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Numerical Treatments of Functional Fredholm Integral Equation in 2D with Discontinuous Kernels

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Abstract

This work proposes a new definition of the functional Fredholm integral equation in 2D of the second kind with discontinuous kernels (FT-DFIE). Furthermore, the work is concerned to study this new equation numerically. The existence of a unique solution of the equation is proved. In addition, the approximate solutions are obtained by two powerful methods Toeplitz Matrix Method (TMM) and Product Nyström Methods (PNM). The given numerical examples showed the efficiency and accuracy of the introduced methods.

Keywords

Functional Integral Equation, TMM, PNM, Hammerstion

1. Introduction

Over the view past years there has been a substantial increase in the use of integral equations on the formulation of solution strategies for scientific and engineering problems. In large measure, this has been due to the work in the engineering and mathematics communities in using integral equation techniques to solve partial differential equations as an alternative to domain-based methods. In practice, approximate methods to solve the problems are needed. So many different methods that have been established can lead us the way to obtain the numerical solution. Those who are interested can review the excellent expositions by Popov [1], Tricomi [2], Hochastadt [3], Green [4] Athkinson [5], Linz [6], Delves and Mohamed [7], Kumar [8] and [9]. In [10], the approximation of solutions for nonlinear functional integral equations was examined by the authors. In [11], the author studied the singular kernel for the functional Volterra integral equation. Jafar and others, in [12], studied the functional integral equations numerically. The authors in [13], studied the Volterra-Hammerstein integral equation. In [14], the authors applied a numerical method for obtaining numerical solutions of Fredholm two-dimensional functional linear integral equations based on the radial basis function. In [15] [16], AL-Bugami studied the two-dimensional integral equations numerically. The authors, in [17] [18] [19], studied the mixed integral equations. In [20], Al-Bugami studied 2D Volterra integral equation with weakly kernels numerically.

In this work, we study the new equation for nonlinear functional integral equation in 2D with discontinuous kernels, which has not been studied before, and we employ the PNM and TMM, which plays an important role in the study of numerical solutions for FT-DFIE. Consider

$$\mu w(s,t)-\lambda f\left(s,t,\int_a^b\int_c^d p(s-u,t-v)w(u,v)\,\mathrm{d}u\mathrm{d}v\right)=g(s,t) \tag{1}$$

The functions $g(s,t), f(s,t,w(s,t))$ are given analytical functions defined, respectively, on $[a,b]\times[c,d]$ and $p(s-u,t-v)$ is the kernel of (1), $p(s-u,t-v)\geq 0$, and $w(x,y)$ is the solution to be determined.

2. Existence and Uniquess of a Solution

The following conditions apply:

(i) $p(s-u,t-v)\in C([a,b]\times[c,d])$, and satisfies:

$$\left[\int_a^b\int_c^d |p(s-u,t-v)|^2\,\mathrm{d}u\mathrm{d}v\right]^{\frac{1}{2}}=A<\infty \quad (A \text{ is a constant})$$

(ii) $g(s,t)$ maintains continuity with its derivatives and belongs to $[a,b]\times[c,d]$ and,

$$\|g(s,t)\|=\max\int_a^b\left[\int_c^d g^2(s,t)\,\mathrm{d}s\right]^{\frac{1}{2}}\mathrm{d}t=M,$$

(iii)

$$\|w(s,t)\|=\left[\int_a^b\int_c^d |w(s,t)|^2\right]^{\frac{1}{2}}\leq C\|w\|_2$$

Equation (1) is written as follows:

$$\bar{W}w(s,t)=\frac{1}{\mu}g(s,t)+Ww(s,t) \tag{2}$$

$$Ww(s,t)=\frac{\lambda}{\mu}f\left(s,t,\int_a^b\int_c^d p(s-u,t-v)w(u,v)\,\mathrm{d}u\mathrm{d}v\right) \tag{3}$$

Theorem 1. The solution to Equation (1) is unique if conditions (i)-(iii) are confirmed in $[a,b]\times[c,d]$.

Lemma 1. Under the condition (i)-(iii), the operator $\bar{W}$), maps the space $[a,b]\times[c,d]$ into itself.

Proof:

From formula (2) and (3), we get

$$\left\|\bar{W}w(s,t)\right\| \le \frac{1}{|\mu|}\left\|g(s,t)\right\| + \left|\frac{\lambda}{\mu}\right|\left\| f\left(s,t,\int_a^b\int_c^d \left|p(s-u,t-v)\right|\left|w(u,v)\right|\mathrm{d}u\mathrm{d}v\right)\right\|$$

$$\left\|\bar{W}w(s,t)\right\| \le \frac{M}{|\mu|} + \left|\frac{\lambda}{\mu}\right|\left\{ f\left(s,t,\int_a^b\int_c^d \left|p(s-u,t-v)\right|^2 \mathrm{d}u\mathrm{d}v\right)\right\}^{\frac{1}{2}} \left\{ f\left(s,t,\int_a^b\int_c^d \left|w(u,v)\right|^2 \mathrm{d}u\mathrm{d}v\right)\right\}^{\frac{1}{2}}$$

$$\left\|\bar{W}w(s,t)\right\| \le \frac{M}{|\mu|} + f\left(s,t,\theta\left\|w(s,t)\right\|\right), \quad \left(\theta = \left|\frac{\lambda}{\mu}\right| AC\right) \tag{4}$$

So, the operator $\bar{W}$ maps the space $[a,b]\times[c,d]$ into itself.

Lemma 2.

The operator $\bar{W}$ is contractive in $[a,b]\times[c,d]$.

Proof:

For $w_1(s,t)$ and $w_2(s,t)$ in the space $[a,b]\times[c,d]$, the formulas (2), (3) lead to

$$\left\|\bar{W}w_1 - \bar{W}w_2\right\| \le \left|\frac{\lambda}{\mu}\right|\left\| f\left(s,t,\int_a^b\int_c^d \left|p(s-u,t-v)\right|\left|w_1(u,v) - w_2(u,v)\right|\mathrm{d}u\mathrm{d}v\right)\right\|$$

Then, we have

$$\left\|\bar{W}w_1 - \bar{W}w_2\right\| \le \left|\frac{\lambda}{\mu}\right| f\left(s,t,\left(\int_a^b\int_c^d \left|p(s-u,t-v)\right|^2 \mathrm{d}u\mathrm{d}v\right)^{\frac{1}{2}} \left(\int_a^b\int_c^d \left|w_1(u,v) - w_2(u,v)\right|^2 \mathrm{d}u\mathrm{d}v\right)^{\frac{1}{2}}\right)$$

Then, we obtain

$$\left\|\left(\bar{W}w_1 - \bar{W}w_2\right)(s,t)\right\| \le f\left(s,t,\theta\left\|w_1(s,t) - w_2(s,t)\right\|\right) \tag{5}$$

3. The Numerical Solutions

3.1. The (TMM)

Consider:

$$\mu w(s,t) - \lambda f\left(s,t,\int_a^b\int_c^d p(s-u,t-v)\,w(u,v)\,\mathrm{d}u\mathrm{d}v\right) = g(s,t) \tag{6}$$

We can be written (6) in the form:

$$\int_0^a\int_0^b p(s-u,t-v)\,w(u,v)\,\mathrm{d}u\mathrm{d}v = \sum_{n=-N}^{N-1}\sum_{m=-M}^{M-1} p(s-u,t-v)\,w(u,v)\,\mathrm{d}u\mathrm{d}v \tag{7}$$

$h = \frac{a}{N}$, we approximate the integral in the Equation (6), if $m=n$, by

$$\begin{aligned}&\int_{nh}^{nh+h}\int_{mh}^{mh+h} p(s-u,t-v)\,w(u,v)\,\mathrm{d}u\mathrm{d}v \\ &= A_{n,m}(s,t)\,w(nh,mh) + B_{n,m}(s,t)\,w(nh+h,mh+h) + R\end{aligned} \tag{8}$$

If $w(u,v) = 1.1,\; uv$ in Equation (8), then:

$$A_{n,m}(s,t)=\frac{1}{h}\left[\frac{(nh+h)(mh+h)I}{nh+mh+h}-\frac{J}{nh+mh+h}\right] \tag{9}$$

$$B_{n,m}(s,t)=\frac{1}{h}\left[\frac{J}{nh+mh+h}-\frac{(nh)(mh)I}{nh+mh+h}\right] \tag{10}$$

where

$$I(s,t)=\int_{nh}^{nh+h}\int_{mh}^{mh+h} p(s-u,t-v)\,\mathrm{d}u\mathrm{d}v$$

$$J(s,t)=\int_{nh}^{nh+h}\int_{mh}^{mh+h} uv\cdot p(s-u,t-v)\,\mathrm{d}u\mathrm{d}v$$

Equation (7) thus becomes

$$\begin{aligned}
&\int_0^a\int_0^b p(s-u,t-v)\phi(u,v)\,\mathrm{d}u\mathrm{d}v\\
&=\sum_{n=-N}^{N-1}\sum_{m=-M}^{M-1}\left[A_{n,m}(s,t)w(nh,mh)+B_{n,m}(s,t)w(nh+h,mh+h)\right]\\
&=\sum_{n=-N}^{N-1}\sum_{m=-M}^{M-1}A_{n,m}(s,t)w(nh,mh)+\sum_{n=-N}^{N}\sum_{m=-M}^{M}B_{(n-1)(m-1)}(s,t)w(nh,mh)\\
&=\sum_{n=-N}^{N}\sum_{m=-M}^{M}D_{n,m}(s,t)w(nh,mh)
\end{aligned} \tag{11}$$

where

$$D_{n,m}(s,t)=\begin{cases}A_{-N}(s,t) & n=m=-N\\ A_n(s,t)+B_{n-1}(s,t) & -N<n=m<N\\ B_{N-1}(s,t) & n=m=N\end{cases}$$

Thus, the Equation (6) becomes:

$$\mu w(s,t)-\lambda f\left(s,t,\sum_{n=-N}^{N}\sum_{m=-M}^{M}D_{n,m}(s,t)w(nh,mh)\right)=g(s,t)$$

If we put $s=kh, t=lh$, then we get:

$$\mu w_{k,l}-\lambda f\left(s,t,\sum_{n=-N}^{N}\sum_{m=-M}^{M}D_{kln,m}w_{nm}\right)=g_{kl},\quad -N\le k\le N,-M\le l\le M \tag{12}$$

where

$$D_{kln,m}=\begin{cases}A_{-N}(kh,lh) & n=m=-N\\ A_n(kh,lh)+B_{n-1}(kh,lh) & -N<n=m<N\\ B_{N-1}(kh,lh) & n=m=N\end{cases} \tag{13}$$

The matrix $D_{kln,m}$ may be written as $D_{kln,m}=G_{kln,m}-E_{kln,m}$, where

$$G_{kln,m}=A_n(kh,lh)+B_{n-1}(kh,lh),\quad -N\le k,l,n,\le N \tag{14}$$

$$E_{kln,m}=\begin{cases}B_{-N-1}(kh,lh) & n=m=-N\\ 0 & -N<n=m<N\\ A_N(kh,lh) & n=m=N\end{cases} \tag{15}$$

3.2. The (PNM)

Consider

$$\mu w(s,t)-\lambda f\left(s,t,\int_a^b\int_c^d p(s-u,t-v)w(u,v)\mathrm{d}u\mathrm{d}v\right)=g(s,t) \tag{16}$$

We can often factor out the singularity in p by writing

$$p(s-u,t-v)=k(s-u,t-v)\overline{p}(s-u,t-v) \tag{17}$$

Equation (16) is expressed as:

$$\mu w(s,t)-\lambda f\left(s,t,\int_0^s\int_0^t k(s-u,t-v)\overline{p}(s-u,t-v)w(u,v)\mathrm{d}u\mathrm{d}v\right)=g(s,t) \tag{18}$$

The integral term in Equation (18) is estimated when $s=s_i,t=t_i$ by

$$\begin{aligned}&\int_0^s\int_0^t k(s_i-u,t_i-v)\overline{p}(s_i-u,t_i-v)w(u,v)\mathrm{d}u\mathrm{d}v\\&\approx\sum_{j=0}^{N}\sum_{i=0}^{M}\kappa_{ij}\kappa_{il}\overline{p}(s_i-u_j,t_i-v_j)w(u_i,v_j)\end{aligned} \tag{19}$$

where κ_{ij},κ_{il} are the weights. Then,

$$\begin{aligned}&\int_0^s\int_0^t k(s_i-u,t_i-v)\overline{p}(s_i-u,t_i-v)w(u,v)\mathrm{d}u\mathrm{d}v\\&\approx\sum_{j=0}^{N}\sum_{i=0}^{M}\int_{u_{2j}}^{u_{2j+2}}\int_{v_{2j}}^{v_{2j+2}}k(s_i-u,t_i-v)\overline{p}(x_i-u,y_i-v)\mathrm{d}u\mathrm{d}v\end{aligned} \tag{20}$$

where $s_i=u_i=t_i=v_i=a+ih, i=0,1,\cdots,N$ with $h=\frac{b-a}{N}$ and N even. Now if we approximate the nonsingular part of the integrand over each interval $[u_{2j},u_{2j+2}],[v_{2l},v_{2l+2}]$ by the second degree Lagrange interpolation polynomial that interpolates it at the points $u_{2j},u_{2j+1},u_{2j+2},v_{2j},v_{2j+1},v_{2j+2}$ we find

$$\begin{aligned}&\int_0^s\int_0^t k(u_i-u,v_i-v)\overline{p}(u_i-u,v_i-v)w(u,v)\mathrm{d}u\mathrm{d}v=\sum_{j=0}^{\frac{N-2}{2}}\sum_{l=0}^{\frac{M-2}{2}}\int_{u_{2j}}^{u_{2j+2}}\int_{v_{2l}}^{v_{2l+2}}k(u_i-u,v_i-v)\\&\times\left\{\frac{(u_{2j+1}-u)(v_{2l+1}-v)(u_{2j+2}-u)(v_{2l+2}-v)}{(2h^2)(2h^2)}\overline{p}(u_i-u_{2j},v_i-v_{2l})w(u_{2j},v_{2l})\right.\\&+\frac{(u-u_{2j})(v-v_{2l})(u_{2j+2}-u)(v_{2l+2}-v)}{(h^2)(h^2)}\overline{p}(u_i-u_{2j+1},v_i-v_{2l+1})w(u_{2j+1},v_{2l+1})\\&\left.+\frac{(u-u_{2j})(v-v_{2l})(u-u_{2j+1})(v-v_{2l+1})}{(2h^2)(2h^2)}\overline{p}(u_i-u_{2j+2},v_i-v_{2l+2})w(u_{2j+2},v_{2l+2})\right\}\mathrm{d}u\mathrm{d}v\\&=\sum_{j=0}^{N}\sum_{l=0}^{M}\kappa_{ij}\kappa_{il}\overline{p}(u_i-u_j,v_i-v_l)w(u_i,v_l)\end{aligned}$$

where $u_j=jh$, $u_{j+1}=(j+1)h$, $u_j-u_{j+1}=v_l-v_{l+1}=-h$, and κ_{ij},κ_{il} are given by

$$\kappa_{i,0}\kappa_{i,0}=\frac{1}{4h^2}\int_{u_0}^{u_2}\int_{v_0}^{v_2}k(u_i-u,v_i-v)(u_1-u)(v_1-v)(u_2-u)(v_2-v)\,\mathrm{d}u\mathrm{d}v$$

$$\kappa_{i,2j+1}\kappa_{i,2l+1}=\frac{1}{h^4}\int_{u_{2j}}^{u_{2j+2}}\int_{v_{2l}}^{v_{2l+2}}k(u_i-u,v_i-v)(u-u_{2j})(v-v_{2l})(u_{2j+2}-u)(v_{2l+2}-v)\,\mathrm{d}u\mathrm{d}v$$

$$\begin{aligned}\kappa_{i,2j}\kappa_{i,2l}&=\frac{1}{4h^4}\int_{u_{2j-2}}^{u_{2j}}\int_{v_{2l-2}}^{v_{2l}}k(u_i-u,v_i-v)(u-u_{2j-2})(v-v_{2j-2})(u-u_{2j-1})(v-v_{2j-1})\,\mathrm{d}u\mathrm{d}v\\&\quad+\frac{1}{4h^4}\int_{u_{2j}}^{u_{2j+2}}\int_{v_{2l}}^{v_{2l+2}}k(u_i-u,v_i-v)(u_{2j+1}-u)(v_{2j+1}-v)(u_{2j+2}-u)(v_{2j+2}-v)\,\mathrm{d}u\mathrm{d}v\end{aligned}\quad(21)$$

$$\kappa_{i,N}\kappa_{i,M}=\frac{1}{4h^4}\int_{u_{N-2}}^{u_N}\int_{v_{M-2}}^{v_M}k(u_i-u,v_i-v)(u-u_{N-2})(v-v_{M-2})(u-u_{N-1})(v-v_{M-1})\,\mathrm{d}u\mathrm{d}v$$

If we define

$$\alpha_{j,i}(u_i,v_i)=\frac{1}{4h^2}\int_{u_{2j-2}}^{u_{2j}}\int_{v_{2j-2}}^{v_{2j}}k(u_i-u,v_i-v)(u-u_{2j-2})(v-v_{2j-2})(u-u_{2j-1})(v-v_{2j-1})\,\mathrm{d}u\mathrm{d}v$$

$$\beta_{j,i}(u_i,v_i)=\frac{1}{4h^2}\int_{u_{2j-2}}^{u_{2j}}\int_{v_{2j-2}}^{v_{2j}}k(u_i-u,v_i-v)(u_{2j-1}-u)(v_{2j-1}-v)(u_{2j}-u)(v_{2j}-v)\,\mathrm{d}u\mathrm{d}v\quad(22)$$

$$\gamma_{j,i}(u_i,v_i)=\frac{1}{4h^2}\int_{u_{2j-2}}^{u_{2j}}\int_{v_{2j-2}}^{v_{2j}}k(u_i-u,v_i-v)(u-u_{2j-2})(v-v_{2j-2})(u_{2j}-u)(v_{2j}-v)\,\mathrm{d}u\mathrm{d}v$$

It follows that

$$\begin{aligned}&\kappa_{i,0}\kappa_{i,0}=\beta_{1,1}(u_i,v_i),\quad \kappa_{i,2j+1}\kappa_{i,2j+1}=4\gamma_{j+1,i+1}(u_i,v_i),\\&\kappa_{i,2j}\kappa_{i,2i}=\alpha_{j,i}(u_i,v_i)+\beta_{j+1,i+1}(u_i,v_i),\quad \kappa_{i,N}\kappa_{i,M}=\alpha_{\frac{N}{2},\frac{M}{2}}(u_i,v_i)\end{aligned}\quad(23)$$

In general, assume $u=u_{2j-2}+\xi h$, $v=v_{2l-2}+\delta h$, $0\le\xi,\delta\le 2$, thus (22) become

$$\alpha_{j,l}(u_i,v_i)=\frac{h}{4}\int_0^2\int_0^2\xi\delta(\xi-1)(\delta-1)p\left(u_i-(u_{2j-2}+\xi h),v_i-(v_{2l-2}+\delta h)\right)\mathrm{d}\xi\mathrm{d}\delta$$

$$\beta_{j,l}(u_i,v_i)=\frac{h}{4}\int_0^2\int_0^2(\xi-1)(\xi-2)(\delta-1)(\delta-2)p\left(u_i-(u_{2j-2}+\xi h),v_i-(v_{2l-2}+\delta h)\right)\mathrm{d}\xi\mathrm{d}\delta\quad(24)$$

$$\gamma_{j,l}(u_i,v_i)=\frac{h}{4}\int_0^2\int_0^2\xi\delta(2-\xi)(2-\delta)p\left(u_i-(u_{2j-2}+\xi h),v_i-(v_{2l-2}+\delta h)\right)\mathrm{d}\xi\mathrm{d}\delta$$

If we define $\psi_k=\int_0^2\int_0^2\xi^k\delta^k p\left(u_i-(u_{2j-2}+\xi h),v_i-(v_{2l-2}+\delta h)\right)\mathrm{d}\xi\mathrm{d}\delta, k=0,1,2$, and let $u_i-u_{2j-2}=(i-2j+2)h$, $v_i-v_{2l-2}=(i-2l+2)h$, we have

$$\psi_k=\int_0^2\int_0^2\xi^k\delta^k k\left((z-\xi)h,(g-\delta)h\right)\mathrm{d}\xi\mathrm{d}\delta,\quad k=0,1,2,\ z=i-2h+2,\ g=i-2i+2\quad(25)$$

Then we get:

$$\begin{aligned}&\mu w(s_i,t_i)-\lambda f\left(s,t,\sum_{j=0}^{N}\sum_{l=0}^{M}\kappa_{ij}\kappa_{il}\bar{k}(s_i-u_j,t_i-v_l)w(u_j,v_l)\right)\\&=g(s_i,t_i),\quad i=0,1,\cdots,N\end{aligned}\quad(26)$$

4. Numerical Examples

We consider two kernels: logarithmic and Carleman. In logartimc kernel we consider λ = 0.001, 0.01, for values of μ = 1, and N = 10, 20 units. In Carleman kernel: we consider $\lambda_1 = 0.02269139783$, $\upsilon_1 = 0.42$, $\upsilon_2 = 0.38$, and $\lambda_2 = 0.03933175622$, $\upsilon_1 = 0.37$, $\upsilon_2 = 0.35$, where N = 10, 20 units. In Tables 1-4: *Aprro. T* $\rightarrow$ approximate solution by TMM, *Error T* $\rightarrow$ *error value by TMM, Aprro. N* $\rightarrow$ approximate solution by PNM, *Error N* $\rightarrow$ *error value by PNM.*

Example 1.

$$w(s,t)-\lambda f\left(s,t,\int_{-1}^{1}\int_{-1}^{1}\ln|s-u|\ln|t-v|\,w(u,v)\,\mathrm{d}u\mathrm{d}v\right)=g(s,t)$$

Exact solution is $w(s,t)=s\cdot t$.

Table 1. The approximate and absolute error values as determined by TMM and PNM at λ = 0.001.

λ	N	x	y	*Aprro. T*	*Error T.*	*Aprro. N*	*Error N.*
0.001	10	−1.00	−1.00	1.00124468	0.00124468	1.00097819	0.000978192
		−0.6	−0.6	0.36093925	0.00093925	0.36056010	0.000560106
		−0.2	−0.2	0.04009519	0.00009519	0.04003873	0.000038738
		0.2	0.2	0.04003274	0.00003274	0.04005675	0.000056752
		0.6	0.6	0.35963143	0.00036857	0.36061358	0.000613584
		1.00	1.00	0.99924413	0.00755862	1.00095182	0.000951828
	20	−1.00	−1.00	1.00112194	0.00112194	1.00099863	0.000998630
		−0.6	−0.6	0.36079723	0.00079723	0.36059097	0.000590971
		−0.2	−0.2	0.04006217	0.00006217	0.04006364	0.000063649
		0.2	0.2	0.03999980	0.19305×10^{-6}	0.04007820	0.000078205
		0.6	0.6	0.35489814	0.00051018	0.36063081	0.000630816
		1.00	1.00	0.99912165	0.00087834	1.00099190	0.000991901

Table 2. The approximate and absolute error values as determined by TMM and PNM at λ = 0.01.

λ	N	x	y	*Aprro. T*	*Error T.*	*Aprro. N*	*Error N.*
0.01	10	−1.00	−1.00	1.01248039	0.01248039	1.00978298	0.009782987
		−0.6	−0.6	0.36944950	0.00944950	0.365595279	0.005595279
		−0.2	−0.2	0.04096513	0.00096513	0.040382566	0.000382566
		0.2	0.2	0.04032618	0.00032618	0.040565268	0.000565268
		0.6	0.6	0.35629240	0.00370759	0.366135449	0.006135497
		1.00	1.00	0.99242529	0.00757470	1.009514701	0.009514701

Continued

20	−1.00	−1.00	1.01123458	0.01123458	1.010039328	0.010039328
	−0.6	−0.6	0.36799784	0.00799784	0.365917381	0.005917382
	−0.2	−0.2	0.04062651	0.00062651	0.040624598	0.000624598
	0.2	0.2	0.03999630	0.36916×10^{-5}	0.040779795	0.000779795
	0.6	0.6	0.35488204	0.00511795	0.366311386	0.006311386
	1.00	1.00	0.99120620	0.008793795	1.009918759	0.009918759

Example 2.

$$w(s,t)-\lambda f\left(s,t,\int_{-1}^{1}\int_{-1}^{1}|s-u|^{-\upsilon_1}|t-v|^{-\upsilon_2}w(u,v)\,\mathrm{d}u\mathrm{d}v\right)=g(s,t)$$

Exact solution is $w(s,t)=s\cdot t$

Table 3. The approximate and absolute error values as determined by TMM and PNM at $\upsilon_1=0.42,\upsilon_2=0.38,\lambda=0.02269139783$.

N	*x*	*y*	*Apro. T*	*Error T.*	*Apro. N*	*Error N.*
10	−1.00	−1.00	1.01794079	0.01794079	1.011727082	0.011727082
	−0.6	−0.6	0.37902404	0.01902404	0.369203272	0.009203237
	−0.2	−0.2	0.04575897	0.00575897	0.043457101	0.003457101
	0.2	0.2	0.04507109	0.00507109	0.043412760	0.003412760
	0.6	0.6	0.36456665	0.00456666	0.368775487	0.008775487
	1.00	1.00	0.99967540	0.00032459	1.013352969	0.013352969
20	−1.00	−1.00	1.01368129	0.01368129	1.010268091	0.010268091
	−0.6	−0.6	0.37297088	0.01297088	0.367643589	0.007643589
	−0.2	−0.2	0.04304212	0.00304212	0.042054967	0.002054967
	0.2	0.2	0.04238472	0.00238472	0.041930781	0.001930781
	0.6	0.6	0.35865605	0.00134394	0.367359820	0.007359820
	1.00	1.00	0.99549767	0.00450232	1.010809209	0.010809309

Table 4. The approximate and absolute error values as determined by TMM and PNM at $\upsilon_1=0.37,\upsilon_2=0.35,\lambda=0.03933175622$.

N	*x*	*y*	*Apro. T*	*Error T.*	*Apro. N*	*Error N.*
10	−1.00	−1.00	1.02362810	0.02362810	1.01575969	0.01575969
	−0.6	−0.6	0.38638462	0.02638462	0.37267699	0.01267699
	−0.2	−0.2	0.04932809	0.00932809	0.04561598	0.00561598
	0.2	0.2	0.04844989	0.00844989	0.04556409	0.00556409
	0.6	0.6	0.36857130	0.00857130	0.37220276	0.01220279
	1.00	1.00	1.00086145	0.00086145	1.01815510	0.01815510

Continued

20	−1.00	−1.00	1.07470912	0.01747091	1.01330710	0.01330710
	−0.6	−0.6	0.37715646	0.01715646	0.37004052	0.01004052
	−0.2	−0.2	0.04482769	0.00482769	0.04316247	0.00316247
	0.2	0.2	0.04400329	0.00400329	0.04299879	0.00299879
	0.6	0.6	0.35958227	0.00041772	0.03696866	0.00968386
	1.00	1.00	0.99484307	0.00515692	1.01410259	0.01410259

5. Conclusions

In order to find the solution of the FT-DHIE of the second kind with noncontinous kernels, this research provided two efficient numerical approaches. TMM and PNM have been introduced for this reason. The correctness and efficacy of the methods are demonstrated by error analysis and a few numerical examples. From previous tables, we found in every instance, the error in evaluating the approximation solution using the PNM is lower than the error in evaluating the approximation solution using the TMM. The FT-DHIE's logarithmic error numbers, Error T. and Error N., are less than error values, Error T. and Error N. of FT-DFIE.

In the future, we will study this equation in nonlinear case and in the different kinds.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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