Journal of Applied Mathematics and Physics, 2023, 11, 2848-2870
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Optimal Portfolio Selection with Delay under the Framework of Uncertainty Theory

Jun Long[1]**, Sanyun Zeng**[2*]

[1]School of Information Technology & Engineering, Guangzhou College of Commerce, Guangzhou, China
[2]School of Accounting, Guangdong University of Foreign Studies, Guangzhou, China
Email: ok54@163.com, *zengsanyun@163.com

How to cite this paper: Long, J. and Zeng, S.Y. (2023) Optimal Portfolio Selection with Delay under the Framework of Uncertainty Theory. *Journal of Applied Mathematics and Physics*, **11**, 2848-2870.
https://doi.org/10.4236/jamp.2023.1110187

Received: September 6, 2023
Accepted: October 17, 2023
Published: October 20, 2023

Abstract

This study focuses on investigating the optimal investment strategy for an optimization problem with delay using the uncertainty theory. The financial market is composed of a risk-free asset and a risk asset with an uncertain price process described by an uncertain differential equation. An optimization problem is assumed that its objective is a nonlinear function of decision variable. By deriving the equation of optimality, an analytical solution is obtained for the optimal delay investment strategy, and the optimal delay value function. Finally, an economic analysis and numerical sensitivity analysis are conducted to evaluate the research results.

Keywords

Delay, Uncertainty Theory, Equation of Optimality, Optimal Value Function, Optimal Investment Strategy

1. Introduction

In actual decision-making processes, investors tend to refer to historical data and information to some extent. When investing in stocks, they not only consider the current price of the stock but also pay attention to relevant information, such as the price trend over a certain period. If a stock has recently performed well (exhibiting an upward trend), investors are more likely to increase their investment proportion in that stock, using a "buying high" strategy. This further attracts more investors to invest in such stocks, leading to further price increases. Conversely, if a stock has recently performed poorly (showing a downward trend), investors are more likely to withdraw funds due to "stop-loss" behavior. This lack of investor attraction results in a further decrease in the stock price. For certain institutions, such as insurance companies, good historical performance brings in more profits, allowing them to distribute part of the income to shareholders. Conversely, poor historical performance may require additional capital infusion to maintain normal operations. Therefore, incorporating historical information into models can help make more reasonable decisions. Based on this, researchers have explored issues related to delay. For instance, [1] [2] [3] [4] discussed stochastic control problems with delay using stochastic maximum principle or dynamic programming methods. [5] considered optimal investment and consumption problems for general investors, assuming that past price information affects future price fluctuations. [6] studied pension fund management problems with delay and derived optimal control strategies under the influence of performance-related additional cash flows. [7] investigated a class of stochastic recursive optimal control problems with delay and solved them using dynamic programming and maximum principles. [8] studied mean-field type stochastic optimal control problems with delay and derived explicit expressions for efficient investment portfolios and efficient frontiers under certain conditions. [9] introduced delay information into the optimal reinsurance and investment problem, obtaining analytical expressions for the optimal strategy, value function, and efficient frontier. [10] researched the maximum principle for time-delay stochastic differential equations and its applications. [11] considered the problem of optimal excess loss reinsurance with delay and derived analytical expressions for the CRRA utility optimal strategies and optimal value functions. [12] studied the optimal excess loss reinsurance problem with delay under the CEV model and obtained closed-form expressions for the optimal strategies and optimal value functions. [13] analyzed the optimal investment and proportional reinsurance problem with delay under the CEV model, deriving analytical expressions for the optimal strategies and value functions in both reinsurance and pure investment scenarios. [14] investigated the dynamic continuous-time assets and liabilities management problem with delay in the mean-variance framework, and derived analytical expressions for the pre-commitment strategies of the mean-variance assets and liabilities management problem with delay. [15] used the conjugate duality approach to study a class of stochastic optimal control problems with delay of state systems described by stochastic differential equations and obtained expressions for the corresponding dual problem. [16] considered the optimal expected-variance reinsurance problem with delay under the dependent-risk model, obtaining analytical expressions for the optimal strategies. [17] proposed an optimal investment portfolio problem with delayed effects on the assets and characterized the optimal strategies using decoupled quadratic forward-backward stochastic differential equations. [18] studied the investment portfolio optimization problem with a new model describing stock price behavior with delayed effects over an infinite time horizon. [19] investigated the time-consistent optimal investment and reinsurance problem in the presence of default risk. And [20] discussed the reinsurance investment problem for insurers as well, but focused on funding requirements associated with historical performance and derived analytical expressions for optimal strategies under specific conditions. [21] studied the robust optimal excess loss reinsurance and investment problem for ambiguity-averse insurers with delay and dependence on risks, obtaining explicit expressions for the optimal excess loss reinsurance and investment strategies. [22] investigated the two-dimensional correlated claims composite Poisson risk model and analyzed the optimal time-consistent investment and reinsurance problem with delay under the expected-variance reinsurance premium principle, obtaining analytical expressions for the optimal time-consistent investment and reinsurance strategies and their corresponding value function. [23] studied the mean-variance investment and reinsurance problem with delay, default risk, and dependent shocks, and solved the problem by using the stochastic control and time consistency theories, deriving explicit expressions for the optimal time-consistent investment and reinsurance strategies, the value function, and the efficient frontier. [24] investigated the robust optimal excess loss reinsurance and investment problem with delay under Value-at-Risk constraints, and derived expressions for the value function and Nash equilibrium strategies. [25] discussed the value function of a state-switching jump-diffusion with delay and derived expressions for the corresponding stochastic optimization problem. [26] studied the zero-sum stochastic control problem with delayed stochastic differential equations in the context of investment and reinsurance, obtaining explicit expressions for the equilibrium investment and reinsurance strategies. [27] examined the optimal investment portfolio problem with delay under the Mean-Variance criterion in a risk-related stochastic volatility model, deriving explicit formulas for the optimal control and corresponding value function. [28] investigated the optimal insurance problem with delayed effects using the stochastic optimal control framework and derived analytical expressions for the optimal strategy of the considered problem.

However, all the aforementioned studies fall within the domain of probability theory. As stated by the founder of uncertainty theory, if there is a lot of historical data and its distribution function is close enough to its frequency, we should use probability theory. However, probability theory cannot be applied if the distribution function deviates significantly from its frequency, or the data is limited, such as a newly formed company or a recently listed stock. Additionally, in cases such as the 2008 financial crisis, recent outbreaks like the COVID-19 pandemic, or even during times of war, where historical data may be abundant, using probability theory for analysis would yield highly inaccurate results. When variables such as stock prices, market demand, or product lifetimes are considered in real-world situations, statisticians have recognized that the obtained distribution may deviate significantly from the assumed distribution function. To address this issue, [29] and [30] introduced the concept of uncertainty theory. After more than a decade of development and refinement, uncertainty theory [31] has become an important branch of mathematics and finds broad applications across various fields, including economics and finance. Statisticians' recognition that "distribution functions may lack sufficient proximity to their frequencies, leading to unreliable results" is an important reason for our interest in combining delayed information with uncertainty theory. To our knowledge, there has been no study on this specifically. Notably, [32] stands as the pioneer in introducing uncertainty theory to optimal control problems within the realm of finance. Building upon this work, [33] extended the application of uncertainty theory by incorporating jump processes into optimal control problems. Since then, there have been many studies, see, [34]-[39], and so on. In recent years, an increasing number of scholars have been applying uncertainty theory to address various financial issues. For instance, [40] developed an uncertain optimal control model to minimize the quadratic loss function in DC pension plans. [41] employed a multi-period mean-variance model to explore DC pension problems considering uncertain returns and wages. In their subsequent work, [42] investigated background-dependent uncertain optimal control systems, applying them to DC pension plans. [43] employed the criterion of uncertain optimistic values to study the optimal control problem of DC pension plans.

However, to the best of our knowledge, no previous studies have focused on financial problems with delay under the framework of uncertainty theory. This research gap has motivated our work. Our study presents several novel contributions. Firstly, we consider a financial market that includes both a risk-free asset and a risk asset with uncertain prices, which are described by an uncertain differential equation. Secondly, we introduce a delay optimization problem with nonlinear decision variables in its objective function and derive the equation of optimality. Thirdly, we provide an analytical solution for the optimal delay investment strategy and the optimal delay value function. Lastly, we discuss the economic implications of our research findings and conduct sensitivity analysis.

The remainder of this paper is structured as follows. Section 2 provides the necessary background on uncertainty theory. Section 3 constitutes the core of our work, where we describe the fundamental assumptions, establish the model, and derive the analytical solution. Section 4 discusses the economic implications and sensitivity analysis, supplemented by numerical examples. Finally, in Section 5, we present our conclusions.

2. Preliminary

In order to enhance comprehension of this paper, it is imperative to review key concepts related to uncertainty theory as introduced in [31]. Let Γ denote a nonempty set, and $\mathcal{L}$ represent a σ-algebra over Γ. Each individual element $\Lambda \in \mathcal{L}$ is referred to as an event.

Definition 1. *A set function* $\mathcal{M}$ *defined on the* σ*-algebra* $\mathcal{L}$ *over* Γ *is called an uncertain measure if it satisfies the following four axioms*:

(i) (*Normality*) $\mathcal{M}\{\Gamma\} = 1$;

(ii) (*Monotonicity*) $\mathcal{M}\{\Lambda_1\} \le \mathcal{M}\{\Lambda_2\}$ *whenever* $\Lambda_1 \le \Lambda_2$ *for* $\Lambda_1, \Lambda_2 \in \mathcal{L}$;

(iii) (*Self-Duality*) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ *for any event* Λ;

(iv) (*Countable Subadditivity*) $\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \le \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}$.

Definition 2. *Let* Γ *be a nonempty set*, $\mathcal{L}$ *the* σ*-algebra over* Γ, *and* $\mathcal{M}$ *be* *an uncertain measure. Then the triplet* $(\Gamma, \mathcal{L}, \mathcal{M})$ *is said to be an uncertainty space.*

Definition 3. *An uncertain variable is a measurable function* ξ *from an uncertainty space* $(\Gamma, \mathcal{L}, \mathcal{M})$ *to the set of real numbers; that is, for any Borel set of real numbers, the set*

$$\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\}, \tag{1}$$

is an event.

Definition 4. *The uncertainty distribution* $\Phi : \mathcal{R} \to [0,1]$ *of an uncertain variable* ξ *is defined by*

$$\Phi(x) = \mathcal{M}\{\xi \le x\}. \tag{2}$$

is an event.

Definition 5. *A canonical process, denoted as* $C(t)$, *is considered uncertain if it satisfies the following conditions:*

(i) $C(0) = 0$ *and almost all sample paths are Lipschitz continuous;*

(ii) $C(t)$ *exhibits stationary and independent increments;*

(iii) *Each increment* $C(s+t) - C(s)$ *is a normal uncertain variable with an expected value of* 0 *and a variance of* t^2. *The uncertainty distribution of this variable is denoted as*

$$\Phi(x) = \left[1 + \exp\left(\frac{-\pi x}{\sqrt{3}t}\right)\right]^{-1}, x \in \mathcal{R}. \tag{3}$$

Theorem 1. ([30]) *Let* ξ *and* η *represent independent uncertain variables with finite expected values. For any real numbers a and b, the following relationship holds:*

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta]. \tag{4}$$

Theorem 2. ([44], *Integration by Parts*) *Suppose* X_t *and* Y_t *are general Liu processes. Then*

$$\mathrm{d}(X_t Y_t) = Y_t \mathrm{d}X_t + X_t \mathrm{d}Y_t. \tag{5}$$

This theorem, along with Definition 5(iii), informs us that the operations of Uncertainty Differential Equations (UDEs) in the framework of uncertainty theory differ from those of Stochastic Differential Equations (SDEs) in the framework of probability theory. UDEs do not involve the product term $\mathrm{d}X_t \mathrm{d}Y_t$, which is present in SDEs. Consequently, the optimality equation in uncertainty theory does not contain second-order partial derivative information, unlike the HJB equation in probability theory, which does include such information.

Theorem 3. ([45]) *Let* $u_{1t}, u_{2t}, v_{1t}, v_{2t}$ *be sample-continuous uncertain processes. Then the linear uncertain differential equation*

$$\mathrm{d}X_t = (u_{1t} X_t + u_{2t})\mathrm{d}t + (v_{1t} X_t + v_{2t})\mathrm{d}C_t, \tag{6}$$

has a solution

$$X_t = U_t\left(X_0 + \int_0^t \frac{u_{2s}}{U_s}\mathrm{d}s + \int_0^t \frac{v_{2s}}{U_s}\mathrm{d}C_s\right), \tag{7}$$

where

$$U_t = \exp\left(\int_0^t u_{1s}\mathrm{d}s + \int_0^t v_{1s}\mathrm{d}C_s\right). \tag{8}$$

3. An Optimal Control Problem with Delay

This section is the core part of this paper. Firstly, we describe the basic assumptions for the optimization problem with delay, and then establish the model whose objective function includes nonlinear decision variables, which helps in the derivation of optimal strategy. In the end, we derive the concrete form of the analytical solution.

3.1. Financial Market

In this study, we examine a financial market that comprises two assets: a risk-free asset (e.g., a bank account or bond) and a risk asset (e.g., stocks).

The price of the risk-free asset at time *t*, denoted as $S_0(t)$, follows an uncertain process described by the equation:

$$\frac{\mathrm{d}S_0(t)}{S_0(t)} = r_0\mathrm{d}t. \tag{9}$$

Here, r_0 represents the constant risk-free interest rate.

The price of the risk asset at time *t*, denoted as $S(t)$, also follows an uncertain process outlined by the equation:

$$\frac{\mathrm{d}S(t)}{S(t)} = \mu_S\mathrm{d}t + \sigma_S\mathrm{d}C(t), \tag{10}$$

where μ_S and σ_S denote the appreciation rate and volatility rate of the risk asset, respectively. $C(t)$ represents a canonical process. Generally, we assume $\mu_S > r_0 > 0$ and $\sigma_S > 0$.

3.2. Wealth Process

Suppose that in the process of fund management, if the past performance is poor, the fund manager may need to seek further financing to adjust the loss so that the final performance target can still be reached. Instead, if past performance has been good, fund managers can pay some of the proceeds to shareholders as bonuses for management or in the form of dividends. The presence of inflows and outflows related to historical information of the wealth gives rise to an optimization problem with delay.

We assume that the fund manager can invest in both the risk-free and risky assets described by Equations (9) and (10), respectively. Let $W(t)$ denote the wealth of the fund at time *t*, w_0 represent the initial wealth, and π_t represent the investment proportion in the risky asset at time *t*. We further define the pointwise performance of the wealth as $Z(t) = W(t-h)$ over the past horizon $[t-h,t]$, and the average wealth $\overline{Y}(t)$ over the period $[t-h,t]$ can be expressed as

$$\bar{Y}(t) = \frac{Y(t)}{\frac{1}{h}\int_{-h}^{0} \mathrm{e}^{\delta s}\mathrm{d}s}, \tag{11}$$

where

$$Y(t) = \int_{-h}^{0} \mathrm{e}^{\delta s} W(t+s)\mathrm{d}s, \tag{12}$$

and $\delta \geq 0$ represents the average parameter, while $h > 0$ represents the delay parameter.

For convenience, we denote the average value of a function $Y(t)$ as $\bar{Y}(t)$ by using a short line above it. Therefore, based on the fact that $G_1 \frac{Y(t)}{\frac{1}{h}\int_{-h}^{0} \mathrm{e}^{\delta s}\mathrm{d}s} = \frac{G_1}{\frac{1}{h}\int_{-h}^{0} \mathrm{e}^{\delta s}\mathrm{d}s} Y(t)$, we have

$$G_1 \bar{Y}(t) = \bar{G}_1 Y(t), \tag{13}$$

which may be used in Equation (15).

In order to derive the equation of optimality in Theorem 5 which needs the first moment information $\left(w - \delta y - \mathrm{e}^{-\delta} z\right)$ of $Y(t)$, we give the following result.

Lemma 1. *For any real functions F and H, if* $F(t) = \int_{-h}^{0} \mathrm{e}^{\delta s} H(t+s)\mathrm{d}s$, *then*

$$\frac{\mathrm{d}F(t)}{\mathrm{d}t} = H(t) - \delta F(t) - \mathrm{e}^{-\delta h} H(t-h). \tag{14}$$

The proof can be seen in **Appendix A**.

Let the function $G\left(t, W(t) - \bar{Y}(t), W(t) - Z(t)\right)$ represent the amount of capital inflow/outflow, which depends on the past performance of the wealth. $W(t) - \bar{Y}(t)$ represents the average performance of the wealth between $t-h$ and *t*, and $W(t) - Z(t)$ indicates the absolute performance of the wealth in the time horizon $[t-h, t]$.

Since the optimization problem with delay is generally infinite-dimensional, we introduce certain assumptions to make the problem solvable. Following the works of [9] [11] and [12], we assume that the amount of capital inflow/outflow is proportional to the past performance of the fund wealth, *i.e.*

$$\begin{aligned} &G\left(t, W(t) - \bar{Y}(t), W(t) - Z(t)\right) \\ &= G_1\left(W(t) - \bar{Y}(t)\right) + G_2\left(W(t) - Z(t)\right) \\ &= G_1 W(t) - G_1 \bar{Y}(t) + G_2\left(W(t) - Z(t)\right) \\ &= G_1 W(t) - \bar{G}_1 Y(t) + G_2\left(W(t) - Z(t)\right). \end{aligned} \tag{15}$$

Furthermore, we make the assumption that G_1 and G_2 satisfy the following parameter conditions.

$$\begin{aligned} &\bar{G}_1 = \delta^2, \\ &G_2 = \delta \mathrm{e}^{-\delta h}. \end{aligned} \tag{16}$$

If we satisfy the condition stated in Equation (16) and set $\delta = h = 0$, we can conclude that the control system (18) will not have any delay.

The condition (16) ensures that the optimality equation has a closed-form solution, which is one of the sufficient conditions for optimizing the control problem with delay. Although these two assumptions may restrict the generality, they make the delayed control problem solvable in a finite-dimensional context.

Hence, when considering a capital inflow/outflow function, the wealth process $W(t)$ can be described by the equation:

$$\begin{aligned}
\mathrm{d}W(t) &= (1-\pi_t)W(t)\frac{\mathrm{d}S_0(t)}{S_0(t)} + \pi_t W(t)\frac{\mathrm{d}S(t)}{S(t)} - G\big(t, W(t)-\bar{Y}(t), W(t)-Z(t)\big)\mathrm{d}t \\
&= (1-\pi_t)W(t)r_0\mathrm{d}t + \pi_t W(t)\left[\mu_S\mathrm{d}t + \sigma_S\mathrm{d}C(t)\right] \\
&\quad -\left[G_1\big(W(t)-\bar{Y}(t)\big) + G_2\big(W(t)-Z(t)\big)\right]\mathrm{d}t \\
&= \left[\pi_t(\mu_S - r_0)W(t) + (r_0 - G_1 - G_2)W(t) + G_1\bar{Y}(t) + G_2 Z(t)\right]\mathrm{d}t \\
&\quad + \pi_t\sigma_S W(t)\mathrm{d}C(t), \\
&:= \left[\pi_t(\mu_S - r_0)W(t) + G_0 W(t) + \bar{G}_1 Y(t) + G_2 Z(t)\right]\mathrm{d}t + \pi_t\sigma_S W(t)\mathrm{d}C(t),
\end{aligned} \tag{17}$$

where $G_0 = r_0 - G_1 - G_2$.

3.3. Optimization Model

In the context of investment management, the managers concerns extend beyond the final wealth at the end of the investment horizon. The manager also takes into account the investment results at different moments, as these outcomes can influence the perception of both team members and superiors. Hence, the fund manager takes into account not only the utility derived from the wealth $W(T)$ at the final time T and the average wealth $\bar{Y}(T)$ in the period $[T-h, T]$, but also the utility derived from investing the funds in the stock market during their tenure. As the interest earned on bank deposits does not contribute to their performance, it is not considered in our model at present. Therefore, our model can be formulated as follows.

$$\begin{aligned}
&J(t,w,y) = \max_{\pi_t\in\Pi} E\left[\int_t^T U\big(\pi_s, W(s)\big)\mathrm{d}s + V(T)\right], \\
&\text{s.t. } \mathrm{d}W(t) = \left[\pi_t(\mu_S - r_0)W(t) + G_0 W(t) + \bar{G}_1 Y(t) + G_2 Z(t)\right]\mathrm{d}t \\
&\qquad\qquad + \pi_t\sigma_S W(t)\mathrm{d}C(t), \\
&\qquad W(t) = w, Y(t) = y, Z(t) = z,
\end{aligned} \tag{18}$$

where U represents the utility gained during the investment period, V represents the utility at the terminal moment, and ρ is the discount rate. Since the optimality equation in the uncertainty theory framework does not incorporate second-order information just like the HJB equation in the probability theory framework, we need to assume that the objective function is a function containing nonlinear decision variables. For example, we can use $U\big(\pi_t, W(t)\big) = \mathrm{e}^{-\rho t}\frac{1}{\gamma}\big(\pi_t W(t)\big)^{\gamma}$ with the risk aversion coefficient γ, and

$$V(T)=\mathrm{e}^{-\rho T}\left[W(T)+\frac{\delta}{h}\left(1-\mathrm{e}^{-\delta h}\right)\bar{Y}(T)\right]^{\gamma}=\mathrm{e}^{-\rho T}\left[W(T)+\delta Y(T)\right]^{\gamma}.$$

3.4. The Solution to the Model

Expanding upon the scholarly contributions by [32], we present the optimality principle and optimality equation for the context of uncertain optimal control.

Theorem 4. (*Principle of Optimality*). *For any* $(t,w,y)\in[0,T]\times R\times R$, *and* $\Delta t>0$, *we have*

$$J(t,w,y)=\max_{\pi_t\in\Pi}E\left[\int_t^{t+\Delta t}U\mathrm{d}s+J\left(t+\Delta t,w+\Delta W(t),y+\Delta Y(t)\right)\right], \tag{19}$$

where $w+\Delta W(t)=W(t+\Delta t)$, $y+\Delta Y(t)=Y(t+\Delta t)$, *and Π is an admissible strategy.*

Furthermore, by incorporating the first moment information of $Y(t)$ as discussed in Lemma 1, it becomes evident that

Theorem 5. (*Equation of Optimality*) *Let* $J(t,w,y)$ *be twice differentiable on* $[0,T]\times R\times R$. *Then we have*

$$0=\max_{\pi_t\in\Pi}\left\{U+J_t+J_w\left[\pi_t\left(\mu_S-r_0\right)W(t)+G_0y+\bar{G}_1y+G_2z\right]+J_y\left(w-\delta y-\mathrm{e}^{-\delta h}z\right)\right\}, \tag{20}$$

where $J_t(t,w,y),J_w(t,w,y)$ *and* $J_y(t,w,y)$ *are the partial derivatives of the function* $J(t,w,y)$ *in t, w and y, respectively.*

To successfully determine the value of $B(t)$ within the optimal delay value function (as detailed in Theorem 6), we present the subsequent lemma as a solution-oriented approach.

Lemma 2. *The solution of the following ODE*

$$f'(t)-kf(t)=g(t), \tag{21}$$

can be expressed as

$$f(t)=f(T)\mathrm{e}^{-k(T-t)}-\int_t^T\mathrm{e}^{-k(s-t)}g(s)\mathrm{d}s. \tag{22}$$

The proof can be seen in **Appendix B**.

Theorem 6. *In relation to the optimization problem* (18), *the optimal strategy for making delay investments can be expressed as*:

$$\pi_t=\left[\mathrm{e}^{\kappa(T-t)}\gamma\left(\mu_S-r_0\right)\right]^{\frac{1}{\gamma-1}}\left(1+\delta\frac{y}{w}\right), \tag{23}$$

nd the optimal value function for delay is provided as:

$$J(t,w,y)=\mathrm{e}^{(\kappa+\rho)(T-t)}\mathrm{e}^{-\rho T}\left(w+\delta y\right)^{\gamma}-\int_t^T\mathrm{e}^{-\rho s}g(s)\mathrm{d}s. \tag{24}$$

where

$$g(s)=\mathrm{e}^{\frac{\gamma}{\gamma-1}\kappa(T-s)}\left(\gamma\left(\mu_S-r_0\right)\right)^{\frac{1}{\gamma-1}}\left(\mu_S-r_0-\gamma\right)\left(w+\delta y\right)^{\gamma}, \tag{25}$$

with $\kappa=\rho-\gamma\left[r_0-\delta\left(1-\frac{\delta^2}{h}\right)\left(\mathrm{e}^{-\delta h}-1\right)\right]$.

The proof can be seen in **Appendix C**.

4. Analysis of Sensitivity and Illustration of Figures

In order to assess the preference level of investors or managers for investment portfolios, we utilize the power utility function as a metric for both economic analysis and numerical sensitivity analysis. Our economic analysis methodology involves utilizing a mathematical model based on portfolio theory within the framework of uncertainty theory to analyze optimal control problems. Our numerical sensitivity analysis approach entails systematically altering the parameter values to observe their impact on the results, while also providing a detailed discussion on the impact of model returns, risk factors, and time delays on optimal investment strategies.

In this section, we provide numerical analyses to illustrate the dynamic behavior of the optimal delay investment strategy and the optimal delay value function. Additionally, we offer numerical examples to demonstrate the effects. Unless specified otherwise, in this section, we show the investment behavior of risk-averse managers, *i.e.*, $\gamma = 0.05 < 1$, and the fundamental parameters are presented as: $T = 10$, $t = 1$, $\mu_S = 0.3$, $r_0 = 0.03$, $\delta = 1.5$, $w = 1$, $\rho = 0.3$, $h = 1$. For simplicity but without loss of generality, we only provide the evolution of the optimal delay investment strategy and the optimal delay value function at three levels of delay wealth $y = 0.1/2/3$.

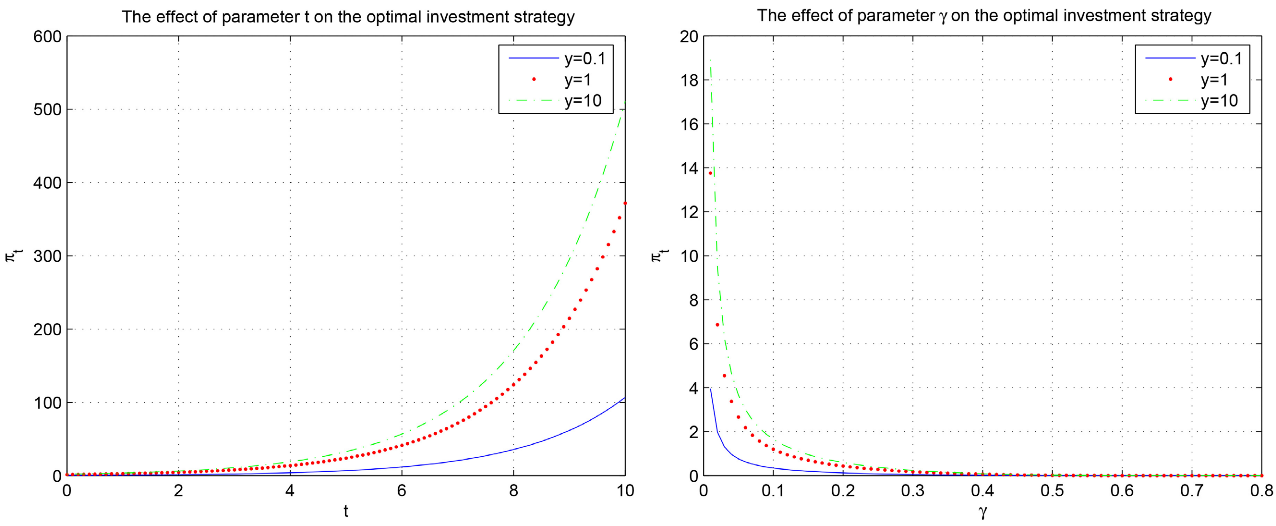
4.1. Analysis of Optimal Investment Strategy

In this subsection, we provide some sensitivity analysis on the effect of the parameters $t, \gamma, \mu_S, w, \delta$, and h on the optimal delay investment strategy. Meanwhile, some numerical examples are provided to illustrate our results.

According to Equation (23) and the risk averse manager hypothesis, $0 < \gamma < 1$, we have

$$\frac{\partial \pi_t}{\partial t} = -\frac{\gamma}{\gamma - 1}\kappa \pi_t > 0,\ \frac{\partial \pi_t}{\partial \mu_S} = \frac{1}{(\gamma - 1)(\mu_S - r_0)}\pi_t < 0,\ \frac{\partial \pi_t}{\partial w} = -\frac{\delta y}{w(w + \delta y)}\pi_t > 0. \quad (26)$$

In the case of $0 < \gamma < 1$, $\frac{\partial \pi_t}{\partial t} > 0$, the optimal delay investment strategy π_t increases over time *t*. However, when the manager is risk-seeking with $\gamma > 1$ and $\frac{\partial \pi_t}{\partial t} < 0$, the optimal delay investment strategy π_t decreases over time *t*. From **Figure 1(a)**, it can be observed that as time progresses, investment portfolio decisions gradually increase, and a larger proportion of investment is allocated to the stock market as more historical information about wealth is considered. This implies that investors become more proactive in adjusting their investment portfolios in response to changing market conditions, the emergence of new information, and evolving economic environments. It reflects the dynamic nature of investment decision-making, wherein investors continually reassess their investment strategies to maximize returns and manage risks. Additionally, having access to a greater amount of historical wealth data enables better evaluation of the risks and returns associated with stock market investments,



(a) (b)

Figure 1. The effect of parameters t and γ on the optimal strategy.

potentially leading to a higher proportion of investment in stocks. **Figure 1(a)** highlights the importance of time and historical information in investment decision-making. It suggests that fund managers tend to adjust their investment portfolios over time, and managers with a greater amount of historical wealth information are more likely to allocate a larger proportion of their investments to the stock market, indicating a potentially higher risk preference and confidence in the long-term performance of the stock market, which is evident in multiple figures below. In **Figure 1(b)**, we observe that the higher the risk aversion coefficient, the smaller the proportion of investment in the stock market. The risk aversion coefficient measures the level of risk that fund managers are willing to accept, and when the coefficient is higher, they are more reluctant to take on risks. Managers with high risk aversion tend to gravitate towards low-risk investment options to avoid losses and preserve capital safety. They may have a preference for stable fixed-income instruments such as bonds or time deposits while reducing their investment allocation in high-risk assets such as stocks. This is because the stock market typically exhibits higher volatility and uncertainty, thereby increasing the risk and potential for losses for investors. Managers with a high degree of risk aversion prioritize capital protection against risks rather than pursuing higher returns. In summary, **Figure 1(b)** reveals the investment behavior of managers with high risk aversion when facing risks. They tend to decrease the proportion of investment in the stock market, seeking lower-risk investment alternatives. This highlights the emphasis fund managers place on risk protection in their decision-making process while balancing the pursuit of returns and capital safety.

In the case of $0<\gamma<1$, $\frac{\partial \pi_t}{\partial \mu_S}<0$, the optimal delay investment strategy decreases as the stock return rate μ_S increases, as shown in **Figure 2(a)**. In **Fig-**

ure 2(a), it is observed that the higher the stock returns, the smaller the proportion of investment in the stock market. This reflects that higher stock returns may lead risk-averse fund managers to anticipate higher risks associated with stocks. Although higher stock returns can yield higher returns, they also imply greater potential volatility and risk. Therefore, some risk-averse fund managers may reduce the proportion of investment in the stock market and seek more stable assets to mitigate risks. Additionally, higher stock returns may also indicate that the stock market is perceived to be overbought, where market prices have already exceeded reasonable levels. In such cases, managers may exercise more caution and reduce their investment proportion in the stock market. In conclusion, Figure 2(a) reveals the investment behavior of risk-averse fund managers when faced with higher stock returns. They tend to decrease the proportion of investment in the stock market, driven by concerns over higher risks or recognition of an overbought state in the stock market. This emphasizes the complexity of decision-making for risk-averse managers when balancing risks and returns. Conversely, when $\gamma > 1$ and $\frac{\partial \pi_t}{\partial \mu_S} > 0$, the optimal delay investment strategy increases as the stock return rate μ_S increases. This observation may explain the risk-seeking behavior observed during bull markets, consistent with the principles of behavioral finance. In Figure 2(b), it is observed that the larger the wealth, the smaller the proportion of investment in the stock market. This has also indirectly led to a reduction in the value of the fund (see Figure 4(b)). This reveals the investment behavior of fund managers on the stock market as wealth increases. With the growth of wealth, they tend to reduce the proportion of investment in the stock market and instead pursue asset diversification and risk diversification. This highlights the complexity faced by fund managers in considering capital protection and risk management in their decision-making processes.

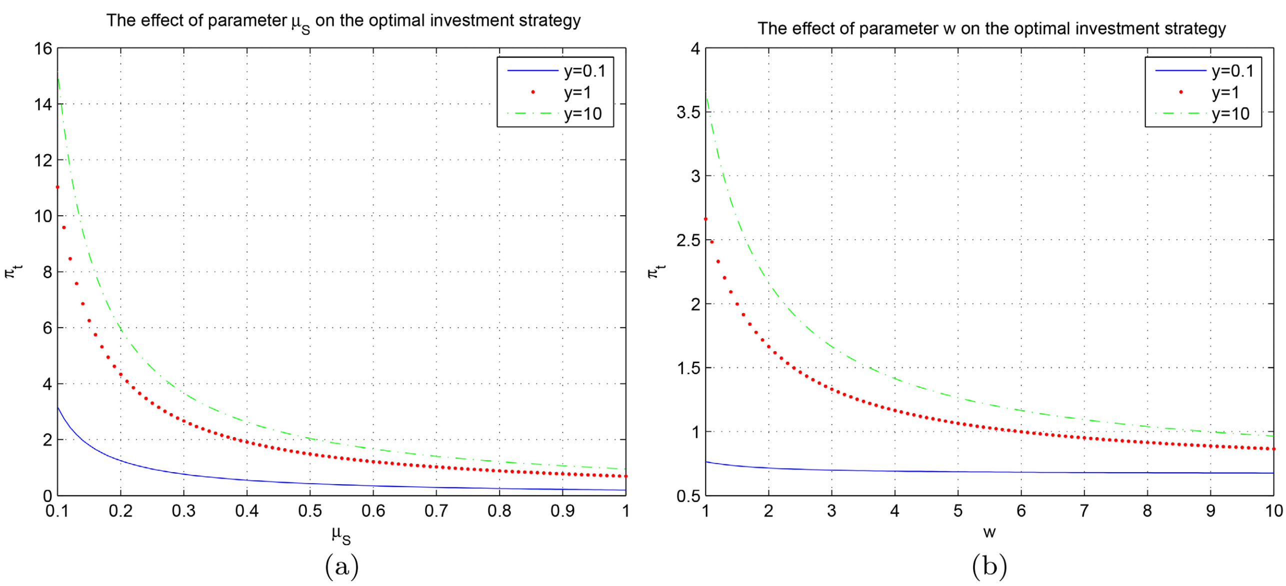


Figure 2. The effect of parameters μ_S and w on the optimal strategy.

In Figure 3(a), it is observed that as the average parameter of historical wealth information increases, the proportion of investment in the stock market initially increases and then decreases. This indicates that in the initial stage, as the average parameter of historical wealth information increases, fund managers tend to increase the proportion of investment in the stock market. This is because they believe that increasing the average parameter can enhance their understanding of the stock market and improve their ability to achieve higher returns. However, with further increases in the average parameter, they may observe a potential neutral or decreasing trend. This suggests that as the parameter increases, fund managers become less confident in the proportion of investment in the stock market and place more emphasis on capital protection and risk management, resulting in a reduction in the proportion of investment in the stock market. In conclusion, Figure 3(a) reveals the impact of the average parameter of historical wealth information on the investment behavior of fund managers in the stock market. It indicates that fund managers tend to increase the proportion of investment in the stock market in the initial stage as the parameter increases, but this proportion gradually decreases with further increases in the parameter. This emphasizes the importance of information consideration and capital protection, as well as the complexity of decision-making in balancing risks and returns for fund managers. In Figure 3(b), it is observed that the more historical wealth information is considered, the larger the proportion of investment in the stock market, which eventually stabilizes at a certain level. This indicates that as the consideration of historical wealth information increases, fund managers gain a deeper understanding of the risks and returns associated with stock market investments. Through analysis and evaluation of historical information, their comprehension of the stock market gradually deepens. Consequently, they may be more willing to allocate a larger proportion of funds to the stock market to

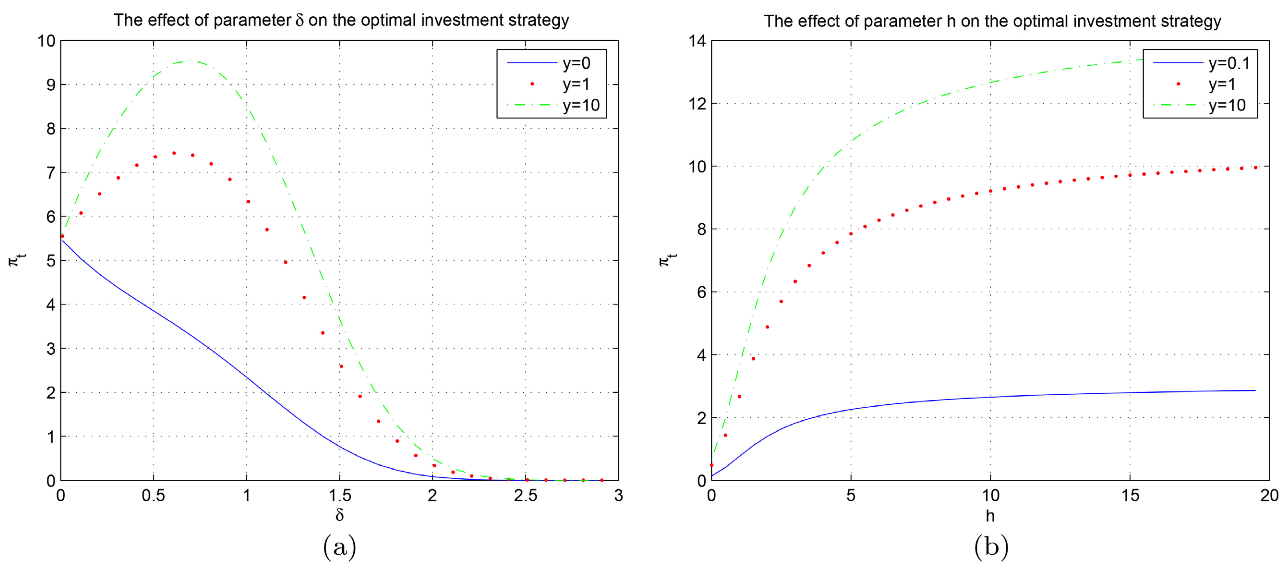


Figure 3. The effect of parameters δ and h on the optimal delay investment strategy.

pursue higher returns. After evaluating the risks and returns, they develop a clear understanding of the suitable investment proportion and maintain portfolio stability at that level. In conclusion, **Figure 3(b)** reveals the impact of historical wealth information on the investment behavior of fund managers in the stock market. It suggests that with the increase in consideration of historical wealth information, fund managers tend to increase the proportion of investment in the stock market, eventually stabilizing at a certain level. It emphasizes the importance of historical information in the decision-making process of fund managers and highlights the potential for the investment proportion to reach a relatively stable level after evaluation.

4.2. Analysis of Optimal Value Function

In this subsection, we investigate the influence of the parameters t, w, δ, h, and γ on the optimal value function.

In **Figure 4(a)**, it is observed that the fund's value function exhibits an inverted U-shaped pattern as time progresses, and considering more historical wealth information leads to a lower value function. This indicates that in the initial stages of investment, the fund faces higher uncertainty and risk, leading to a decrease in its value. However, with the accumulation of experience and improvement in investment strategies by the fund manager, the fund's value increases, resulting in higher returns. On the other hand, considering more historical information reveals greater risks and challenges, resulting in a lower value for the fund. This suggests that the additional historical information exposes more risks and consequently lowers the value of the fund. Other figures below reflect this trend. Overall, **Figure 4(a)** reveals the complex economic relationship between the fund's value function, time, and the consideration of historical wealth information. As time progresses, the fund's value function exhibits different stages of changes, while considering more historical wealth information

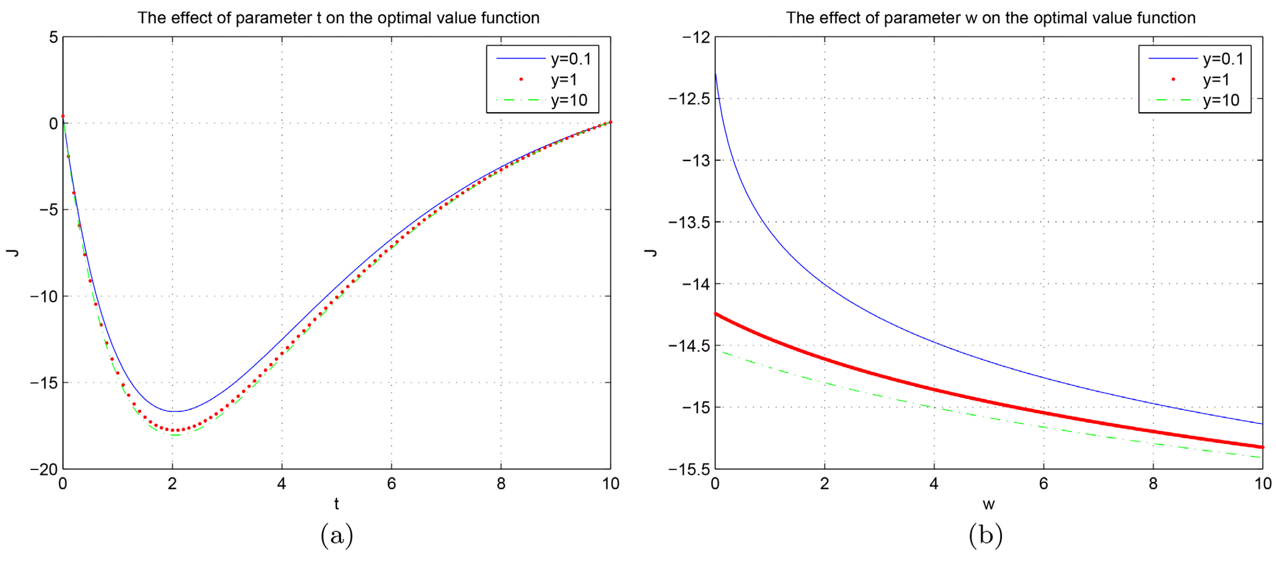
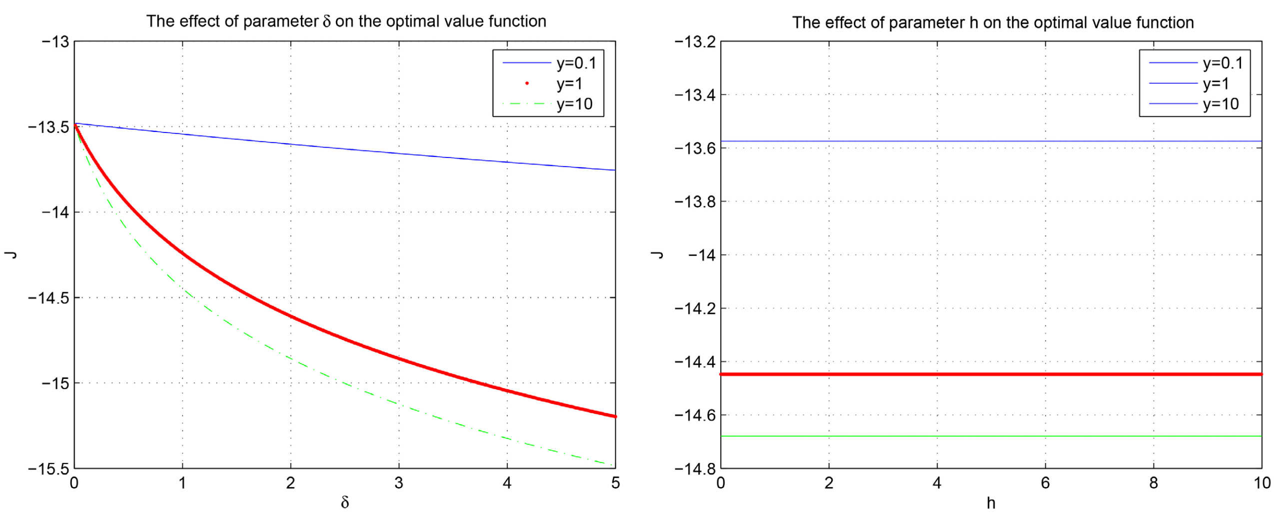


Figure 4. The effect of parameters t and w on the optimal value function.

negatively affects the fund's value. This highlights the interplay between the fund's value function, time, and historical wealth information and underscores the challenges faced by fund managers in their decision-making processes.

In Figure 5(a), we observe that the valuation of funds decreases as the average parameter considering wealth history information increases. The increase in the average parameter signifies a greater focus on historical information, which may limit the flexibility and adaptability of funds. In rapidly changing financial markets, historical data is inadequate in fully predicting future changes and risks. Excessive reliance on historical information can result in funds missing out on new opportunities or being unable to adapt to new market conditions, thus impacting their value. Additionally, the increase in the average parameter considering wealth history information may also heighten the sensitivity of funds to market risks. As the parameter increases, funds may become more susceptible to market volatility and adverse events, potentially negatively affecting their value. Overall, Figure 5(a) reveals the relationship between the value of funds and the consideration of wealth history information. When the average parameter considers wealth history information increases, the value of funds may decrease. This underscores the importance of flexibility in incorporating new information and opportunities and emphasizes the need for fund managers to weigh historical information against the uncertainty of future markets in their decision-making. From Figure 5(b), it can be observed that while historical data is crucial for understanding asset price fluctuations and market trends, the value of funds is not sensitive to the consideration of the time span for wealth historical information. This time lag parameter can only reflect one aspect of the fund value, and fund managers need to integrate other factors to comprehensively assess and make decisions in order to enhance the value of the fund.

From Figure 6(a), it can be observed that for fund managers who are risk-averse ($0<\gamma<1$), the higher the risk aversion coefficient, the greater the



(a) (b)

Figure 5. The effect of parameters δ and h on the optimal value function.

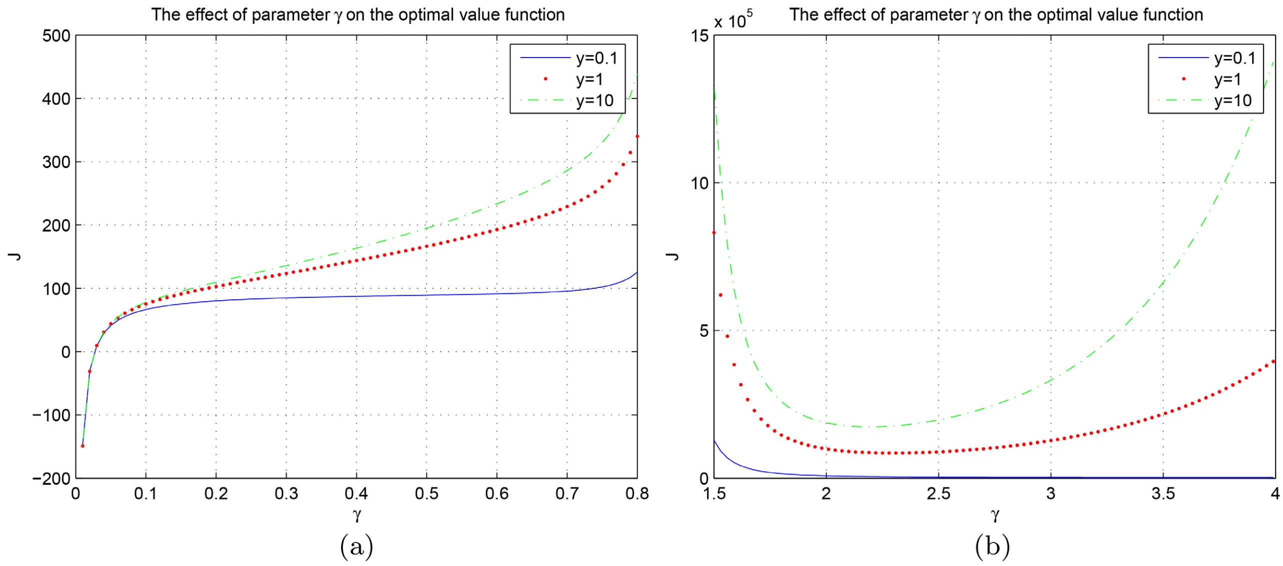


Figure 6. The effect of parameters $0<\gamma<1$ and $\gamma>1$ on the optimal value function.

value of the fund, and the more sensitive it is to risk aversion coefficients close to 0 or 1. This implies that fund managers who exhibit a more cautious attitude towards risk and are more inclined to avoid it tend to have higher-value funds. When the risk aversion coefficient approaches 0, fund managers strongly dislike risk and adopt a more conservative and asset-protective investment strategy to ensure the safety and stability of capital. This can have a positive impact on the value of the fund. Conversely, when the risk aversion coefficient approaches 1, fund managers are closer to risk neutrality, and they may choose more aggressive investment strategies in pursuit of higher returns. The higher sensitivity of fund managers to the risk aversion coefficient indicates their greater focus on risk management and return control, aiming to maximize the value of the fund. Therefore, in risk management and investment decision-making, the risk aversion coefficient is one of the primary factors that decision-makers need to consider. From Figure 6(b), it can be observed that for fund managers with risk preferences ($\gamma>1$), the value of the fund exhibits a U-shaped pattern with increasing risk aversion coefficient. This can be explained by the fact that fund managers, with higher risk aversion coefficients, tend to adopt more aggressive investment strategies to pursue higher returns. However, as the risk aversion coefficient continues to increase, fund managers' risk tolerance gradually reaches its limit, and they become unwilling to take on excessive risk. As a result, the value of the fund decreases to a minimum at a certain point and then begins to rise. This U-shaped relationship reveals the impact of adjusting risk aversion levels on the value of funds for risk-seeking fund managers. An appropriate level of risk aversion coefficient allows fund managers to seek higher returns while avoiding excessive risk, thereby maintaining a favorable value for the fund. However, excessively high or low risk aversion coefficients can lead to a decrease in the value of the fund. Therefore, fund managers need to carefully evaluate the impact of risk aversion coefficients in their investment decisions in order to maximize the value of the fund at the optimal risk level.

5. Conclusions

Suppose that in the process of fund management, if the past performance is poor, the fund manager may need to seek further financing to adjust the loss so that the final performance target can still be reached. Instead, if past performance has been good, fund managers can pay some of the proceeds to shareholders as bonuses for management or in the form of dividends. The presence of inflows and outflows related to historical information of the wealth gives rise to an optimization problem with delay. Inspired by [9] [11] and [12], we study the time-delay optimal portfolio problem under the framework of uncertainty theory. This study assumes a financial market consisting of risk-free assets and risky assets, with the price process of risky assets following a general uncertain process model. As the optimality equation in the uncertainty theory framework which is not similar to the HJB equation in the probability theory framework, does not incorporate second-order information, which may pose a challenge to the optimization problem aimed at terminal wealth, we assume that the objective function is a nonlinear function of decision variable. We hypothesize that a fund manager's primary concern lies in the immediate utility of wealth invested in the stock market during their tenure, potentially impacting the perception of their team members or leadership towards them. Furthermore, as the interest accrued from bank deposits does not contribute to their performance, we will temporarily exclude the interest gained from bank deposits from our analysis. Therefore, we let $U\left(\pi_t, W(t)\right) = \mathrm{e}^{-\rho t}\frac{1}{\gamma}\left(\pi_t W(t)\right)^{\gamma}$ in problem (18) with the boundary condition $V(T) = \mathrm{e}^{-\rho T}\left[W(T) + \delta Y(T)\right]^{\gamma}$. Subsequently, we derive the optimal equation with delay and provide the optimal investment strategy and value function. Finally, we conduct sensitivity analysis and numerical demonstrations.

The results of this study have some practical significance and guiding significance. First, in terms of the practical significance of the research findings, previous studies on delayed information within the framework of probability theory may produce unreliable results if there is limited historical data or if the data's distribution function is not sufficiently close to its frequency. However, conducting the research using the uncertainty theory can address these issues. Thus, this study contributes to and complements the existing research on delayed information within the probability theory framework. Second, in relation to the guidance implications in practical applications, 1) The findings of this study provide valuable guidance for decision-making by investors and fund managers. 2) The results of this study have significant implications for areas such as risk management, asset allocation, and investment decisions involving delayed information. Finally, concerning future improvements and expansion, we believe that within the uncertainty theory framework, the integration of delayed information with DC pension plans, insurance and reinsurance issues, and asset liability management can be explored, among other potential areas of study.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Elsanosi, I., Ø ksendal, B. and Sulem, A. (2000) Some Solvable Stochastic Control Problems with Delay. *Stochastics*: *An International Journal of Probability and Stochastic Processes*, **71**, 69-89. https://doi.org/10.1080/17442500008834259

[2] Ø'ksendal, B. and Sulem, A. (2000) A Maximum Principle for Optimal Control of Stochastic Systems with Delay, with Applications to Finance. Preprint Series. Pure Mathematics. http://urn.nb.no/URN:NBN:no-8076

[3] Elsanosi, I. and Larssen, B. (2001) Optimal Consumption under Partial Observations for a Stochastic System with Delay. Preprint Series. Pure Mathematics http://urn.nb.no/URN:NBN:no-8076

[4] Larssen, B. (2002) Dynamic Programming in Stochastic Control of Systems with Delay. *Stochastics*: *An International Journal of Probability and Stochastic Processes*, **74**, 651-673. https://doi.org/10.1080/1045112021000060764

[5] Chang, M.H., Pang, T. and Yang, Y. (2011) A Stochastic Portfolio Optimization Model with Bounded Memory. *Mathematics of Operations Research*, **36**, 604-619. https://doi.org/10.1287/moor.1110.0508

[6] Federico, S. (2011) A Stochastic Control Problem with Delay Arising in a Pension Fund Model. *Finance and Stochastics*, **15**, 421-459. https://doi.org/10.1007/s00780-010-0146-4

[7] Shi, J. (2013) Two Different Approaches to Stochastic Recursive Optimal Control Problem with Delay and Applications.

[8] Shen, Y., Meng, Q. and Shi, P. (2014) Maximum Principle for Mean-Field Jump-Diffusion Stochastic Delay Differential Equations and Its Application to Finance. *Automatica*, **50**, 1565-1579. https://doi.org/10.1016/j.automatica.2014.03.021

[9] Shen, Y. and Zeng, Y. (2014) Optimal Investment-Reinsurance with Delay for Mean-Variance Insurers: A Maximum Principle Approach. *Insurance*: *Mathematics and Economics*, **57**, 1-12. https://doi.org/10.1016/j.insmatheco.2014.04.004

[10] Shen, Y. and Zeng, Y. (2015) Optimal Investment-Reinsurance Strategy for Mean-Variance Insurers with Square-Root Factor Process. *Insurance*: *Mathematics and Economics*, **62**, 118-137. https://doi.org/10.1016/j.insmatheco.2015.03.009

[11] A, C. and Li, Z. (2015) Optimal Investment and Excess-of-Loss Reinsurance Problem with Delay for an Insurer under Hestons SV Model. *Insurance*: *Mathematics and Economics*, **61**, 181-196. https://doi.org/10.1016/j.insmatheco.2015.01.005

[12] A, C., Lai, Y. and Shao, Y. (2018) Optimal Excess-of-Loss Reinsurance and Investment Problem with Delay and Jump-Diffusion Risk Process under the CEV Model. *Journal of Computational and Applied Mathematics*, **342**, 317-336. https://doi.org/10.1016/j.cam.2018.03.035

[13] A, C. and Shao, Y. (2020) Optimal Investment and Reinsurance Problem with Delay under the CEV Model. *Operations Research Transactions*, **24**, 73-87. (In Chinese)

[14] A, C., Shen, Y. and Zeng, Y. (2022) Dynamic Asset-Liability Management Problem

in a Continuous-Time Model with Delay. *International Journal of Control*, **95**, 1315-1336. https://doi.org/10.1080/00207179.2020.1849807

[15] Wang, Z., Hodge, D.J. and Le, H. (2017) Conjugate Duality in Stochastic Controls with Delay. *Advances in Applied Probability*, **49**, 1011-1036. https://doi.org/10.1017/apr.2017.32

[16] Yang, X., Liang, Z. and Zhang, C. (2017) Optimal Mean-Variance Reinsurance with Delay and Multipleclasses of Dependent Risks. *Scientia Sinica Mathematica*, **47**, 723-756. (In Chinese) https://doi.org/10.1360/SCM-2016-0388

[17] Sheu, S.J., Sun, L.H. and Zhang, Z. (2018) Portfolio Optimization with Delay Factor Models.

[18] Pang, T. and Yong, Y. (2019) A New Stochastic Model for Stock Price with Delay Effects. In 2019 *Proceedings of the Conference on Control and Its Applications*, Society for Industrial and Applied Mathematics, Philadelphia, 110-117. https://doi.org/10.1137/1.9781611975758.17

[19] Wang, S., Rong, X. and Zhao, H. (2019) Optimal Time-Consistent Reinsurance-Investment Strategy with Delay for an Insurer under a Defaultable Market. *Journal of Mathematical Analysis and Applications*, **474**, 1267-1288. https://doi.org/10.1016/j.jmaa.2019.02.016

[20] Deng, C., Bian, W. and Wu, B. (2020) Optimal Reinsurance and Investment Problem with Default Risk and Bounded Memory. *International Journal of Control*, **93**, 2982-2994. https://doi.org/10.1080/00207179.2019.1573320

[21] Zhang, Y. and Zhao, P. (2019) Robust Optimal Excess-of-Loss Reinsurance and Investment Problem with Delay and Dependent Risks. *Discrete Dynamics in Nature and Society*, **2019**, Article ID: 6805351. https://doi.org/10.1155/2019/6805351

[22] Li, S. and He, Y. (2020) Optimal Time-Consistent Investment and Reinsurance Strategy under Time Delay and Risk Dependent Model. *Mathematical Problems in Engineering*, **2020**, Article ID: 9368346. https://doi.org/10.1155/2020/9368346

[23] Li, S. and Qiu, Z. (2021) Optimal Time-Consistent Investment and Reinsurance Strategies with Default Risk and Delay under Hestons SV Model. *Mathematical Problems in Engineering*, **2021**, Article ID: 8834842. https://doi.org/10.1155/2021/8834842

[24] He, X., Gu, A. and Yao, H. (2022) Stochastic Differential Reinsurance and Investment Games with Delay under VaR Constraints? *Communications in Statistics—Theory and Methods*. https://doi.org/10.1080/03610926.2022.2103149

[25] Llemit, D. and Escaner IV, J.M. (2021) Value Functions in a Regime Switching Jump Diffusion with Delay Market Model. *AIMS Mathematics*, **6**, 11595-11609. https://doi.org/10.3934/math.2021673

[26] Zhu, H., Zhong, H. and Bin, N. (2021) Non-Zero-Sum Investment and Reinsurance Game with Delay Effect and Mean-Variance Utility. *Operations Research Transactions*, **25**, 35-54. (In Chinese)

[27] Zhao, Y., Mi, H. and Xu, L. (2022) Robust Optimal Investment Problem with Delay under Hestons Model. *Methodology and Computing in Applied Probability*, **24**, 1271-1296.

[28] Ma, A., Zhang, C. and Wang, Y. (2023) Optimal Consumption and Investment Problem under 4/2-CIR Stochastic Hybrid Model. *Mathematics*, **11**, Article No. 3695. https://doi.org/10.3390/math11173695

[29] Liu, B. (2004) Uncertainty Theory: An Introduction to Its Axiomatic Foundation. Physica-Verlag, Heidelberg.

[30] Liu, B. (2010) Uncertainty Theory: A Branch of Mathematics for Modeling Human Uncertainty. Springer-Verlag, Berlin.

[31] Liu, B. (2020) Uncertainty Theory. 5th Edition, Springer, Berlin.

[32] Zhu, Y. (2010) Uncertain Optimal Control with Application to a Portfolio Selection Model. *Cybernetics and Systems*: *An International Journal*, **41**, 535-547. https://doi.org/10.1080/01969722.2010.511552

[33] Deng, L. and Zhu, Y. (2012) Uncertain Optimal Control with Jump. *ICIC Express Letters*, *Part B*: *Applications*, **3**, 419-424.

[34] Jiang, Y., Yan, H. and Zhu, Y. (2016) Optimal Control Problem for Uncertain Linear Systems with Multiple Input Delays. *Journal of Uncertainty Analysis and Applications*, **4**, Article No. 5. https://doi.org/10.1186/s40467-016-0048-x

[35] Shu, Y. and Zhu, Y. (2018) Optimal Control for Multi-Stage and Continuous-Time Linear Singular Systems. *International Journal of Systems Science*, **49**, 1419-1434. https://doi.org/10.1080/00207721.2018.1454534

[36] Shu, Y. (2021) Optimal Control for Discrete-Time Descriptor Noncausal Systems. *Asian Journal of Control*, **23**, 1885-1899. https://doi.org/10.1002/asjc.2343

[37] Gu, Y. and Zhu, Y. (2022) Optimal Control for Parabolic Uncertain System Based on Wavelet Transformation. *Axioms*, **11**, Article No. 453. https://doi.org/10.3390/axioms11090453

[38] Chen, X. and Jin, T. (2023) Optimal Control for a Multistage Uncertain Random System. *IEEE Access*, **11**, 2105-2117. https://doi.org/10.1109/ACCESS.2023.3234068

[39] Chen, X., Li, F., Yuan, D., Wang, J. and Shao, Y. (2023) Optimal Control Problems Subject to Uncertain Random Discrete-Time Noncausal Systems. *Chaos*, *Solitons & Fractals*, **173**, Article ID: 113604. https://doi.org/10.1016/j.chaos.2023.113604

[40] Gao, J. and Wu, Y. (2018) Optimal Portfolio Strategy of a Defined Contribution Pension Plan under the Uncertainty Theory. *Mathematics in Practice and Theory*, **48**, 97-106.

[41] Liu, Y., Yang, M., Zhai, J. and Bai, M. (2018) Portfolio Selection of the Defined Contribution Pension Fund with Uncertain Return and Salary: A Multi-Period Mean-Variance Model. *Journal of Intelligent & Fuzzy Systems*, **34**, 2363-2371. https://doi.org/10.3233/JIFS-171440

[42] Liu, W., Wu, W., Tang, X. and Hu, Y. (2022) Optimal Control of Background-Based Uncertain Systems with Applications in DC Pension Plan. *Entropy*, **24**, Article No. 734. https://doi.org/10.3390/e24050734

[43] Wang, X. and Zhu, Y. (2022) Optimal Control of Defined Contribution Pension Plan under Uncertain Optimistic Value Criterion. *International Mathematical Forum*, **17**, 25-32. https://doi.org/10.12988/imf.2022.912302

[44] Liu, B. (2009) Some Research Problems in Uncertainty Theory. *Journal of Uncertain Systems*, **3**, 3-10.

[45] Chen, X. and Liu, B. (2010) Existence and Uniqueness Theorem for Uncertain Differential Equations. *Fuzzy Optimization and Decision Making*, **9**, 69-81. https://doi.org/10.1007/s10700-010-9073-2

Appendix A: The Proof of Lemma 1

Let $u = t + s$, then,

$$\begin{aligned} F(t) &= \int_{-h}^{0} \mathrm{e}^{\delta s} H(t+s)\,\mathrm{d}s \\ &= \int_{t-h}^{t} \mathrm{e}^{\delta(u-t)} H(u)\,\mathrm{d}u \\ &= \mathrm{e}^{-\delta t}\left[\int_{0}^{t} \mathrm{e}^{\delta u} H(u)\,\mathrm{d}u - \int_{0}^{t-h} \mathrm{e}^{\delta u} H(u)\,\mathrm{d}u\right] \end{aligned} \tag{27}$$

By the derivation method of the integral upper limit function, it can be obtained

$$\begin{aligned} \frac{\mathrm{d}F(t)}{\mathrm{d}t} &= \frac{\mathrm{d}\left(\mathrm{e}^{-\delta t}\right)}{\mathrm{d}t}\left[\int_{0}^{t} \mathrm{e}^{\delta u} H(u)\,\mathrm{d}u - \int_{0}^{t-h} \mathrm{e}^{\delta u} H(u)\,\mathrm{d}u\right] \\ &\quad + \mathrm{e}^{-\delta t}\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{0}^{t} \mathrm{e}^{\delta u} H(u)\,\mathrm{d}u - \int_{0}^{t-h} \mathrm{e}^{\delta u} H(u)\,\mathrm{d}u\right] \\ &= -\delta \mathrm{e}^{-\delta t}\left[\int_{0}^{t} \mathrm{e}^{\delta u} H(u)\,\mathrm{d}u - \int_{0}^{t-h} \mathrm{e}^{\delta u} H(u)\,\mathrm{d}u\right] \\ &\quad + \mathrm{e}^{-\delta t}\left[\mathrm{e}^{\delta t} H(t) - \mathrm{e}^{\delta(t-h)} H(t-h)\right] \\ &= H(t) - \delta F(t) - \mathrm{e}^{-\delta h} H(t-h). \end{aligned} \tag{28}$$

Appendix B: The Proof of Lemma 2

Firstly, we solve the ODE

$$f'(t) - kf(t) = 0. \tag{29}$$

It is easy to get

$$f(t) = P\mathrm{e}^{kt}, \tag{30}$$

where P is any real number.

Secondly, transform P into a function of t, and assume that

$$f(t) = P(t)\mathrm{e}^{kt}. \tag{31}$$

Taking the derivative, we have

$$f'(t) = P'(t)\mathrm{e}^{kt} + kP(t)\mathrm{e}^{kt}. \tag{32}$$

Substituting Equations (31) and (32) into Equation (21) yields that $P'(t) = \mathrm{e}^{-kt} g(t)$. Further, it is easy to see that

$$P(t) = P(T) - \int_{t}^{T} \mathrm{e}^{-ks} g(s)\,\mathrm{d}s. \tag{33}$$

Plugging Equation (33) into Equation (31), we have

$$f(t) = \left[P(T) - \int_{t}^{T} \mathrm{e}^{-ks} g(s)\,\mathrm{d}s\right]\mathrm{e}^{kt}. \tag{34}$$

Taking $t = T$ yields that $P(T) = f(T)\mathrm{e}^{-kT}$.

Therefore, we obtain

$$\begin{aligned} f(t) &= \left[f(T)\mathrm{e}^{-kT} - \int_{t}^{T} \mathrm{e}^{-ks} g(s)\,\mathrm{d}s\right]\mathrm{e}^{kt} \\ &= f(T)\mathrm{e}^{-k(T-t)} - \int_{t}^{T} \mathrm{e}^{-k(s-t)} g(s)\,\mathrm{d}s. \end{aligned} \tag{35}$$

Appendix C: The Proof of Theorem 6

By the first-order conditions of (20), we obtain the optimal solution for problem (18)

$$\pi_t = \frac{1}{w}\left[e^{\rho t} J_w \left(\mu_S - r_0\right)\right]^{\frac{1}{\gamma-1}}. \tag{36}$$

Substituting Equation (36) into Equation (20), we get

$$\begin{aligned} 0 = J_t &- \frac{1}{\gamma} e^{-\rho t}\left[e^{\rho t} J_w \left(\mu_S - r_0\right)\right]^{\frac{\gamma}{\gamma-1}} + J_w \left(\mu_S - r_0\right)\left[e^{\rho t} J_w \left(\mu_S - r_0\right)\right]^{\frac{1}{\gamma-1}} \\ &+ J_w \left(G_0 w + \bar{G}_1 y + G_2 z\right) + J_y \left(w - \delta y - e^{-\delta h} z\right). \end{aligned} \tag{37}$$

We conjecture that $J(t,w,y) = \left[A(t)(w+\delta y)^{\gamma} + B(t)\right]e^{-\rho t}$, then

$$\begin{aligned} J_t &= \left[A'(w+\delta y)^{\gamma} + B' - A\rho(w+\delta y)^{\gamma} - B\rho\right]e^{-\rho t}, \\ J_w &= A\gamma(w+\delta y)^{\gamma-1} e^{-\rho t}, \\ J_y &= A\gamma\delta(w+\delta y)^{\gamma-1} e^{-\rho t}. \end{aligned} \tag{38}$$

Substituting Equation (38) into Equation (36), we have

$$\pi_t w = \left[A\gamma\left(\mu_S - r_0\right)\right]^{\frac{1}{\gamma-1}}(w+\delta y). \tag{39}$$

By substituting Equation (38) into Equation (37), we have

$$\begin{aligned} 0 = e^{-\rho t}\Big\{&\left[A' - A\left(\rho - \gamma\left(G_0 + \delta\right)\right)\right](w+\delta y)^{\gamma} \\ &B' - B\rho - A^{\frac{\gamma}{\gamma-1}}\left(\gamma\left(\mu_S - r_0\right)\right)^{\frac{1}{\gamma-1}}\left(\mu_S - r_0 - \gamma\right)(w+\delta y)^{\gamma} \\ &+ A\gamma(w+\delta y)^{\gamma-1}\left[\left(\bar{G}_1 - \delta^2\right)y + \left(G_2 - \delta e^{-\delta h}\right)z\right]\Big\}. \end{aligned} \tag{40}$$

By assumption condition (16), *i.e.*, $\bar{G}_1 - \delta^2 = 0$ and $G_2 - \delta e^{-\delta h} = 0$, the upper formula becomes

$$\begin{aligned} 0 = &\left[A' - A\left(\rho - \gamma\left(G_0 + \delta\right)\right)\right](w+\delta y)^{\gamma} \\ &+ B' - B\rho - A^{\frac{\gamma}{\gamma-1}}\left(\gamma\left(\mu_S - r_0\right)\right)^{\frac{1}{\gamma-1}}\left(\mu_S - r_0 - \gamma\right)(w+\delta y)^{\gamma}. \end{aligned} \tag{41}$$

In order to ensure the above formula to be true for any w and y, we assume that

$$A' - A\left(\rho - \gamma\left(G_0 + \delta\right)\right) = 0, \tag{42}$$

and

$$B' - B\rho = A^{\frac{\gamma}{\gamma-1}}\left(\gamma\left(\mu_S - r_0\right)\right)^{\frac{1}{\gamma-1}}\left(\mu_S - r_0 - \gamma\right)(w+\delta y)^{\gamma}. \tag{43}$$

From Equation (42) we can derive that

$$A(t) = e^{\left[\rho - \gamma\left(G_0 + \delta\right)\right](T-t)}. \tag{44}$$

Substituting Equation (44) into Equation (43), the ODE (43) can be expressed as

$$B'(t)-B(t)\rho=g(t), \tag{45}$$

where

$$g(t)=\mathrm{e}^{\frac{\gamma}{\gamma-1}[\rho-\gamma(G_0+\delta)](T-t)}\left(\gamma\left(\mu_S-r_0\right)\right)^{\frac{1}{\gamma-1}}\left(\mu_S-r_0-\gamma\right)\left(w+\delta y\right)^{\gamma}. \tag{46}$$

By Lemma 1, we have

$$B(t)=B(T)\mathrm{e}^{-\rho(T-t)}-\int_t^T \mathrm{e}^{-\rho(s-t)}g(s)\mathrm{d}s. \tag{47}$$

Notice that Equation (16) $\bar{G}_1=\delta^2$, $G_2=\delta\mathrm{e}^{-\delta h}$, and $\bar{G}_1=\dfrac{G_1}{\frac{1}{h}\int_{-h}^{0}\mathrm{e}^{\delta s}\mathrm{d}s}$, we have

$$\begin{aligned}
&G_1=-\frac{\mathrm{e}^{-\delta h}-1}{h}\delta^3,\\
&G_0=r_0-\mathrm{e}^{-\delta h}\delta+\frac{\mathrm{e}^{-\delta h}-1}{h}\delta^3.
\end{aligned} \tag{48}$$

Plugging G_0 into Equations (44) and (46) yield

$$\begin{aligned}
&A(t)=\mathrm{e}^{\kappa(T-t)},\\
&g(t)=\mathrm{e}^{\frac{\gamma}{\gamma-1}\kappa(T-t)}\left(\gamma\left(\mu_S-r_0\right)\right)^{\frac{1}{\gamma-1}}\left(\mu_S-r_0-\gamma\right)\left(w+\delta y\right)^{\gamma}.
\end{aligned} \tag{49}$$

where

$$\kappa=\rho-\gamma\left[r_0-\delta\left(1-\frac{\delta^2}{h}\right)\left(\mathrm{e}^{-\delta h}-1\right)\right]. \tag{50}$$

Combining Equation (39) with Equation (41) yields

$$\pi_t=\left[\mathrm{e}^{\kappa(T-t)}\gamma\left(\mu_S-r_0\right)\right]^{\frac{1}{\gamma-1}}\left(1+\delta\frac{y}{w}\right), \tag{51}$$

Plugging Equations (49) and (41) in to $J(t,w,y)=\left[A(t)(w+\delta y)^{\gamma}+B(t)\right]\mathrm{e}^{-\rho t}$, we get the optimal value function

$$J(t,w,y)=\mathrm{e}^{(\kappa+\rho)(T-t)}\mathrm{e}^{-\rho T}\left(w+\delta y\right)^{\gamma}+\mathrm{e}^{-\rho T}B(T)-\int_t^T \mathrm{e}^{-\rho s}g(s)\mathrm{d}s. \tag{52}$$

By the boundary condition $V(T)=\mathrm{e}^{-\rho T}\left[W(T)+\frac{\delta}{h}\left(1-\mathrm{e}^{-\delta h}\right)\bar{Y}(T)\right]^{\gamma}=\mathrm{e}^{-\rho T}\left[W(T)+\delta Y(T)\right]^{\gamma}$ of problem (18), we have $B(T)=0$. Therefore, the value function can be rewritten as

$$J(t,w,y)=\mathrm{e}^{(\kappa+\rho)(T-t)}\mathrm{e}^{-\rho T}\left(w+\delta y\right)^{\gamma}-\int_t^T \mathrm{e}^{-\rho s}g(s)\mathrm{d}s. \tag{53}$$

The proof is completed. □