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Novel Method to Deal with Interval Quadratic Equations via Sign-Variation Analysis

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Abstract

In this article, analytical results are obtained apparently for the first time in the literature, for the lower and upper bounds of the roots of quadratic equations when two or all three coefficients *a*, *b*, *c* constitute an interval, with a method called the sign-variation analysis. The results are compared with the parametrization technique offered by Elishakoff and Miglis, and with the solution yielded by minimization and maximization commands of the Maple software. Solutions for some interval word problems are also provided to edulcorate the methodology. This article only focuses on the real roots of those quadratic equations, complex solutions being beyond this investigation.

Keywords

Analytical Results, Quadratic Equations, Bounds, Sign-Variation Analysis, Interval Word Problems

1. Introduction

Fuzzy-sets based analysis often involves solution of interval variables, as demonstrated in papers by Dourado *et al.* [1], Mo *et al.* [2], Tian *et al.* [3], and Liao *et al.* [4]. In this conjunction, the solution of interval quadratic equations is of paramount importance (see, for example, anonymous, 2022 [5]). Alolyan [6] apparently was the first who investigated the fuzzy quadratic equations. In their book *A History of Mathematics*, Carl B. Boyer and Uta C. Merzbach [7] stated that "[Quadratic] equations have been handled effectively by the Babylonians in some of the oldest problem texts", approximately around 400 AD. This problem has also been tackled by the Indian mathematician Brahmagupta, in Chapter 18 of Brahmasphuṭasiddhānta, composed in 628 CE, according to Kim Pfloker in *The Mathematics of Egypt*, *Mesopotamia*, *China*, *India*, *and Islam* by Victor J.

Katz [8]. Indeed, Brahmagupta gave the solution to the general quadratic equation defined as follows

$$ax^2 + bx = c \tag{1}$$

where a, b, c are coefficients, and x denotes an unknown. He stated his recipe as "Diminish by the middle [number] the square-root of the *rupas* (*rupas* refer to the constant c) multiplied by four times the square and increased by the square of the middle [number]; divide the remainder by twice the square. [The result is] the middle [number]." This leads us to the following expressions of the roots of the quadratic equation defined in Equation (1), which are:

$$x_{1,2} = \frac{\pm\sqrt{4ac + b^2} - b}{2a} \tag{2}$$

This is equivalent to the well-known modern formula that we use, since, in Equation (1), c is on the right side of the quadratic equation. Note that we set that $x_1 < x_2$. This implies that we need to be cautious whether $a < 0$ or $a > 0$, because a cannot be zero in Equation (2).

Indeed, when $a > 0$, to respect $x_1 < x_2$, we will have to consider that:

$$x_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \text{ and } x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

On the other hand, when $a < 0$, to get $x_1 < x_2$, we have to consider that:

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

The mathematical explanation of the previous statement will be provided in **Appendix A**.

2. Which Methodology to Use?

In this paper, we treat the cases of quadratic equations in which two of the three coefficients a, b and c constitute intervals. This implies that finding the formulas of x_1 and x_2 becomes a much harder task for some of the methods developed in our previous studies. Indeed, the method used by Elishakoff and Daphnis [9] cannot lead to the establishment of formulas for those problems, while Ioakimidis [10] reports that "quantifier elimination cannot be performed in a reasonable CPU time [so] all the present results were confined to only one interval parameter". We cannot consider these methods in the rest of this article. Elishakoff and Miglis [11] developed a special parametrization technique as a cure for the dependency problem suffered by the classical interval analysis, that they referred to as interval parametrization. This method is a computational tool, rather than being an analytical technique.

This paper aims then to solve that issue, by offering a method that yields analytical formulas for cases in which two or all three coefficients of the quadratic equation are defined as intervals.

Interval analysis is extremely instrumental in case of uncertainty analysis when probability density functions or membership functions are unknown. Still, if we know the lower and upper bounds, we can use either least favourable or most favourable designs.

This method will be called sign-variation method in the rest of the article and is based upon the partial derivatives of the expressions of the roots of a quadratic equation. Indeed, we will consider every possible sign variation of each partial derivative of a root of the quadratic equation. Afterwards, we will be able to know how the root's value evolves in regard to the variable that has been used to partially derive the root's expression. The method will be developed in the next section, and the results yielded by the sign-variation method will be compared to the results obtained via the interval parametrization technique developed by Elishakoff and Miglis [11], but also to the direct approach with the Maple software. The direct approach comprises of the straightforward minimization and the maximization of expressions utilizing Maple's *Minimize* and *Maximize* commands [12]. Analogous methodology was applied by Landowski [13], whereas Sevastjanov and Dymova [14] developed a fuzzy solution of interval linear equation, while Mamehrashi [15] utilizing the context of fuzzy sets, too.

3. The Sign-Variation Method

We first introduce an example of using this method for a particular interval quadratic equation, and will provide in the Appendices B and C the results obtained for every possible subcase. Let us consider a quadratic equation with two variables serving as intervals, a and b, and c being a crisp quantity. We also state here that $a > 0, b < 0$ and $c < 0$, and that the roots of this equation are real, so $\sqrt{b^2 - 4ac} > 0$. Since a and b constitute intervals, we have:

$$a = \left[\underline{a}; \overline{a}\right] \tag{3}$$

and

$$b = \left[\underline{b}; \overline{b}\right] \tag{4}$$

In Equations (3) and (4), the lower bar designates the smallest value of the variable, whereas the upper bar denotes the greatest value of the variable, and those values will be separated by a semi column in the rest of this study.

We will start by considering the first root x_1, which is:

$$x_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \tag{5}$$

Since x_1 depends on a, b and c, we can consider the following partial derivatives:

$$pda_1(a,b,c) = \frac{\partial x_1}{\partial a} = \frac{c}{a\sqrt{b^2 - 4ac}} - \frac{-b - \sqrt{b^2 - 4ac}}{2a^2} \tag{6}$$

$$pdb_1(a,b,c) = \frac{\partial x_1}{\partial b} = \frac{-1 - \dfrac{b}{\sqrt{b^2 - 4ac}}}{2a} \tag{7}$$

We will not consider $pdc_1(a,b,c) = \dfrac{\partial x_1}{\partial c}$ here because it is not useful for our current example, as c is a crisp quantity. However, it will be used when c constitutes an interval quantity in the **Appendix B** and **Appendix C**.

Let us now draw the sign-variation tables of both pda_1 and pdb_1 related to our problem, so considering that $a > 0, b < 0$ and $c < 0$.

For $pda_1(a,b,c) = \frac{\partial x_1}{\partial a}$, considering $b < 0$ and $c < 0$, the signs of the partial derivative are, for different regions of variation of the parameter a:

Using Table 1, we see that $\partial x_1 / \partial a$ is positive when $a > 0$. That means that, regarding the variable a, if a is increasing, then x_1 is also increasing. Consequently, if we want to minimize x_1 regarding a, we need to use $\underline{a}$, thus using $\overline{a}$ will result in maximizing x_1.

We can process to doing the same thing with the variable b.

For $pdb_1(a,b,c) = \frac{\partial x_1}{\partial b}$, considering $a > 0$ and $c < 0$, Table 2 is obtained:

As shown in Table 2, $\partial x_1 / \partial b$ is negative when $b < 0$. That means that, regarding the variable b, if b is increasing, then x_2 is decreasing. So, if we want to minimize x_1 regarding b, we need to use $\overline{b}$, so using $\underline{b}$ will result in maximizing x_1.

Now combining what we obtained above yields the following expressions:

$$\underline{x}_1 = \frac{-\overline{b} - \sqrt{\overline{b}^2 - 4\underline{a}c}}{2\underline{a}} \tag{8}$$

$$\overline{x}_1 = \frac{-\underline{b} - \sqrt{\underline{b}^2 - 4\overline{a}c}}{2\overline{a}} \tag{9}$$

Let us now conduct analogous analysis with the root x_2, which is:

$$x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \tag{10}$$

The corresponding partial derivatives of x_2 in regards to a and b are:

Table 1. Sign variation of $pda_1(a,b,c) = \frac{\partial x_1}{\partial a}$ with $b < 0$ and $c < 0$.

a	$\frac{b^2}{4c} < a < 0$	$a = 0$	$0 < a < +\infty$
$\frac{\partial x_1}{\partial a}$	+	Undefined	+

Table 2. Sign variation of $pdb_1(a,b,c) = \frac{\partial x_1}{\partial b}$ with $a > 0$ and $c < 0$.

b	$-\infty < b < +\infty$
$\frac{\partial x_1}{\partial c}$	$-$

$$pda_2(a,b,c)=\frac{\partial x_2}{\partial a}=\frac{-c}{a\sqrt{b^2-4ac}}-\frac{-b+\sqrt{b^2-4ac}}{2a^2} \tag{11}$$

$$pdb_2(a,b,c)=\frac{\partial x_2}{\partial b}=\frac{-1+\frac{b}{\sqrt{b^2-4ac}}}{2a} \tag{12}$$

Let us now draw the sign-variation tables of pda_1 and pdb_1 related to our problem, so considering that $a>0, b<0$ and $c<0$, as we did for x_1:

For $pda_2(a,b,c)=\frac{\partial x_2}{\partial a}$, considering $b<0$ and $c<0$:

As seen in **Table 3**, $\partial x_2/\partial a$ is negative when $a>0$. That means that, regarding the variable a, if a is increasing, then x_2 is decreasing. Consequently, if we want to minimize x_2 regarding a, we need to use $\underline{a}$. Using $\overline{a}$ will then maximize x_2.

For $pdb_2(a,b,c)=\frac{\partial x_2}{\partial b}$, considering $a>0$ and $c<0$:

Now, with the variable b, we see in **Table 4** that $\partial x_2/\partial b$ is also negative when $b<0$. That means that, regarding the variable b, if b is increasing, then x_2 is also decreasing. So, if we want to minimize x_2 regarding b, we need to use $\overline{b}$, thus using $\underline{b}$ will result in maximizing x_2.

We then obtain the following formulas for x_2:

$$\underline{x}_2=\frac{-\overline{b}+\sqrt{\overline{b}^2-4\overline{a}c}}{2\overline{a}} \tag{13}$$

$$\overline{x}_2=\frac{-\underline{b}+\sqrt{\underline{b}^2-4\underline{a}c}}{2\underline{a}} \tag{14}$$

We now obtained the formulas for both roots of a quadratic equation with two variables, namely a and b, serving as intervals, in the specific case in which $a>0, b<0$ and $c<0$.

Table 3. Sign variation of $pda_2(a,b,c)=\frac{\partial x_2}{\partial a}$ with $b<0$ and $c<0$.

a	$\frac{b^2}{4c}<a<0$	$a=0$	$0<a<+\infty$
$\frac{\partial x_2}{\partial a}$	–	Undefined	–

Table 4. Sign variation of $pdb_2(a,b,c)=\frac{\partial x_2}{\partial b}$ with $a>0$ and $c<0$.

b	$-\infty<b<+\infty$
$\frac{\partial x_2}{\partial b}$	–

Let us now compare the results yielded by this method with the direct approach from minimization and maximization commands of the Maple software, by considering a numerical example of that interval quadratic equation.

As we need to satisfy $a > 0, b < 0, c < 0$, and $\sqrt{b^2 - 4ac} > 0$, we consider the following example:

$$a = [\underline{a}; \overline{a}] = [5;6], \; b = [\underline{b}; \overline{b}] = [-11;-10], \; c = -1 \tag{15}$$

The direct approach using Equation (15) yields:

$$\underline{x}_1 = -0.095445 \left(\text{at } \underline{a} = 5 \text{ and } \overline{b} = -10\right), \overline{x}_1 = -0.086799 \left(\text{at } \overline{a} = 6 \text{ and } \underline{b} = -11\right) \tag{16}$$

$$\overline{x}_2 = 1.761294 \left(\text{at } \overline{a} = 6 \text{ and } \overline{b} = -10\right), \underline{x}_2 = 2.287434 \left(\text{at } \underline{a} = 5 \text{ and } \underline{b} = -11\right) \tag{17}$$

The sign-variation formulas using Equations (9), (10), (13), (14) and (15) give the following results:

$$\underline{x}_1 = -0.095445, \; \overline{x}_1 = -0.086799 \tag{18}$$

$$\underline{x}_2 = 1.761294, \; \overline{x}_2 = 2.287434 \tag{19}$$

As is seen, we obtain the same results, with the same values for parameters *a* and *b*. We will extend this to every subcase, and provide Tables C1-C3 in **Appendix C** that summarize all results for every possible subcase, for the cases in which two or all the parameters constitute intervals. We will also provide multiple examples of different subcases to compare the results given by those formulas with the other methods, namely interval parametrization and direct approach from minimization and maximization commands of the Maple software.

One might argue that considering the case of *a* as an interval, then *b* also, and then assuming that combining the results from both separated cases leads to obtaining the formulas when both variables are considered as intervals might be questionable. This method actually works because the sign of both $\partial x_n / \partial a$ and $\partial x_n / \partial c$, $n \in \{1;2\}$ is constant whatever the values of the other variables are. Consequently, when we consider $\partial x_n / \partial b$, $n \in \{1;2\}$, it ensures that the tables shown above will remain valid whatever the value *a* and *c* will take in each subcase.

One could also point out that some of the signs obtained in the sign-variation tables are not obvious to determine. We will then provide in the **Appendix D**, the proofs that these obtained signs are valid, namely using Mathematica commands, in order to be on the safe side and prevent an error.

The sign-variation method also works for the case when only one parameter *a*, *b* or *c* is considered as an interval, but we need to consider every subcase possible. It is then less optimal to use than quantifier elimination method [10] for example, that has been the best way to yield analytical formulas when only one parameter *a*, *b* or *c* of quadratic equations constitutes an interval. We will, at the end of our article in the conclusion, draw Table C3 that summarizes the advantages and the drawbacks of each method.

As is seen the present methodology is based upon heavy use of sensitivity derivatives.

This method is advantageous over the parametrization technique by Elishakoff and Miglis [11] since the latter is a purely numerical technique, whereas the current methodology constitutes an analytical approach. It is also superior to the technique developed by Elishakoff and Daphnis [9] since the latter study is not able to cover all possible cases of interval equations.

4. Two Coefficients Serving as Intervals

4.1. Coefficients *a* and *b* Constitute Intervals

We start with reporting results via the sign-variation analysis, considering now the case where a and b constitute intervals,

$$a = \left[\underline{a};\overline{a}\right] \tag{20}$$

and

$$b = \left[\underline{b};\overline{b}\right] \tag{21}$$

and c as a crisp quantity. We also consider here $a < 0, b > 0$ and $c < 0$ for this example. We assume that $a \neq 0$ and $\sqrt{b^2 - 4ac} > 0$ too. As $a < 0$, we will make sure to adapt the formulas of x_1 and x_2 to make sure that we respect $x_1 < x_2$.

Using the **Table B3**, **Table B10**, **Table B13**, **Table B20** defined in the **Appendix B**, we can apply the sign-variation method to find the formulas of minimized and maximized values of both roots of the equation.

For x_1 first, using **Table B3**, the value $\partial x_1/\partial a$ is positive when $a < 0, b > 0$ and $c < 0$. Consequently, if we want to minimize x_1 regarding a, we need to use $\underline{a}$, thus using $\overline{a}$ will result in maximizing x_1.

The value $\partial x_1/\partial b$ in **Table B10** is also positive when $a < 0, b > 0$ and $c < 0$. So, if we want to minimize x_1 regarding b, we need to use $\underline{b}$, thus using $\overline{b}$ leads to maximize x_1. Those results yield the following formulas:

$$\underline{x}_1 = \frac{-\underline{b} + \sqrt{\underline{b}^2 - 4\underline{a}c}}{2\underline{a}} \tag{22}$$

$$\overline{x}_1 = \frac{-\overline{b} + \sqrt{\overline{b}^2 - 4\overline{a}c}}{2\overline{a}} \tag{23}$$

Now, let us consider the second root. $\partial x_2/\partial a$ is negative in **Table B13** when $a < 0, b > 0$ and $c < 0$. To minimize x_2 regarding a, we need to use $\overline{a}$, maximizing x_2 implies using $\underline{a}$.

The value $\partial x_2/\partial b$ is also negative in **Table B20** when $a < 0, b > 0$ and $c < 0$. To minimize x_2 regarding to b, we need to use $\overline{b}$, using $\underline{b}$ will result in maximizing x_2.

We then obtain the following formulas:

$$\underline{x}_2 = \frac{-\overline{b} - \sqrt{\overline{b}^2 - 4\overline{a}c}}{2\overline{a}} \tag{24}$$

$$\overline{x}_2 = \frac{-\underline{b} - \sqrt{\underline{b}^2 - 4\underline{a}c}}{2\underline{a}} \tag{25}$$

All the other cases will be summarized in the **Appendix C** in **Table C1**, since this method can give us the formulas for any subcase.

Let us now consider the same problem, using the interval parametrization analysis by Elishakoff and Miglis (2012), with the same interval for the parameters a and b given in Equations (20) and (21), and the parameter c remaining a deterministic quantity.

We need to introduce a few quantities in order to parametrize both intervals: the average value of a denoted a_{ave} and the deviation value of a denoted a_{dev}, and the same thing for b, so the average value of b named b_{ave} and the deviation value of b which is b_{dev}, as follows:

$$a_{ave} = \frac{\overline{a} + \underline{a}}{2} \tag{26}$$

$$a_{dev} = \frac{\overline{a} - \underline{a}}{2} \tag{27}$$

$$b_{ave} = \frac{\overline{b} + \underline{b}}{2} \tag{28}$$

$$b_{dev} = \frac{\overline{b} - \underline{b}}{2} \tag{29}$$

We can now write a and b as:

$$a = a_{ave} + a_{dev} t_1 \tag{30}$$

$$b = b_{ave} + b_{dev} t_2 \tag{31}$$

with

$$t_1 = [-1;1] \tag{32}$$

$$t_2 = [-1;1] \tag{33}$$

As a consequence, the roots of this equation can be written, using Equations (30) and (31):

$$x_1 = \frac{-(b_{ave} + b_{dev} t_2) + \sqrt{(b_{ave} + b_{dev} t_2)^2 - 4c(a_{ave} + a_{dev} t_1)}}{2(a_{ave} + a_{dev} t_1)} \tag{34}$$

$$x_2 = \frac{-(b_{ave} + b_{dev} t_2) - \sqrt{(b_{ave} + b_{dev} t_2)^2 - 4c(a_{ave} + a_{dev} t_1)}}{2(a_{ave} + a_{dev} t_1)} \tag{35}$$

Now that the expressions have been defined for each method considered above, we evaluate them using a numerical example.

Since we established formulas for $a < 0, b > 0$ and $c < 0$, let us consider an example:

$$a = [\underline{a}; \overline{a}] = [-6;-5],\ b = [\underline{b}; \overline{b}] = [10;11],\ c = -1 \tag{36}$$

The direct approach gives the following results, using Equation (36):

$$\underline{x}_1 = 0.095012 \left(\text{at } \overline{a} = -5 \text{ and } \overline{b} = 11\right),\ \overline{x}_1 = 0.106850 \left(\text{at } \underline{a} = -6 \text{ and } \underline{b} = 10\right) \tag{37}$$

$$\underline{x}_2 = 1.559816 \left(\text{at } \underline{a} = -6 \text{ and } \underline{b} = 10\right),\ \overline{x}_2 = 2.104987 \left(\text{at } \overline{a} = -5 \text{ and } \overline{b} = 11\right) \tag{38}$$

The sign-variation method formulas using Equations (22)-(25) and (36) read:

$$\underline{x}_2 = 0.095012,\ \overline{x}_2 = 0.106850 \tag{39}$$

$$\underline{x}_1 = 1.559816,\ \overline{x}_1 = 2.104987 \tag{40}$$

Let us now use the interval parametrization formulas in Equations (34)-(36), which result in:

$$\underline{x}_1 = 0.095012\left(\text{at } t_1 = 1 \text{ and } t_2 = 1\right),\ \overline{x}_1 = 0.106850\left(\text{at } t_1 = -1 \text{ and } t_2 = -1\right) \tag{41}$$

$$\underline{x}_2 = 1.559816\left(\text{at } t_1 = -1 \text{ and } t_2 = -1\right),\ \overline{x}_2 = 2.104987\left(\text{at } t_1 = 1 \text{ and } t_2 = 1\right) \tag{42}$$

All the results obtained by the methods above match with each other.

4.2. Coefficients *b* and *c* Constitute Intervals

We start by dealing with sign-variation method, considering here that the parameters b and c constitute intervals. They can then be written as follows:

$$b = \left[\underline{b};\overline{b}\right] \tag{43}$$

and

$$c = \left[\underline{c};\overline{c}\right] \tag{44}$$

and a is a deterministic quantity. We also consider here $a < 0, b < 0$ and $c > 0$ for this example, which means $a \neq 0$. We assume that $\sqrt{b^2 - 4ac} > 0$ too. The formulas for the other subcases are presented in **Table C2** in the **Appendix C**.

Using **Table B9**, for x_1, the value $\partial x_1 / \partial b$ is positive when $b < 0$. We want to minimize x_1 with respect to b, we need to use $\underline{b}$, thus using $\overline{b}$ will result in maximizing x_1. The value $\partial x_1 / \partial c$ is also positive when $c > 0$ according to **Table B7**. So, if we want to minimize x_1 with respect to c, we need to use $\underline{c}$, thus using $\overline{c}$ will result in maximizing x_1.

These results yield the following formulas for x_1:

$$\underline{x}_1 = \frac{-\underline{b} + \sqrt{\underline{b}^2 - 4a\underline{c}}}{2a} \tag{45}$$

$$\overline{x}_1 = \frac{-\overline{b} + \sqrt{\overline{b}^2 - 4a\overline{c}}}{2a} \tag{46}$$

Now, in **Table B18**, we get that the partial derivative $\partial x_2 / \partial b$ is positive when $b < 0$. To minimize x_2 with respect to b, we need to use $\underline{b}$, maximizing x_2 implies using $\overline{b}$.

Table B16 shows that the value $\partial x_2 / \partial c$ is negative when $c > 0$. To minimize x_2 regarding c, we need to use $\overline{c}$, using $\underline{c}$ will result in maximizing x_2. This results in the formulas:

$$\underline{x}_2 = \frac{-\underline{b} - \sqrt{\underline{b}^2 - 4a\overline{c}}}{2\overline{a}} \tag{47}$$

$$\overline{x}_2 = \frac{-\overline{b} - \sqrt{\overline{b}^2 - 4a\underline{c}}}{2a} \tag{48}$$

We will now consider the same problem, using interval parametrization analysis. The parameters b and c still constitute the intervals defined in Equations (43) and (44) and a remains acrisp quantity. We need to introduce some quantities to parametrize both intervals: the average value of c denoted c_{ave} and the deviation value of c denoted c_{dev}. Proceeding analogously for b, the average value of b is b_{ave} and the deviation value of b is b_{dev}, as follows:

$$b_{ave} = \frac{\overline{b} + \underline{b}}{2} \tag{49}$$

$$b_{dev} = \frac{\overline{b} - \underline{b}}{2} \tag{50}$$

$$c_{ave} = \frac{\overline{c} + \underline{c}}{2} \tag{51}$$

$$c_{dev} = \frac{\overline{c} - \underline{c}}{2} \tag{52}$$

b and c are now:

$$b = b_{ave} + b_{dev} t_1 \tag{53}$$

$$c = c_{ave} + c_{dev} t_2 \tag{54}$$

with

$$t_1 = [-1;1] \tag{55}$$

$$t_2 = [-1;1] \tag{56}$$

The roots of this equation can now be written, using Equations (53) and (54):

$$x_1 = \frac{-(b_{ave} + b_{dev} t_1) + \sqrt{(b_{ave} + b_{dev} t_1)^2 - 4a(c_{ave} + c_{dev} t_2)}}{2a} \tag{57}$$

$$x_2 = \frac{-(b_{ave} + b_{dev} t_1) - \sqrt{(b_{ave} + b_{dev} t_1)^2 - 4a(c_{ave} + c_{dev} t_2)}}{2a} \tag{58}$$

Now that the formulas have been established for the considered methods, we can proceed to evaluate them numerically, being related to the interval parameters b and c, respecting that $a < 0, b < 0$ and $c > 0$.

Let us for instance consider the following values:

$$a = -4,\ b = [\underline{b}; \overline{b}] = [-11; -10],\ c = [\underline{c}; \overline{c}] = [1; 2] \tag{59}$$

The direct approach gives the following results using Equation (59):

$$\underline{x}_1 = -2.921165 \left(\text{at } \underline{b} = -11 \text{ and } \overline{c} = 2\right), \overline{x}_1 = -2.596291 \left(\text{at } \overline{b} = -10 \text{ and } \underline{c} = 1\right) \tag{60}$$

$$\underline{x}_2 = 0.088087 \left(\text{at } \underline{b} = -11 \text{ and } \underline{c} = 1\right), \overline{x}_2 = 0.186141 \left(\text{at } \overline{b} = -10 \text{ and } \overline{c} = 2\right) \tag{61}$$

The sign-variation method formulas in Equations (45)-(48) yield, using Equation (59):

$$\underline{x}_1 = -2.921165,\ \overline{x}_1 = -2.596291 \tag{62}$$

$$\underline{x}_2 = 0.088087,\ \overline{x}_2 = 0.186141 \tag{63}$$

Let us now use the interval parametrization formulas with Equations (57)-(59), which result in:

$$\underline{x}_1 = -2.921165(\text{at } t_1 = -1 \text{ and } t_2 = 1), \overline{x}_1 = -2.596291(\text{at } t_1 = 1 \text{ and } t_2 = -1) \quad (64)$$

$$\underline{x}_2 = 0.088087(\text{at } t_1 = -1 \text{ and } t_2 = -1), \overline{x}_2 = 0.186141(\text{at } t_1 = 1 \text{ and } t_2 = 1) \quad (65)$$

Again, the results match with each other, with also the same values for the parameters *b* and *c*.

4.3. Coefficients *a* and *c* Serve as Intervals

Let us start by using the sign-variation method, with the coefficients *a* and *c* of a quadratic equation serving as interval parameters, defined as follows

$$a = [\underline{a}; \overline{a}] \quad (66)$$

and

$$c = [\underline{c}; \overline{c}] \quad (67)$$

and *b* as a deterministic quantity.

Looking at **Table 1**, **Table 3**, **Tables B1-B6** and **Tables B10-B16** in the **Appendix B**, we can notice that, as long as $\sqrt{b^2 - 4ac} > 0$ and $a \neq 0$, the sign of both derivatives $\partial x_n / \partial a$ and $\partial x_n / \partial c$, $n \in \{1;2\}$ is constant whatever the values of the other variables are. Indeed, $\partial x_1 / \partial a$ and $\partial x_1 / \partial c$ are both positive when they are defined, while $\partial x_2 / \partial a$ and $\partial x_2 / \partial c$ are both negative as long as they are defined. That means that, to minimize x_1 regarding *a*, we need to use $\underline{a}$, thus using $\overline{a}$ will result in maximizing x_1, and it is the same for *c*.

For x_2, it is just the exact opposite as $\partial x_2 / \partial a$ and $\partial x_2 / \partial c$ are both negative. Minimizing x_2 implies using $\overline{a}$ and $\overline{c}$ whereas maximizing x_2 means using $\underline{a}$ and $\underline{c}$.

If $\sqrt{b^2 - 4ac} > 0$ and $a \neq 0$, we can then establish general formulas for this casein which *a* and *c* are considered as intervals. We still need to take the sign of *a* into account to respect $x_1 < x_2$. When $a > 0$, we obtain the following formulas:

$$\underline{x}_1 = \frac{-b - \sqrt{b^2 - 4\underline{a}\underline{c}}}{2\underline{a}} \quad (68)$$

$$\overline{x}_1 = \frac{-b - \sqrt{b^2 - 4\overline{a}\overline{c}}}{2\overline{a}} \quad (69)$$

$$\underline{x}_2 = \frac{-b + \sqrt{b^2 - 4\overline{a}\overline{c}}}{2\overline{a}} \quad (70)$$

$$\overline{x}_2 = \frac{-b + \sqrt{b^2 - 4\underline{a}\underline{c}}}{2\underline{a}} \quad (71)$$

Whereas, when $a < 0$, the formulas now become:

$$\underline{x}_1 = \frac{-b + \sqrt{b^2 - 4\underline{a}\underline{c}}}{2\underline{a}} \quad (72)$$

$$\overline{x}_1 = \frac{-b + \sqrt{b^2 - 4\overline{a}\overline{c}}}{2\overline{a}} \quad (73)$$

$$\underline{x}_2 = \frac{-b - \sqrt{b^2 - 4\overline{a}\,\overline{c}}}{2\overline{a}} \tag{74}$$

$$\overline{x}_2 = \frac{-b - \sqrt{b^2 - 4\underline{a}\,\underline{c}}}{2\underline{a}} \tag{75}$$

We now consider the same problem but using the interval parametrization method. The coefficients a and c of a quadratic equation serve as interval parameters, as defined in Equations (66) and (67), while b remains a crisp quantity.

We introduce the average value of a denoted a_{ave} and the deviation value of a denoted a_{dev}, and also the average value of c denoted c_{ave} and the deviation value of c denoted c_{dev}, as follows:

$$a_{ave} = \frac{\overline{a} + \underline{a}}{2} \tag{76}$$

$$a_{dev} = \frac{\overline{a} - \underline{a}}{2} \tag{77}$$

$$c_{ave} = \frac{\overline{c} + \underline{c}}{2} \tag{78}$$

$$c_{dev} = \frac{\overline{c} - \underline{c}}{2} \tag{79}$$

a and c are now:

$$a = a_{ave} + a_{dev} t_1 \tag{80}$$

$$c = c_{ave} + c_{dev} t_2 \tag{81}$$

with

$$t_1 = [-1;1] \tag{82}$$

$$t_2 = [-1;1] \tag{83}$$

The roots of this equation can now be written, using Equations (80) and (81), when $a > 0$:

$$x_1 = \frac{-b - \sqrt{b^2 - 4(a_{ave} + a_{dev} t_1)(c_{ave} + c_{dev} t_2)}}{2(a_{ave} + a_{dev} t_1)} \tag{84}$$

$$x_2 = \frac{-b + \sqrt{b^2 - 4(a_{ave} + a_{dev} t_1)(c_{ave} + c_{dev} t_2)}}{2(a_{ave} + a_{dev} t_1)} \tag{85}$$

On the other hand, when $a < 0$, Equations (80) and (81) lead to:

$$x_1 = \frac{-b + \sqrt{b^2 - 4(a_{ave} + a_{dev} t_1)(c_{ave} + c_{dev} t_2)}}{2(a_{ave} + a_{dev} t_1)} \tag{86}$$

$$x_2 = \frac{-b - \sqrt{b^2 - 4(a_{ave} + a_{dev} t_1)(c_{ave} + c_{dev} t_2)}}{2(a_{ave} + a_{dev} t_1)} \tag{87}$$

Since we established formulas for general cases, let us consider multiple numerical examples, to prove that the formulas found above for the sign-variation method are indeed generalized for a quadratic equation with the parameters a

and c serving as intervals. First, we consider:

$$a = [\underline{a}; \overline{a}] = [-4; -3], \; b = 10, \; c = [\underline{c}; \overline{c}] = [1; 2] \tag{88}$$

The direct approach gives the following results using Equation (88):

$$\underline{x}_1 = -0.189255 (\text{at } \overline{a} = -3 \text{ and } \overline{c} = 2), \overline{x}_1 = -0.096291 (\text{at } \underline{a} = -4 \text{ and } \overline{c} = 1) \tag{89}$$

$$\underline{x}_2 = 2.596291 (\text{at } \underline{a} = -4 \text{ and } \underline{c} = 1), \overline{x}_2 = 3.522588 (\text{at } \overline{a} = -3 \text{ and } \overline{c} = 2) \tag{90}$$

The sign-variation method formulas yield, using Equations (72)-(75) and (88):

$$\underline{x}_1 = -0.189255, \; \overline{x}_1 = -0.096291 \tag{91}$$

$$\underline{x}_2 = 2.596291, \; \overline{x}_2 = 3.522588 \tag{92}$$

Let us now use the interval parametrization formulas with Equations (86)-(88), which result in:

$$\underline{x}_2 = -0.189255 (\text{at } t_1 = 1 \text{ and } t_2 = 1), \overline{x}_2 = -0.096291 (\text{at } t_1 = -1 \text{ and } t_2 = -1) \tag{93}$$

$$\underline{x}_1 = 2.596291 (\text{at } t_1 = -1 \text{ and } t_2 = -1), \overline{x}_1 = 3.522588 (\text{at } t_1 = 1 \text{ and } t_2 = 1) \tag{94}$$

The results obtained using each presented method match with each other, leading also to the same values for the parameters a and c.

Let us try with another example to prove that these formulas found by sign-variation are working for any subcase, as long as $\sqrt{b^2 - 4ac} > 0$ and $a \neq 0$. We consider:

$$a = [\underline{a}; \overline{a}] = [2; 3], \; b = -10, \; c = [\underline{c}; \overline{c}] = [-1; 1] \tag{95}$$

The direct approach yields using Equation (95):

$$\underline{x}_1 = -0.098076 (\text{at } \underline{a} = 2 \text{ and } \underline{c} = -1), \overline{x}_1 = 0.103195 (\text{at } \overline{a} = 3 \text{ and } \overline{c} = 1) \tag{96}$$

$$\underline{x}_2 = 3.230139 (\text{at } \overline{a} = 3 \text{ and } \overline{c} = 1), \overline{x}_2 = 5.098076 (\text{at } \underline{a} = 2 \text{ and } \underline{c} = -1) \tag{97}$$

The sign-variation method formulas result in, using Equations (68)-(71) and (95):

$$\underline{x}_1 = -0.098076, \; \overline{x}_1 = 0.103195 \tag{98}$$

$$\underline{x}_2 = 3.230139, \; \overline{x}_2 = 5.098076 \tag{99}$$

Let us use the interval parametrization formulas with Equations (84), (85) and (95) to obtain:

$$\underline{x}_1 = -0.098076 (\text{at } t_1 = -1 \text{ and } t_2 = -1), \overline{x}_1 = 0.103195 (\text{at } t_1 = 1 \text{ and } t_2 = 1) \tag{100}$$

$$\underline{x}_2 = 3.230139 (\text{at } t_1 = 1 \text{ and } t_2 = 1), \overline{x}_2 = 5.098076 (\text{at } t_1 = -1 \text{ and } t_2 = -1) \tag{101}$$

We again get coinciding results between all methods, even with different intervals and a particular interval for c that has both negative and positive values in it.

4.4. Some Generalized Babylonian Problems Involving Interval Quadratic Equations

Let us consider the same problem that we used in our previous article, that was focused on one coefficient as an interval. According to Katz ([8], p.23), Babylonians "applied to various standard problems such as finding the length and width of a rectangle, given the semi perimeter and the area. For example, consider the problem: $x+y=6\frac{1}{2}$, $xy=7\frac{1}{2}$ from tablet 4663". Let us now consider the interval problem with the following equations:

$$x+y=\left[6\frac{1}{4};6\frac{3}{4}\right] \tag{102}$$

and

$$xy=\left[7\frac{1}{4};7\frac{3}{4}\right] \tag{103}$$

because, in reality, measurements are inherently associated with error.

Equations (102) and (103) yield the quadratic equation:

$$ay^2+\left[\underline{b};\overline{b}\right]y+\left[\underline{c};\overline{c}\right]=-y^2+\left[6.25;6.75\right]y-\left[7.75;7.25\right]=0 \tag{104}$$

We are in a situation where b and c are intervals, we can get the adapted formulas in **Table C2** in the **Appendix B** that will solve our problem. Sign-variation method yields:

$$\underline{x}_1=\frac{-\overline{b}+\sqrt{\overline{b}^2-4a\overline{c}}}{2a}=1.340 \tag{105}$$

$$\overline{x}_1=\frac{-\underline{b}+\sqrt{\underline{b}^2-4a\underline{c}}}{2a}=1.705 \tag{106}$$

$$\underline{x}_2=\frac{-\underline{b}-\sqrt{\underline{b}^2-4a\underline{c}}}{2a}=4.545 \tag{107}$$

$$\overline{x}_2=\frac{-\overline{b}-\sqrt{\overline{b}^2-4a\overline{c}}}{2a}=5.409 \tag{108}$$

Now, using the direct approach with Maple, we obtain from Equation (104):

$$\underline{x}_1=1.340\left(\text{at } \overline{b}=6.75 \text{ and } \overline{c}=-7.25\right), \overline{x}_1=1.705\left(\text{at } \underline{b}=6.25 \text{ and } \underline{c}=-7.75\right) \tag{109}$$

$$\underline{x}_2=4.545\left(\text{at } \underline{b}=6.25 \text{ and } \underline{c}=-7.75\right), \overline{x}_2=5.409\left(\text{at } \overline{b}=6.75 \text{ and } \overline{c}=-7.25\right) \tag{110}$$

We remark that the results from sign-variation method and from the computer evaluation are the same here. This means that Babylonians need a rectangle that is between 1.34 and 1.705 m wide, and between 4.545 and 5.409 m long to meet their requirements.

4.5. Kinematics Problem Using Interval Quadratic Equations

Let us also consider the other word problem. This is a classic and uniformly available Grade 12 Kinematics problem: "A pedestrian is running at a maximum speed of 6.0 m/s to catch a bus stopped by a traffic light. When the pedestrian is 25 m from the bus, the light changes and the bus accelerates uniformly at 1.0 m/s^2. Find either (a) how far the pedestrian must run to catch the bus, or (b) the pedestrian's frustration distance (closest approach)."

For the deterministic problem, we obtained the following quadratic equation:

$$\frac{1}{2}t^2 - 6t + 25 = 0 \tag{111}$$

Let us now adapt the problem to obtain a quadratic equation with two coefficients as intervals.

The pedestrian, seeing that he/she will miss the bus if his running speed doesn't increase, gets some impetus from the situation, and now runs at an estimated speed between 7.0 m/s and 7.5 m/s. There is also traffic in front of the bus, so the bus cannot accelerate uniformly, and now accelerates between 0.5 m/s^2 and 0.9 m/s^2. We now get the following quadratic equation:

$$[\underline{a};\overline{a}]t^2 + [\underline{b};\overline{b}]t + c = [0.25;0.45]t^2 + [-7.5;-7]t + 25 = 0 \tag{112}$$

The discriminant of Equation (112) is:

$$\begin{aligned}\Delta = b^2 - 4ac &= [-7.5;-7]^2 - 4\times[0.25;0.45]\times 25 \\ &= [49;56.25] - [25;45] = [4;31.25]\end{aligned} \tag{113}$$

The pedestrian is catching the bus if the discriminant is positive, which is the case here.

We can use the formulas that have been established to solve this problem. Using Table C1 in the **Appendix C**, we get:

$$\underline{x}_1 = \frac{-\underline{b} - \sqrt{\underline{b}^2 - 4\underline{a}c}}{2\underline{a}} = 3.82 \tag{114}$$

$$\overline{x}_1 = \frac{-\overline{b} - \sqrt{\overline{b}^2 - 4\overline{a}c}}{2\overline{a}} = 5.55 \tag{115}$$

$$\underline{x}_2 = \frac{-\overline{b} + \sqrt{\overline{b}^2 - 4\overline{a}c}}{2\overline{a}} = 10 \tag{116}$$

$$\overline{x}_2 = \frac{-\underline{b} + \sqrt{\underline{b}^2 - 4\underline{a}c}}{2\underline{a}} = 26.18 \tag{117}$$

Now, using the direct approach with Maple, we obtain from Equation (112):

$$\underline{x}_1 = 3.82\left(\text{at } \underline{a} = 0.25 \text{ and } \underline{b} = -7.5\right), \overline{x}_1 = 5.55\left(\text{at } \overline{a} = 0.45 \text{ and } \overline{b} = -7\right) \tag{118}$$

$$\underline{x}_2 = 10\left(\text{at } \overline{a} = 0.45 \text{ and } \overline{b} = -7\right), \overline{x}_2 = 26.18\left(\text{at } \underline{a} = 0.25 \text{ and } \underline{b} = -7.5\right) \tag{119}$$

Again, we get the same results by both methods, as it should be.

The results obtained mean that the pedestrian will be able to catch the bus, and will have a time window that varies between 4.45 and 22.36 seconds ([10 - 5.55; 26.18 - 3.82]).

We only considered some subcases in these Sections, and we presented the method to obtain the formulas for any case. However, we provide in **Appendix C** Table C1 and Table C2 that give the expressions of the roots of quadratic equations with two variables considered as intervals, for every subcase possible.

We will now consider the more complex cases in which all variables are intervals.

5. All Coefficients Serving as Intervals

Now, we consider the case in which all the coefficients of a quadratic equation are serving as intervals. When it comes to interval parametrization, as it needs to be computed in order to give results, we will not get analytical formulas that could directly lead to solving the problem. In order to get analytical formulas, we will then keep using the sign-variation method.

Let us first consider the case with all the parameters *a*, *b* and *c* constituting intervals, with the sign-variation method.

$$a = \left[\underline{a}; \overline{a}\right] \tag{120}$$

$$b = \left[\underline{b}; \overline{b}\right] \tag{121}$$

$$c = \left[\underline{c}; \overline{c}\right] \tag{122}$$

Let us consider here $a > 0, b < 0$ and $c < 0$ for this example. We assume that $a \neq 0$ and $\sqrt{b^2 - 4ac} > 0$ too.

We can apply the sign-variation analysis method in order to find the formulas of minimized and maximized values of both roots of the equation, the same way we did previously.

For x_1 first, as **Table 1** shows, the value $\partial x_1 / \partial a$ is positive when $a > 0$. Consequently, if we want to minimize x_1 regarding *a*, we need to use $\underline{a}$, thus using $\overline{a}$ will result in maximizing x_1.

The value $\partial x_1 / \partial b$ is negative when $b < 0$ according to **Table 2**. So, if we want to minimize x_1 with respect to *b*, we need to use $\overline{b}$, then we need to use $\underline{b}$ to maximize x_1.

The value $\partial x_1 / \partial c$ is positive when $c < 0$ in **Table B6** in **Appendix B**. To minimize x_1 with respect to *c*, we need to use $\underline{c}$, using $\overline{c}$ will result in maximizing x_1. These results together yield the following formulas:

$$\underline{x}_1 = \frac{-\overline{b} - \sqrt{\overline{b}^2 - 4\underline{a}\underline{c}}}{2\underline{a}} \tag{123}$$

$$\overline{x}_1 = \frac{-\underline{b} - \sqrt{\underline{b}^2 - 4\overline{a}\overline{c}}}{2\overline{a}} \tag{124}$$

For x_2, **Table 3** shows that the value $\partial x_2 / \partial a$ is negative when $a > 0$. Consequently, if we want to minimize x_2 regarding *a*, we need to use $\overline{a}$, thus using $\underline{a}$ will result in maximizing x_2.

The derivative $\partial x_2 / \partial b$ is negative when $b < 0$ as demonstrates **Table 4**. So, if we want to minimize x_2 regarding *b*, we need to use $\overline{b}$, then we need to use $\underline{b}$ to maximize x_2.

In **Table B16**, the derivative $\partial x_2 / \partial c$ is negative when $c < 0$. To minimize x_2 regarding *c*, we need to use $\overline{c}$, using $\underline{c}$ will result in maximizing x_2. These propositions yield the formulas:

$$\underline{x}_2 = \frac{-\overline{b} + \sqrt{\overline{b}^2 - 4\overline{a}\overline{c}}}{2\overline{a}} \tag{125}$$

$$\overline{x}_2 = \frac{-\underline{b} + \sqrt{\underline{b}^2 - 4\underline{a}\underline{c}}}{2\underline{a}} \tag{126}$$

Let us consider another case: $a < 0, b > 0$ and $c < 0$. We still assume that $a \neq 0$ and $\sqrt{b^2 - 4ac} > 0$ too.

The derivative $\partial x_1 / \partial a$ is positive when $a < 0$ as **Table B3** shows. Consequently, if we want to minimize x_1 regarding a, we need to use $\underline{a}$, thus using $\overline{a}$ will result in maximizing x_1.

The derivative $\partial x_1 / \partial b$ is positive when $b > 0$ as seen in **Table B9**. So, if we want to minimize x_1 regarding b, we need to use $\underline{b}$, then we need to use $\overline{b}$ to maximize x_1.

Table B5 shows that the value $\partial x_1 / \partial c$ is positive when $c < 0$. To minimize x_1 with respect to c, we need to use $\underline{c}$, using $\overline{c}$ will result in maximizing x_1.Those results yield the following formulas:

$$\underline{x}_1 = \frac{-\underline{b} + \sqrt{\underline{b}^2 - 4\underline{a}\underline{c}}}{2\underline{a}} \tag{127}$$

$$\overline{x}_1 = \frac{-\overline{b} + \sqrt{\overline{b}^2 - 4\overline{a}\overline{c}}}{2\overline{a}} \tag{128}$$

For x_2, **Table B13** demonstrates that the value $\partial x_2 / \partial a$ is negative when $a < 0$. Consequently, if we want to minimize x_2 regarding a, we need to use $\overline{a}$, thus using $\underline{a}$ will result in maximizing x_2.

The derivative $\partial x_2 / \partial b$ is negative when $b > 0$ in **Table B20**. So, if we want to minimize x_2 with respect to b, we need to use $\overline{b}$, then we need to use $\underline{b}$ to maximize x_2.

In **Table B15**, we see that $\partial x_2 / \partial c$ is also negative when $c < 0$. To minimize x_2 regarding c, we need to use $\overline{c}$, using $\underline{c}$ will result in maximizing x_2. This leads us to the following formulas:

$$\underline{x}_2 = \frac{-\overline{b} - \sqrt{\overline{b}^2 - 4\overline{a}\overline{c}}}{2\overline{a}} \tag{129}$$

$$\overline{x}_2 = \frac{-\underline{b} - \sqrt{\underline{b}^2 - 4\underline{a}\underline{c}}}{2\underline{a}} \tag{130}$$

We cannot consider every possible case, so we provide in the **Appendix C Table C3** that contains the results for every possible subcase when all coefficients constitute intervals.

Let us now consider the same problem defined above, with the parameters a, b and c still constituting the intervals defined in Equations (120), (121) and (122):

We introduce the average value of a denoted a_{ave} and the deviation value of a denoted a_{dev}. We proceed analogously for b and c, so the average values of b and c denoted respectively b_{ave} and c_{ave} and the deviation values of b and c denoted b_{dev} and c_{dev}, as follows:

$$a_{ave} = \frac{\overline{a} + \underline{a}}{2} \tag{131}$$

$$a_{dev} = \frac{\overline{a} - \underline{a}}{2} \quad (132)$$

$$b_{ave} = \frac{\overline{b} + \underline{b}}{2} \quad (133)$$

$$b_{dev} = \frac{\overline{b} - \underline{b}}{2} \quad (134)$$

$$c_{ave} = \frac{\overline{c} + \underline{c}}{2} \quad (135)$$

$$c_{dev} = \frac{\overline{c} - \underline{c}}{2} \quad (136)$$

a, $b$and c become now:

$$a = a_{ave} + a_{dev} t_1 \quad (137)$$

$$b = b_{ave} + b_{dev} t_2 \quad (138)$$

$$c = c_{ave} + c_{dev} t_3 \quad (139)$$

using

$$t_1 = [-1;1] \quad (140)$$

$$t_2 = [-1;1] \quad (141)$$

$$t_3 = [-1;1] \quad (142)$$

The roots of this equation can now be written, using Equations (137), (138) and (139), with $a > 0$:

$$x_1 = \frac{-(b_{ave} + b_{dev} t_2) - \sqrt{(b_{ave} + b_{dev} t_2)^2 - 4(a_{ave} + a_{dev} t_1)(c_{ave} + c_{dev} t_3)}}{2(a_{ave} + a_{dev} t_1)} \quad (143)$$

$$x_2 = \frac{-(b_{ave} + b_{dev} t_2) + \sqrt{(b_{ave} + b_{dev} t_2)^2 - 4(a_{ave} + a_{dev} t_1)(c_{ave} + c_{dev} t_3)}}{2(a_{ave} + a_{dev} t_1)} \quad (144)$$

The roots of this equation are, when $a < 0$, using Equations (137), (138) and (139):

$$x_1 = \frac{-(b_{ave} + b_{dev} t_2) + \sqrt{(b_{ave} + b_{dev} t_2)^2 - 4(a_{ave} + a_{dev} t_1)(c_{ave} + c_{dev} t_3)}}{2(a_{ave} + a_{dev} t_1)} \quad (145)$$

$$x_2 = \frac{-(b_{ave} + b_{dev} t_2) - \sqrt{(b_{ave} + b_{dev} t_2)^2 - 4(a_{ave} + a_{dev} t_1)(c_{ave} + c_{dev} t_3)}}{2(a_{ave} + a_{dev} t_1)} \quad (146)$$

We defined the expressions for both methods with $a > 0, b < 0$ and $c < 0$, we can now proceed to evaluate their results numerically. Let us consider the following example:

$$a = [\underline{a}; \overline{a}] = [3;4],\ b = [\underline{b}; \overline{b}] = [-11;-10],\ c = [\underline{c}; \overline{c}] = [-2;-1] \quad (147)$$

The direct approach gives the following results using Equation (147):

$$\begin{aligned} &\underline{x}_1 = -0.189255\left(\text{at } \underline{a} = 3, \overline{b} = -10 \text{ and } \underline{c} = -2\right), \\ &\overline{x}_1 = -0.088087\left(\text{at } \overline{a} = 4, \underline{b} = -11 \text{ and } \overline{c} = -1\right) \end{aligned} \quad (148)$$

$$\underline{x}_2 = 2.596291\left(\text{at } \overline{a} = 4, \overline{b} = -10 \text{ and } \overline{c} = -1\right),$$
$$\overline{x}_2 = 3.840266\left(\text{at } \underline{a} = 3, \underline{b} = -11 \text{ and } \underline{c} = -2\right) \tag{149}$$

The sign-variation method formulas yield, using Equations (123)-(126) and (147):

$$\underline{x}_1 = -0.189255,\ \overline{x}_1 = -0.088087 \tag{150}$$

$$\underline{x}_2 = 2.596291,\ \overline{x}_2 = 3.840266 \tag{151}$$

Let us now use the interval parametrization formulas with Equations (143), (144) and (147):

$$\underline{x}_1 = -0.189255\left(\text{at } t_1 = -1, t_2 = 1 \text{ and } t_3 = -1\right),$$
$$\overline{x}_1 = -0.088087\left(\text{at } t_1 = 1, t_2 = -1 \text{ and } t_3 = 1\right) \tag{152}$$

$$\underline{x}_2 = 2.596291\left(\text{at } t_1 = 1, t_2 = 1 \text{ and } t_3 = 1\right),$$
$$\overline{x}_2 = 3.522588\left(\text{at } t_1 = -1, t_2 = -1 \text{ and } t_3 = -1\right) \tag{153}$$

These results match with to those found using both other methods.

Let us try with the other subcase detailed above, $a < 0, b > 0$ and $c < 0$. We define:

$$a = \left[\underline{a}; \overline{a}\right] = \left[-4, -3\right],\ b = \left[\underline{b}; \overline{b}\right] = \left[10, 11\right],\ c = \left[\underline{c}; \overline{c}\right] = \left[-2, -1\right] \tag{154}$$

The direct approach gives the following results using Equation (154):

$$\underline{x}_1 = 0.093282\left(\text{at } \overline{a} = -3, \overline{b} = 11 \text{ and } \overline{c} = -1\right),$$
$$\overline{x}_1 = 0.219223\left(\text{at } \underline{a} = -4, \underline{b} = 10 \text{ and } \underline{c} = -2\right) \tag{155}$$

$$\underline{x}_2 = 2.280776\left(\text{at } \underline{a} = -4, \underline{b} = 10 \text{ and } \underline{c} = -2\right),$$
$$\overline{x}_2 = 3.573384\left(\text{at } \overline{a} = -3, \overline{b} = 11 \text{ and } \overline{c} = -1\right) \tag{156}$$

The sign-variation method formulas yield using Equations (127)-(130) and (154):

$$\underline{x}_1 = 0.093282,\ \overline{x}_1 = 0.219223 \tag{157}$$

$$\underline{x}_2 = 2.280776,\ \overline{x}_2 = 3.573384 \tag{158}$$

Let us use the interval parametrization formulas with Equations (145), (146) and (154), to get:

$$\underline{x}_1 = 0.093282\left(\text{at } t_1 = 1, t_2 = 1 \text{ and } t_3 = 1\right),$$
$$\overline{x}_1 = 0.219223\left(\text{at } t_1 = -1, t_2 = -1 \text{ and } t_3 = -1\right) \tag{159}$$

$$\underline{x}_2 = 2.280776\left(\text{at } t_1 = -1, t_2 = -1 \text{ and } t_3 = -1\right),$$
$$\overline{x}_2 = 3.573384\left(\text{at } t_1 = 1, t_2 = 1 \text{ and } t_3 = 1\right) \tag{160}$$

We again get corresponding results between all methods.

5.1. Kinematics Problem Using Interval Quadratic Equations

We will again consider the same problem that was dealt with in the previous part but in interval setting. "A pedestrian is running at his maximum speed of 6.0

m/s to catch a bus stopped by a traffic light. When the pedestrian is 25 m from the bus, the light changes and the bus accelerates uniformly at 1.0 m/s^2. Find either (a) how far the pedestrian must run to catch the bus, or (b) the pedestrian's frustration distance (closest approach)".

For the deterministic problem, we obtained the following quadratic equation:

$$\frac{1}{2}t^2 - 6t + 25 = 0 \tag{161}$$

Let us now adapt the problem to obtain a quadratic equation with all coefficients as intervals.

The pedestrian, seeing that she/he will miss the bus if she/he doesn't run faster, gets some impetus from the situation, as it were, and now runs at an estimated speed between 7.0 m/s and 7.5 m/s. There is also traffic in front of the bus, so the bus cannot accelerate uniformly, and now accelerates between 0.5 m/s^2 and 0.9 m/s^2. Since the pedestrian was running and started to be stressed from the situation, she/he could not really give an exact value for how far the bus is. Therefore, the latter is not really 25m away, but its distance from the pedestrian is uncertain, being between 20 and 25 m from the pedestrian's initial position.

We now get the following quadratic equation:

$$[\underline{a};\overline{a}]t^2 + [\underline{b};\overline{b}]t + [\underline{c};\overline{c}] = [0.25;0.45]t^2 + [-7.5;-7]t + [20;25] = 0 \tag{162}$$

The discriminant of Equation (162) is:

$$\begin{aligned}\Delta = b^2 - 4ac &= [-7.5;-7]^2 - 4\times[0.25;0.45]\times[20;25] \\ &= [49;56.25] - [20;45] = [4;36.25]\end{aligned} \tag{163}$$

The pedestrian is catching the bus if the discriminant is positive, which is the case here.

Using sign-variation formulas in **Table C3**, we get the following results, using Equation (162):

$$\underline{x}_1 = \frac{-\underline{b} - \sqrt{\underline{b}^2 - 4\underline{a}\underline{c}}}{2\underline{a}} = 2.96 \tag{164}$$

$$\overline{x}_1 = \frac{-\overline{b} - \sqrt{\overline{b}^2 - 4\overline{a}\overline{c}}}{2\overline{a}} = 5.55 \tag{165}$$

$$\underline{x}_2 = \frac{-\overline{b} + \sqrt{\overline{b}^2 - 4\overline{a}\overline{c}}}{2\overline{a}} = 10 \tag{166}$$

$$\overline{x}_2 = \frac{-\underline{b} + \sqrt{\underline{b}^2 - 4\underline{a}\underline{c}}}{2\underline{a}} = 27.04 \tag{167}$$

Let us again use the direct approach from Maple, using Equation (162):

$$\begin{aligned}&\underline{x}_1 = 2.96\left(\text{at } \underline{a} = 0.25, \underline{b} = -7.5 \text{ and } \underline{c} = 20\right), \\ &\overline{x}_1 = 5.55\left(\text{at } \overline{a} = 0.45, \overline{b} = -7 \text{ and } \overline{c} = 25\right)\end{aligned} \tag{168}$$

$$\begin{aligned}&\underline{x}_2 = 10\left(\text{at } \overline{a} = 0.45, \overline{b} = -7 \text{ and } \overline{c} = 25\right), \\ &\overline{x}_2 = 27.04\left(\text{at } \underline{a} = 0.25, \underline{b} = -7.5 \text{ and } \underline{c} = 20\right)\end{aligned} \tag{169}$$

The results obtained match again with the sign-variation method.

The pedestrian will then be able to catch the bus, and will have a time window that varies between 4.45 and 24.08 seconds ([10 - 5.55; 27.04 - 2.96]).

5.2. Electronics Problem Using Interval Quadratic Equations

Let us consider the following electrical engineering problem [16]. "A 100W (Watts) lamp is connected to a 20 Ω (Ohm) resistor and a 120 V (Volt) power supply as shown on Figure 1.

Find the current I in Amperes."

Let us solve the problem in its crisp setting first. Using Kirchhoff Voltage laws for Equation (170) and Ohm laws for Equation (171), we get the following:

$$V_P = V_L + V_R = 120 \tag{170}$$

$$V_R = 20I \tag{171}$$

For the lamp, we also have Power = Current × Voltage, which leads to:

$$P_L = V_L \times I = 100\text{ W} \Rightarrow V_L = \frac{100}{I} \tag{172}$$

Substituting Equation (171) and (172) into Equation (170) results in:

$$120 = \frac{100}{I} + 20I \tag{173}$$

We now multiply Equation (173) by I to obtain the following quadratic Equation (174)

$$20I^2 - 120I + 100 = 0 \tag{174}$$

We will set that the resistance of the resistor is associated to the parameter a, the opposite of the power supply voltage to b and the lamp's power to the coefficient c. This way, Equation (174) now becomes:

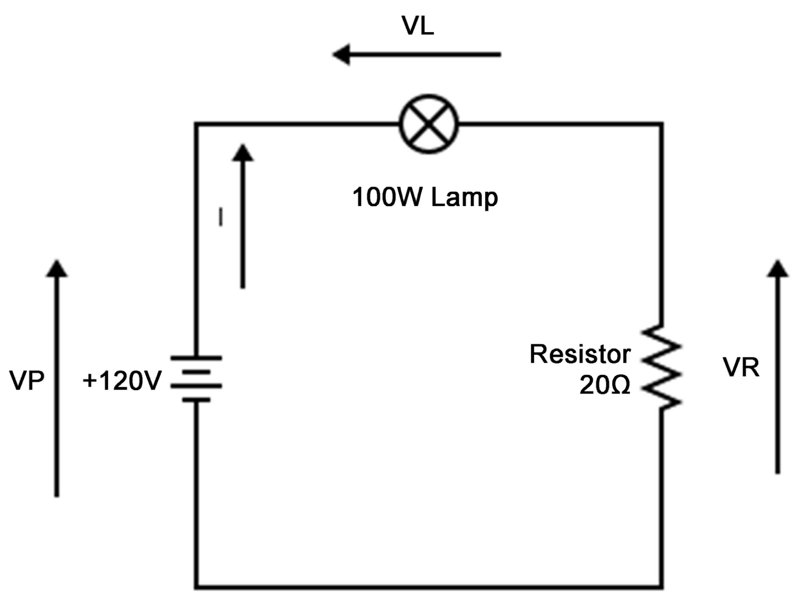

Figure 1. Electrical circuit.

$$aI^2 + bI + c = 0 \tag{175}$$

with $a = 20$, $b = -120$ and $c = 100$. Let us now solve this quadratic equation that yields the following roots:

$$I_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{120 - \sqrt{120^2 - 4 \times 20 \times 100}}{2 \times 20} = \frac{120 - 80}{40} = 1\text{ A} \tag{176}$$

$$I_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{120 + \sqrt{120^2 - 4 \times 20 \times 100}}{2 \times 20} = \frac{120 + 80}{40} = 5\text{ A} \tag{177}$$

In this circuit, the current can either be 1 A or 5 A, depending on the lamp's voltage.

Actually, manufacturers always indicate that their components are subject to uncertainty.

Let us consider that the manufacturer of the power supply indicates that its voltage has a 10% uncertainty around its mean value of 120 V. The company that produces the lamps considers that its product is very well checked, and that the uncertainty of the lamp's power is only 3% around the mean value of 100 W. Last but not least, the resistor's manufacturer states that its indicated resistance is 20 Ω ± 5%. This leads to a change in the *a*, *b* and *c* values mentioned before, which now take the following values:

$$a = [\underline{a}; \overline{a}] = [0.95 \times 20; 1.05 \times 20] = [19; 21]\,\Omega \tag{178}$$

$$b = [\underline{b}; \overline{b}] = [0.90 \times 120; 1.10 \times 120] = [108; 132]\text{ V} \tag{179}$$

$$c = [\underline{c}; \overline{c}] = [0.97 \times 100; 1.03 \times 100] = [97; 103]\text{ W} \tag{180}$$

Solving the problem in such a setting implies now solving the quadratic Equation (175) in which all coefficients constitute the interval variables defined in Equations (178)-(180), as follows:

$$[\underline{a}; \overline{a}] I^2 + [\underline{b}; \overline{b}] I + [\underline{c}; \overline{c}] = [19; 21] I^2 + [-132; -108] I + [97; 103] = 0 \tag{181}$$

The discriminant of Equation (181) is:

$$\begin{aligned} \Delta = b^2 - 4ac &= [-132; -108]^2 - 4 \times [19; 21] \times [97; 103] \\ &= [11664; 17424] - [7372; 8652] = [3012; 10052] \end{aligned} \tag{182}$$

The discriminant is always positive, we can apply the sign-variation method.

Using sign-variation formulas in **Table C3**, we get the following results, using Equation (181):

$$\underline{I}_1 = \frac{-\underline{b} - \sqrt{\underline{b}^2 - 4\underline{a}\underline{c}}}{2\underline{a}} = 0.835 \tag{183}$$

$$\overline{I}_1 = \frac{-\overline{b} - \sqrt{\overline{b}^2 - 4\overline{a}\overline{c}}}{2\overline{a}} = 1.265 \tag{184}$$

$$\underline{I}_2 = \frac{-\overline{b} + \sqrt{\overline{b}^2 - 4\overline{a}\overline{c}}}{2\overline{a}} = 3.878 \tag{185}$$

$$\overline{I}_2 = \frac{-\underline{b} + \sqrt{\underline{b}^2 - 4\underline{a}\,\underline{c}}}{2\underline{a}} = 6.112 \tag{186}$$

Let us also use the direct approach from Maple, using Equation (181):

$$\begin{aligned} &\underline{I}_1 = 0.835\left(\text{at } \underline{a} = 19, \underline{b} = -132 \text{ and } \underline{c} = 97\right), \\ &\overline{I}_1 = 1.265\left(\text{at } \overline{a} = 21, \overline{b} = -108 \text{ and } \overline{c} = 103\right) \end{aligned} \tag{187}$$

$$\begin{aligned} &\underline{I}_2 = 3.878\left(\text{at } \overline{a} = 21, \overline{b} = -108 \text{ and } \overline{c} = 103\right), \\ &\overline{I}_2 = 6.112\left(\text{at } \underline{a} = 19, \underline{b} = -132 \text{ and } \underline{c} = 97\right) \end{aligned} \tag{188}$$

Considering the uncertainties on the component's nominal values, we now have a current's value that is either between 0.835 A and 1.265 A, or between 3.878 A and 6.112 A, again depending on the voltage of the lamp.

6. Conclusions

This article offers a method that leads to obtaining analytical formulas for the roots of quadratic equations in which two or all three coefficients serve as intervals. As seen, the results yielded by the sign-variation method using numerical examples keep matching with those obtained using a direct approach that is based on *Minimize* and *Maximize* commands of the Maple software, also with those obtained using the interval parametrization method introduced by Elishakoff and Miglis [11]. Let us now draw a table that summarizes the advantages and the drawbacks of the classic interval analysis method used by Elishakoff and Daphnis (2015), the quantifier elimination used by Ioakimidis [10], the interval parametrization method by Elishakoff and Miglis [11], and the sign-variation method presented in this article.

Table 5 aims to show which method is the most adapted when solving a problem involving quadratic equations in which one, two or even all three of the coefficients could serve as intervals.

Table 5. Advantages and drawbacks of the presented methods that deal with interval quadratic equations.

Method used	One coefficient serves as an interval	Two coefficients serve as intervals	All three coefficients serve as intervals
Classic interval analysis	Works, but often leads to errors, yields analytical formulas, done by hand	Does not work	Does not work
Quantifier elimination	Works, yields analytical formulas without much effort, cannot be done by hand	Does not work because CPU time gets too important	Does not work because CPU time gets too important
Interval Parametrization	Works, no analytical formulas, cannot be done by hand as it is a numerical only	Works, no analytical formulas, cannot be done by hand as it is a numerical only	Works, no analytical formulas, cannot be done by hand as it is a numerical only
Sign variation analysis	Works, takes some effort to consider every subcase (24 subcases), yields analytical formulas, could be done by hand	Works, takes some effort to consider every subcase (24 subcases), yields analytical formulas, could be done by hand	Works, does not take much effort to consider every subcase (8 subcases), yields analytical formulas, could be done by hand

Obtaining analytical formulas seems remarkable to those problems because then they could be used by hand for any similar problem. Thus, quantifier elimination method seems to be the most adapted method to solve problems dealing with quadratic equations in which one coefficient serves as an interval, whereas sign-variation method appears as the only analytical way to deal with quadratic equations in which two or three of the coefficients constitute intervals.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] De P. Dourado, A., Lobato, F.S., Cavalini Jr., A.Ap and Steffen Jr., V. (2019) Fuzzy Reliability-Based Optimization for Engineering System. *International Journal of Fuzzy Systems*, **21**, 1418-1429. https://doi.org/10.1007/s40815-019-00655-5

[2] Mo, H., Zhao, X. and Wang, F.-Y. (2019) Application of Interval Type-2 Fuzzy Sets in Unmanned Vehicle Visual Guidance. *International Journal of Fuzzy Systems*, **21**, 1661-1668. https://doi.org/10.1007/s40815-019-00680-4

[3] Tian, D., Wang, Y. and Yu, T. (2020) Fuzzy Risk Assessment Based on Interval Numbers and Assessment Distributions. *International Journal of Fuzzy Systems*, **22**, 1142-1157. https://doi.org/10.1007/s40815-020-00837-6

[4] Liao, H., Wang, J., Tang, M. and Al-Barakati, A. (2023) An Overview of Interval Analysis Techniques and Their Fuzzy Extensions in Multi-Criteria Decision-Making: What's Going on and What's Next? *International Journal of Fuzzy Systems*, **25**, 2081-2108. https://doi.org/10.1007/s40815-022-01448-z

[5] https://engineering-computer-science.wright.edu/sites/engineering-computer-science.wright.edu/files/page/attachments/2_Quadratic_Equations_in_Engineering_1.pdf

[6] Alolyan, I. (2007) Real Roots of Quadratic Interval Polynomials. *International Journal of Mathematical Analysis*, **1**, 1041-1050.

[7] Boyer, C.B. and Merzbach, U.C. (1991) A History of Mathematics. 2nd Edition, Wiley, New York, 30-31.

[8] Katz, V.J. (2007) The Mathematics of Egypt, Mesopotamia, China, India, and Islam: Princeton University Press, Princeton, NJ, 428-434. https://doi.org/10.1515/9780691235394

[9] Elishakoff, I. and Daphnis, A. (2015) Exact Enclosures of Roots of Interval Quadratic Equations by Sridhara's and Fagnano's or Modified Fagnano's Formulas. *Applied Mathematics and Computation*, **271**, 1024-1037. https://doi.org/10.1016/j.amc.2015.09.005

[10] Ioakimidis, N. (2019) Sharp Enclosures of the Real Roots of the Classical Parametric Quadratic Equation with One Interval Coefficient by the Method of Quantifier Elimination.
https://www.researchgate.net/publication/333005417_Sharp_enclosures_of_the_real_roots_of_the_classical_parametric_quadratic_equation_with_one_interval_coefficient_by_the_method_of_quantifier_elimination

[11] Elishakoff, I. and Miglis, Y. (2012) Novel Parameterized Intervals May Lead to Sharp Bounds. *Mechanics Research Communications*, **44**, 1-8.
https://doi.org/10.1016/j.mechrescom.2012.04.004

[12] (2022) Wolfram Documentation Center.
https://reference.wolfram.com/language/ref/Minimize.html

[13] Landowski, M. (2007) RDM Interval Method for Solving Quadratic Interval Equation. *PrzeglądElektrotechniczny*, **93**, 65-68.

[14] Sevastjanov, P. and Dymova, L. (2007) Fuzzy Solution of Interval Linear Equation. *International Conference on Parallel Processing and Applied Mathematics*, Springer, Berlin, Heidelberg, 1392-1399. https://doi.org/10.1007/978-3-540-68111-3_147

[15] Mamehrashi, K. (2021) A New Method for Solving Interval and Fuzzy Quadratic Equations of Dual Form. *UKH Journal of Science and Engineering*, **5**, 81-89.
https://doi.org/10.25079/ukhjse.v5n2y2021.pp81-89

[16] https://engineering-computer-science.wright.edu/sites/engineering-computer-science.wright.edu/files/page/attachments/2_Quadratic_Equations_in_Engineering_1.pdf

Appendices

Appendix A: Why Do the Root's Formulas Depend on the Sign of Coefficient *a*?

As stated in the Introduction part, we have to adapt which of the two root's expression is x_1 and which is x_2 depending on the sign of a, in order to respect $x_1 < x_2$.

Indeed, let us consider both expressions of the root of a quadratic equation, which are:

$$r_1 = \frac{-b-\sqrt{b^2-4ac}}{2a} \text{ and } r_2 = \frac{-b+\sqrt{b^2-4ac}}{2a} \tag{189}$$

In order to better understand when $r_1 < r_2$, we need to study their respective signs.

No matter what is the sign of a, we always have $-b-\sqrt{b^2-4ac} < -b+\sqrt{b^2-4ac}$, since $\sqrt{b^2-4ac}$ is a positive quantity.

We shall now consider to take the sign of a into account. As said before, a cannot be zero otherwise r_1 and r_2 do not exist.

If $a > 0$:

$$-b-\sqrt{b^2-4ac} < -b+\sqrt{b^2-4ac} \text{ becomes}$$
$$-b-\sqrt{b^2-4ac}\big/2a < -b+\sqrt{b^2-4ac}\big/2a$$

We now consider that $a < 0$:

$$-b-\sqrt{b^2-4ac} < -b+\sqrt{b^2-4ac} \text{ turns into}$$
$$-b+\sqrt{b^2-4ac}\big/2a < -b-\sqrt{b^2-4ac}\big/2a$$

Thus, if $a > 0$ then, to respect $x_1 < x_2$, we shall write the roots as follows:

$$x_1 = \frac{-b-\sqrt{b^2-4ac}}{2a} \text{ and } x_2 = \frac{-b+\sqrt{b^2-4ac}}{2a} \tag{190}$$

While, if we have $a < 0$, the formulas of the roots should be written as:

$$x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a} \tag{191}$$

Appendix B: Sign-Variation Tables of the Remaining Partial Derivatives of the Roots of Interval Quadratic Equations

Let us proceed to establish the sign-variation tables for the root x_1, defined in Equation (5), for the subcases that were not studied in the article. We need to introduce the last partial derivative of x_1 that hasn't been considered yet, in regard to the variable c, as follows:

$$pdc_1(a,b,c) = \frac{\partial x_1}{\partial c} = \frac{1}{\sqrt{b^2-4ac}} \tag{192}$$

For $pda_1(a,b,c) = \dfrac{\partial x_1}{\partial a}$, considering $b > 0$ and $c > 0$:

Table B1. Sign variation of $pda_1(a,b,c) = \frac{\partial x_1}{\partial a}$ with $b > 0$ and $c > 0$.

a	$-\infty < a < 0$	$a = 0$	$0 < a < \frac{b^2}{4c}$
$\frac{\partial x_1}{\partial a}$	+	Undefined	+

For $pda_1(a,b,c) = \frac{\partial x_1}{\partial a}$, considering $b < 0$ and $c > 0$:

Table B2. Sign variation of $pda_1(a,b,c) = \frac{\partial x_1}{\partial a}$ with $b < 0$ and $c > 0$.

a	$-\infty < a < 0$	$a = 0$	$0 < a < \frac{b^2}{4c}$
$\frac{\partial x_1}{\partial a}$	+	Undefined	+

For $pda_1(a,b,c) = \frac{\partial x_1}{\partial a}$, considering $b > 0$ and $c < 0$:

Table B3. Sign variation of $pda_1(a,b,c) = \frac{\partial x_1}{\partial a}$ with $b > 0$ and $c < 0$.

a	$\frac{b^2}{4c} < a < 0$	$a = 0$	$0 < a < +\infty$
$\frac{\partial x_1}{\partial a}$	+	Undefined	+

For $pdc_1(a,b,c) = \frac{\partial x_1}{\partial c}$, considering $a > 0$ and $b > 0$:

Table B4. Sign variation of $pdc_1(a,b,c) = \frac{\partial x_1}{\partial c}$ with $a > 0$ and $b > 0$.

c	$-\infty < c < \frac{b^2}{4a}$
$\frac{\partial x_1}{\partial c}$	+

For $pdc_1(a,b,c) = \frac{\partial x_1}{\partial c}$, considering $a < 0$ and $b > 0$:

Table B5. Sign variation of $pdc_1(a,b,c) = \dfrac{\partial x_1}{\partial c}$ with $a < 0$ and $b > 0$.

c	$\dfrac{b^2}{4a} < c < +\infty$
$\dfrac{\partial x_1}{\partial c}$	$+$

For $pdc_1(a,b,c) = \dfrac{\partial x_1}{\partial c}$, considering $a > 0$ and $b < 0$:

Table B6. Sign variation of $pdc_1(a,b,c) = \dfrac{\partial x_1}{\partial c}$ with $a > 0$ and $b < 0$.

c	$-\infty < c < \dfrac{b^2}{4a}$
$\dfrac{\partial x_1}{\partial c}$	$+$

For $pdc_1(a,b,c) = \dfrac{\partial x_1}{\partial c}$, considering $a < 0$ and $b < 0$:

Table B7. Sign variation of $pdc_1(a,b,c) = \dfrac{\partial x_1}{\partial c}$ with $a < 0$ and $b < 0$.

c	$\dfrac{b^2}{4a} < c < +\infty$
$\dfrac{\partial x_1}{\partial c}$	$+$

As we can see from Table 1 and Tables B1-B7, for both $\partial x_1/\partial a$ and $\partial x_1/\partial c$, when they are defined, their sign remains constantly positive whatever the other variables could be.

Let us now establish the tables for $\partial x_1/\partial b$, with the missing subcases:

For $pdb_1(a,b,c) = \dfrac{\partial x_1}{\partial b}$, considering $a > 0$ and $c > 0$:

Table B8. Sign variation of $pdb_1(a,b,c) = \dfrac{\partial x_1}{\partial b}$ with $a > 0$ and $c > 0$.

b	$-\infty < b < -\sqrt{-4ac}$	$-\sqrt{-4ac} < b < \sqrt{-4ac}$	$\sqrt{-4ac} < b < +\infty$
$\dfrac{\partial x_1}{\partial b}$	$+$	Undefined	$-$

For $pdb_1(a,b,c) = \dfrac{\partial x_1}{\partial b}$, considering $a < 0$ and $c > 0$:

Table B9. Sign variation of $pdb_1(a,b,c) = \dfrac{\partial x_1}{\partial b}$ with $a < 0$ and $c > 0$.

b	$-\infty < b < +\infty$
$\dfrac{\partial x_1}{\partial c}$	+

For $pdb_1(a,b,c) = \dfrac{\partial x_1}{\partial b}$, considering $a < 0$ and $c < 0$:

Table B10. Sign variation of $pdb_1(a,b,c) = \dfrac{\partial x_1}{\partial b}$ with $a < 0$ and $c < 0$.

b	$-\infty < b < -\sqrt{-4ac}$	$-\sqrt{-4ac} < b < \sqrt{-4ac}$	$\sqrt{-4ac} < b < +\infty$
$\dfrac{\partial x_1}{\partial b}$	–	Undefined	+

When it comes to $\partial x_1/\partial b$, we see in Table 2 and Tables B7-B9 that the variations are more complex than for the other variables in Table 1 and Tables B1-B7.

Let us now proceed to do the same thing for the other root, x_2, that is defined in Equation (10). Since x_2 also depends on a, b and c, we can consider this last partial derivative:

$$pdc_2(a,b,c) = \frac{\partial x_2}{\partial c} = \frac{-1}{\sqrt{b^2 - 4ac}} \tag{193}$$

As we did for the previous root, we will proceed to creating the sign-variation tables of each of those functions, considering the missing subcases that were not treated in the article.

For $pda_2(a,b,c) = \dfrac{\partial x_2}{\partial a}$, considering $b > 0$ and $c > 0$:

Table B11. Sign variation of $pda_2(a,b,c) = \dfrac{\partial x_2}{\partial a}$ with $b > 0$ and $c > 0$.

a	$-\infty < a < 0$	$a = 0$	$0 < a < \dfrac{b^2}{4c}$
$\dfrac{\partial x_2}{\partial a}$	–	Undefined	–

For $pda_2(a,b,c) = \dfrac{\partial x_2}{\partial a}$, considering $b < 0$ and $c > 0$:

Table B12. Sign variation of $pda_2(a,b,c)=\frac{\partial x_2}{\partial a}$ with $b<0$ and $c>0$.

a	$-\infty<a<0$	$a=0$	$0<a<\frac{b^2}{4c}$
$\frac{\partial x_2}{\partial a}$	–	Undefined	–

For $pda_2(a,b,c)=\frac{\partial x_2}{\partial a}$, considering $b>0$ and $c<0$:

Table B13. Sign variation of $pda_2(a,b,c)=\frac{\partial x_2}{\partial a}$ with $b>0$ and $c<0$.

a	$\frac{b^2}{4c}<a<0$	$a=0$	$0<a<+\infty$
$\frac{\partial x_2}{\partial a}$	–	Undefined	–

For $pdc_2(a,b,c)=\frac{\partial x_2}{\partial c}$, considering $a>0$ and $b>0$:

Table B14. Sign variation of $pdc_2(a,b,c)=\frac{\partial x_2}{\partial c}$ with $a>0$ and $b>0$.

c	$-\infty<c<\frac{b^2}{4a}$
$\frac{\partial x_2}{\partial c}$	–

For $pdc_2(a,b,c)=\frac{\partial x_2}{\partial c}$, considering $a<0$ and $b>0$:

Table B15. Sign variation of $pdc_2(a,b,c)=\frac{\partial x_2}{\partial c}$ with $a<0$ and $b>0$.

c	$\frac{b^2}{4a}<c<+\infty$
$\frac{\partial x_2}{\partial c}$	–

For $pdc_2(a,b,c)=\frac{\partial x_2}{\partial c}$, considering $a>0$ and $b<0$:

Table B16. Sign variation of $pdc_2(a,b,c) = \dfrac{\partial x_2}{\partial c}$ with $a > 0$ and $b < 0$.

c	$-\infty < c < \dfrac{b^2}{4a}$
$\dfrac{\partial x_2}{\partial c}$	$-$

For $pdc_2(a,b,c) = \dfrac{\partial x_2}{\partial c}$, considering $a < 0$ and $b < 0$:

Table B17. Sign variation of $pdc_2(a,b,c) = \dfrac{\partial x_2}{\partial c}$ with $a < 0$ and $b < 0$.

c	$\dfrac{b^2}{4a} < c < +\infty$
$\dfrac{\partial x_2}{\partial c}$	$-$

As we can see from **Table 3** and **Tables B11-B17** for both $\partial x_2/\partial a$ and $\partial x_2/\partial c$, when they are defined, their sign remains constantly negative whatever the other variables could be.

Let us now build the tables for $\partial x_2/\partial b$, with the missing subcases:

For $pdb_2(a,b,c) = \dfrac{\partial x_2}{\partial b}$, considering $a > 0$ and $c > 0$:

Table B18. Sign variation of $pdb_2(a,b,c) = \dfrac{\partial x_2}{\partial b}$ with $a > 0$ and $c > 0$.

b	$-\infty < b < -\sqrt{-4ac}$	$-\sqrt{-4ac} < b < \sqrt{-4ac}$	$\sqrt{-4ac} < b < +\infty$
$\dfrac{\partial x_2}{\partial b}$	$-$	Undefined	$+$

For $pdb_2(a,b,c) = \dfrac{\partial x_2}{\partial b}$, considering $a < 0$ and $c > 0$:

Table B19. Sign variation of $pdb_2(a,b,c) = \dfrac{\partial x_2}{\partial b}$ with $a < 0$ and $c > 0$.

b	$-\infty < b < +\infty$
$\dfrac{\partial x_2}{\partial b}$	$+$

For $pdb_2(a,b,c) = \dfrac{\partial x_2}{\partial b}$, considering $a < 0$ and $c < 0$:

Table B20. Sign variation of $pdb_2(a,b,c) = \frac{\partial x_2}{\partial b}$ with $a<0$ and $c<0$.

b	$-\infty < b < -\sqrt{-4ac}$	$-\sqrt{-4ac} < b < \sqrt{-4ac}$	$\sqrt{-4ac} < b < +\infty$
$\frac{\partial x_2}{\partial b}$	+	Undefined	−

We can analogously denote that, for $\partial x_2/\partial b$, we get in **Table 4** and **Tables B18-B20** variations that are more complex than for the other variables in **Table 3** and **Tables B11-B17**.

Appendix C: Summary of the Analytic Formulas for the Roots of Interval Quadratic Equations

Now that we have built all the sign-variations tables of every partial derivative of both roots of the quadratic equations, we can proceed to establishing the analytic formulas.

Since there is a lot of subcases, and this implies a lot of formulas, we had to summarize the results in tables, that will be established for every subcase possible, when two but also when all the coefficients of quadratic equations constitute intervals.These formulas always require that $a \neq 0$ and $\sqrt{b^2-4ac} > 0$ too, as we are only dealing with the real roots of interval quadratic equations. As seen in Section 4.3, the formulas established for a quadratic equation in which a and c constitute intervals are the same whatever the signs of *a*, *b* and *c* are. The following tables that will be presented are then used when *a* and *b*, or when *b* and *c*, or when *a*, *b* and *c* serve as intervals.

Please note that the expressions of x_1 and x_2 should change whether *a* is a positive or negative quantity, as stated in the Introduction part of this study.

We will start in **Table C1** by considering that *a* and *b* constitute intervals, and that *c* is a crisp quantity.

Let us give an example on how to read this table. We consider each subcase possible for the values of *a*, *b* and *c*, and we give for each root which value of the parameters will lead to obtaining the minimum values and maximum values of the root. For example, when $a>0, b>0$ and $c>0$, the minimum value of the first root x_1 will be obtained using $\overline{b}$ and $\underline{a}$, as stated in the cell marked with the star sign * in **Table C1**.

Table C1. Sign-variation formulas for x_1 and x_2 when *a* and *b* constitute intervals.

Sign of *a*	Sign of *b*	Sign of *c*	$\underline{x}_1$	$\overline{x}_1$	$\underline{x}_2$	$\overline{x}_2$
$a>0$	$b>0$	$c>0$	$\underline{a};\overline{b}$ *	$\overline{a};\underline{b}$	$\overline{a};\underline{b}$	$\underline{a};\overline{b}$
$a>0$	$b<0$	$c>0$	$\underline{a};\underline{b}$	$\overline{a};\overline{b}$	$\overline{a};\overline{b}$	$\underline{a};\underline{b}$
$a<0$	$b>0$	$c>0$	$\underline{a};\underline{b}$	$\overline{a};\overline{b}$	$\overline{a};\underline{b}$	$\underline{a};\overline{b}$
$a<0$	$b<0$	$c>0$	$\underline{a};\underline{b}$	$\overline{a};\overline{b}$	$\overline{a};\underline{b}$	$\underline{a};\overline{b}$
$a>0$	$b>0$	$c<0$	$\underline{a};\overline{b}$	$\overline{a};\underline{b}$	$\overline{a};\overline{b}$	$\underline{a};\underline{b}$

Continued

$a>0$	$b<0$	$c<0$	$\underline{a};\overline{b}$	$\overline{a};\underline{b}$	$\overline{a};\overline{b}$	$\underline{a};\underline{b}$
$a<0$	$b>0$	$c<0$	$\underline{a};\underline{b}$	$\overline{a};\overline{b}$	$\overline{a};\overline{b}$	$\underline{a};\underline{b}$
$a<0$	$b<0$	$c<0$	$\underline{a};\overline{b}$	$\overline{a};\underline{b}$	$\overline{a};\underline{b}$	$\underline{a};\overline{b}$

This leads us to the following expression for the lowest value of the first root $\underline{x}_1$:

$$\underline{x}_1 = \frac{-\overline{b} - \sqrt{\overline{b}^2 - 4\underline{a}\underline{c}}}{2\underline{a}}$$

Let us now establish **Table C2**, in which b and c are now the coefficients of quadratic equations that serve as intervals, as a is a deterministic quantity here. This Table should be read analogously to **Table C1**.

Table C2. Sign-variation formulas for x_1 and x_2 when b and c constitute intervals.

Sign of a	Sign of b	Sign of c	$\underline{x}_1$	$\overline{x}_1$	$\underline{x}_2$	$\overline{x}_2$
$a>0$	$b>0$	$c>0$	$\overline{b};\underline{c}$	$\underline{b};\overline{c}$	$\underline{b};\overline{c}$	$\overline{b};\underline{c}$
$a>0$	$b<0$	$c>0$	$\underline{b};\underline{c}$	$\overline{b};\overline{c}$	$\overline{b};\overline{c}$	$\underline{b};\underline{c}$
$a<0$	$b>0$	$c>0$	$\underline{b};\underline{c}$	$\overline{b};\overline{c}$	$\underline{b};\overline{c}$	$\overline{b};\underline{c}$
$a<0$	$b<0$	$c>0$	$\underline{b};\underline{c}$	$\overline{b};\overline{c}$	$\underline{b};\overline{c}$	$\overline{b};\underline{c}$
$a>0$	$b>0$	$c<0$	$\overline{b};\underline{c}$	$\underline{b};\overline{c}$	$\overline{b};\overline{c}$	$\underline{b};\underline{c}$
$a>0$	$b<0$	$c<0$	$\overline{b};\underline{c}$	$\underline{b};\overline{c}$	$\overline{b};\overline{c}$	$\underline{b};\underline{c}$
$a<0$	$b>0$	$c<0$	$\underline{b};\underline{c}$	$\overline{b};\overline{c}$	$\overline{b};\overline{c}$	$\underline{b};\underline{c}$
$a<0$	$b<0$	$c<0$	$\overline{b};\underline{c}$	$\underline{b};\overline{c}$	$\underline{b};\overline{c}$	$\overline{b};\underline{c}$

We can finally consider the case when all coefficients of a quadratic equation constitute intervals in **Table C3**, that should again be read analogously to **Table C1**.

Table C3. Sign-variation formulas for x_1 and x_2 when a, b and c constitute intervals.

Sign of a	Sign of b	Sign of c	$\underline{x}_1$	$\overline{x}_1$	$\underline{x}_2$	$\overline{x}_2$
$a>0$	$b>0$	$c>0$	$\underline{a};\overline{b};\underline{c}$	$\overline{a};\underline{b};\overline{c}$	$\overline{a};\underline{b};\overline{c}$	$\underline{a};\overline{b};\underline{c}$
$a>0$	$b<0$	$c>0$	$\underline{a};\underline{b};\underline{c}$	$\overline{a};\overline{b};\overline{c}$	$\overline{a};\overline{b};\overline{c}$	$\underline{a};\underline{b};\underline{c}$
$a<0$	$b>0$	$c>0$	$\underline{a};\underline{b};\underline{c}$	$\overline{a};\overline{b};\overline{c}$	$\overline{a};\underline{b};\overline{c}$	$\underline{a};\overline{b};\underline{c}$
$a<0$	$b<0$	$c>0$	$\underline{a};\underline{b};\underline{c}$	$\overline{a};\overline{b};\overline{c}$	$\overline{a};\underline{b};\overline{c}$	$\underline{a};\overline{b};\underline{c}$
$a>0$	$b>0$	$c<0$	$\underline{a};\overline{b};\underline{c}$	$\overline{a};\underline{b};\overline{c}$	$\overline{a};\overline{b};\overline{c}$	$\underline{a};\underline{b};\underline{c}$
$a>0$	$b<0$	$c<0$	$\underline{a};\overline{b};\underline{c}$	$\overline{a};\underline{b};\overline{c}$	$\overline{a};\overline{b};\overline{c}$	$\underline{a};\underline{b};\underline{c}$
$a<0$	$b>0$	$c<0$	$\underline{a};\underline{b};\underline{c}$	$\overline{a};\overline{b};\overline{c}$	$\overline{a};\overline{b};\overline{c}$	$\underline{a};\underline{b};\underline{c}$
$a<0$	$b<0$	$c<0$	$\underline{a};\overline{b};\underline{c}$	$\overline{a};\underline{b};\overline{c}$	$\overline{a};\underline{b};\overline{c}$	$\underline{a};\overline{b};\underline{c}$

Appendix D: Justifications of the Signs Obtained in Tables 1-B20

One could argue that some of the signs of the partial derivatives obtained in Tables 1-4, Tables B1-B19 are not obvious and should deserve a deeper explanation. We will then provide justifications in this last part of the Appendices to make sure that these signs are actually valid.

In order to justify the signs obtained, we will use a computerized method, with the software Mathematica, with the *Minimize* and *Maximize* commands.

As stated in the Document Center of Mathematica [12], those formulas should be used as follows:

$$\text{Minimize}[f, cons, x] \text{ and } \text{Maximize}[f, cons, x] \tag{c1}$$

where *f* corresponds to the function that needs to be minimized/maximized under the constraints stated in *cons*, in regard to the variable *x*. We will use the traditional form defined as *//tf* at the end of each command to get clearer results, as follows:

$$tf = \text{Traditional Form}; \tag{c2}$$

We will check the signs for $pda_1(a,b,c)=\frac{\partial x_1}{\partial a}$, $pdb_1(a,b,c)=\frac{\partial x_1}{\partial b}$, $pda_2(a,b,c)=\frac{\partial x_2}{\partial a}$ and $pdb_2(a,b,c)=\frac{\partial x_2}{\partial b}$, since finding the signs of $pdc_1(a,b,c)=\frac{\partial x_1}{\partial c}$ and $pdc_2(a,b,c)=\frac{\partial x_2}{\partial c}$ does not yield any particular issue when looking at their formulas.

We define the functions to study as follows in the Mathematica software:

$$\begin{aligned} pda1 &= c/\left(a*\left(b*b-4*a*c\right)^{0.5}\right) \\ &-\left(\left(-b-\left(b*b-4*a*c\right)^{0.5}\right)/\left(2*\left(a*a\right)\right)\right); pda1//tf \end{aligned} \tag{c3}$$

$$pdb1 = \left(-1-\left(b\right)/\left(\left(b*b-4*a*c\right)^{0.5}\right)\right)/\left(2*a\right); pdb1//tf \tag{c4}$$

$$\begin{aligned} pda2 &= -c/\left(a*\left(b*b-4*a*c\right)^{0.5}\right) \\ &-\left(\left(-b+\left(b*b-4*a*c\right)^{0.5}\right)/\left(2*\left(a*a\right)\right)\right); pda2//tf \end{aligned} \tag{c5}$$

$$pdb2 = \left(-1+\left(b\right)/\left(\left(b*b-4*a*c\right)^{0.5}\right)\right)/\left(2*a\right); pdb2//tf \tag{c6}$$

We can now proceed to look for the minimum or maximum values of these functions, considering multiple subcases for the variables *a*, *b* and *c*. We will consider the minimum value of the function when the subcase in the corresponding table depicts that the sign of the function is positive and will look for the maximum value of the function when the subcase in the corresponding table depicts that the sign of the function is negative. Indeed, if we obtain that the minimum value of the function is 0, then it shows that the function is positive, whereas if we get that the maximum value of the function is 0, then we can argue that the function is negative.

Let us now do the whole process for two different cases as examples.

We will first consider $pda_1(a,b,c)=\frac{\partial x_1}{\partial a}$, and, as seen in Table 1, Table B1, Table B7 and Table B3, pda_1 is always positive when defined (namely when $a\neq 0$). Let us for example verify Table 1.

We need to set the adapted constraints for our command, they will be $b<0$ and $c<0$ (subcase studied in Table 1), $a\neq 0$ (so that pda_1 is defined), and $\sqrt{b^2-4ac}>0$ (since we are only considering real roots). Since we want to verify that pda_1 is positive, we will use the Minimize command, and we will use it with respect to the variable *a* because we considered pda_1.

This leads us to entering the following command in Mathematica:

$$t1 = Minimize\left[pda1, a\neq 0 \wedge c<0 \wedge b<0 \wedge (b*b-4*a*c)>0, a\right]//tf \quad \text{(c7)}$$

which outputs the following results:

$$\left\{\begin{cases} 0. & c<0\wedge b<0 \\ \infty & \text{True} \end{cases}, \{a\to \text{Indeterminate}\}\right\}$$

As we can see, when respecting $b<0$ and $c<0$, we get that the minimum value of $pda_1(a,b,c)=\frac{\partial x_1}{\partial a}$ is 0. We can then argue that pda_1 is positive when $b<0$ and $c<0$.

We can proceed identically for the rest of the Tables related to pda_1, namely Tables B5-B7.

We will enter the same commands, and just change one of the constraints, which will become $b>0$ and $c>0$ for Table 5, $b>0$ and $c>0$ for Table B1 and $b>0$ and $c>0$ for Table B2.

As we want to prove that pda_1 is positive when $b>0$ and $c>0$, $b>0$ and $c>0$ and $b>0$ and $c>0$, we use the following commands 8, 9 and 10:

$$t2 = Minimize\left[pda1, a\neq 0 \wedge c>0 \wedge b<0 \wedge (b*b-4*a*c)>0, a\right]//tf \quad \text{(c8)}$$

which returns

$$\left\{\begin{cases} 0. & c>0\wedge b<0 \\ \infty & \text{True} \end{cases}, \{a\to \text{Indeterminate}\}\right\}$$

$$t3 = Minimize\left[pda1, a\neq 0 \wedge c<0 \wedge b>0 \wedge (b*b-4*a*c)>0, a\right]//tf \quad \text{(c9)}$$

which outputs

$$\left\{\begin{cases} 0. & c<0\wedge b>0 \\ \infty & \text{True} \end{cases}, \{a\to \text{Indeterminate}\}\right\}$$

$$t4 = Minimize\left[pda1, a\neq 0 \wedge c>0 \wedge b>0 \wedge (b*b-4*a*c)>0, a\right]//tf \quad \text{(c10)}$$

which leads to

$$\left\{\begin{cases} 0. & c>0\wedge b>0 \\ \infty & \text{True} \end{cases}, \{a\to \text{Indeterminate}\}\right\}$$

These commands (8), (9) and (10) justify the signs found in Table 1, Tables

B1-B3.

We can process analogously for $pda_2(a,b,c)=\frac{\partial x_2}{\partial a}$, which is always a negative quantity when defined (namely when $a \neq 0$). We will just input the same formulas that have been used for pda_1, but with Maximize commands because we want to show that pda_2 is negative in Table 3, Tables B11-B13. This leads us to using the following commands (11)-(14) in Mathematica:

$$t5 = Maximize\left[pda2, a \neq 0 \wedge c > 0 \wedge b > 0 \wedge (b*b-4*a*c) > 0, a\right] //tf \quad \text{(c11)}$$

which returns

$$\left\{ \begin{cases} 0. & c > 0 \wedge b > 0 \\ -\infty & \text{True} \end{cases}, \{a \to \text{Indeterminate}\} \right\}$$

$$t6 = Maximize\left[pda2, a \neq 0 \wedge c < 0 \wedge b > 0 \wedge (b*b-4*a*c) > 0, a\right] //tf \quad \text{(c12)}$$

which outputs

$$\left\{ \begin{cases} 0. & c < 0 \wedge b > 0 \\ -\infty & \text{True} \end{cases}, \{a \to \text{Indeterminate}\} \right\}$$

$$t7 = Maximize\left[pda2, a \neq 0 \wedge c > 0 \wedge b < 0 \wedge (b*b-4*a*c) > 0, a\right] //tf \quad \text{(c13)}$$

which leads to

$$\left\{ \begin{cases} 0. & c > 0 \wedge b < 0 \\ -\infty & \text{True} \end{cases}, \{a \to \text{Indeterminate}\} \right\}$$

$$t8 = Maximize\left[pda2, a \neq 0 \wedge c < 0 \wedge b < 0 \wedge (b*b-4*a*c) > 0, a\right] //tf \quad \text{(c14)}$$

which yields the following results

$$\left\{ \begin{cases} 0. & c < 0 \wedge b < 0 \\ -\infty & \text{True} \end{cases}, \{a \to \text{Indeterminate}\} \right\}$$

Commands (11), (12), (13) and (14) justify the signs of pda_2 shown in Table 3, Tables B10-B12.

Let us now study pdb_1 and pdb_2. We will proceed analogously to the method we used for pda_1 and pda_2.However, since the sign of pdb_1 and pdb_2 can change whether b is a negative or a positive quantity, as in Table B8, Table B10, Table B18, and Table B20, we will need to adjust the constraints and add one in regard to the sign of b. We also do not need to set $a \neq 0$ anymore since we will set either $a > 0$ or $a < 0$. We will start with Table 2, Table 4, Table B9 and Table B19 because the sign of the functions in those Tables does not change with the value of b, and will finish by considering Table B8, Table B10, Table B18 and Table B20 that need more commands to be justified.

Let us start with the sign of pdb_1 in Table 2, that is negative. We input the following command:

$$t9 = Maximize\left[pdb1, a > 0 \wedge c < 0 \wedge (b*b-4*a*c) > 0, b\right] //tf \quad \text{(c15)}$$

Command (15) yields the following results:

$$\left\{\begin{cases} 0. & c<0 \wedge a>0 \\ -\infty & \text{True} \end{cases}, \{b \to \text{Indeterminate}\}\right\}$$

In Table 4, pdb_2 is negative. We then use command (16) in Mathematica:

$$t10 = Maximize\left[pdb2, a>0 \wedge c<0 \wedge \left(b*b-4*a*c\right)>0, b\right]//tf \quad \text{(c16)}$$

which outputs:

$$\left\{\begin{cases} 0. & c<0 \wedge a>0 \\ -\infty & \text{True} \end{cases}, \{b \to \text{Indeterminate}\}\right\}$$

Table B9 demonstrates that pdb_1 is a positive quantity. We use the following command (17):

$$t11 = Minimize\left[pdb1, a<0 \wedge c>0 \wedge \left(b*b-4*a*c\right)>0, b\right]//tf \quad \text{(c17)}$$

Command (17) leads to the following results:

$$\left\{\begin{cases} 0. & c>0 \wedge a<0 \\ \infty & \text{True} \end{cases}, \{b \to \text{Indeterminate}\}\right\}$$

Table B18 also points out that pdb_2 is a positive quantity. We can input in Mathematica:

$$t12 = Minimize\left[pdb2, a<0 \wedge c>0 \wedge \left(b*b-4*a*c\right)>0, b\right]//tf \quad \text{(c18)}$$

which yields the following results:

$$\left\{\begin{cases} 0. & c>0 \wedge a<0 \\ \infty & \text{True} \end{cases}, \{b \to \text{Indeterminate}\}\right\}$$

Now that we have treated the simpler cases, let us draw our attention on Table B8, Table B10, Table B18, and Table B20, that will each need two commands. Indeed, we need to consider that the sign of b has an influence on the sign of the functions, which was not the case before.

Table B8 shows that pdb_1 is positive when b is negative, and pdb_1 is negative when b becomes positive. We then need to enter commands (19) and (20) in Mathematica:

$$t13 = Minimize\left[pdb1, a>0 \wedge c>0 \wedge b<0 \wedge \left(b*b-4*a*c\right)>0, b\right]//tf \quad \text{(c19)}$$

Command (19) results in:

$$\left\{\begin{cases} 0. & c>0 \wedge a>0 \\ \infty & \text{True} \end{cases}, \{b \to \text{Indeterminate}\}\right\}$$

Command (20) is the following:

$$t14 = Maximize\left[pdb1, a>0 \wedge c>0 \wedge b>0 \wedge \left(b*b-4*a*c\right)>0, b\right]//tf \quad \text{(c20)}$$

that yields:

$$\left\{\begin{cases} -\frac{1}{a} & c>0 \wedge a>0 \\ -\infty & \text{True} \end{cases}, \{b \to \text{Indeterminate}\}\right\}$$

$-1/a$ being a negative quantity here, and also being the maximum of pdb_1

when $a > 0, b > 0$ and $c > 0$, we can then conclude that pdb_1 is indeed negative here.

In **Table B10**, pdb_1 is negative when b is negative, and pdb_1 is positive when b gets positive. We will enter the following commands (21) and (22) in Mathematica:

$$t15 = Maximize\left[pdb1, a < 0 \wedge c < 0 \wedge b < 0 \wedge \left(b * b - 4 * a * c\right) > 0, b\right] / / tf \quad \text{(c21)}$$

which outputs:

$$\left\{ \begin{cases} 0. & c < 0 \wedge a < 0 \\ -\infty & \text{True} \end{cases}, \{b \to \text{Indeterminate}\} \right\}$$

Command (22) being:

$$t16 = Minimize\left[pdb1, a < 0 \wedge c < 0 \wedge b > 0 \wedge \left(b * b - 4 * a * c\right) > 0, b\right] / / tf \quad \text{(c22)}$$

Command (22) gives the following results:

$$\left\{ \begin{cases} -\dfrac{1}{a} & c < 0 \wedge a < 0 \\ \infty & \text{True} \end{cases}, \{b \to \text{Indeterminate}\} \right\}$$

$-1/a$ is here a positive number. Since it is the maximum of pdb_1 when $a < 0, b > 0$ and $c < 0$, we can then conclude that pdb_1 is positive here.

In **Table B18**, we denote that pdb_2 is a negative quantity when b is negative, whereas it becomes a positive one when b is also positive. We can then input the following command (23):

$$t17 = Maximize\left[pdb2, a > 0 \wedge c > 0 \wedge b < 0 \wedge \left(b * b - 4 * a * c\right) > 0, b\right] / / tf \quad \text{(c23)}$$

which outputs:

$$\left\{ \begin{cases} -\dfrac{1}{a} & c > 0 \wedge a > 0 \\ \infty & \text{True} \end{cases}, \{b \to \text{Indeterminate}\} \right\}$$

$-1/a$, here, is a positive quantity since $a > 0$. Since it is the maximum of pdb_2, pdb_2 is indeed negative here.

We now type command (24), which is:

$$t18 = Minimize\left[pdb2, a > 0 \wedge c > 0 \wedge b > 0 \wedge \left(b * b - 4 * a * c\right) > 0, b\right] / / tf \quad \text{(c24)}$$

Command (24) gives the following results:

$$\left\{ \begin{cases} 0. & c > 0 \wedge a > 0 \\ \infty & \text{True} \end{cases}, \{b \to \text{Indeterminate}\} \right\}$$

To conclude, our last Table to justify is **Table B20**. We observe that pdb_2 is a positive quantity when b is negative, and it becomes negative when b gets positive. This leads to command (25):

$$t19 = Minimize\left[pdb2, a < 0 \wedge c < 0 \wedge b < 0 \wedge \left(b * b - 4 * a * c\right) > 0, b\right] / / tf \quad \text{(c25)}$$

which results in:

$$\left\{\begin{cases} -\frac{1}{a} & c<0\wedge a<0 \\ \infty & \text{True} \end{cases}, \{b \to \text{Indeterminate}\}\right\}$$

$-1/a$ is here positive. Being the maximum of pdb_1, the latter is indeed positive here.

We then type the following command (26):

$$t20 = Maximize\left[pdb2, a<0 \wedge c<0 \wedge b>0 \wedge \left(b*b-4*a*c\right)>0, b\right]//tf \quad \text{(c26)}$$

Command (26) yields the following results:

$$\left\{\begin{cases} 0. & c<0\wedge a<0 \\ -\infty & \text{True} \end{cases}, \{b \to \text{Indeterminate}\}\right\}$$

Using all the provided commands above in the Mathematica software, we justified the signs obtained in **Tables 1-4**, **Tables B1-B3**, **Tables B8-B13** and **Tables B18-B20** since some of them are not immediate results.