-ni+1)\cdot((\lambda-1)ni)!}{((\lambda-1)ni)!}$$

$$\frac{(\lambda ni)!}{((\lambda-1)ni)!} = \frac{i\cdot(\lambda n)\cdot(\lambda ni-1)\cdots i\cdot(\lambda n-1)\cdots i\cdot(\lambda n-2)\cdots i\cdot(\lambda n-n+1)\cdots(\lambda ni-ni+1)\cdot((\lambda-1)ni)!}{((\lambda-1)ni)!}$$

Thus, $\frac{(\lambda ni)!}{((\lambda-1)ni)!}$ contains all the factors of $\lambda n, \lambda n-1, \lambda n-2, \cdots, \lambda n-n+1$ in $\frac{(\lambda n)!}{((\lambda-1)n)!}$.

These factors make up all the consecutive integers in the range of $\lambda n \ge p > (\lambda-1)n$ in $\frac{(\lambda n)!}{((\lambda-1)n)!}$. Thus, $\frac{(\lambda ni)!}{((\lambda-1)ni)!}$ contains $\frac{(\lambda n)!}{((\lambda-1)n)!}$.

Referring to the definition, all prime numbers in $\frac{(\lambda ni)!}{((\lambda-1)ni)!}$ in the ranges of $\lambda ni \ge p > \lambda n$ and $(\lambda-1)n > p$ do not contribute to

$\Gamma_{\lambda n\ge p>(\lambda-1)n}\left\{\frac{(\lambda ni)!}{((\lambda-1)ni)!}\right\}$, nor does i for $\lambda-1\ge i\ge 1$. Only the prime numbers in the prime factorization of $\frac{(\lambda ni)!}{((\lambda-1)ni)!}$ in the range of $\lambda n\ge p>(\lambda-1)n$ present in $\Gamma_{\lambda n\ge p>(\lambda-1)n}\left\{\frac{(\lambda ni)!}{((\lambda-1)ni)!}\right\}$. Since $\frac{(\lambda n)!}{((\lambda-1)n)!}$ is the product of all the consecutive integers in this range,

$$\Gamma_{\lambda n\ge p>(\lambda-1)n}\left\{\frac{(\lambda ni)!}{((\lambda-1)ni)!}\right\}=\Gamma_{\lambda n\ge p>(\lambda-1)n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\}.$$

Referring to (2.4.4), $\Gamma_{\lambda n\ge p>(\lambda-1)n}\left\{\frac{(\lambda ni)!}{((\lambda-1)ni)!}\right\}=\Gamma_{\lambda n\ge p>(\lambda-1)n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\}>1$. (2.4.5)

Referring to (2.4.1), (2.4.3), (2.4.4), and (2.4.5), when $26\ge\lambda\ge 3$ and $n\ge n_0\ge 24$,

if $f_8(n,\lambda)>1$, then $f_8(m,\lambda)>1$ and $\Gamma_{\lambda m\ge p>m}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}>1$;

when $\Gamma_{\lambda m\ge p>m}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}>1$, then $\Gamma_{\lambda n\ge p>(\lambda-1)n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\}>1$.

Thus, when $26\ge\lambda\ge 3$ and $n\ge n_0\ge 24$, if $f_8(n,\lambda)>1$, then $\Gamma_{\lambda n\ge p>(\lambda-1)n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\}>1$, referring to (1.3), there exists at least a prime number p such that $(\lambda-1)n<p\le\lambda n$.

Thus, **Lemma 4** is proven.

Lemma 5: When $n\ge\lambda-2\ge 25$, there exists at least a prime number p such that $(\lambda-1)n<p\le\lambda n$. (2.5)

Proof:

Referring to (2.2), when $n\ge\lambda-2\ge 9$, $f_3(n,\lambda)=\frac{2\lambda^2\cdot\left(\frac{\lambda-1}{4}\cdot e\right)^{n-1}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+3}}$ is an increasing function with respect to the product of λn and with respect to n.

Let integers $n_1=25$, and $\lambda_1=27$. Then

$$f_3(n_1,\lambda_1)=\frac{2\cdot 27^2\cdot\left(\frac{27-1}{4}\cdot e\right)^{25-1}}{(25\cdot 27)^{\frac{\sqrt{675}}{3}+3}}\approx\frac{1.2495E+33}{9.7843E+32}>1.$$

When $m=n=n_1=\lambda-2=\lambda_1-2=25$, $f_3(m,\lambda)=f_3(n,\lambda)=f_3(n_1,\lambda_1)>1$.

From (2.2), when $m=n=\lambda-2\ge 25$, $f_3(n,\lambda)>1$ and $f_3(m,\lambda)>1$ since $f_3(n,\lambda)$ is an increasing function with respect to the product of λn. When $m\ge n\ge\lambda-2\ge 25$, then $f_3(n,\lambda)>1$ and $f_3(m,\lambda)>1$ since $f_3(n,\lambda)$ is also an increasing function with respect to n.

Referring to (2.2.1), when $m \ge n \ge \lambda - 2 \ge 25$,

$$\Gamma_{\lambda n \ge p > n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\} > f_3(n,\lambda) \ge f_3(n_1,\lambda_1) > 1 \text{; and}$$

$$\Gamma_{\lambda m \ge p > m}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\} > f_3(m,\lambda) \ge f_3(n,\lambda) > 1. \quad (2.5.1)$$

Referring to (2.4.2), when $m \ge n \ge \lambda - 2 \ge 25$, every prime number in

$\Gamma_{\lambda n \ge p > n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\}$ and in $\Gamma_{\lambda m \ge p > m}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}$ has a power of 0 or 1. (2.5.2)

Referring to (2.5.1), (2.4.3), (2.4.4), and (2.4.5), when $m \ge n \ge \lambda - 2 \ge 25$,

$\Gamma_{\lambda n \ge p > n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\} > 1$, and $\Gamma_{\lambda m \ge p > m}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\} > 1$, then

$\Gamma_{\lambda n \ge p > (\lambda-1)n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\} > 1$; referring to (1.3), there exists at least a prime number p such that $(\lambda-1)n < p \le \lambda n$.

Thus, **Lemma 5** is proven. It was proven in [[5], pp 1324-1329] with more details.

3. A Prime Number between (λ − 1)*n* and λ*n* When 26 ≥ λ ≥ 3 and *n* ≥ λ − 2

Proposition 1: For $\lambda = 3$, when $n \ge \lambda - 2$, there exists at least a prime number p such

$$\text{that } (\lambda-1)n < p \le \lambda n. \quad (3.1)$$

Proof:

Referring to (2.3) for $\lambda = 3$, when $n \ge n_0 \ge 24$, $f_8(n_0,\lambda)$ is a strictly increasing function on n_0.

Let $n_0 = 83$. When $\lambda = 3$ and $n \ge n_0 = 83$,

$$f_8(n_0,\lambda) = \frac{2\lambda^2 \cdot \left(\frac{\lambda}{4}\cdot\left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{n_0-1}}{(\lambda n_0)^{\frac{\sqrt{\lambda n_0}}{3}+3}} = \frac{18\cdot 1.6875^{83-1}}{249^{\frac{\sqrt{249}}{3}+3}} \approx \frac{7.7493E+19}{6.2000E+19} > 1.$$

Thus, for $\lambda = 3$, when $n \ge 83$, $n > \lambda - 2$ and $f_8(n,\lambda) \ge f_8(n_0,\lambda) > 1$; then, referring to (2.4), there exists at least a prime number p such that $(\lambda-1)n < p \le \lambda n$.

For $\lambda = 3$, when $82 \ge n \ge 1 = \lambda - 2$, Table 1 shows that there exists a prime number p such that $(\lambda-1)n < p \le \lambda n$.

Thus, (3.1) is proven.

Proposition 2: For $5 \ge \lambda \ge 4$, when $n \ge \lambda - 2$, there exists at least a prime number p such that

$$(\lambda-1)n < p \le \lambda n. \quad (3.2)$$

Table 1. For $82 \ge n \ge 1$ and $\lambda = 3$, there is a prime number p such that $2n < p \le 3n$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
p	3	5	7	11	13	17	19	19	23	23	29	29	31	31
n	15	16	17	18	19	20	21	22	23	24	25	26	27	28
p	37	37	41	41	43	43	47	47	53	53	59	59	61	61
n	29	30	31	32	33	34	35	36	37	38	39	40	41	42
p	67	67	71	71	73	73	79	79	83	83	89	89	97	97
n	43	44	45	46	47	48	49	50	51	52	53	54	55	56
p	101	101	103	103	107	107	109	109	113	113	127	127	131	131
n	57	58	59	60	61	62	63	64	65	66	67	68	69	70
p	139	139	149	149	151	151	157	157	163	163	167	167	173	173
n	71	72	73	74	75	76	77	78	79	80	81	82		
p	179	179	181	181	191	191	193	193	197	197	199	199		

Proof:

From (2.3), for $5 \ge \lambda \ge 4$, when $n \ge n_0 \ge 24$, $f_8(n_0, \lambda)$ is a strictly increasing function on n_0.

Let $n_0 = 40$. When $n \ge n_0 = 40$ and $5 \ge \lambda \ge 4$, $f_8(n, \lambda) \ge f_8(n_0, \lambda) > 1$. The calculations are below.

When $\lambda = 4$, $$f_8(n_0, \lambda) = \frac{2\lambda^2 \cdot \left(\frac{\lambda}{4} \cdot \left(\frac{\lambda}{\lambda - 1}\right)^{\lambda - 1}\right)^{n_0 - 1}}{(\lambda n_0)^{\frac{\sqrt{\lambda n_0}}{3} + 3}} \approx \frac{32 \cdot 2.3703^{40-1}}{160^{\frac{\sqrt{160}}{3} + 3}} \approx \frac{1.3258E + 16}{8.0503E + 15} > 1.$$

When $\lambda = 5$, $$f_8(n_0, \lambda) = \frac{2\lambda^2 \cdot \left(\frac{\lambda}{4} \cdot \left(\frac{\lambda}{\lambda - 1}\right)^{\lambda - 1}\right)^{n_0 - 1}}{(\lambda n_0)^{\frac{\sqrt{\lambda n_0}}{3} + 3}} \approx \frac{50 \cdot 3.0518^{40-1}}{200^{\frac{\sqrt{200}}{3} + 3}} \approx \frac{7.8968E + 18}{5.6253E + 17} > 1.$$

Thus, for $5 \ge \lambda \ge 4$, when $n \ge 40$, $n > \lambda - 2$ and $f_8(n, \lambda) > f_8(n_0, \lambda) > 1$; then, referring to (2.4), there exists at least a prime number p such that $(\lambda - 1)n < p \le \lambda n$.

For $5 \ge \lambda \ge 4$, when $39 \ge n \ge \lambda - 2$, Table 2 shows that there exists a prime number p such that $(\lambda - 1)n < p \le \lambda n$.

Thus, (3.2) is proven.

Proposition 3: For $11 \ge \lambda \ge 6$, when $n \ge \lambda - 2$, there exists at least a prime number p such that

$$(\lambda - 1)n < p \le \lambda n. \tag{3.3}$$

Proof:

Referring to (2.3), for $11 \ge \lambda \ge 6$, when $n \ge n_0 \ge 24$, $f_8(n_0, \lambda)$ is a strictly increasing function with respect to n_0.

Let $n_0 = 28$. When $n \ge n_0 = 28$ and $11 \ge \lambda \ge 6$, $f_8(n, \lambda) \ge f_8(n_0, \lambda) > 1$.

Table 2. For $5 \ge \lambda \ge 4$ and $39 \ge n \ge \lambda - 2$, there is a prime number p such that $(\lambda - 1)n < p \le \lambda n$.

n	2	3	4	5	6	7	8	9	10	11	12	13	14
$3n$	6	9	12	15	18	21	24	27	30	33	36	39	42
p	7	11	13	17	19	23	29	31	31	37	37	41	43
$4n$	8	12	16	20	24	28	32	36	40	44	48	52	56
p		13	17	23	29	31	37	41	43	47	53	59	61
$5n$		15	20	25	30	35	40	45	50	55	60	65	70
n	15	16	17	18	19	20	21	22	23	24	25	26	27
$3n$	45	48	51	54	57	60	63	66	69	72	75	78	81
p	47	53	59	59	61	61	67	67	71	73	79	79	83
$4n$	60	64	68	72	76	80	84	88	92	96	100	104	108
p	67	71	73	79	79	83	89	97	101	103	107	109	113
$5n$	75	80	85	90	95	100	105	110	115	120	125	130	135
n	28	29	30	31	32	33	34	35	36	37	38	39	
$3n$	84	87	90	93	96	99	102	105	108	111	114	117	
p	89	89	97	97	101	101	103	107	109	113	127	131	
$4n$	112	116	120	124	128	132	136	140	144	148	152	156	
p	113	127	131	137	139	149	151	157	163	173	179	191	
$5n$	140	145	150	155	160	165	170	175	180	185	190	195	

The calculations are below.

When $\lambda = 6$, $$f_8(n_0,\lambda) = \frac{2\lambda^2 \cdot \left(\frac{\lambda}{4}\cdot\left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{n_0-1}}{(\lambda n_0)^{\frac{\sqrt{\lambda n_0}}{3}+3}} \approx \frac{72\cdot 3.7325^{28-1}}{168^{\frac{\sqrt{168}}{3}+3}} \approx \frac{2.0014E+17}{1.9515E+16} > 1.$$

When $\lambda = 7$, $$f_8(n_0,\lambda) = \frac{2\lambda^2 \cdot \left(\frac{\lambda}{4}\cdot\left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{n_0-1}}{(\lambda n_0)^{\frac{\sqrt{\lambda n_0}}{3}+3}} \approx \frac{98\cdot 4.4128^{28-1}}{196^{\frac{\sqrt{196}}{3}+3}} \approx \frac{2.5033E+19}{3.7501E+17} > 1.$$

When $\lambda = 8$, $$f_8(n_0,\lambda) = \frac{2\lambda^2 \cdot \left(\frac{\lambda}{4}\cdot\left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{n_0-1}}{(\lambda n_0)^{\frac{\sqrt{\lambda n_0}}{3}+3}} \approx \frac{128\cdot 5.0930^{28-1}}{224^{\frac{\sqrt{224}}{3}+3}} \approx \frac{1.5686E+21}{5.9689E+18} > 1.$$

When $\lambda = 9$, $$f_8(n_0,\lambda) = \frac{2\lambda^2 \cdot \left(\frac{\lambda}{4}\cdot\left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{n_0-1}}{(\lambda n_0)^{\frac{\sqrt{\lambda n_0}}{3}+3}} \approx \frac{162\cdot 5.7730^{28-1}}{252^{\frac{\sqrt{252}}{3}+3}} \approx \frac{5.8527E+22}{8.1510E+19} > 1.$$

When $\lambda=10$, $f_8(n_0,\lambda)=\frac{2\lambda^2\cdot\left(\frac{\lambda}{4}\cdot\left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{n_0-1}}{(\lambda n_0)^{\frac{\sqrt{\lambda n_0}}{3}+3}}\approx\frac{200\cdot 6.4529^{28-1}}{280^{\frac{\sqrt{280}}{3}+3}}\approx\frac{1.4602E+24}{9.7946E+20}>1$.

When $\lambda=11$, $f_8(n_0,\lambda)=\frac{2\lambda^2\cdot\left(\frac{\lambda}{4}\cdot\left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{n_0-1}}{(\lambda n_0)^{\frac{\sqrt{\lambda n_0}}{3}+3}}\approx\frac{242\cdot 7.1328^{28-1}}{308^{\frac{\sqrt{308}}{3}+3}}\approx\frac{2.6414E+25}{1.0560E+22}>1$.

Thus, for $11\ge\lambda\ge 6$, when $n\ge 28$, $n>\lambda-2$ and $f_8(n,\lambda)\ge f_8(n_0,\lambda)>1$; then, referring to (2.4), there exists at least a prime number p such that $(\lambda-1)n<p\le\lambda n$.

For $11\ge\lambda\ge 6$, when $27\ge n\ge\lambda-2$, Table 3 shows that there exists a prime number p such that $(\lambda-1)n<p\le\lambda n$.

Thus, (3.3) is proven.

Proposition 4: For $26\ge\lambda\ge 12$, when $n\ge\lambda-2$, there exists at least a prime number p such that

$$(\lambda-1)n<p\le\lambda n. \tag{3.4}$$

Proof:

Referring to (2.3) for $26\ge\lambda\ge 12$, when $n\ge n_0\ge 24$, $f_8(n_0,\lambda)$ is a strictly increasing function with respect to n_0. Let $n_0=25$. When $n\ge n_0=25$ and $\lambda=12$, then $f_8(n,\lambda)\ge f_8(n_0,\lambda)$ and

$$f_8(n_0,\lambda)=\frac{2\lambda^2\cdot\left(\left(\frac{\lambda}{4}\right)\cdot\left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{n_0-1}}{(\lambda n_0)^{\frac{\sqrt{\lambda n_0}}{3}+3}}\approx\frac{288\cdot 7.8126^{25-1}}{300^{\frac{\sqrt{300}}{3}+3}}\approx\frac{7.7000E+23}{5.4078E+21}>1.$$

For $n\ge n_0=25$ and $26\ge\lambda\ge 13$, then $f_8(n,\lambda)\ge f_8(n_0,\lambda)$. It can be seen from Table 4 that the values of $f_8(n_0,\lambda)$ as well as $f_8(n,\lambda)$ are all greater than 1.

(Detailed calculations are in **Appendix**.)

Thus, for $26\ge\lambda\ge 12$, when $n\ge 25>\lambda-2$, $f_8(n,\lambda)\ge f_8(n_0,\lambda)>1$; then, referring to (2.4), there exists at least a prime number p such that $(\lambda-1)n<p\le\lambda n$.

For $13\ge\lambda\ge 12$ and $24\ge n\ge\lambda-2$, Table 5 shows that there exists a prime number p such that $(\lambda-1)n<p\le\lambda n$.

For $26\ge\lambda\ge 14$ and $24\ge n\ge\lambda-2$, Table 6 shows that there exists a prime number p such that $(\lambda-1)n<p\le\lambda n$.

Thus, (3.4) is proven.

Combining (3.1), (3.2), (3.3), and (3.4), when $26\ge\lambda\ge 3$ and $n\ge\lambda-2$, there exists at least a prime number p such that $(\lambda-1)n<p\le\lambda n$. (3.5)

4. A Prime Number between *kn* and (*k* + 1)*n* When *n* ≥ *k* − 1

Proposition 5: For two positive integers n and k, when $n\ge k-1$, there exists at least a prime number p such that $kn<p\le(k+1)n$. (4.1)

Table 3. For $11 \ge \lambda \ge 6$ and $27 \ge n \ge \lambda - 2$, there is a prime number p such that $(\lambda - 1)n < p \le \lambda n$.

	n	4	5	6	7	8	9	10	11	12	13	14	15
	$5n$	20	25	30	35	40	45	50	55	60	65	70	75
$\lambda = 6$	p	23	29	31	37	41	47	53	59	61	67	71	79
	$6n$	24	30	36	42	48	54	60	66	72	78	84	90
$\lambda = 7$	p		31	37	47	53	59	61	67	79	83	89	97
	$7n$		35	42	49	56	63	70	77	84	91	98	105
$\lambda = 8$	p			47	53	59	67	79	83	89	97	101	109
	$8n$			48	56	64	72	80	88	96	104	112	120
$\lambda = 9$	p				59	67	79	83	89	97	107	113	127
	$9n$				63	72	81	90	99	108	117	126	135
$\lambda = 10$	p					79	83	97	107	113	127	137	139
	$10n$					80	90	100	110	120	130	140	150
$\lambda = 11$	p						97	107	113	127	137	149	157
	$11n$						99	110	121	132	143	154	165
	n	16	17	18	19	20	21	22	23	24	25	26	27
	$5n$	80	85	90	95	100	105	110	115	120	125	130	135
$\lambda = 6$	p	83	89	97	101	103	107	113	127	131	137	139	149
	$6n$	96	102	108	114	120	126	132	138	144	150	156	162
$\lambda = 7$	p	101	107	113	127	131	137	139	149	151	157	163	167
	$7n$	112	119	126	133	140	147	154	161	168	175	182	189
$\lambda = 8$	p	113	127	131	137	149	149	151	163	173	179	191	199
	$8n$	128	136	144	152	160	168	176	184	192	200	208	216
$\lambda = 9$	p	139	137	149	163	173	179	191	199	211	223	227	229
	$9n$	144	153	162	171	180	189	198	207	216	225	234	243
$\lambda = 10$	p	157	163	173	179	191	199	211	223	227	229	239	251
	$10n$	160	170	180	190	200	210	220	230	240	250	260	270
$\lambda = 11$	p	163	179	191	199	211	223	227	239	251	257	269	281
	$11n$	176	187	198	209	220	231	242	253	264	275	286	297

Table 4. When $n \ge n_0 = 25$ and $26 \ge \lambda \ge 13$, then $f_8(n,\lambda) \ge f_8(n_0,\lambda) > 1$.

λ	13	14	15	16	17	18	19	20	21	22	23	24	25	26
$f_8(n_0)$	156	157	145	125	102	78.7	58.5	41.9	29.1	19.7	13.0	8.35	5.29	3.29

Table 5. For $13 \ge \lambda \ge 12$ and $24 \ge n \ge \lambda - 2$, there is a prime number p such that $(\lambda - 1)n < p \le \lambda n$.

	n	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
	$11n$	110	121	132	143	154	165	176	187	198	209	220	231	242	253	264
$\lambda = 12$	p	113	127	137	149	157	167	179	191	199	211	223	241	251	269	277
	$12n$	120	132	144	156	168	180	192	204	216	228	240	252	264	276	288
$\lambda = 13$	p		137	149	157	179	191	199	211	223	241	251	269	277	283	293
	$13n$		143	156	169	182	195	208	221	234	247	260	273	286	299	312

Table 6. For $26 \ge \lambda \ge 14$ and $24 \ge n \ge \lambda - 2$, there is a prime number p such that $(\lambda - 1)n < p \le \lambda n$.

	n	12	13	14	15	16	17	18	19	20	21	22	23	24
	$13n$	156	169	182	195	208	221	234	247	260	273	286	299	312
$\lambda = 14$	p	157	179	191	199	211	223	241	251	269	277	293	307	317
	$14n$	168	182	196	210	224	238	252	266	280	294	308	322	336
$\lambda = 15$	p		191	199	211	227	241	269	277	293	307	317	331	347
	$15n$		195	210	225	240	255	270	285	300	315	330	345	360
$\lambda = 16$	p			211	227	241	269	277	293	307	317	331	347	373
	$16n$			224	240	256	272	288	304	320	336	352	368	384
$\lambda = 17$	p				241	269	277	293	307	331	347	373	389	401
	$17n$				255	272	289	306	323	340	357	374	391	408
$\lambda = 18$	p					277	293	307	331	347	373	389	401	419
	$18n$					288	306	324	342	360	378	396	414	432
$\lambda = 19$	p						307	331	347	373	389	401	419	449
	$19n$						323	342	361	380	399	418	437	456
$\lambda = 20$	p							347	373	389	401	419	449	461
	$20n$							360	380	400	420	440	460	480
$\lambda = 21$	p								389	419	433	461	467	491
	$21n$								399	420	441	462	483	504
$\lambda = 22$	p									433	461	467	491	521
	$22n$									440	462	484	506	528
$\lambda = 23$	p										467	491	521	541
	$23n$										483	506	529	552
$\lambda = 24$	p											521	541	563
	$24n$											528	552	576
$\lambda = 25$	p												563	587
	$25n$												575	600
$\lambda = 26$	p													613
	$26n$													624

Proof:

Referring to (2.5), when $n \geq \lambda - 2 \geq 25$, there exists at least a prime number *p* such that $(\lambda - 1)n < p \leq \lambda n$. This statement is the same as that when $\lambda \geq 27$ and $n \geq \lambda - 2$, there exists at least a prime number *p* such that $(\lambda - 1)n < p \leq \lambda n$.

Referring to (3.5), when $26 \geq \lambda \geq 3$ and $n \geq \lambda - 2$, there exists at least a prime number *p* such that $(\lambda - 1)n < p \leq \lambda n$.

Thus, when $\lambda \geq 3$ and $n \geq \lambda - 2$, there is at least a prime number *p* such that $(\lambda - 1)n < p \leq \lambda n$.

Let integer $k = \lambda - 1$, then for $n \geq 1$ and $k \geq 2$, when $n \geq k - 1$, there exists at least a prime number *p* such that $kn < p \leq (k+1)n$. (4.2)

The Bertrand-Chebyshev's theorem points out that for $n \geq 1$ and $k = 1$, there exists at least a prime number *p* such that $kn < p \leq (k+1)n$. (4.3)

From (4.2) and (4.3), we can conclude that for $n \geq 1$ and $k \geq 1$, when $n \geq k - 1$, there exists at least a prime number *p* such that $kn < p \leq (k+1)n$. Thus, **Proposition 5** becomes a theorem, **Theorem 4.1**, and Bertrand-Chebyshev's theorem is a special case of this theorem.

In the field of prime number distribution, an important theorem is the prime number theorem, $\pi(N) \sim \frac{N}{\ln(N)}$, where $\pi(N)$ is the number of distinct prime numbers less than or equal to a natural number *N*. The prime number theorem provides the approximate number of prime numbers relative to the natural numbers, while Theorem 4.1 shows that when $n \geq k - 1$, the prime number exists in the interval between *kn* and (*k* + 1)*n*, that is, Theorem 4.1 provides the approximate locations of prime numbers among natural numbers. Using this theorem, Legendre's conjecture [7] and several other conjectures can be easily proven. The method of proving Theorem 4.1 can also help to study and solve some difficult problems in number theory such as the other three Landau problems [8].

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Aigner, M. and Ziegler, G. (2014) Proofs from THE BOOK. 5th Edition, Springer, Berlin. https://doi.org/10.1007/978-3-662-44205-0

[2] El Bachraoui, M. (2006) Prime in the Interval [2n, 3n]. *International Journal of Contemporary Mathematical Sciences*, **1**, 617-621. https://doi.org/10.12988/ijcms.2006.06065

[3] Loo, A. (2011) On the Prime in the Interval [3n, 4n]. https://arxiv.org/abs/1110.2377

[4] Shevelev, V., Greathouse IV, C. and Moses, P. (2013) On Intervals (kn, (kn + 1)n) Containing a Prime for All n > 1. *Journal of Integer Sequences*, **16**, Article 13.7.3.

[5] Yu, W. (2023) The Proofs of Legendre's Conjecture and Three Related Conjectures. *Journal of Applied Mathematics and Physics*, **11**, 1319-1336. https://doi.org/10.4236/jamp.2023.115085

[6] Wikipedia. Proof of Bertrand's Postulate. https://en.wikipedia.org/wiki/Proof_of_Bertrand%27s_postulate

[7] Wikipedia. Legendre's Conjecture. https://en.wikipedia.org/wiki/Legendre%27s_conjecture

[8] Wikipedia. Landau's problems. https://en.wikipedia.org/wiki/Landau%27s_problems

Appendix

Calculation of $f_8(n_0,\lambda)$ when $n_0 = 25$ and $26 \ge \lambda \ge 13$.

When $\lambda = 13$, $f_8(n_0,\lambda) = \dfrac{2\lambda\cdot\left(\frac{\lambda}{4}\cdot\left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{n_0-1}}{(\lambda n_0)^{\frac{\sqrt{\lambda n_0}}{3}+3}} \approx \dfrac{338\cdot 8.4924^{25-1}}{325^{\frac{\sqrt{325}}{3}+3}} \approx \dfrac{6.6934E+24}{4.2676E+22} \approx 156 > 1$.

When $\lambda = 14$, $f_8(n_0,\lambda) = \dfrac{2\lambda\cdot\left(\frac{\lambda}{4}\cdot\left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{n_0-1}}{(\lambda n_0)^{\frac{\sqrt{\lambda n_0}}{3}+3}} \approx \dfrac{392\cdot 9.1721^{25-1}}{350^{\frac{\sqrt{350}}{3}+3}} \approx \dfrac{4.9265E+25}{3.1424E+23} \approx 157 > 1$.

When $\lambda = 15$, $f_8(n_0,\lambda) = \dfrac{2\lambda\cdot\left(\frac{\lambda}{4}\cdot\left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{n_0-1}}{(\lambda n_0)^{\frac{\sqrt{\lambda n_0}}{3}+3}} \approx \dfrac{450\cdot 9.8518^{25-1}}{375^{\frac{\sqrt{375}}{3}+3}} \approx \dfrac{3.1448E+26}{2.1746E+24} \approx 145 > 1$.

When $\lambda = 16$, $f_8(n_0,\lambda) = \dfrac{2\lambda\cdot\left(\frac{\lambda}{4}\cdot\left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{n_0-1}}{(\lambda n_0)^{\frac{\sqrt{\lambda n_0}}{3}+3}} \approx \dfrac{512\cdot 10.5315^{25-1}}{400^{\frac{\sqrt{400}}{3}+3}} \approx \dfrac{1.7743E+27}{1.4234E+25} \approx 125 > 1$.

When $\lambda = 17$, $f_8(n_0,\lambda) = \dfrac{2\lambda\cdot\left(\frac{\lambda}{4}\cdot\left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{n_0-1}}{(\lambda n_0)^{\frac{\sqrt{\lambda n_0}}{3}+3}} \approx \dfrac{578\cdot 11.2112^{25-1}}{425^{\frac{\sqrt{425}}{3}+3}} \approx \dfrac{8.9862E+27}{8.8497E+25} \approx 102 > 1$.

When $\lambda = 18$, $f_8(n_0,\lambda) = \dfrac{2\lambda\cdot\left(\frac{\lambda}{4}\cdot\left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{n_0-1}}{(\lambda n_0)^{\frac{\sqrt{\lambda n_0}}{3}+3}} \approx \dfrac{648\cdot (11.8909)^{25-1}}{450^{\frac{\sqrt{450}}{3}+3}} \approx \dfrac{4.1374E+28}{5.2574E+26} \approx 78.7 > 1$.

When $\lambda = 19$, $f_8(n_0,\lambda) = \dfrac{2\lambda\cdot\left(\frac{\lambda}{4}\cdot\left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{n_0-1}}{(\lambda n_0)^{\frac{\sqrt{\lambda n_0}}{3}+3}} \approx \dfrac{722\cdot 12.5705^{25-1}}{475^{\frac{\sqrt{475}}{3}+3}} \approx \dfrac{1.7498E+29}{2.9904E+27} \approx 58.5 > 1$.

When $\lambda = 20$, $f_8(n_0,\lambda) = \dfrac{2\lambda\cdot\left(\frac{\lambda}{4}\cdot\left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{n_0-1}}{(\lambda n_0)^{\frac{\sqrt{\lambda n_0}}{3}+3}} \approx \dfrac{800\cdot 13.2502^{25-1}}{500^{\frac{\sqrt{500}}{3}+3}} \approx \dfrac{6.8616E+29}{1.6366E+27} \approx 41.9 > 1$.

When $\lambda = 21$, $f_8(n_0,\lambda) = \dfrac{2\lambda\cdot\left(\frac{\lambda}{4}\cdot\left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{n_0-1}}{(\lambda n_0)^{\frac{\sqrt{\lambda n_0}}{3}+3}} \approx \dfrac{882\cdot 13.9298^{25-1}}{525^{\frac{\sqrt{525}}{3}+3}} \approx \dfrac{2.5127E+30}{8.6291E+28} \approx 29.1 > 1$.

When $\lambda = 22$, $f_8(n_0,\lambda) = \dfrac{2\lambda\cdot\left(\frac{\lambda}{4}\cdot\left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{n_0-1}}{(\lambda n_0)^{\frac{\sqrt{\lambda n_0}}{3}+3}} \approx \dfrac{968\cdot 14.6094^{25-1}}{550^{\frac{\sqrt{550}}{3}+3}} \approx \dfrac{8.6507E+30}{4.4015E+29} \approx 19.7 > 1$.

When $\lambda = 23$, $f_8\left(n_0,\lambda\right) = \dfrac{2\lambda\cdot\left(\dfrac{\lambda}{4}\cdot\left(\dfrac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{n_0-1}}{\left(\lambda n_0\right)^{\frac{\sqrt{\lambda n_0}}{3}+3}} \approx \dfrac{1058\cdot 15.2891^{25-1}}{575^{\frac{\sqrt{575}}{3}+3}} \approx \dfrac{2.8161E+31}{2.1742E+30} \approx 13.0 > 1$.

When $\lambda = 24$, $f_8\left(n_0,\lambda\right) = \dfrac{2\lambda\cdot\left(\dfrac{\lambda}{4}\cdot\left(\dfrac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{n_0-1}}{\left(\lambda n_0\right)^{\frac{\sqrt{\lambda n_0}}{3}+3}} \approx \dfrac{1152\cdot 15.9687^{25-1}}{600^{\frac{\sqrt{600}}{3}+3}} \approx \dfrac{8.7081E+31}{1.0425E+31} \approx 8.35 > 1$.

When $\lambda = 25$, $f_8\left(n_0,\lambda\right) = \dfrac{2\lambda\cdot\left(\dfrac{\lambda}{4}\cdot\left(\dfrac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{n_0-1}}{\left(\lambda n_0\right)^{\frac{\sqrt{\lambda n_0}}{3}+3}} \approx \dfrac{1250\cdot 16.6483^{25-1}}{625^{\frac{\sqrt{625}}{3}+3}} \approx \dfrac{2.5691E+32}{4.8599E+31} \approx 5.29 > 1$.

When $\lambda = 26$, $f_8\left(n_0,\lambda\right) = \dfrac{2\lambda\cdot\left(\dfrac{\lambda}{4}\cdot\left(\dfrac{\lambda}{\lambda-1}\right)^{\lambda-1}\right)^{n_0-1}}{\left(\lambda n_0\right)^{\frac{\sqrt{\lambda n_0}}{3}+3}} \approx \dfrac{1352\cdot 17.3279^{25-1}}{650^{\frac{\sqrt{650}}{3}+3}} \approx \dfrac{7.2594E+32}{2.2080E+32} \approx 3.29 > 1$.