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Generalization of Inequalities in Metric Spaces with Applications

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Abstract

In this paper, which serves as a continuation of earlier work, we generalize the idea of inequalities in metric spaces and use them to demonstrate that the incomplete metric space can be used to obtain a Banach space.

Keywords

Metric Spaces, Banach Space, Inequalities

1. Introduction

This paper aims to generalize some inequalities in metric spaces by providing an explanation for the fact that every normed space is a metric space, while the converse is not always true. We also present applications of these concepts in metric spaces, supported by relevant results. The utilization of the parallelogram law, a fundamental property of Hilbert spaces, has enabled several researchers, including Kirk [1], Reich [2], Lim [3], Zalinescu [4], Poffald and Reich [5], Prus and Smarzewski [6], Xu [7], Gornicki [8], and Takahashi [9], to establish equalities and inequalities in metric spaces and successfully solve various problems.

We present an introduction to some of the fundamental properties of a metric space. In essence, a metric space is defined as a non-empty set X such that to each $x, y \in X$ there corresponds a non-negative number called the distance between x and y. The concept of a metric space was initially introduced in 1906 and further developed in 1914. Additionally, a general inequality concerning polygonal inequality that holds true in metric spaces was established in [10].

A distance on a non-empty set X is defined as a function $d: X \times X \to [0, \infty]$ if the following properties are satisfied:

(i) $d(x, y) = 0$ iff $x = y$.

(ii) $d(x,y)=d(y,x)$ for all $x,y\in X$ (symmetry).

(iii) $d(x,y)\le d(x,z)+d(z,y)$ for any $x,y,z\in X$ (triangle inequality).

When these properties are met, the pair (*X*, *d*) forms a metric space. One of the main goals of this article is to define metric spaces for specific types of spaces, ensuring that all requirements of a metric space are fulfilled.

2. Basic Definitions

We begin by recalling certain fundamental properties of real numbers.

For all $x,y,z\in\mathbb{R}$,

i) $|x-y|\ge 0$; $|x-y|=0$ iff $x=y$;

ii) $|x-y|=|y-x|$;

iii) $|x-y|\ge|x-z|+|z-y|$.

To generalize these properties, let (X,d) be a metric space and $x=\xi_1,\xi_2,\cdots,\xi_m$. Then, we have.

$$d(x,y)\le d(x,\xi_1)+d(\xi_1,\xi_2)+\cdots+d(\xi_m,y) \tag{1}$$

$$d(x,y)\le d(x,\xi_1)+d(\xi_1,y) \tag{2}$$

$$d(\xi_1,y)\le d(\xi_1,\xi_2)+d(\xi_{2,}y) \tag{3}$$

$$d(x,\xi_1)+d(\xi_1,y)\le d(x,\xi_1)+d(\xi_1,\xi_2)+d(\xi_2,y) \tag{4}$$

$$d(x,y)\le d(x,\xi_1)+d(\xi_1,\xi_2)+d(\xi_2,y) \tag{5}$$

$$d(\xi_2,y)\le d(\xi_2,\xi_3)+d(\xi_3,y) \tag{6}$$

Thus, $d(x,y)$ satisfies the properties of a metric space.

Definition 1.2. Let $\alpha:X\to X$. A point *x* is said to be an *α*-fixed point of a mapping of $F:X\to X$ if $\alpha\circ x=\alpha\circ F(x)$.

Definition 2.2. (*α*-weakly isotone increasing) Let $(X,\le)$ be a partially ordered set, $\alpha:X\to X$, and F,G be two self-mappings of *X*. The mapping *F* is said to be *G*, *α*-weakly isotone increasing if for all $x\in X$, we have $(\alpha\circ F)x\le(\alpha\circ G)x\le(\alpha\circ F)(\alpha\circ G)(\alpha\circ F)x$.

3. Some Concepts to Prove a Metric Space

Let *X* be a set of ordered pairs of real numbers $\{x=(\xi_1,\xi_2):\xi_i\in\mathbb{R}\}$; we define a metric *d* on $\mathbb{R}^2$ as

$$d(x,y)=\sqrt{(\xi_1-\eta_1)^2+(\xi_2-\eta_2)^2} \tag{7}$$

where $x=(\xi_1,\xi_2)$ and $y=(\eta_1,\eta_2)\in\mathbb{R}^2$. Moreover, for Euclidean space $\mathbb{R}^n$, $\mathbb{R}^n:\{(x=\xi_1,\xi_2,\cdots,\xi_i)\}$, ξ_i are real. Additionally, for $C^n:z=\{(\mu_1,\mu_2,\cdots,\mu_i)\}$, z_i are complex numbers, and

$$d(x,y)=\sqrt{(\xi_1-\eta_1)^2+(\xi_2-\eta_2)^2+\cdots+(\xi_n-\eta_n)^n} \quad \text{on} \quad \mathbb{R}^n \tag{8}$$

and

$$d_1(x,y)=\sqrt{(\xi_1-\eta_1)^2+(\xi_2-\eta_2)^2+\cdots+(\xi_n-\eta_n)^n} \quad \text{on} \quad C^n \tag{9}$$

where $x=\xi_1,\xi_2,\cdots,\xi_n$, $y=\eta_1,\eta_2,\cdots,\eta_n$. Then, $d_1(x,y)=|\xi_1-\eta_1|+|\xi_2-\eta_2|$. To satisfy the triangle inequality, let $z=(\mu_1,\mu_2)$; then,

$$\begin{aligned} d_1(x,y) &= |\xi_1-\mu_1+\mu_1-\eta_1|+|\xi_2-\mu_2+\mu_2-\eta_2| \\ &\le |\xi_1-\mu_1|+|\xi_2-\mu_2|+|\mu_1-\eta_1|+|\mu_2-\eta_2| \\ &= d_1(x,z)+d_1(z,y) \end{aligned} \tag{10}$$

Therefore, $(\mathbb{R}^2,d_1)$ is also a metric space. Let $X:\{x=(\xi_1,\xi_2,\cdots,\xi_n):\xi_i\in\mathbb{R} \text{ or } C\}$ be a set of bounded sequences of real or complex numbers such that $|\xi_j|\le M \quad \forall j$, which are all bounded. Then, we also say $|\xi_j|\le M_x \quad \forall j$; hence, $L^\infty=\{x=(\xi_i)_{i=1}^\infty:\sup_i|\xi_i|<\infty\}$.

We define the distance as $d(x,y)=\sup_i|\xi_i-\eta_i|$, where $x=(\xi_i)_{i=1}^\infty\in L^\infty$ and $y=(\eta_i)_{i=1}^\infty\in L^\infty$. Let $z=(\mu_i)_{i=1}^\infty\in L^\infty$,

$$\begin{aligned} d(x,y) &= \sup_i|\xi_i-\eta_i| = \sup_i|\xi_i-\mu_i+\mu_i-\eta_i| \\ &\le \sup_i|\xi_i-\mu_i|+\sup_i|\mu_i-\eta_i| = d(x,z)+d(z,y) \end{aligned} \tag{11}$$

Thus, (L^∞,d) is a metric space.

Example 1.3. Suppose that S consists of the set of all bounded and unbounded sequences of complex numbers. Let the metric d be defined as $d(x,y)=\sum_{i=1}^\infty \frac{1}{2^i}\frac{|\xi_i-\mu_i|}{1+|\xi_i-\mu_i|}$, which is convergent and finite. To prove $d(x,y)\le d(x,z)+d(z,y)$, let $z=\mu_i\in S$. We consider the function $f(t)=\frac{t}{1+t}$, where $t\in\mathbb{R}$. Since $f'(t)=\frac{t}{(1+t)^2}>0$, the function $f(t)$ is increasing. Based on the inequality, $|a+b|\le|a|+|b|$, we have $f(|a+b|)=f(|a|+|b|)$, which implies

$$\frac{|a+b|}{1+|a+b|}\le\frac{|a|+|b|}{1+|a|+|b|}\le\frac{|a|}{1+|a|}+\frac{|b|}{1+|b|}$$

Setting $a=\xi_i-\eta_i$ and $b=\mu_i-\eta_i$, we obtain

$$\begin{aligned} \frac{|\xi_i-\eta_i|}{1+|\xi_i-\eta_i|} &\le \frac{|\xi_i-\mu_i|}{1+|\xi_i-\mu_i|}+\frac{|\mu_i-\eta_i|}{1+|\mu_i-\eta_i|} = \sum_{i=1}^\infty\frac{1}{2^i}\frac{|\xi_i-\eta_i|}{1+|\xi_i-\eta_i|} \\ &\le \sum_{i=1}^\infty\frac{1}{2^i}\frac{|\xi_i-\mu_i|}{1+|\xi_i-\mu_i|}+\sum_{i=1}^\infty\frac{1}{2^i}\frac{|\mu_i-\eta_i|}{1+|\mu_i-\eta_i|} \end{aligned} \tag{12}$$

Because $d(x,y)\le d(x,z)+d(z,y)$, we conclude that (S,d) is a metric space.

Example 2.3. Consider the L^b space for $p\ge1$, where $L^b:\{x=(\xi_1,\xi_2,\cdots,\xi_i,\cdots)\}$ such that $\sum_{i=1}^\infty|\xi_i|^b<\infty$ and ξ_i are scalars.

Result 1. (Holder's inequality). Let $x=\xi_j\in L^b$ and $y=\eta_j\in L^b$. Then, the product of these sequences satisfies

$$\sum_{j=1}^\infty|\xi_j\eta_j|\le\left(\sum_{k=1}^\infty|\xi_k|^p\right)^{\frac{1}{p}}\left(\sum_{m=1}^\infty|\eta_m|^q\right)^{\frac{1}{q}} \tag{13}$$

where $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Let $\bar{\xi}_j$ and $\bar{\eta}_j$ be two sequences such that $\sum_{j=1}^{\infty}\left|\bar{\xi}_j\right|^p = 1$ and $\sum_{j=1}^{\infty}\left|\bar{\eta}_j\right|^q = 1$. Taking $\alpha = \left|\bar{\xi}_j\right|$ and $\beta = \left|\bar{\eta}_j\right|$ as real positive numbers, we use the inequality $\alpha\beta = \frac{\alpha^p}{p} + \frac{\beta^q}{q}$ to obtain

$$\sum_{j=1}^{\infty}\left|\bar{\xi}\bar{\eta}\right| \le \frac{1}{p}\sum_{j=1}^{\infty}\left|\bar{\xi}_j\right|^p + \frac{1}{q}\sum_{j=1}^{\infty}\left|\bar{\eta}_j\right|^q \le \frac{1}{p} + \frac{1}{q} = 1 \tag{14}$$

Let us derive Holder's inequality. Suppose that $x = \xi_j \in L^b$ and $y = \eta_j \in L^b$ are non-zero elements with

$$\bar{\xi}_j = \frac{\xi_j}{\left(\sum_{k=1}^{\infty}\left|\xi_k\right|^p\right)^{\frac{1}{p}}} \text{ and } \bar{\eta}_j = \frac{\eta_j}{\left(\sum_{m=1}^{\infty}\left|\eta_m\right|^q\right)^{\frac{1}{q}}} \tag{15}$$

Clearly, the sequences $\bar{\xi}_j$ and $\bar{\eta}_j$ satisfy (13). Hence, using (13), we obtain $\sum_{j=1}^{\infty}\left|\xi_j\eta_j\right| \le \left(\sum_{k=1}^{\infty}\left|\xi_k\right|^p\right)^{\frac{1}{p}}\left(\sum_{m=1}^{\infty}\left|\eta_m\right|^q\right)^{\frac{1}{q}}$, which is finite.

Result 2. (Minkowski inequality). Let $x = \xi_j \in L^b$, $y = \eta_j \in L^b$, and $p \ge 1$. Then,

$$\left(\sum_{j=1}^{\infty}\left|\xi_j + \eta_j\right|^p\right)^{\frac{1}{p}} \le \left(\sum_{k=1}^{\infty}\left|\xi_k\right|^p\right)^{\frac{1}{p}} + \left(\sum_{m=1}^{\infty}\left|\eta_m\right|^p\right)^{\frac{1}{p}}$$

Proof. By setting $p = 1$ and $\left|\xi_j + \eta_j\right| \le \left|\xi_j\right| + \left|\eta_j\right|$ and applying the triangle inequality, we obtain

$$\sum_{j=1}^{\infty}\left|\xi_j + \eta_j\right| \le \sum_{j=1}^{\infty}\left|\xi_j\right| + \sum_{j=1}^{\infty}\left|\eta_j\right| \tag{16}$$

For simplicity, let $\xi_j + \eta_j = \omega_j$; then,

$$\left|\omega_j\right| = \left|\xi_j + \eta_j\right|^p = \left|\xi_j + \eta_j\right|^{p-1} \le \left|\xi_j\right|\left|\omega_j\right|^{p-1} + \left|\eta_j\right|\left|\omega_j\right|^{p-1} \tag{17}$$

By choosing $j = 1, 2, \cdots, n$ (any fixed value of *n*), $x = \xi_j \in L^b$ and $\left|\omega_j\right|^{p-1} \in L^q$ because

$$\left(\left|\omega_j\right|^{p-1}\right)^q = \left|\omega_j\right|^{(p-1)q} \tag{18}$$

Because $\sum_{j=1}^{\infty}\left|\omega_j\right|^{(p-1)q} = \sum_{j=1}^{\infty}\left|\omega_j\right|^p < \infty$, we can apply the Holder's inequality to obtain

$$\sum_{j=1}^{n}\left|\xi_j\right|\left|\omega_j\right|^{p-1} \le \left(\sum_{k=1}^{n}\left|\xi_k\right|^p\right)^{\frac{1}{p}}\left(\left(\sum_{m=1}^{n}\left|\omega_m\right|^{p-1}\right)^q\right)^{\frac{1}{q}} = \left(\sum_{k=1}^{n}\left|\xi_k\right|^p\right)^{\frac{1}{p}}\left(\sum_{m=1}^{n}\left|\omega_m\right|^p\right)^{\frac{1}{q}}$$

$$\sum_{j=1}^{n}\left|\xi_j\right|\left|\omega_j\right|^{p-1} \le \left(\sum_{k=1}^{\infty}\left|\xi_k\right|^p\right)^{\frac{1}{p}}\left(\sum_{m=1}^{\infty}\left|\omega_m\right|^p\right)^{\frac{1}{q}} \tag{19}$$

Then, we obtain

$$\begin{aligned}\sum_{j=1}^{n}\left|\omega_j\right|^p &\le \sum_{j=1}^{n}\left(\left|\xi_j\right| + \left|\eta_j\right|\right)\left|\omega_j\right|^{p-1} \\ &\le \left(\left(\sum_{k=1}^{n}\left|\xi_k\right|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n}\left|\eta_k\right|^p\right)^{\frac{1}{p}}\right)\left(\sum_{m=1}^{n}\left|\omega_m\right|^p\right)^{\frac{1}{q}}\end{aligned} \tag{20}$$

Taking the limit as $n \to \infty$, we obtain

$$\left(\sum_{j=1}^{\infty}\left|\omega_j\right|^p\right)^{1-\frac{1}{q}} \le \left(\left(\sum_{k=1}^{\infty}\left|\xi_k\right|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{\infty}\left|\eta_k\right|^p\right)^{\frac{1}{p}}\right) \tag{21}$$

Thus,

$$\left(\sum_{j=1}^{\infty}\left|\omega_j\right|^p\right)^{\frac{1}{p}} \le \left(\left(\sum_{k=1}^{\infty}\left|\xi_k\right|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{\infty}\left|\eta_k\right|^p\right)^{\frac{1}{p}}\right) \tag{22}$$

Finally, we conclude that

$$\left(\sum_{j=1}^{\infty}\left|\xi_{j+}\eta_j\right|^p\right)^{\frac{1}{p}} \le \left(\sum_{k=1}^{\infty}\left|\xi_k\right|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{\infty}\left|\eta_k\right|^p\right)^{\frac{1}{p}} \tag{23}$$

Theorem 3.3. A mapping T of a metric space (X,d) into a metric space (X,d) is continuous if and only if the inverse image of any open subset of Y is an open subset of *X*.

Definition 3.4. (Complete metric space) A continuous metric space (X,d) is said to be complete if every Cauchy sequence in X converges to an element of *X*.

Example 3.5. Let $(\mathbb{R}^n, d)$ be a complete metric space.

$$L_\infty d(x,y) = \sqrt{\sum_{i=1}^{n}\left|\xi_i - \mu_i\right|^2} \tag{24}$$

where $x = \xi_1, \xi_2, \cdots, \xi_n$, $y = \eta_1, \eta_2, \cdots, \eta_n \in \mathbb{R}^n$. Now, we are ready to state our main result.

Example 3.6. A Banach space under the norm defined by $\|x\|_2 = \left[\sum_{i=1}^{n}\left|\xi_i\right|^2\right]^{\frac{1}{2}}$, where $x = (\xi_i)_{i=1}^{n} = (\xi_1, \xi_2, \cdots, \xi_n) \in \mathbb{R}^n$, $\xi_i \in \mathbb{R}$ for all *i*.

Proof. To prove that $\|x\|_2$ is a normed linear space, we prove the properties of a norm:

(i) $\|x\| = \left[\sum_{i=1}^{n}\left|\xi_i\right|^2\right]^{\frac{1}{2}} \ge 0$, $\xi_i \in \mathbb{R}$ for all *i*. $\|x\| = 0$ implies that $\left[\sum_{i=1}^{n}\left|\xi_i\right|^2\right]^{\frac{1}{2}} = 0$; then, $\sum_{i=1}^{n}\left|\xi_i\right|^2 = 0$, $\left|\xi_i\right|^2 = 0$, $\left|\xi_i\right| = 0$, and $\xi_i = 0$.

(ii) $\|x+y\|^2 = \left[\sum_{i=1}^{n}\left|\xi_i + \eta_i\right|^2\right]^{\frac{1}{2}} = \sum_{i=1}^{n}\left|\xi_i + \eta_i\right| + \left|\xi_i + \eta_i\right|$.

$\|x+y\|^2 \le \sum_{i=1}^{n}\left(\left|\xi_i\right| + \left|\eta_i\right|\right)\left|\xi_i + \eta_i\right| = \sum_{i=1}^{n}\left|\xi_i\right|\left|\xi_i + \eta_i\right| + \sum_{i=1}^{n}\left|\eta_i\right|\left|\xi_i + \eta_i\right|$ $= \sum_{i=1}^{n}\left|\xi_i(\xi_i + \eta_i)\right| + \sum_{i=1}^{n}\left|\eta_i(\xi_i + \eta_i)\right| \le \|x\|\|x+y\| + \|y\|\|x+y\|$ implies that $\|x+y\|^2 \le \|x+y\|\left(\|x+y\|\right)$. Then, $\|x+y\| \le \|x\| + \|y\|$.

(iii) $\|\alpha x\| = \left[\sum_{i=1}^{n}\left|\alpha\xi_i\right|^2\right]^{\frac{1}{2}} = \left[\sum_{i=1}^{n}\left|\alpha\right|^2\left|\xi_i\right|^2\right]^{\frac{1}{2}} = \left|\alpha\right|\left[\sum_{i=1}^{n}\left|\xi_i\right|^2\right]^{\frac{1}{2}} = \left|\alpha\right|\|x\|$. Hence, it is a normed space.

Proof of completeness: Let $\langle x_m \rangle$ be a Cauchy sequence in $\mathbb{R}^2$. Then, for any $\varepsilon > 0$, $\exists n_0 \in N$ such that $\|x_m - x_n\| < \varepsilon$ $\exists m, r \ge n_0$ and $x_m, x_r \in \mathbb{R}^2$.

$x_m = \left(\xi_1^{(m)}, \xi_2^{(m)}, \cdots, \xi_i^{(m)}, \cdots, \xi_n^{(m)}\right)$ and $x_n = \left(\xi_1^{(r)}, \xi_2^{(r)}, \cdots, \xi_i^{(r)}, \cdots, \xi_n^{(r)}\right)$,

$\xi_i^{(m)}, \xi_i^{(r)} \in \mathbb{R}$, $\forall i$ so $\left[\sum_{i=1}^{n}\left|\xi_i^{(m)} - \xi_i^{(r)}\right|^2\right]^{\frac{1}{2}} < \varepsilon$, $\forall m,r \ge n_0$.

$$\sum_{i=1}^{n}\left|\xi_i^{(m)} - \xi_i^{(r)}\right|^2 < \varepsilon^2, \quad \forall m,r \ge n_0$$

$$\left|\xi_i^{(m)} - \xi_i^{(r)}\right|^2 < \varepsilon^2, \quad \forall m,r \ge n_0$$

$$\left|\xi_i^{(m)} - \xi_i^{(r)}\right| < \varepsilon, \quad \forall m,r \ge n_0$$

Because $\left\langle \xi_i^{(m)} \right\rangle$ is a Cauchy sequence in $\mathbb{R}$, $\xi_i^{(m)} \to \xi_i \in \mathbb{R}$, and $\mathbb{R}$ is complete.

Let $x = \xi_1, \xi_2, \cdots, \xi_i, \cdots, \xi_n \in \mathbb{R}$, $\xi_i \in \mathbb{R}$, $\forall i$. Now, $\|x_m - x\| = \left[\sum_{i=1}^{n}\left|\xi_i^{(m)} - \xi_i\right|^2\right]^{\frac{1}{2}}$, where $\xi_i^{(m)} \to \xi_i$, $\forall i$; then, $\xi_i^{(m)} - \xi_i \to 0$ as $m \to \infty$, and $x_m - x \to 0$ as $m \to \infty$ implies that $x_m \to x \in \mathbb{R}^2$, so $\mathbb{R}^n$ is a complete space.

Lemma 3.7. Let (L_∞, d_∞) be a complete metric space $L_\infty : \left\{x = (\xi_i), \xi_i \in \mathbb{R}^n \text{ or } C : \sup_i |\xi_i| < \infty\right\}$. $d_\infty d(x,y) = \sup_i |\xi_i - \mu_i|$, where $x = (\xi_i)_{i=1}^{\infty}$ and $y = (\eta_i)_{i=1}^{\infty} \in L_\infty$.

Claim. We consider L_∞. Given $x_m = \left(\xi_i^m\right)_{i=1}^{\infty}$ as a Cauchy sequence in L_∞, for given $\varepsilon > 0$, $\exists N(\varepsilon)$ such that for $n \ge N$, $d_\infty(x_m, x_n) < \varepsilon$, $\exists m,r \ge N$, $\sup_i \left|x_m - \mu_i^{(r)}\right| < \varepsilon$. For each fixed $\left|\xi_i^{(m)} - \mu_i^{(r)}\right| < \varepsilon$, we consider $\left(\xi_i^{(1)}, \xi_i^{(2)}, \cdots\right)$. x_i behaves as a real or complex Cauchy sequence because $x_m = (\eta_1, \eta_2, \cdots, \eta_n, \cdots)$. To show $x \in L_\infty$, we obtain $\sup_i \left|\xi_i^{(m)} - \mu_i^{(r)}\right| < \varepsilon$ for $m,r \ge N$, each *i*, and let $r \to \infty$. Thus, $d_\infty(x_m, x) = \sup_i \left|\xi_i^{(m)} - \mu_i^{(r)}\right| < \varepsilon$, $x_m \to x$ because

$$|\xi_i| = \left|\xi_i - \xi_i^{(m)}\right| + \left|\xi_i^{(m)}\right| < \varepsilon + k_m \tag{25}$$

where $k_m = \sup_i \left|\xi_i^{(m)}\right| < \infty$, $x_m \in L_\infty$. (L_∞, d_∞) is a complete metric space. Alternative proofs do exist.

Result 3. Every normed space is a metric space, but the converse need not be true in general.

Let S be s set of sequences (bounded or unbounded) of real or complex numbers and define $d(x,y) = \sum_{i=1}^{\infty}\frac{1}{2^i}\frac{|\xi_i - \eta_i|}{1+|\xi_i - \eta_i|}$, where $x = (\xi_i)_{i=1}^{\infty} \in X$ and $y = (\eta_i)_{i=1}^{\infty} \in X$. Clearly, (S,d) is a metric space. The question is whether it is a normed space? The answer is no. If it were a normed space, we could define $(x,0) = \|x\| = \sum_{i=1}^{\infty}\frac{1}{2^i}\frac{|x_i|}{1+|x_i|}$. In that case, we would have the following:

$$\|\alpha x\| = \sum_{i=1}^{\infty}\frac{1}{2^i}\frac{|\alpha x_i|}{1+|\alpha x_i|} = |\alpha|\|x\| \tag{26}$$

$\|\alpha x\| \ne |\alpha|\|x\|$ which fails to satisfy the norm property. So, $(S, \|.\|)$ is not a normed space, but it is a metric space.

Lemma 3.8. Consider L^p space and $p > 1$:

$L^p : \left\{x = (\xi_i)_{i=1}^{\infty}, \xi_i \in \mathbb{R} \text{ or } \mathbb{C} : \sup_i |\xi_i|^p < \infty\right\}$. Define $\|x\|_{L^p} = \left(\sum_{i=1}^{\infty}|\xi_i|^p\right)^{\frac{1}{p}}$. Therefore, $\left(L^p, \|\cdot\|_p\right)$ is a normed space, so

$$d(x,y)=\left(\sum_{i=1}^{\infty}\left|\xi_i-\eta_i\right|^p\right)^{\frac{1}{p}}=\|x-y\|_{L^p} \tag{27}$$

where $x=(\xi_i)\in L^p$ and $y=(\eta_i)\in L^p$. Thus, $\left(L^p,d\right)$ is a complete metric space. Hence, $\left(L^p,\|\cdot\|_p\right)$ is a Banach space.

Theorem 2.9. Let $(X,\leq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X,d) is a complete metric space. Let $\alpha: X\to X$ and F and G be two self-mappings of (X,d) such that for comparable $x,y\in X$,

$$\begin{aligned}&\xi d\left(\alpha\circ F(x),\alpha\circ G(y)\right)+\eta d\left(\alpha\circ F(x),\alpha(x)\right)+\mu d\left(\alpha(y),\alpha\circ G(y)\right)\\&-\min\left\{d\left(\alpha\circ F(x),\alpha(y)\right),d\left(\alpha\circ G(y),\alpha(x)\right)\right\}\\&\leq k\max\left\{d\left(\alpha(x),\alpha(y)\right),d\left(\alpha\circ F(x),\alpha(x)\right),d\left(\alpha(y),\alpha\circ G(y),\frac{1}{2}d\left(\alpha\circ F(x),\alpha(y)\right)\right)\right\}\end{aligned}$$

For $\xi,\eta,\mu>0$, $k>0$, and $\xi>k$, we assume the following:

(i) F is G,α-weakly increasing and

(ii) X is regular.

Then, F and G have a unique α-fixed point.

Proof. Let $x_0\in X$. From the sequence x_n with respect to α, we obtain $x_{2n+2}=\alpha\circ F(x_{2n+1})=F_\alpha(x_{2n+1})$ and $x_{2n+1}=\alpha\circ G(x_{2n})=G_\alpha(x_{2n})$ for $n=1,2,\cdots$. Let $d_n=d\left(\alpha\circ(x_n)\right)$, $\left(\alpha\circ(x_{n+1})\right)>0$, $n=1,2,\cdots$. Because G_α is F_α-weakly increasing, we have

$$\begin{aligned}x_1&\leq\alpha\circ G(x_0)\leq\alpha\circ F\left(\alpha\circ G(x_0)\right)=\alpha\circ F(x_1)=x_2\leq(\alpha\circ G)\left(\alpha\circ F\left(\alpha\circ G(x_0)\right)\right)\\&=\alpha\circ G\left(\alpha\circ F(x_1)\right)=\alpha\circ G(x_2)=x_3\leq\alpha\circ G(x_1)\leq\alpha\circ F\left(\alpha\circ G(x_2)\right)\\&=\alpha\circ F(x_3)=x_4\leq(\alpha\circ G)\alpha\circ F\left(\alpha\circ G(x_2)\right)=(\alpha\circ G)\left(\alpha\circ F(x_3)\right)=x_5\end{aligned} \tag{28}$$

By continuing this process, we obtain $x_1\leq x_2\leq x_3\leq\cdots\leq x_n\leq x_{n+1}\leq\cdots$ so $x_{2n}\leq x_{2n+1}$, $\forall n=1,2,\cdots$. Now, with $x=x_{2n+1}$ and $y=x_{2n}$, we have

$$\begin{aligned}&\left[\xi d\left(\alpha\circ F(x_{2n+1})\right),\alpha\circ G(x_{2n})+\eta d\left(\alpha\circ F(x_{2n+1}),x_{2n+2}\right)\right]\\&+\mu d\left(x_{2n},\alpha\circ G(x_{2n})\right)-\min\left\{d\left(\alpha\circ F(x_{2n+1}),x_{2n}\right),d\left(\alpha\circ G(x_{2n}),x_{2n+1}\right)\right\}\\&\leq k\max\left\{d\left(x_{2n+1},x_{2n}\right),d\left(\alpha\circ F(x_{2n+1}),x_{2n+1}\right),d\left(x_{2n},\alpha\circ G(x_{2n}),\frac{1}{2}d\left(\alpha\circ F(x_{2n+1}),x_{2n}\right)\right)\right\}\end{aligned} \tag{29}$$

or

$$\begin{aligned}&\left[\xi d\left(x_{2n+2},x_{2n+1}\right)+\eta d\left(x_{2n+2},x_{2n+1}\right)+\mu d\left(x_{2n},x_{2n+1}\right)\right.\\&\left.-\min\left\{d\left(x_{2n},x_{2n+2}\right),d\left(x_{2n+1},x_{2n+1}\right)\right\}\right]\\&\leq k\max\left\{d\left(x_{2n+1},x_{2n}\right),d\left(x_{2n+2},x_{2n+1}\right),d\left(x_{2n},x_{2n+1}\right),\frac{1}{2}d\left(x_{2n+1},x_{2n}\right)\right\}\end{aligned} \tag{30}$$

or

$$\xi d_{2n+1}+\eta d_{2n+1}+\mu d_{2n}-\min\left\{d_{2n},d_{2n+1},0\right\}\leq k\max\left\{d_{2n},d_{2n+1},\frac{1}{2}\left(d_{2n},d_{2n+1}\right)\right\} \tag{31}$$

Letting $H=\max\{d_{2n},d_{2n+1}\}$, we have $d_{2n}\le H$, and $d_{2n+1}\le H$ implies that $\frac{1}{2}(d_{2n},d_{2n+1})\le H$. Therefore,

$\max\left\{d_{2n},d_{2n+1},\frac{1}{2}(d_{2n},d_{2n+1})\right\}\le H=\max\{d_{2n},d_{2n+1}\}$. From Equation (30),

$(\xi+\eta)d_{2n+1}+\mu d_{2n}\le k\max\{d_{2n},d_{2n+1}\}$ if $d_{2n}\le d_{2n+1}$. Then,

$(\xi+\eta)d_{2n+1}+\mu d_{2n}\le kd_{2n+1}$, so $(\xi+\eta-k)d_{2n+1}\le -\mu d_{2n}$, $d_{2n+1}\le gd_{2n}$, where $g=\frac{-\mu}{\xi+\eta-f}\le 1$, and if $d_{2n+1}\le d_{2n}$, then $(\xi+\eta)d_{2n+1}+\mu d_{2n}\le gd_{2n}$ implies that $d_{2n+1}\le\frac{g-\mu}{\xi+\eta}d_{2n}$ and $d_{2n+1}\le gd_{2n}$, where $g=\frac{f-\mu}{\xi+\eta}d_{2n}<1$. Therefore, $d_{2n+1}\le gd_{2n}\le g^2d_{2n-1}\le\cdots\le g^{2n+1}d_0\to 0$ as $n\to\infty$. Thus, $\{x_n\}$ is a Cauchy sequence in *X*, *X* is complete, and there exists a point $z\in X$ such that $\{x_{2n}\}$ converges to *z*. Hence, $\lim(\alpha\circ F)(x_{2n+1})=x_{2n+1}=z$ and $\lim(\alpha\circ F)(x_{2n+1})=z$. Because $\{x_{2n}\}$ is a nondecreasing sequence, if *X* is regular, it follows that $x_{2n}\le z$, $\forall n$. Now, if we put $x=x_{2n+1}$ and $y=z$, we obtain

$$
\begin{aligned}
&\Big[\xi d(\alpha\circ F)x_{2n+1},((\alpha\circ G),z)+\eta d\left(x_{2n+1},(\alpha\circ F)x_{2n+1}\right)+\mu d\left(z,(\alpha\circ G)z\right)\\
&-\min\left\{d\left((\alpha\circ G)(x_{2n+1}),z\right),d\left((\alpha\circ G)z,x_{2n+1}\right)\right\}\Big]\\
&\le k\max\left\{d\left(x_{2n+1},z\right),d\left((\alpha\circ F)(x_{2n+1}),(x_{2n+1}),d\left(z,(\alpha\circ G)z\right),\frac{1}{2}d\left((\alpha\circ F)(x_{2n+1}),z\right)\right)\right\}.
\end{aligned}
$$

Finally, we arrive at the following conclusion:

$$
\begin{aligned}
&\xi d\left(z,\alpha\circ F\right)(z)+\eta d(z,z)+\mu d\left(z,\alpha\circ G(z)\right)-\min\left\{d(z,z),d\left(z,\alpha\circ G(u)\right)\right\}\\
&\le k\max\left\{d(z,z),d(z,z),d\left(z,\alpha\circ G(z),\frac{1}{2}d(z,z)\right)\right\}.
\end{aligned}
$$

Alternatively, we can simplify it as $(\xi+\mu-g)d\left(z,\alpha\circ G(z)\right)\le 0$ or $\alpha\circ G(z)=z$, given that $\xi>1+g$. Thus, *z* is a fixed point of *G*. Additionally, using similar reasoning with $x=z$, $y=x_{2n}$, we obtain $\alpha\circ z=\alpha\circ F(z)$. Hence, *z* is a common α-fixed point of *F* and *G*.

Conflicts of Interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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References

[1] Goebel, K. and Kirk, W.A. (1973) A Fixed-Point Theorem for Transformations

Whose Iterates Have Uniform Lipschitz Constants. *Studia Mathematica*, **47**, 135-140. https://doi.org/10.4064/sm-47-2-134-140

[2] Reich, S. (1978) An Iterative Procedure for Constructing Zeros of Accretive Sets in Banach Spaces. *Nonlinear Analysis*, **2**, 85-92. https://doi.org/10.1016/0362-546X(78)90044-5

[3] Lim, T.C. (1983) Fixed-Point Theorems for Uniformly Lipschitzian Mappings in L_p-Spaces. *Nonlinear Analysis*, **7**, 555-563. https://doi.org/10.1016/0362-546X(83)90044-5

[4] Poffald, I.E. and Reich, S. (1986) An Incomplete Cauchy Problem. *Journal of Mathematical Analysis and Applications*, **113**, 514-543. https://doi.org/10.1016/0022-247X(86)90323-9

[5] Prus, B. and Smarzewski, R. (1987) Strongly Unique Best Approximation and Centers in Uniformly Convex Spaces. *Journal of Mathematical Analysis and Applications*, **121**, 10-21. https://doi.org/10.1016/0022-247X(87)90234-4

[6] Xu, H.K. (1991) Inequalities in Banach Spaces with Applications. *Nonlinear Analysis*, **16**, 1127-1138. https://doi.org/10.1016/0362-546X(91)90200-K

[7] Gornicki, J. (1996) Fixed Points of Involutions. *Mathematica Japanica*, **43**, 151-155.

[8] Takahasi, W. (1970) A Convexity in Metric Spaces and Non-Expansive Mapping I. *Kodai Mathematical Seminar Reports*, **22**, 142-149.

[9] Ra, A.C.M. and Reurings, M.C.B. (2004) A Fixed-Point Theorem in Partially Ordered Sets and Some Applications to Matrix Equations. *Proceedings of the American Mathematical Society*, **132**, 1435-1443. https://doi.org/10.1090/S0002-9939-03-07220-4

[10] Choudhury, B.S. and Kundu, A. (2010) A Coupled Coincidence Point Result in Partially Ordered Metric Spaces for Compatible Mappings. *Nonlinear Analysis TMA*, **73**, 2524-253. https://doi.org/10.1016/j.na.2010.06.025